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INFEM: An Inference-Finite Element Program For Flow Problems

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INFEM: An Inference-Finite Element Program for Flow Problems

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1. Introduction

In the present work an inference model producing the input data to the main analytical scheme on the basis of field information is demonstrated.The main body of data concerning the physical parameters are taken from the field of rock mechanics.

Krumbein (3), describes the fundamental techniques used in defining linear inference models. Such techniques are for example the method of least squares, fitting a polynomial of two variables, etc. However, all these techniques fail to provide an evaluation of how good the estimation is. They also exhibit operational difficulties for the nonstationary case. A different inference scheme is proposed by Matheron (5). It is based on a model originally suggested by Krige (4), and is particularly well suited to treat nonstationary cases. A similar moving average technique is presented hereafter, that generates the spatial distribution of the physical properties of the rock media, defined in the previous chapter. Also the interfacing of the Inference model with the analytical Finite Element model is shown below.

3.2 Justification of the Moving Average Technique

Drill hole samples most often produce extreme values which are erratic in their spatial distribution. Figure 3.1 gives an illustration of the variability of some commonly investigated physical parameters. Therefore there is a need for models generating smooth spatial distributions of these parameters. Two such models most commonly used are based on trend surface estimates and moving average estimates.

It is generally observed that moving average estimates are superior to trend surface estimates. In the former case the estimation is exclusively based on adjacent informational sets, the more distant sets exerting no influence at all. In the case of the trend surface estimates, the estimation is based on all known data. Moving average estimates tend to be more stable than corresponding polynomial trend estimates obtained from the same number of data points, especially if the sample points are sparce, Whitten (7). Davis (2), observed that moving average schemes in two dimensions have not been significantly tested yet. Indeed the time series analysis from which this method is derived is not as well developed at the present time, as the regression analysis, for example, on which the trend-surface method is based.

Among the moving average methods, the Kriging (names by French geomathematicians in honor of Krige) seems to be the most advantageous. More specifically the Kriging technique makes optimal use of the given data and provides a measure of the variance of the estimation made at

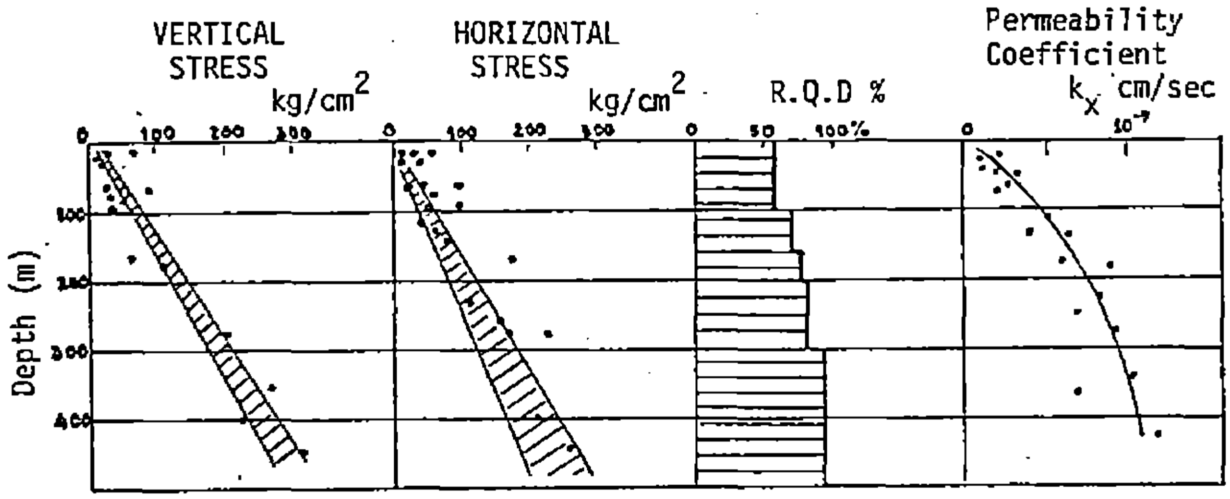

FIGURE [illegible].1 VARIABILITY OF PHYSICAL PARAMETERS

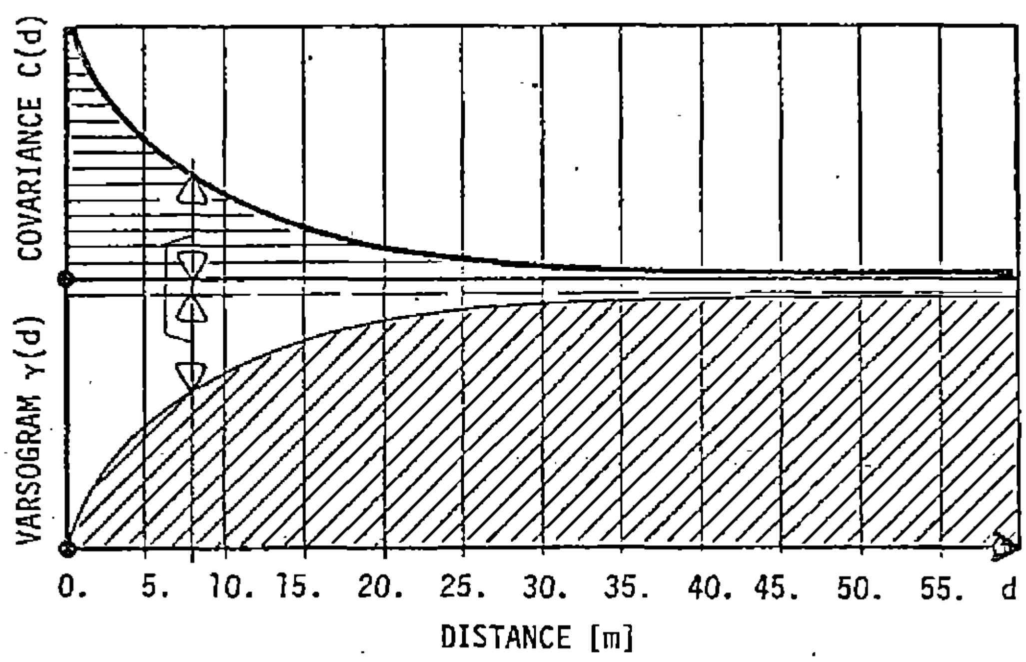

FIGURE [illegible].2 VARIOGRAM AND COVARIANCE VS. DISTANCE any specific location. Both these features are essential if the results from the Inference model are to provide the input for an uncertainty analysis. However it is believed that a general method incorporating parts of both techniques may give superior results.

3 Implementation of the Inference Model in a Two-Dimensional Geometric Space

3.1 Statistical Model and Corresponding Assumptions

To determine the values of a rock property $Z(x,y)$ a number of measurements are made on rock samples from bore holes. The set of points where observations are made is indexed by β and Z_β represents the measured value of the random rock property at point β. $\hat{Z}(x_0,y_0)$ is the estimation of property Z at a particular point (x_0,y_0) in the media, evaluated from the measured Z_β values.

One simple way to obtain this estimation is to define $\hat{Z}$ in terms of the known values Z_β according to a linear combination as follows:

$$\hat{Z}(x_0,y_0) = \sum_{\beta=1}^{n} b_\beta \cdot Z_\beta \qquad (3.1)$$

where: Z_β are the known data points,

b_β are unknown weight coefficients to be determined by the Inference model.

