Solution of Gas Pulsation Equations in Compressor Suction and Discharge Lines by Mixed-Mode Approach

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INTRODUCTION

In simulation and design of reciprocating compressors, the gas pulsations in the suction and discharge lines are important factors. The importance is evidenced by the fact that two sessions of the 1974 Purdue Compressor Technology Conference were devoted to this subject, and a total of 12 papers were presented on various phases of this topic. Singh and Soedel [1] presented two papers in which they reviewed the analyses of compressor lines pulsation. They discuss briefly various solution techniques under two general categories of lumped and distributed parameter approach. One of the techniques listed under the second category is the so-called method of characteristics. This paper deals with a modified method of characteristics by employing a hybrid approach.

No single method of analysis or technique of solution can be said to be more advantageous than other techniques. Some methods are more complicated than others; and some yield more accurate results. The investigator or the engineer simply weighs the marginal benefit accrued from employing a more complicated formulation against the marginal effort needed to solve such a formulation, and bases his decision upon this intuitive cost-benefit analysis. Here, we wish to show how to overcome some of the difficulties associated with the adoption of a set of non-linear hyperbolic partial differential equations as representing gas pulsation formulation. For small compressors approximate techniques to assess the gas pulsation effects are appropriate. However, when large compressors are under study, exact techniques are warranted in the solution of unsteady gas flow equations in spite of the extra effort required. This modified method of characteristics is such an exact technique.

EXACT METHODS

There are two basic procedures which are used to provide exact solutions to the non-linear hyperbolic partial differential equations of unsteady gas pulsation in compressors. They are the method of characteristics [2] and the finite-difference method [3,4]. A purely characteristic method finds special curves in the solution domain, called characteristic curves, along which partial differential equations reduce to ordinary differential equations. Certain relations are satisfied between the dependent and independent variables which are sufficient to determine a solution in the domain of interest. The characteristics method does not lend itself easily to digital computation. The main difficulty lies in the fact that the computation front is irregular, and a solution at any instant of time thus requires interpolation which may often prove formidable. Additional complications arise in the case of general nonhomentropic flows, where the three sets of characteristics (derived from the three main equations of continuity, momentum and energy) do not commonly intersect.

Numerical integration based on the finite-difference method would appear to be the natural choice for solution on digital computers, but the representation of shock waves with a finite difference method leads to difficulties. Moreover, there is no correspondence between the physical process and the computation procedure, and the troubles that arise can be traced to this fact.

Here, the best aspects of the characteristics and the finite-difference methods have been combined in a mixed-mode or hybrid approach. The difficulty in numerical interpretation inherent in the method of characteristics can be overcome by recourse to interpolation. A rectangular grid may be superimposed on the characteristic node and the values of dependent variables interpolated from the nodes to grid points. The computation can then proceed systematically [5].

In each method a choice is made on the selection of dependent variables. Some researchers employ the finite-difference method and make use of density, velocity and pressure as dependent variables [6]; while others, including O'Shea [7], employ the hybrid method and make use of the so-called Riemann variables as dependent variables. The former set of variables is suitable for purely finite-difference methods, where the actual variables are used; while the Riemann variables are best suited to the graphical method of characteristics. The Riemann variables, each, are functions of velocity and speed of sound. This double dependence creates problems when a numerical integration is attempted.

Originally the present author employed O'Shea's approach. The results obtained for an almost instant opening and closing of the ports or valves were satisfactory [8]; but when actual port or valve opening and closing times were incorporated, difficulties were encountered due to the problem in the convergence of boundary condition equations. The author, then, adopted a modified set of characteristic variables originally suggested by Matthews [9]. Each variable is only a function of exactly one fluid property, namely pressure, entropy or velocity. The problems associated with other techniques were eliminated by this approach [5].

BASIC EQUATIONS

The physical process occurring in a compressor suction or discharge line is that of unsteady compressible fluid flow. The conservation laws of mass, momentum and energy, restricted to one-dimensional flow, are applied to this process. The one-dimensional approach used in this analysis is justified by the fact that the length to hydraulic diameter ratio of compressor lines is usually very high and the viscosity of the working fluids (refrigerants or air) is normally so small that the transverse effects are negligible compared to the longitudinal ones.

