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THE TIME RESPONSE OF A CONTINUOUS GAS COLUMN TO A NONHARMONIC FORCING FLOW AT ITS ENTRANCE

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INTRODUCTION

The purpose of this paper is to report an addition to the work of the reference [1], where the pressure to entrance displacement relationship of an anechoic pipe was developed. This was needed in order to allow a combination of Helmholtz resonator type models [2,3] with an anechoic termination, as it may occur in certain refrigeration compressor systems. In this paper, the case was addressed where a volume-neck type manifold terminates in a long but finite pipe, which in turn terminates in a collection tank. This combination occurs often in air compressors. A long pipe cannot any longer be treated as a neck and a continuous approach is needed.

In the following, the special case of a gas column of uniform crossection was treated as example. The approach can without fundamental difficulty be generalized to include piecewise continuous gas columns and gas columns of varying crossection.

The paper is primarily mathematical in nature and is directed to those that attack compressor gas oscillations in the time domain using Helmholtz resonator models or linear wave travel approaches. For a discussion of these and other time domain approaches, the reader is refered to reference [4].

SETTING UP THE GOVERNING EQUATION

Let us consider a pipe of uniform crossection. The termination is a large collection volume or tank. At the entrance of the system, we wish to know the pressure as function of entrance mass flow.

The equation of motion for gas particles for uniform or piecewise uniform pipes is well-known:

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = 0 \tag{1}$$

If we designate by x = 0 the entrance to the pipe system and by x = L the exit into the large collection volume, we have as boundary conditions

$$\frac{\partial \xi}{\partial x}(L,t) = 0 \tag{2}$$

$$\rho_o A\, \xi(o,t) = \int_o^t \dot{M}\, dt \tag{3}$$

To solve this problem, we must make the following substitution:

$$\xi(x,t) = \xi(o,t) + \eta(x,t) \tag{4}$$

Then

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2} = \frac{1}{c^2 \rho_o A}\frac{d\dot{M}}{dt} \tag{5}$$

The boundary conditions become

$$\frac{\partial \eta}{\partial x}(L,t) = 0 \tag{6}$$

$$\eta(o,t) = 0 \tag{7}$$

Therefore, we have to solve Equation (5), with Equations (6) and (7) as boundary conditions. The particle displacement solution is then given by Equation (4). The pressure solution is given by

$$p(x,t) = -c^2 \rho_o \frac{\partial \xi(x,t)}{\partial x} \tag{8}$$

Note that if the collection tank is not large, boundary condition (6) modifies to

$$\frac{\partial \eta}{\partial x}(L,t) - \frac{A}{V}\eta(L,t) = 0 \tag{9}$$

EIGENVALUES OF THE GAS COLUMN

Solving the homogeneous part of Equation (5), we recognize solutions of the form

$$\eta(x,t) = \bar{\eta}(x)e^{i\omega t} \qquad (10)$$

Substituting this into

$$\frac{\partial^2\eta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} = 0 \qquad (11)$$

gives

$$\frac{d^2\bar{\eta}}{dt^2} + \left(\frac{\omega}{c}\right)^2 \bar{\eta} = 0 \qquad (12)$$

its solution is

$$\bar{\eta}(x) = A \sin\frac{\omega}{c}x + B\cos\frac{\omega}{c}x \qquad (13)$$

Equations (6) and (7) become

$$\frac{\partial\bar{\eta}}{\partial x}(L) = 0 \qquad (14)$$

$$\bar{\eta}(o) = 0 \qquad (15)$$

Substituting (13) into (15) gives

$$B = 0 \qquad (16)$$

Substituting (13) into (14) gives

$$\cos\frac{\omega}{c}L = 0 \qquad (17)$$

This is satisfied whenever

$$\frac{\omega L}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots \qquad (18)$$

or

$$\omega_n = (2n-1)\frac{\pi}{2}\frac{c}{L} \qquad (n = 1,2,\ldots) \quad (19)$$

The associated eigenfunction (mode shape) is, from (13),

$$\bar{\eta}_n(x) = \sin\frac{\omega_n}{c}x \qquad (20)$$

SOLUTION BY MODAL EXPANSION

We are now able to solve Equation (5) by modal expansion. Let

$$\eta(x,t) = \sum_{n=1}^{\infty} q_n(t)\bar{\eta}_n(x) \qquad (21)$$

Substituting this into Equation (5) and making use of Equation (12) gives

$$-\sum^{\infty}\left(\frac{d^2q_n}{dt^2} + \omega_n^2 q_n\right)\bar{\eta}_n(x) = \frac{1}{\rho_o A}\frac{d\dot{M}}{dt} \qquad (22)$$

Because of orthogonality of the eigen functions

$$\int_{x=0}^{L}\bar{\eta}_n(x)\bar{\eta}_n(x)\,ds = \begin{cases} 0 & ; \; n \neq m \\ \int_0^L \bar{\eta}_n^2(x)\,dx & ; \; n = m \end{cases} \qquad (23)$$

We may multiply (22) by $\bar{\eta}_m(x)$ and integrate. This allows us to remove the summation and Equation (22) becomes

$$\frac{d^2q_n}{dt^2} + \omega_n^2 q_n = Q_n(t) \qquad (24)$$

where

$$Q_n(t) = -\frac{1}{\rho_o A N_n}\frac{d\dot{M}}{dt}\int_o^L \bar{\eta}_n(x)\,dx \qquad (25)$$

$$N_n = \int_o^L \bar{\eta}_n^2(x)\,dx \qquad (26)$$

For zero initial conditions, the solution of Equation (24) is

$$q_n(t) = \frac{1}{\omega_n}\int_o^t Q_n(\tau)\sin\omega_n(t-\tau)\,dt \qquad (27)$$