An alternate approach would be to estimate the mean value of $Z(x,y)$, namely $\bar{Z}(x,y)$ at point (x_0,y_0), according to a linear combination:

$$\hat{\bar{Z}}(x_0,y_0) = \sum_{\beta=1}^{n} a_\beta \bar{Z}_\beta \qquad (3.2)$$

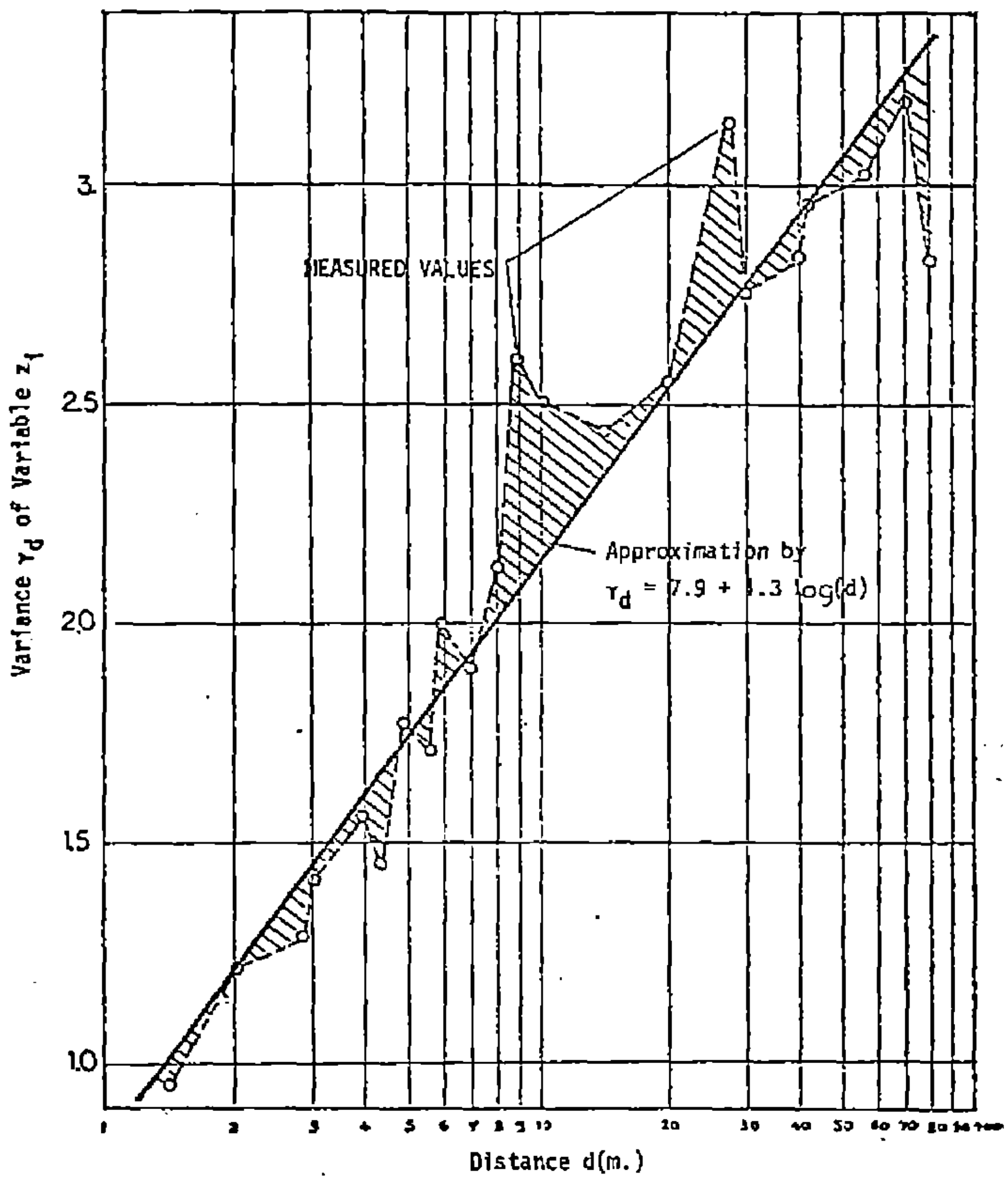

FIGURE 4 APPROXIMATE VARIOGRAM FUNCTION FROM ACTUAL DATA

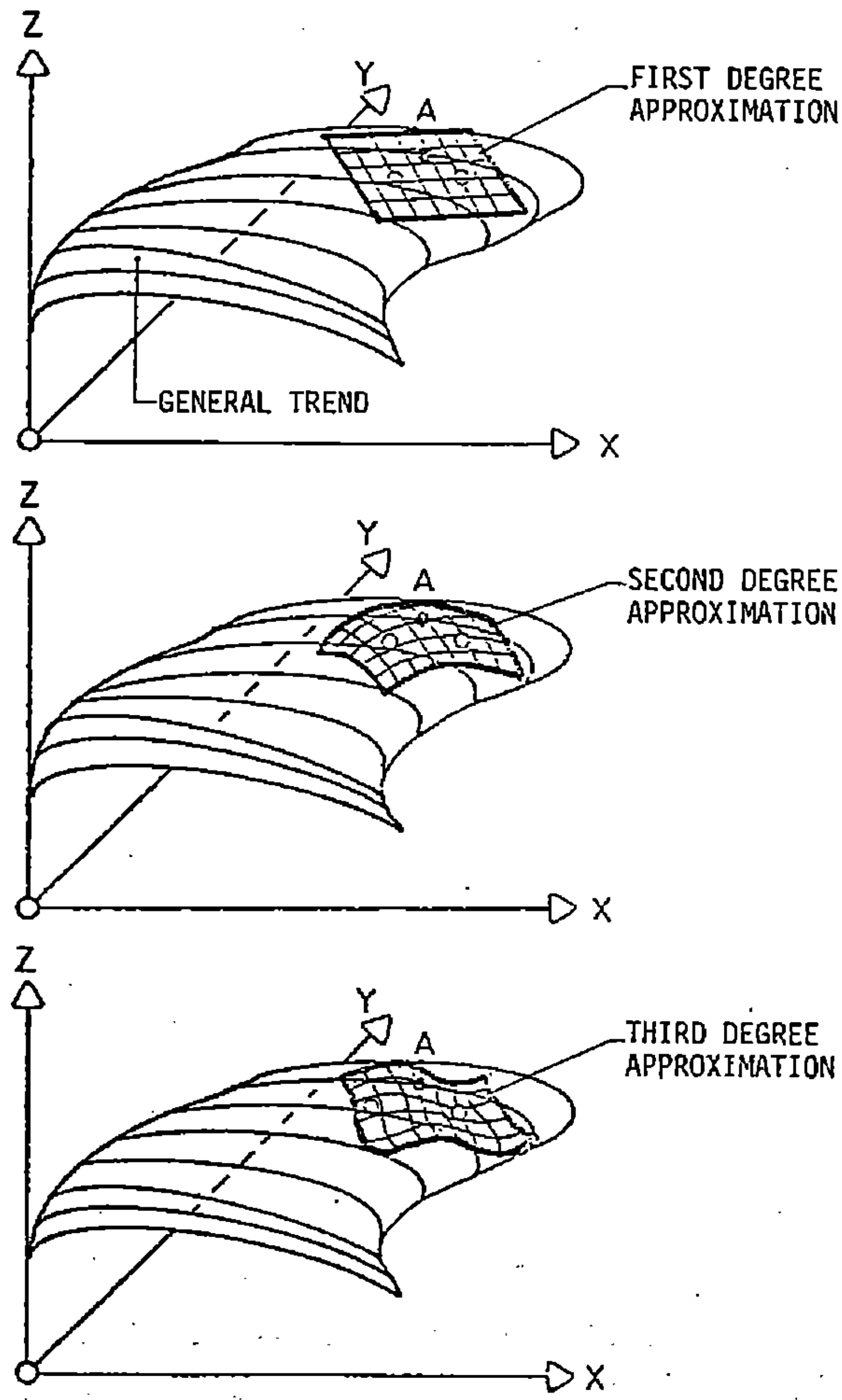

FIGURE 3 LOCAL APPROXIMATIONS OF GENERAL TRENDS OF TWO INDEPENDENT VARIABLES

where: $\bar{Z}_\beta$ is the mean-value at the known data points,

a_β are the unknown weight coefficients.

The random variable $Z(x,y)$ in turn can be expressed as follows:

$$Z(x,y) = \bar{Z}(x,y) + FZ(x,y) \quad (3.3)$$

where: $\bar{Z}(x,y)$ is the mean value, and

$FZ(x,y)$ is the fluctuating term around the mean.

This model leads to two possible groups of assumptions. They concern the first and second moments of the random variable $Z(x,y)$. The first group is:

$$E[Z(x,y)] = \bar{Z}(x,y) \quad (3.4)$$

$$E[Z(x_1,y_1), Z(x_2,y_2)] = \bar{Z}(x_1,y_1) \cdot \bar{Z}(x_2,y_2) + C((x_1,y_1), (x_2,y_2)) \quad (3.5)$$

where: $\bar{Z}(x,y)$ is the mean value, and

$C(x_1,x_2)$ is the covariance of the variable $Z(x,y)$.

These assumptions are in many cases too restrictive and need to be replaced by more flexible ones. This can be realized by focusing on the rate of change of the random variable $Z(x,y)$, as follows:

$$E[Z(x_1,y_1) - Z(x_2,y_2)] = \bar{Z}(x_1,y_1) - \bar{Z}(x_2,y_2) \quad (3.6)$$

$$E[(Z(x_1,y_1) - Z(x_2,y_2))^2] = 2\,\gamma((x_1,y_1), (x_2,y_2)) \quad (3.7)$$

where: $\gamma(x_1,x_2)$ is the variogram or semivariance of the difference $Z(x_1y_1) - Z(x_2,y_2)$.