The general equations of mass, momentum, and energy conservation for a non-steady one-dimensional flow of an ideal gas are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u.u + p) = -f \tag{2}$$

$$\frac{\partial}{\partial t}\left[\rho\left(h + \frac{u^2}{2} - \frac{p}{\rho}\right)\right] + \frac{\partial}{\partial x}\left[\rho u\left(h + \frac{u^2}{2}\right)\right] = q \tag{3}$$

Here, ρ, u, p and h have their usual fluid dynamics interpretation of density, velocity, pressure and specific enthalpy; while f is the friction force per unit volume, and q is the heat transfer to unit volume of fluid. These equations are a set of three non-linear hyperpolic partial differential equations with two independent variables, x and t. Mathematically speaking, such equations have no closed-form solution, but the existence and uniqueness of their solution have been proved provided that appropriate boundary conditions are specified [5].

Equations (1), (2), and (3), together with the following thermodynamic relations

$$a^2 = \gamma RT \tag{4}$$

$$p = \rho RT \tag{5}$$

$$Tds = dh - \frac{dp}{\rho} \tag{6}$$

$$h = C_p T + h_r \tag{7}$$

Constitute the system of governing equations. If three operators S, M, and N are defined

$$S = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{8}$$

$$M = S - a\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x} \tag{9}$$

$$N = S + a\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x} \tag{10}$$

the three conservation equations can be written in the following forms:

$$Mm = D - E\frac{\partial s}{\partial x} + G \tag{11}$$

$$Nn = D + E\frac{\partial s}{\partial x} - G \tag{12}$$

$$Ss = F \tag{13}$$

where

$$m = a - \frac{\gamma-1}{2}u \tag{14}$$

$$n = a + \frac{\gamma-1}{2}u \tag{15}$$

$$D = \frac{a}{p}\,\frac{\gamma-1}{2}(q + uf) \tag{16}$$

$$E = a^2/2\,C_p \tag{17}$$

$$F = R(q + uf)/p \tag{18}$$

$$G = \frac{a^2}{p}\left(\frac{\gamma-1}{2\gamma}\right)f \tag{19}$$

Equations (11) through (13) are in the characteristic coordinate system relevant to the well-known "method of characteristics." The operators M, N and S are directional derivative operators along curves in the (x,t) plane with slopes of 1/(u-a), 1/(u+a), and 1/u respectively. These curves are the characteristic curves; m and n are the so-called Riemann variables, and s is the entropy. Modified characteristic variables are defined in such a way that (unlike m, n, and s) each modified variable is a function of only one flow parameter:

$$U = \frac{\gamma-1}{2}u \tag{20}$$

$$P = (p)^{\frac{\gamma-1}{2\gamma}} \tag{21}$$

$$\sigma = \exp\left[(s-s_*)/2C_p\right] \tag{22}$$

where s_* is a constant, so chosen that:

$$a = P\sigma \quad (23)$$

The three conservation differential equations can now be written compactly in terms of the new variables as:

$$M(P\sigma - U) = D - E\frac{\partial s}{\partial x} + G \quad (24)$$

$$N(P\sigma + U) = D + E\frac{\partial s}{\partial x} - G \quad (25)$$

$$S\sigma = \sigma F \quad (26)$$

where

$$P\sigma - U = m \quad (27)$$

$$P\sigma + U = n \quad (28)$$

<u>Particular Cases of Equations</u>

The quantities D, G, and F are only functions of friction and heat transfer effects. When these effects are neglected, the conservation equations (24) through (26) become:

$$M(P\sigma - U) = -E\frac{\partial s}{\partial x} \quad (29)$$

$$N(P\sigma + U) = E\frac{\partial s}{\partial x} \quad (30)$$

$$S\sigma = 0 \quad (31)$$

In this paper, the effects of friction and heat transfer have been neglected, and hence equations (29) through (31) are the governing equations.