Since the inputs response of (24) is

$$g(t) = \frac{1}{\omega_n}\sin\omega_n t \qquad (28)$$

Equation (21) becomes

$$\eta(x,t) = -\sum_{n=1}^{\infty}\bar{\eta}_n(x)\int_o^L\bar{\eta}_n(x)\,dx\,\frac{1}{\omega_n N_n}$$

$$\int_o^t \left(\frac{d\dot{M}}{dt}\right)_{t=\tau}\sin\omega_n(t-\tau)\,d\tau \qquad (29)$$

Equation (8) becomes, utilizing (4) and (30)

$$p(x,t) = c^2 \rho_o \sum_{n=1}^{\infty} \frac{\partial \bar{\eta}_n(x)}{\partial x} \int_o^L \bar{\eta}_n(x)\,dx \, \frac{1}{\omega_n N_n}$$

$$\int_o^t \left(\frac{d\dot{M}}{dt}\right)_{t=\tau} \sin \omega_n (t-\tau)\,d\tau \qquad (30)$$

This equation relates the input mass flow rate $\dot{M}$ at the pipe entrance to the pressure at any position x along the pipe.

Note, that since

$$\frac{d\dot{M}}{dt} = \frac{1}{\rho_o A} \frac{d\xi(o,t)}{dt} \qquad (31)$$

Equation (30) can also be interpreted as relating pressure in the pipe to input displacement.

Let us now substitute Equation (20).

Equation (30) becomes

$$p(x,t) = \frac{2c^2\rho_o}{L} \sum_{n=1}^{\infty} \frac{\cos \frac{\omega_n}{c} x}{\omega_n}$$

$$\int_o^t \left(\frac{d\dot{M}}{dt}\right)_{t=\tau} \sin \omega_n (t-\tau)\,d\tau \qquad (32)$$

At the entrance of the pipe, the pressure is

$$P(o,t) = \frac{2c^2\rho_o}{L} \sum_{n=1}^{\infty} \frac{1}{\omega_n}$$

$$\int_o^t \left(\frac{d\dot{M}}{dt}\right)_{t=\tau} \sin \omega_n (t-\tau)\,d\tau \qquad (33)$$

Equations (32) and (33) can now be combined with time domain Helmholtz resonator models or can be used in their own right.

If we need to consider damping, it is best to work in terms of a modal damping coefficient ζ_n. This coefficient varies with the mode number n and becomes larger as n increases. This means that higher frequency modes are damped more. The best way to find ζ_n is by measurement or by assuming values based on experience. Considering modal damping, Equation (24) becomes

$$\frac{d^2 q_n}{dt^2} + 2\zeta_n \omega_n \frac{dq_n}{dt} + \omega_n^2 q_n = Q_n(t) \qquad (34)$$

The impulse response is

$$g(t) = \frac{1}{\omega_n \sqrt{1-\zeta_n^2}} e^{-\zeta_n \omega_n t} \sin \sqrt{1-\zeta^2}\,\omega_n t \qquad (35)$$

and thus

$$p(x,t) = \frac{2c^2\rho_o}{L} \sum_{n=1}^{\infty} \frac{\cos \frac{\omega_n}{c} x}{\omega_n \sqrt{1-\zeta_n^2}}$$

$$\int_o^t \left(\frac{d\dot{M}}{dt}\right)_{t=\tau} e^{-\zeta_n \omega_n t} \sin \sqrt{1-\zeta^2}\,(t-\tau)\,d\tau$$

(36)

The integral can easily be evaluated for many standard forms of mass flow rate input. For arbitrary inputs, the solution has to be set up as a numerical integration.

CLOSURE

First, the wave equation was set up for the case of a mass flow input at one end of a uniform pipe. It was solved by defining a new variable and by using the technique of modal expansion.

The resulting equation is suited to be combined with lumped parameter models of the Helmholtz resonator type since it is formulated in the time domain.

The equation is also a valuable tool to investigate discharge and suction tuning for increased thermodynamic efficiency.

The fact that the solution is given in terms of a series of the natural modes might also be of interest. Modal expansion belongs to the tools of trade of the structural vibration analyst, but it is rarely found in fluid and gas mechanics. It may give new insights into suction and discharge system design, just because it looks at the same problem with a different set of glasses.

The need for working in the time domain, expecially when thermodynamic efficiency is to be investigated, does not have to be argued.

It is essential that the nonlinear behaviour of automatic valves be coupled to the discharge of suction system directly in order to explore best the mutual dependence.

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NOMENCLATURE

ξ = total particle displacement [m]

x = coordinate [m]

L = length of pipe [m]

c = speed of sound [m/sec]

t = time [sec]

A = pipe crossection [m^2]

V = collection volume at exit of pipe [m^3]

ρ_o = Mean mass density [$Nsec^2/m^4$]

$\dot{M}$ = mass flow rate at pipe entrance [Nsec/m]

η = gas displacement [m]

p = pressure variation about mean pressure [N/m^2]

ω = frequency [rad/sec]

ω_n = natural frequency [rad/sec]

$\bar{\eta}_n$ = mode shape [-]

q_n = modal participation factor [m]

ζ_n = damping coefficient