Interestingly the variogram $\gamma(d)$ and the covariance $C(d)$ are both defined as functions of the distance between two locations and can be related by Eq. 3.8, as shown in Appendix A:

$$\gamma(d) = C(o) - C(d). \tag{3.8}$$

As shown in Figure 2, $\gamma(d)$ and $C(d)$ are complementary.

Both above sets of assumptions concern the stochastic model. An additional set of assumptions is needed to specify the nature of the randomness of the variable $Z(x,y)$, in particular its mean and its variance.

Locally at a point (x,y) the mean $\bar{Z}(x,y)$ is approximated as a linear combination of known functions:

$$\bar{Z}(x,y) = \sum_{i=1}^{k} a_i \, f^i(x,y) \tag{3.9}$$

where: a_i are unknown weight coefficients,

$f^i(x,y)$ are apriori known functions locally approximating $\bar{Z}(x,y)$.

The global trend analysis of most real cases, Davis (2), can be represented by a polynomial equation of the first, second or third order. Therefore it can be intuitively seen that locally the trend can be best approximated by quadratic functions as shown in Figure 3. A corresponding norm can be defined for a best approximation with respect to the encountered trend.

The covariance $C(x_1,x_2)$ can be computed from field measurements, and can be represented by the following expression:

$$C(x_1,x_2) = k\, e^{-\alpha d} \tag{3.10}$$

where: d = distance between x_1 and x_2,

α, k = fitting parameters.

In many cases the variogram $\gamma(d)$ is more suitable than the covariance $C(d)$ to define the distances over which, realizations of the random variable Z are interdependent. Thus a range of maximum allowable sampling interval can be evaluated. The expressions most commonly used for $\gamma(d)$ are:

$$\gamma(d) = \beta + 3\alpha \log(d) \tag{3.11}$$

or

$$\gamma(d) = \alpha(d) \tag{3.12}$$

where: α and β are fitting parameters.

The first expression is often referred to as DeWij's model, while the second as Linear model. The selection of the adequate expression for $\gamma(d)$ is difficult since it requires a very careful treatment of the data obtained from field investigation. Matheron (5) gives a case study of such a treatment as illustrated in Figure 4. In the present study both expressions were tested.

3.2 Identification of the Best Estimator

Summarizing, up to this point two sets of model assumptions are given, Eqs. (3.4) and (3.5) and Eqs. (3.6) and (3.7), concerning the random variables $Z(x,y)$. Both these sets of assumptions can be used to identify estimators of both the random variable $\hat{Z}(x,y)$ and its mean $\hat{Z}(x,y)$.

The first model, (Eqs. (3.4) and (3.5)) is used to obtain the best estimator for the random variable $\hat{Z}(x,y)$ itself. Both models identify the best estimators among all possible functions satisfying the hypothesis concerning the randomness of the rock media, Eq. (3.9). This is done by minimizing the variance of the estimation, Eqs. (3.5), respectively (3.7), subject to the first moment constraints, Eqs. (3.4), respectively (3.6). The Lagrange Multipliers approach is used for this constrained optimization problem. The computations carried out in Appendix B, lead to the following results:

FOR MODEL 1 concerning the estimation of the mean $\bar{Z}(x,y)$ (from Eqs. (3.4) and (3.5)).

The method of Lagrange Multipliers leads to a set of n + k equations with n + k unknowns, namely the weight coefficients:

$$\begin{aligned} &\sum_{\beta} a_{\ell}^{\beta} C_{\alpha\beta} - \sum_{\ell} \mu_{\ell} f_{\alpha}^{\ell} = 0 \quad ; \quad \alpha,\beta = 1,\ldots n \\ &\sum_{\alpha} a_{\ell}^{\alpha} f_{\alpha}^{\ell} - \delta_{\ell}^{\alpha} = 0 \quad ; \quad \ell = 1,\ldots k \end{aligned} \tag{3.13}$$

where: a_{ℓ}^{β} are the unknown weight coefficients,

μ_{ℓ} are the Lagrange Multipliers,

δ_{ℓ}^{α} is the Kronecker delta.

The variance of the estimation is given by:

$$E[\hat{\bar{Z}}(x,y)]^2 = \sum_{\ell} \mu_{\ell} f^{\ell}(x,y) \quad ; \quad \ell = 1,\ldots k \tag{3.14}$$

FOR MODEL 2 concerning the estimation of the random variable $Z(x,y)$ (from Eqs. (3.6) and (3.7)).

Similarly the following system is obtained (Appendix B).

$$\begin{aligned} &\sum_{\beta} b^{\beta} \gamma_{\alpha\beta} + \sum_{\ell} \mu_{\ell} f_{\alpha}^{\ell} = \gamma(x_{\alpha},x) \quad ; \quad \alpha,\beta = 1,\ldots n \\ &\sum_{\alpha} b^{\alpha} f_{\alpha}^{\ell} = f^{\ell}(x) \quad ; \quad \ell = 1,\ldots k \\ &\sum_{\alpha} b^{\alpha} = 1 \end{aligned} \tag{3.15}$$

where: b^{β} are the unknown weight coefficients,

μ_{ℓ} are the Lagrange Multipliers.

The variance of the estimation is given by:

$$E[Z-\hat{Z}]^2 = \sum_{\alpha} b^{\alpha} \gamma(x_{\alpha},x) + \sum_{\ell} \mu_{\ell} f_{\alpha}^{\ell} \quad ; \quad \alpha = 1,\ldots n, \quad \ell = 1,\ldots k \tag{3.16}$$

4 Interfacing the Inference Model with the Analytical Model

4.1 Uncertainty Analysis in the Analytical Model

As mentioned previously the analytical model is treated using the finite element technique which provides us with the transfer mechanism between a set of inputs {F} and a set of generally unknown outputs {u}. The general solution is given by the following relation in matrix form:

$$\{u\} = [K]^{-1} \{F\} \tag{4.17}$$

where: {K} is generally known as stiffness matrix and is defined as a function of the random variables $Z_1, Z_2, \ldots, Z_n$,

{F} is the loading term.

Applying now the first order uncertainty analysis as described by Papoulis ( ), the first and second moments of the unknown vector {u} are obtained as follows:

FIRST MOMENT

$$E[\{u(Z_1,Z_2)\}] = \{u(\bar{Z}_1,\bar{Z}_2)\} + \frac{1}{2}\left[\sigma^2_{Z_1} \frac{\partial^2\{u(\bar{Z}_1,\bar{Z}_2)\}}{\partial Z_1{}^2} + \ldots\right] \tag{4.18}$$

where: $\sigma^2_{Z_1}$ is the variance of the physical parameter Z_1.

The second part of the right hand side can be neglected being a very small quantity.

SECOND MOMENT

$$E[\{u(Z_1,Z_2)\}^2] \approx \sigma^2_{Z_1}\left(\frac{\partial\{u(\bar{Z}_1,\bar{Z}_2)\}}{\partial Z_1}\right)^2 + \sigma^2_{Z_2}\left(\frac{\partial\{u(\bar{Z}_1,\bar{Z}_2)\}}{\partial Z_2}\right)^2 + 2\,\frac{\partial\{u(\bar{Z}_1,\bar{Z}_2)\}}{\partial Z_1}\,\frac{\partial\{u(\bar{Z}_1,\bar{Z}_2)\}}{\partial Z_2}\,\mathrm{cov}(Z_1,Z_2). \tag{4.19}$$

The partial derivatives in Eq. (4.19) are obtained from the following system:

$$[K(Z_1,Z_2)]\,\frac{\partial\{u(Z_1,Z_2)\}}{\partial Z_i} = \frac{\partial\{F\}}{\partial Z_i} - \frac{\partial[K(Z_1,Z_2)]}{\partial Z_i}\{u\}; \quad i = 1,2 \qquad (4.20)$$

The solution is obtained by the classical finite element methodology, ( ).

4.2 Coupling of the Inference Model with the Analytical Uncertainty Model

The estimates of the spatial mean $\hat{Z}_1(x,y)$, $\hat{Z}_2(x,y)$ and the variances $\sigma_{Z_1}(x,y)$, $\sigma_{Z_2}(x,y)$ provided by the inference model are substituted in the statistical relations of the dependent random variable $\{u\}$, Eqs. (4.18) and (4.19). Thus the first and second moments, as well as the coefficient of variation of $\{u\}$ can be evaluated. The coefficient of variation in particular being an essential statistical property, can be used to evaluate the performance of the analytical model.