BOUNDARY CONDITIONS

A compressor valve may be closed, partly open, or fully open. If it is open, the flow may be directed either into or out of the cylinder, and may be wholly or partly subsonic. The equations for the five simple cases are listed below. For more complicated cases of the boundary conditions for partly open valves, the reader is referred to reference [5].

(i) Closed valve:

$$U = 0 \quad (32)$$

(ii) Subsonic outflow through a fully-open valve:

$$P = P_{ex} \quad (33)$$

(iii) Sonic outflow through a fully-open valve:

$$U = \frac{(\gamma-1)}{2} P\sigma \quad (34)$$

(iv) Subsonic inflow through a fully-open valve:

$$U = \sigma_{ex}\sqrt{\frac{(\gamma-1)}{2}(P_{o,ex}-P)} \quad (35)$$

(v) Sonic inflow through a fully-open valve:

$$P = P_{o,ex}\sqrt{\frac{2}{\gamma+1}} \quad (36)$$

$$U = P_{o,ex}\,\sigma_{ex}\,\frac{(\gamma-1)}{2}\sqrt{\frac{2}{\gamma+1}} \quad (37)$$

HYBRID METHOD OF INTEGRATION

As mentioned earlier, this method utilizes the advantageous aspects of the finite-difference and modified characteristics method. The dependent variables are chosen as P, U, and σ; and equations (29) through (31) have to be integrated. Figure 1 provides an illustration of the geometrical relations between the points which have to be considered during a forward step in time. Figure 1a concerns a point in the central part of the grid, while Figure 1b represents a point near the left-hand boundary.

For a central-region point (Figure 1a), the task is to calculate the values of P, U, and σ at point P, given the values of these variables at points A, B and C. First the characteristic lines DP, EP and FP are located, having the slopes $1/(u+a)$, $1/u$, and $1/(u-a)$, respectively. Then the values of P, U, and σ for the points D, E and F are obtained from those at points A, B, and C by interpolation. Finally, the variations in the following quantities are calculated: (a) the change in the value of $(P\sigma + U)$ along the line DP, by integrating the right-hand side of equation (30); (b) the change in the value of $(P\sigma - U)$ along the line FP, by integrating the right-hand side of equation (29); and (c) the change in the value of σ along the line EP, by integrating the right-hand side of equation (31). From the resulting values, the desired values of P, U, and σ for point P are obtained.

For boundaries, the procedure is modified by the fact that part of the information implied by the above relations is replaced by information deriving from boundary conditions. Consider outflow to the left (Figure 1b); instead of knowledge of $(P\sigma + U)$ transmitted by the DP characteristic, the appropriate P versus U relations corresponding to the relevant boundary condition equations would replace the information derived from the change in the value of $(P\sigma + U)$ along the DP characteristic.

<u>The Main Steps in Calculations</u>

It is easy to see what operations have to be performed in order that the calculation should be advanced by one time interval. The main ones are listed here:

(i) Calculate the slopes of the characteristics for each point. These can be estimates only because, firstly, u and a are not uniform over the space and time intervals; and secondly, the u and a values which prevail at P cannot be known until the forward step has been completed.

(ii) Choose the value of the time interval Δt.

This interval should not be so large that the "heels" of the characteristics, D, E and F, lie outside the range AC.

(iii) Calculate the positions of points D, E and F.

(iv) Calculate the values of P, U and σ at D, E and F by interpolation.

(v) Calculate the increments of $(P\sigma + U)$, σ, and $(P\sigma - U)$ from the relevant equations.

(vi) Hence, calculate the new values of P, U and σ for all points P, directly if they lie in the center of the grid, or with the aid of boundary condition formulae if they lie on the extremities of the suction or discharge lines. In the latter case, iteration is often necessary.