For testing purposes, an example was treated borrowed from the field of underground confined flow, as shown in section 6. The output of the analysis provides the first and second moments of the unknown hydraulic head $\{u\}$, at any specified location of the domain under consideration.

.5 Description of the Algorithm

The above described procedure is summarized by the computation steps of the flow chart of Fig. 5. The geometric domain under investigation is divided using a rectangular mesh common for the Inference

Model and the analytical model (Finite element mesh). The computations are carried out at each node using a number among the known realizations $Z_\beta(x,y)$. Therefore a zone of influence, characteristic of the media are depending on the covariance $C(x_1,x_2)$ and/or the variogram $\gamma(d)$ is defined at each node of the mesh. For computational efficiency, 'n' known points are selected within this zone, to determine the estimation of the random variable $Z(x,y)$ at that particular node. A polynomial comporting k terms, approximates the trend of $Z(x,y)$ in the neighborhood of the node at hand. Thus, at every node (x,y), a system of $n + k$ equations permits to determine the estimator $\hat{Z}(x,y)$. The outcome of the procedure depends on the number of known points and the density of the provided information $Z_\beta(x,y)$.

In case of insufficient information in the domain of interest, the original assumptions are violated. However, in this case the values of variance of the estimation indicate the poor performance of the inference and give the exact location where more information is needed.

The flow chart of Figure 5 gives the sequence in which the computations are performed by program INFMOD. The mean values of the physical parameters evaluated by the inference model INFMOD are introduced at each node of the finite element mesh. Thereafter, the computations are performed in a conventional way taking into account the prescribed boundary conditions. The mean values of the unknown vector $\{u\}$ are determined, as well as the vectors $\frac{\partial\{u\}}{\partial Z_i}$, $i = 1,2$. Finally the variance of the unknown vector $\{u\}$ is computed at each node of the mesh.

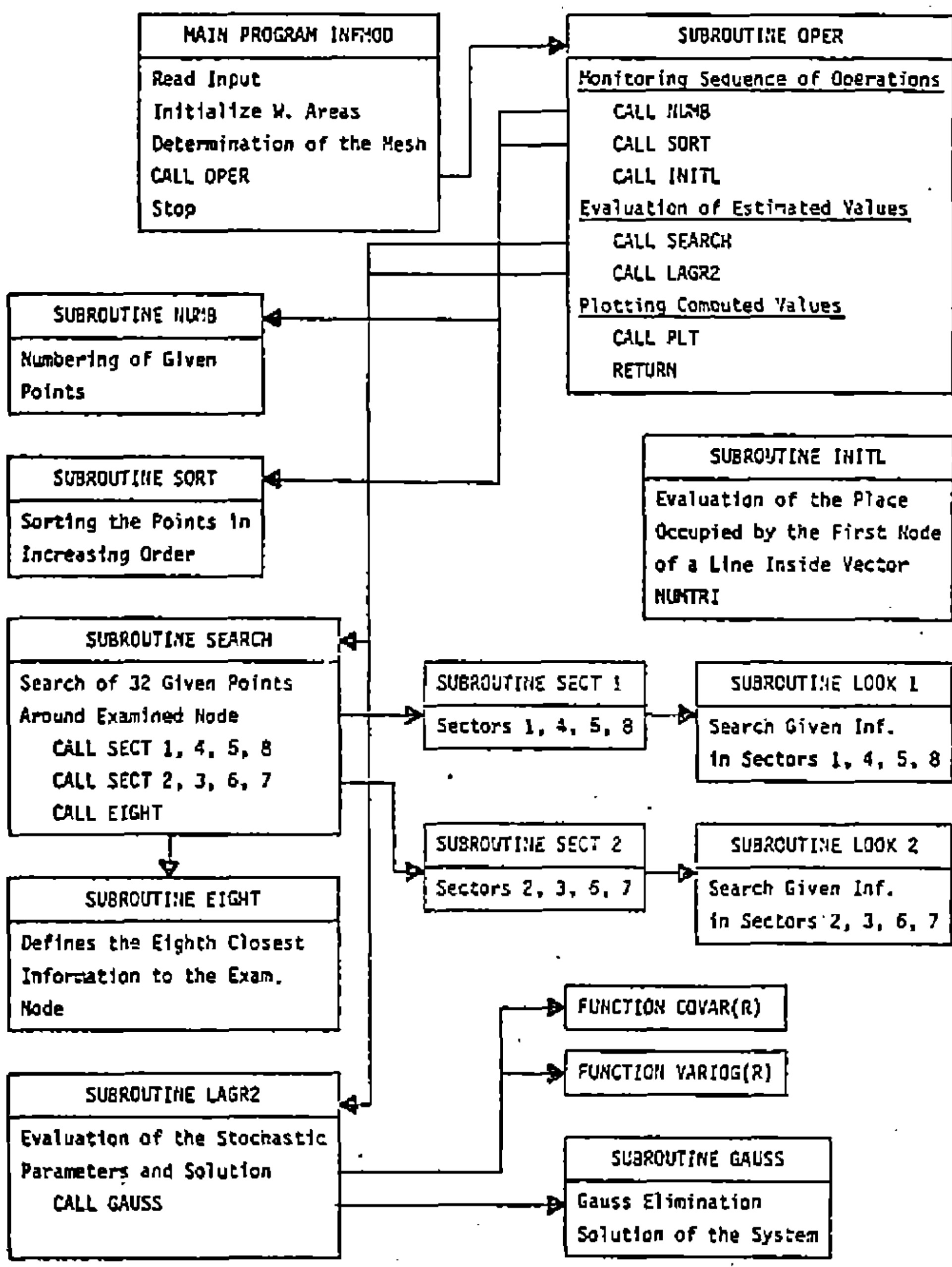

FIGURE 5 FLOW CHART OF PROGRAM INFMOD (INFERENCE MODEL)

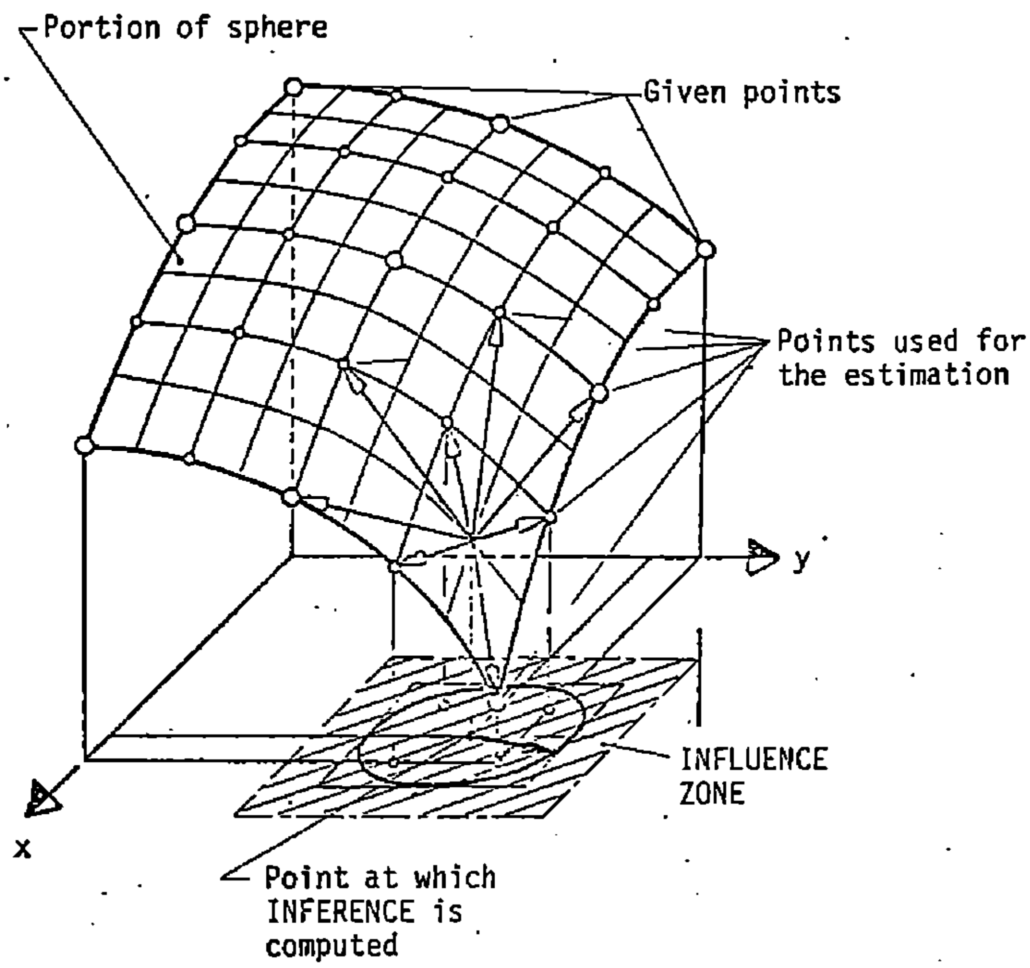

FIGURE - 6 ILLUSTRATION OF PROBLEM 1 (SPHERICAL TREND)

As expected, increasing amount of information, produces a smaller variance of the estimates. The case of largest variance is reported in Table 1.

It is to be mentioned that the maximum error obtained is less than 0.1 percent and that the estimated first and second moments are symmetrically distributed with respect to the existing plane of symmetry.

Another interesting feature is that the spatial distribution of the error is similar to that of the variance, justifying the use of the estimated variance as an indicator of the error.

2. Tests of the statistical assumptions

These tests concerned (a) the assumptions on the apriori known function f(y) characterizing the behavior of the mean $\bar{Z}(x,y)$, and (b) the assumption on the general form of the variogram obtained either from site investigation or considered apriori. The tests were performed on the square 400 x 400m of Problem 1 and over a region 50 x 20m of the flow problem (Problem 2).

Several functions f(x,y) were tested in the above mentioned regions and the most satisfactory results from the point of view of both accuracy and efficiency were obtained for the following quadratic function:

$$f(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy \qquad (5.21)$$

This form was also suggested from a trend analysis (sect. 2), applied to the data of the flow problem as illustrated in Figure 8.

The following expressions for the variogram were studied:

$$\gamma(d) = \alpha \log |d| \qquad (5.22)$$

$$\gamma(d) = \alpha |d| \qquad (5.23)$$

TABLE .1 RESULTS OF INFERENCE MODEL FOR PROBLEM 1

NUMBER OF GIVEN INFORMATION	TYPE OF MODEL	MAX ERROR	EXPECTED VALUE	MAX VARIANCE	MAX COEFFICIENT OF VARIATION
9 points	MODEL 1	0.056	3.18	.81	0.28
	MODEL 2	0.049	3.45	1.08	0.30
25 points	MODEL 1	0.011	3.33	.35	0.17
	MODEL 2	0.0047	3.61	0.569	0.21
81 points	MODEL 1	0.002	3.40	0.21	0.135
	MODEL 2	0.0007	3.62	0.249	0.14

TABLE 2 RESULTS OF INFERENCE MODEL FOR PROBLEM 2

COORDINATES		ESTIMATED MEAN	ESTIMATED VARIANCE	COEFFICIENT OF VARIATION
x	y	10^{-3} cm/sec	10^{-5} cm/sec	10^{-2}
5.000	5.000	.013	.002	3.651
10.000	5.000	.008	.002	6.137
15.000	5.000	.002	.003	32.673
20.000	5.000	.015	.001	2.167
25.000	5.000	.027	.002	1.813
30.000	5.000	.008	.001	4.449
35.000	5.000	.030	.004	2.041
40.000	5.000	.026	.000	.805
45.000	5.000	.018	.001	1.698
50.000	5.000	.171	.071	1.559
0	10.000	.097	.029	1.736
5.000	10.000	.026	.001	.905
10.000	10.000	.012	.002	3.290
15.000	10.000	.080	.000	.000
20.000	10.000	.053	.004	1.198
25.000	10.000	.061	.001	.603
30.000	10.000	.065	.010	1.534
35.000	10.000	.003	.002	13.735
40.000	10.000	.025	.004	2.696
45.000	10.000	.091	.003	.590
50.000	10.000	.121	.262	4.229
0	15.000	.503	.026	.323
5.000	15.000	.335	.002	.119
10.000	15.000	.254	.004	.262
15.000	15.000	.331	.001	.069
20.000	15.000	.460	.003	.121
25.000	15.000	.454	.001	.053
30.000	15.000	.418	.003	.120
35.000	15.000	.502	.001	.066
40.000	15.000	.315	.002	.133
45.000	15.000	.542	.002	.079
50.000	15.000	.599	.082	.479