Integration of Governing Equations

To obtain flow parameters for point P (Figure 1a), equations (29) through (31) have to be integrated along their respective characteristics. The integration yields:

$$P_P\sigma_P - U_P = P_F\sigma_F - U_F - \int_F^P E \frac{\partial s}{\partial x} dt \tag{38}$$

$$P_P\sigma_P + U_P = P_D\sigma_D - U_D - \int_D^P E \frac{\partial s}{\partial x} dt \tag{39}$$

$$\sigma_P = \sigma_E \tag{40}$$

The expression for the integral could easily be obtained, because the definitions reveal:

$$\int_D^P E \frac{\partial s}{\partial x} dt \cong \int_D^P P^2\sigma \frac{\partial \sigma}{\partial x} dt =$$

$$\int_D^P (P\sigma)P \frac{\partial \sigma}{\partial x} \left(\frac{dt}{dx}\right)dx \tag{41}$$

and $(dt/dx) \cong 1/a = 1/P\sigma$ (42)

hence

$$\int_D^P E \frac{\partial s}{\partial x} dt \cong \int_D^P P d\sigma = \overline{P}(\sigma_P - \sigma_D) \tag{43}$$

But the appropriate average $\overline{P}$ is Pp; therefore, equations (38) through (40) take the integrated forms of:

$$P_P\sigma_P - U_P = P_F\sigma_F - U_F - P_P\,(\sigma_P - \sigma_F) \tag{44}$$

$$P_P\sigma_P + U_P = P_D\sigma_D + U_D + P_P\,(\sigma_P - \sigma_D) \tag{45}$$

$$\sigma_P = \sigma_E \tag{46}$$

These three relations could be solved simultaneously to yield the values of P, U and σ at point P, given the values of these parameters at points D, E and F, which are easily obtained by interpolation along the ABC line. Once P, U, and σ are known, the values of different fluid variables could be computed by equations (20) through (23). Another time step Δt could now be taken and the solution proceeds as far as desired.

Stability Considerations

The characteristic method of solution in which the solution proceeds along the characteristic curves is unconditionally stable. The stability of the hybrid method is, however, conditional. The Courant-Friedrichs-Lewy stability criterion is given by reference [10] as:

$$\frac{\Delta t}{\Delta x} \leqslant \frac{1}{(a + |u|)_{max}} \tag{47}$$

This condition ensures that point P (Figure 1a) lies inside the zone of dependence of line AC. In this case, the PD, PE and PF characteristics will surely have their "heels" in the interval AC. The calculation of the time step Δt and hence the speed by which the solution could proceed is dependent on relation (47).

The Computer Program

The program is divided into ten subroutines to facilitate the jobs of both user and programmer. It was written in Fortran IV language and was executed on an IBM Model 50 computer. A sample program for a basic unsteady flow process is given by Hai [5].

APPLICATIONS

The algorithm presented in this paper is of such a general nature that it can be used for a variety of unsteady incompressible flow problems. In fact, the author had the opportunity to apply the above numerical scheme to the safety analysis of nuclear reactor vessels under hypothetical core disruptive accidents (HCDA). He was able to predict the pressure loadings on the FFTF (Fast Flux Test Facility) pressure vessel penetration seals due to an HCDA [11].

Another application of the above algorithm is in the performance analysis of dynamic pressure exchangers. A pressure exchanger is a direct energy exchange device that utilizes one-dimensional wave action for the transfer of mechanical energy between two gas flows which are at different pressure levels. The primary (driving) gas exerts its pressure forces directly to compress the secondary (driven) gas. In effect, the DPE contains in one unit the mechanisms of a conventional compressor-turbine set. The DPE usually consists of a cylindrical rotor with a plurality of straight axial cells arranged uniformly around its periphery. The rotor rotates between two stators, each of which has a specified number of ports to accomodate the interacting gases, and each is connected to the appropriate ducting (see Figure 2). For example, dividers are three-port pressure exchangers and have a number of practical applications. Advantage can be taken of their high pressure (HP) stream in pressure-boosting operations for air or gas compression plants or on the output of steam boilers [12]. Advantage can also be taken from the cooling potential of the low pressure (LP) stream for applications such as the cooling of high speed aircrafts by the use of ram air after expansion in dividers. In general, the use of dividers has been suggested in all pressure-boosting applications where erosion problems hinder the use of turbo-machines. A pressure exchanger divider can also be used as an "expansion engine" in a vapor-compression refrigerator in which case good use is made of the erosion resistance potentialities. Another suggestion would be the use of the divider as a device for alleviating pressure losses in natural gas distribution mains [13]. Figure 3 shows a divider simplified wave diagram with the circuit diagram of a compressor-turbine set.

The performance analysis of dividers have been obtained by the numerical scheme described in this paper. These have been discussed in greater detail in reference [13]. Here, two examples are given and the results are compared with those of Kentfield [14]. Figure 4 indicates that lines of constant M_M have the same qualitative variation as those given by Kentfield. Also, for the same P_{HO}/P_{MO}, the values of P_{LO}/P_{MO} predicted by the present analysis are seen to lie between the values given by Kentfield's approximate method and his isentropic analysis for an infinitely narrow cell. Here, M_M, P_{HO}, P_{LO}, and P_{MO} denote the Mach number in the medium pressure port, and the stagnation (or total) pressure in the high, low and medium pressure ports, respectively. Figure 5 shows that, for constant M_M and constant P_{HO}/P_{MO} (i.e., when the divider operates as a pressure-boosting device), the prediction of the values of β ($\beta = \dot{m}_H/\dot{m}_M$, where $\dot{m}_H$ and $\dot{m}_M$ are the mass flow rates through the high and medium pressure ports, respectively) by the present analysis are somewhat lower than those of Kentfield's isentropic analysis. This means that the present analysis predicts a higher value of $\dot{m}_M$ necessary to maintain a given $\dot{m}_H$ at a given value of P_{HO}/P_{MO}. This difference increases with increasing values of M_M.

The fact that the estimates obtained from the present analysis show higher input values of $\dot{m}_M$ for the same output (P_{HO}/P_{MO}) when compared with Kentfield's isentropic results is because the present analysis takes into account the irreversibilities caused by entropy differences and these in turn increase the required mass flow rate $\dot{m}_M$.

CONCLUSIONS

In this paper, a mixed-mode (hybrid) computerized method for the integration of the non-linear hyperbolic partial differential equations of unsteady, one-dimensional compressible fluid flow was presented. The method combines the advantageous aspects of the finite-difference method and the modified method of characteristics. The algorithm is quite general and has applications in the analysis of gas pulsation in the suction and discharge lines of reciprocating compressors, in the safety analysis of nuclear reactor vessels under hypothetical core disruptive accidents, and in the performance analysis of pressure exchanger dividers. The method is capable of handling non-homentropic flows where temperature discontinuities exist between the two flows.

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NOMENCLATURE

a	velocity of sound
C_p	specific heat at constant pressure
f	friction force per unit volume
h	specific enthalpy
p	absolute pressure
q	heat transfer to unit volume of fluid
R	the gas constant
s	specific entropy
T	absolute temperature
t	time coordinate
u	velocity in x-direction
x	distance coordinate in flow direction
δ	dimensionless cell width
γ	ratio of specific heats
$\frac{\partial}{\partial}$	partial derivative operator
$\frac{d}{d}$	material derivative operator

Subscripts

ex	external condition
H	high pressure fluid
L	low pressure fluid
M	medium pressure fluid
max	maximum
0	stagnation (or total) condition
r	reference condition

Abbreviations

DPE	dynamic pressure exchanger
HP	high pressure
LP	low pressure
MP	medium pressure

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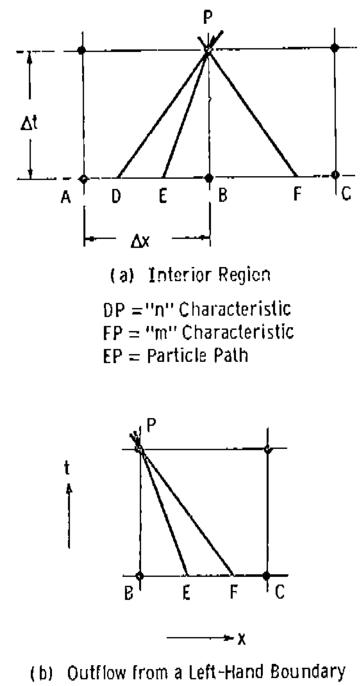