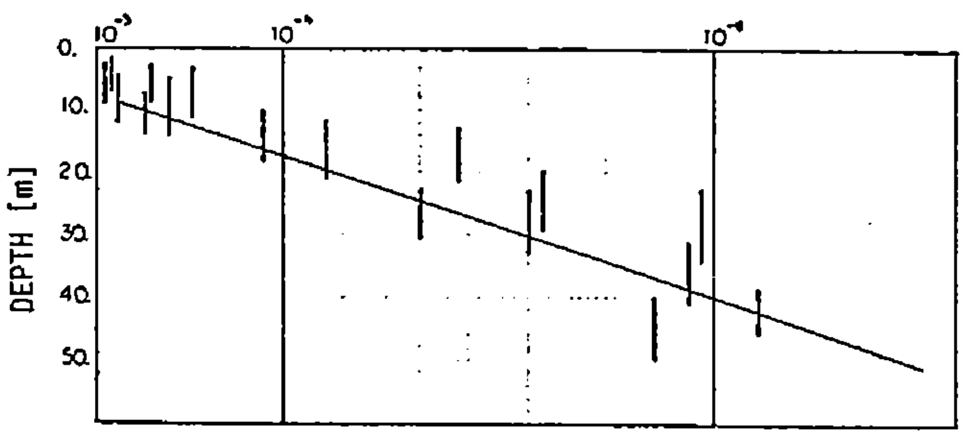

FIGURE 8 GENERAL TREND FUNCTION OF THE PERMEABILITY

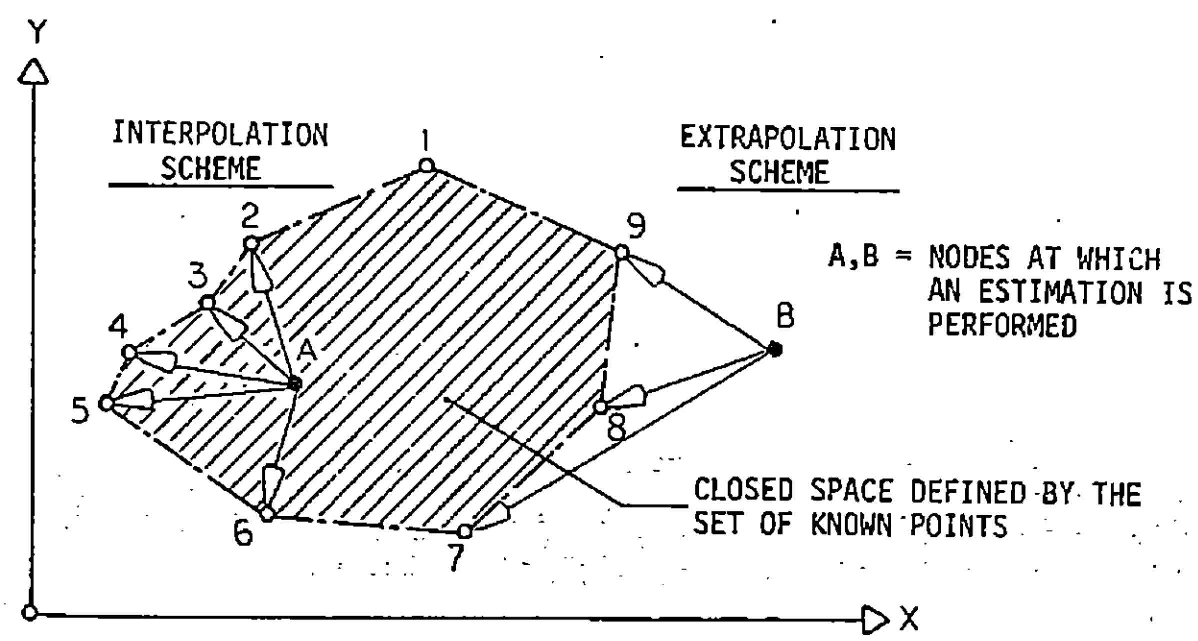

FIGURE 9 TOPOLOGICAL CONSIDERATIONS FOR THE ESTIMATION NODE

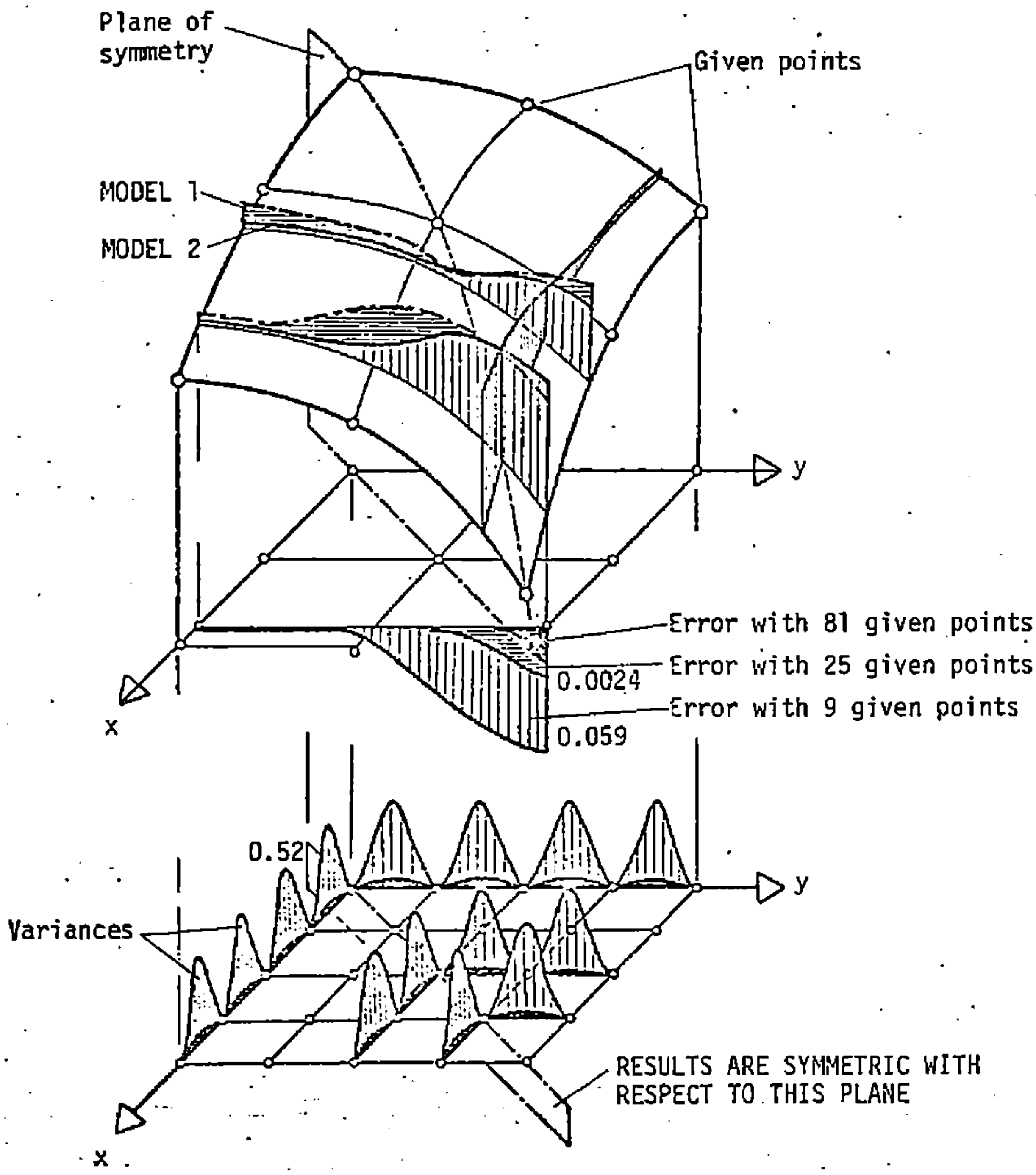

FIGURE 7 ESTIMATED STATISTICAL CHARACTERISTICS OF PROBLEM I

6 Example of Application and Discussion

As said in section 4 the example of application is borrowed from the field of underground confined flow. Two groups of tests are conducted to explore the limitations and the applicability of the general algorithm. One group concerns the inference model alone, while the other deals with the performance of the finite element model.

FIRST GROUP

The three different tests related to the Inference model, in connection with two different problems, produced the following results:

1. Tests of the statistical convergence

This test was conducted under the assumption of statistical isotrophy and for a uniform spatial distribution of the given information points. More specifically in Problem 1 a domain defined by a square mesh of 400 x 400m was examined and the random variable $Z(x,y)$ was assumed to possess a realization lying on a portion of a sphere over this domain, as shown in Figure 6. The domain was divided into squares of 25 x 25m having 289 nodes where the computations were performed. The given information was located first on nine points as shown in Figure 6, then on twenty-five points and finally on eighty-one points.

The estimation at the 288 nodes of the mesh was performed by both Inference models defined previously (Sect. 3). Interestingly enough in the region of great variability of the random variable $Z(x,y)$ the two models show a good agreement in their estimation, as illustrated in Figure 7.

SECOND GROUP

The following tests, related to the coupling of the uncertainty analysis using F.E.M. and the inference model were performed:

1. Tests for the convergence of the statistical characteristics of {u} as a function of the size of the element's mesh.

For the First Moment this was done automatically through a mesh generation subroutine. For the Second Moment, a denser mesh was considered only around the flow barriers, where the hydraulic gradient was important.

A satisfactory convergence was observed, as shown in Table 2.

2. Tests of the effect of the boundary conditions on the statistical characteristics of {u}.

Several differential hydraulic heads were considered and their results plotted in Figure 12. They display expected responses, namely as the differential head increases, the variance of the unknown quantity {u} increases also, showing that the flow becomes more variable reaching eventually a state of nonlaminar flow. Interestingly enough the variances seem to be directly related to the hydraulic gradient, Figure 13.

7 Remarks on the Applicability of the Method

Becker, Hazen and Scott (1) have produced statistical evidence that data selected from random samples produce more realistic information. Attention then should be given to the randomness of the sample points, as well as the uniformity of the sample volume. The Inference model could be used to advantage here as follows:

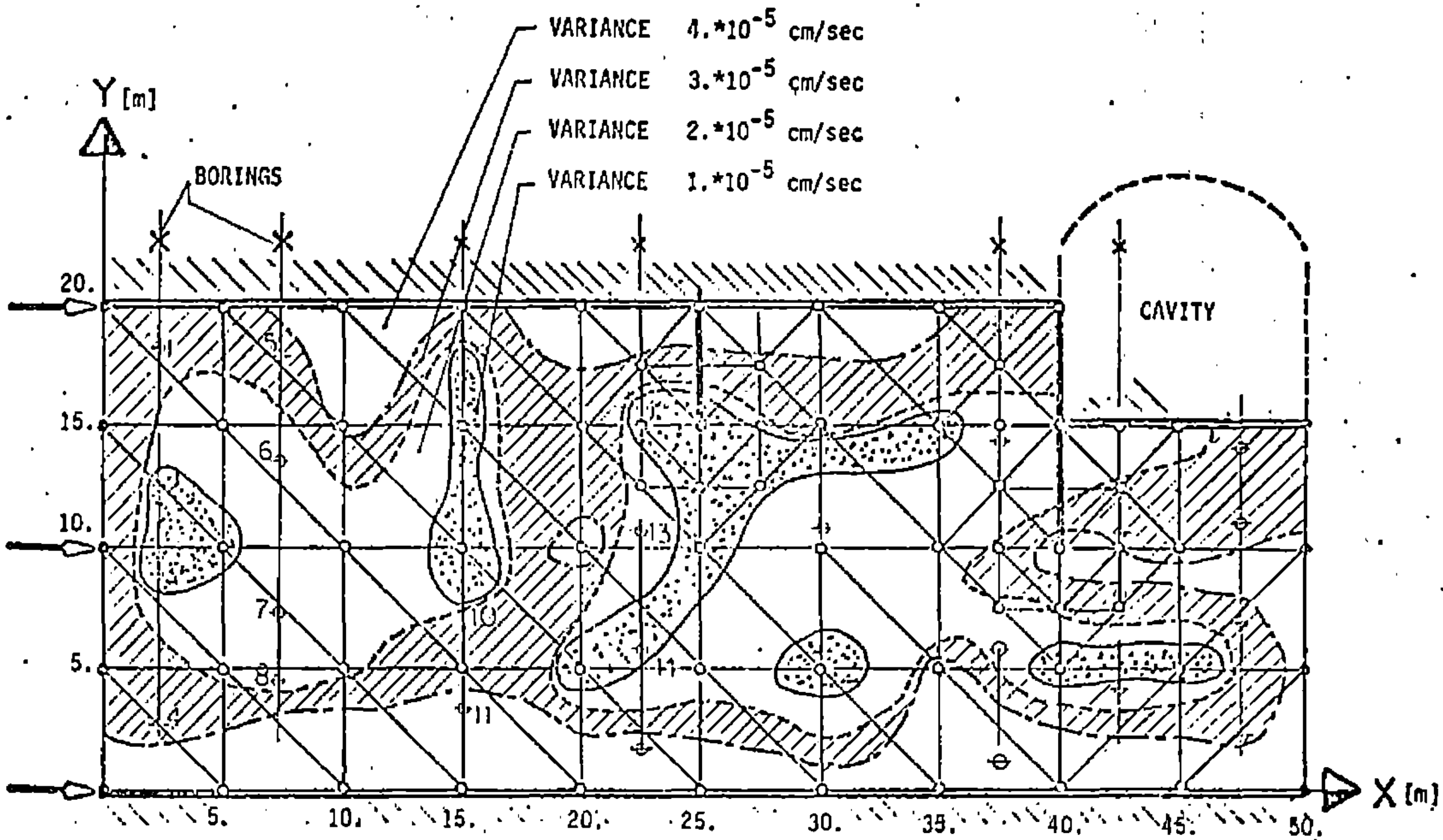

FIGURE 11 ESTIMATED STATISTICAL CHARACTERISTICS OF PROBLEM 2 (HORIZONTAL FLOW)

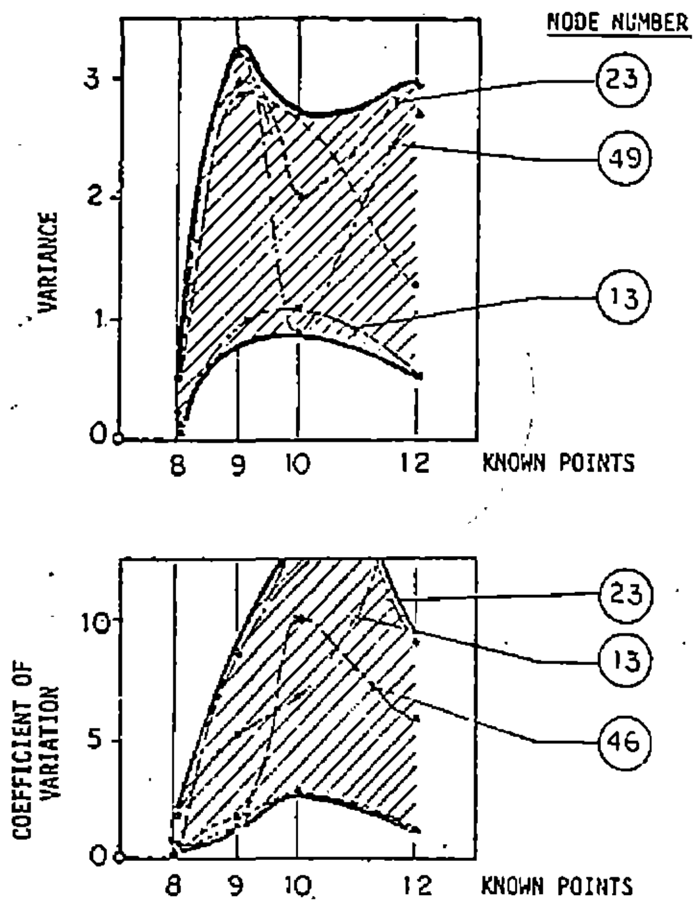

FIGURE 10 ESTIMATED VARIANCES VS. THE NUMBER OF KNOWN POINTS

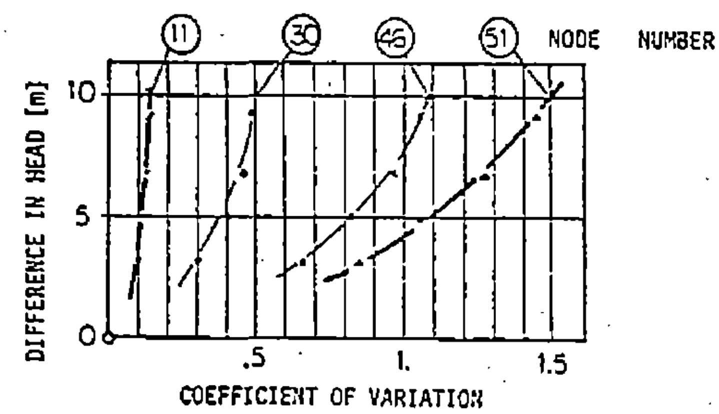

FIGURE 12 EFFECT OF BOUNDARY CONDITIONS ON ESTIMATED STATISTICAL VALUES

1. Randomness of data points

In the Inference model, the data randomness is automatically tested through the use of the functions describing the statistical constraints, Eqs. 13 and 15. Indeed, if the provided information does not follow a random pattern, numerical instability is induced which shows that complementary information is needed.

2. Uniform sample volume

It is recognized that a basic requirement for the successful use of statistics in sampling is the uniformity of the sample volume. The Inference model accounts for this size effect through the use of the notion of variogram, (sect. 2). The variogram permits to determine the range of significant statistical inference around a given point. On the other hand [illegible] was seen that a one meter boring was sufficient to produce a good statistical description of the physical characteristics of the rock. Therefore, it is proposed here, to use the average values obtained from one meter of borings as the input information to the Inference model, as illustrated in Figure 14.

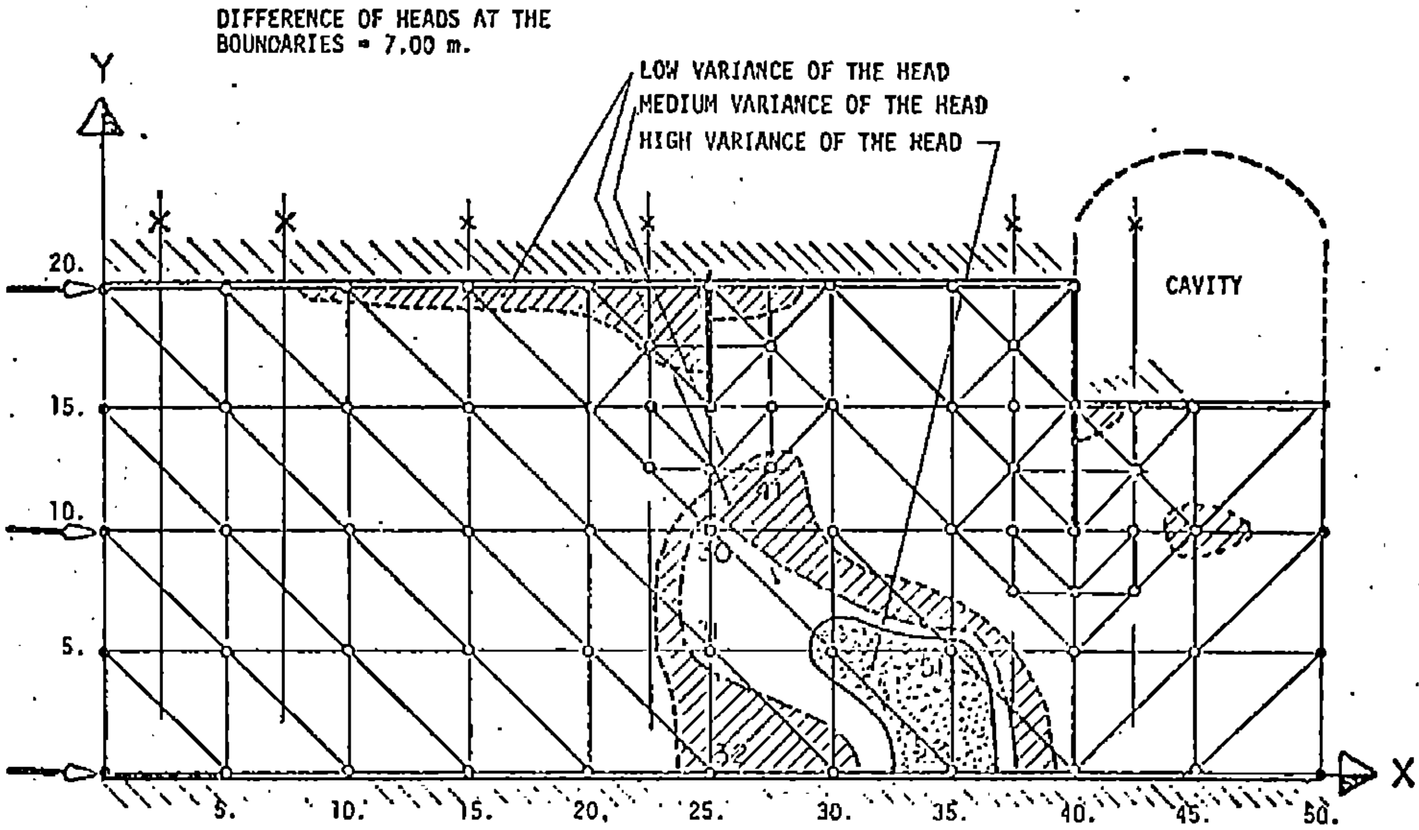

FIGURE 13 RESULTS OF THE INFERENCE MODEL COUPLED WITH THE F.E.M. (HORIZONTAL FLOW)