Figure 1: Grid System and Characteristics

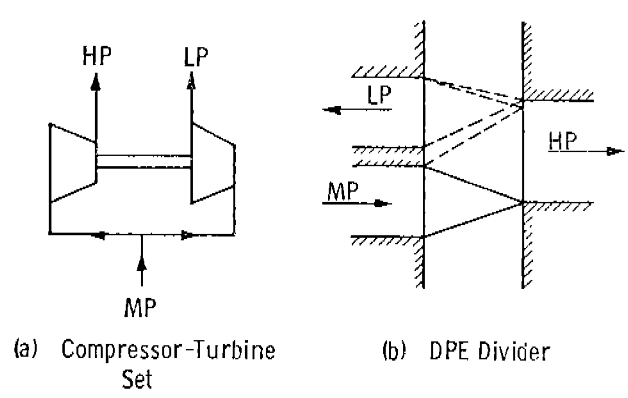

Figure 3: Divider Flow Diagram Analogy

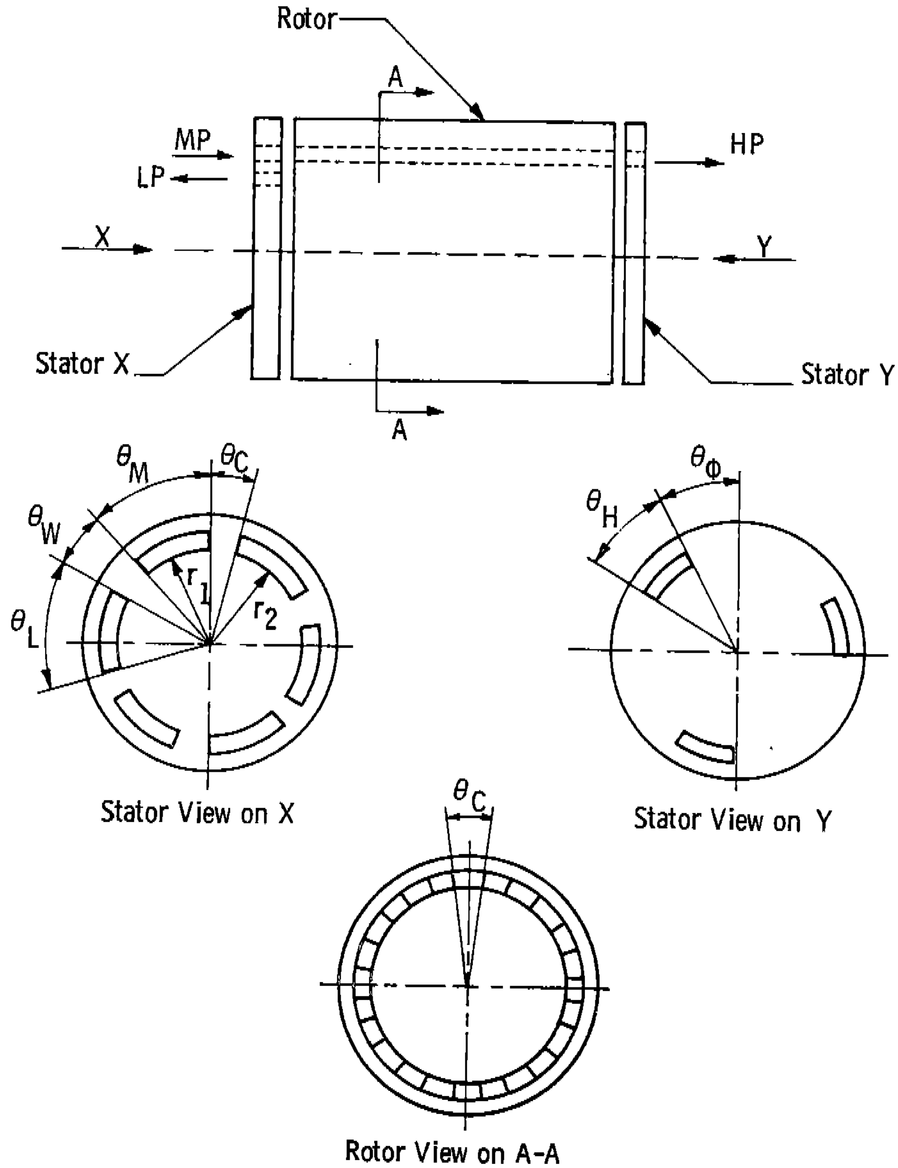

Figure 2: Diagrammatic Layout of a DPE Divider

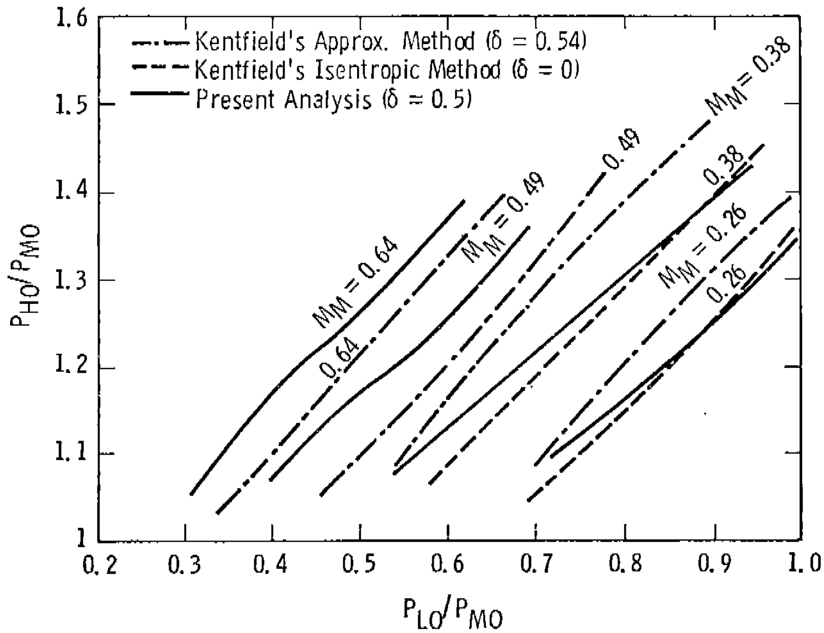

Figure 4: Divider Variation of M_M with P_{HO}/P_{MO} And P_{LO}/P_{MO}

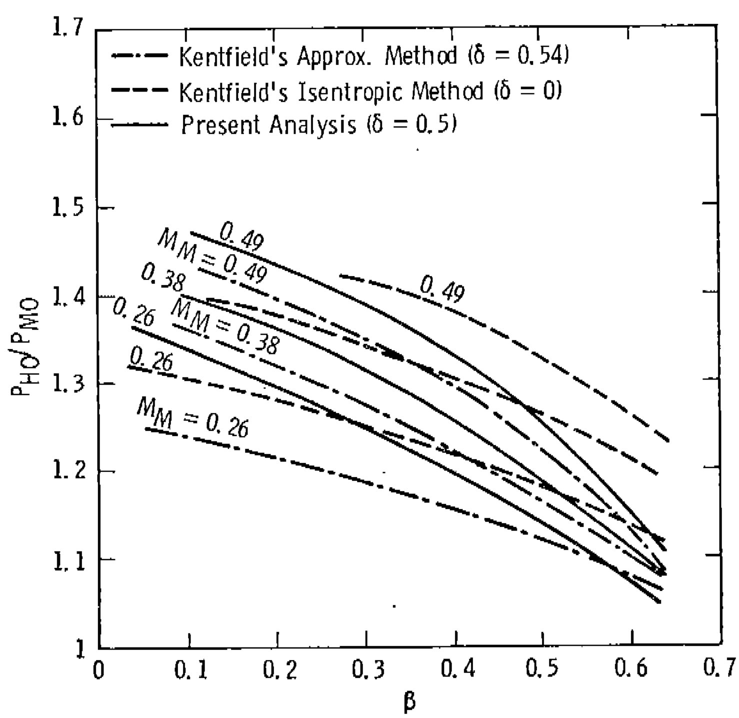

Figure 5: Divider Variation of β with M_M And P_{HO}/P_{MO}