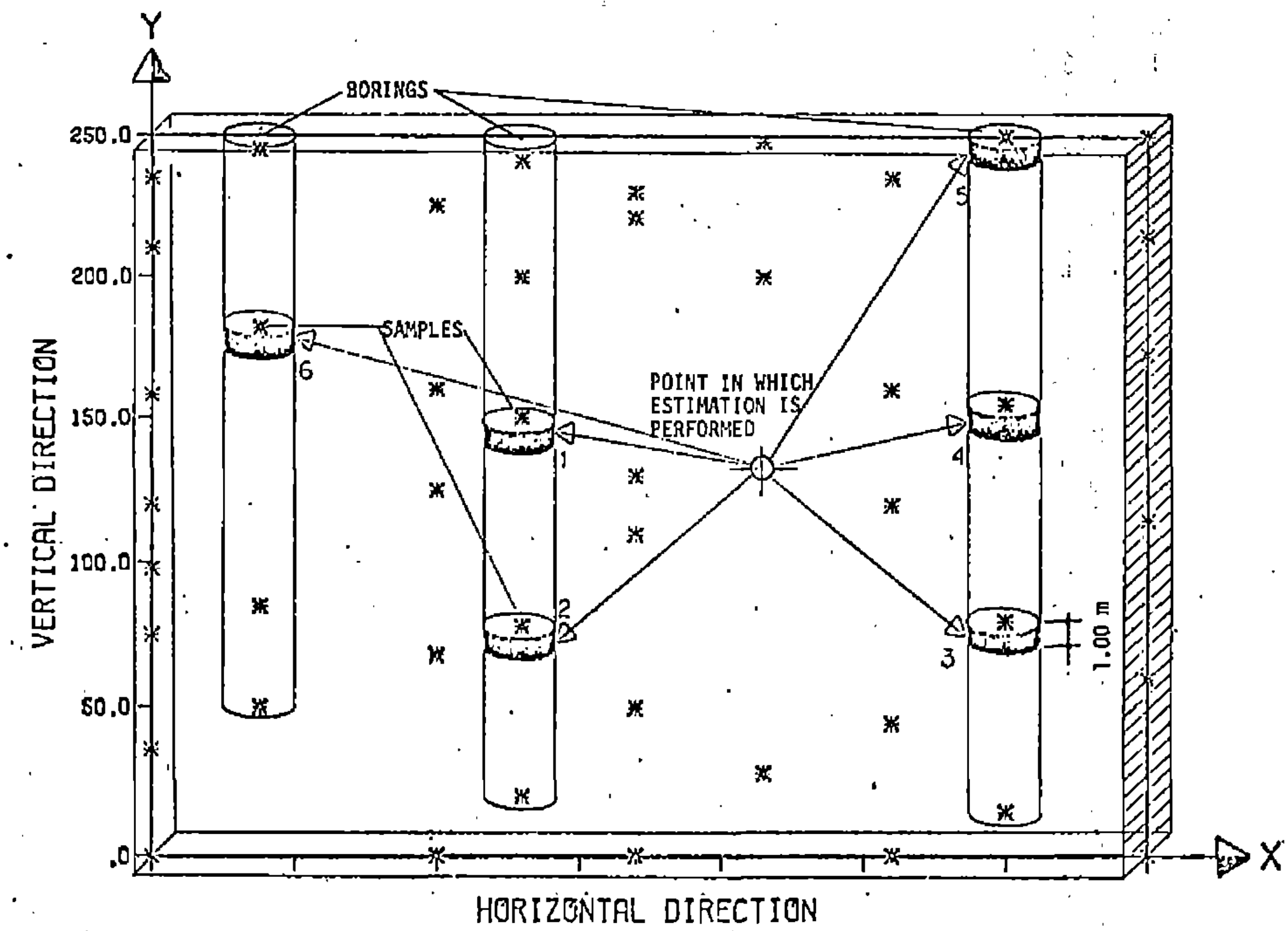

FIGURE 14 ILLUSTRATION OF THE ESTIMATION PROCEDURES

REFERENCES

[1] Becker, R. M. et al., "Particle Statistics of Infinite Populations as Applied to Mine Sampling," Bureau of Mines, Rept. of Investigations 5669, 1961.

[2] Davis, J. C., "Statistics and Data Analysis in Geology", J. Wiley, 1973.

[3] Krumbein, W. C., "Analytic Geology", McGraw-Hill Company, 1963.

[4] Krige, D. G., "Two-Dimensional Weighted Moving Average Trend Surfaces For Ore Evaluation", Proc. Symposium on Mathematical Statistics and Computer Applications in Ore Evaluation, Johannesburg, 1966, pp. 13-38.

[5] Matheron, G., "Le Krigeage Universel", Les cahiers du centre de Morphologie Mathematique de Foutainedean, Fascicule 1, 1969.

[6] Popoulis, A., "Probability, Random Variables and Stochastic Processes", McGraw-Hill Book Co., New York, 1965.

[7] Whitten, E. H. T., "The General Linear Equation in Prediction of Gold Content in Witwatersand Rocks", South Africa, Proc., Symposium on Mathematical Statistics and Computer Applications in Ore Evaluation, Johannesburg, 1966.

Appendix A

Computations Related to the Inference Model

1. Relationship between Covariance and Variogram

Using the general expression of the second statistical moment one can obtain

$$E\left[\left[(Z_x-Z_y) - (\hat{Z}_x-\hat{Z}_y)\right]^2\right] = E\left[(Z_x-Z_y)^2\right] + E\left[(\hat{Z}_x-\hat{Z}_y)^2\right] - 2E\left[(Z_x-Z_y)(\hat{Z}_x-\hat{Z}_y)\right] =$$

$$= E(Z_x-Z_y)\ E(Z_x-Z_y) + C(Z_x-Z_y,Z_x-Z_y) + E(\hat{Z}_x-\hat{Z}_y)\ E(\hat{Z}_x-\hat{Z}_y) + C(\hat{Z}_x-\hat{Z}_y,\hat{Z}_x-\hat{Z}_y) - 2E(\hat{Z}_x-\hat{Z}_y)\ E(Z_x-Z_y) - 2C(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y)$$

where: C = the covariance of variables Z and $\hat{Z}$

which after a few simplifications leads to

$$E\left[(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y)^2\right] = C(Z_x-Z_y,Z_x-Z_y) + C(\hat{Z}_x-\hat{Z}_y,\hat{Z}_x-\hat{Z}_y) - 2C(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y) =$$

$$= 2C(Z_x-Z_y,Z_x-Z_y) - 2C(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y)$$

Making use of the general statistical assumptions given in section (A.3) we obtain

$$\frac{E\left[\left[(Z_x-Z_y) - (\hat{Z}_x-\hat{Z}_y)\right]^2\right]}{2} = C(0) - C(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y)$$

which leads to the expression of the variogram γ

$$\gamma[(Z_x-Z_y),(\hat{Z}_x-\hat{Z}_y)] = C(0) - C(Z_x-Z_y,\hat{Z}_x-\hat{Z}_y)$$

or

$$\gamma(d) = C(0) - C(d) \qquad (A.1)$$

2. Optimization Using Lagrange Multipliers

The problem consists to minimize $E[(z-\hat{z})^2]$ the variance with the constraint $E[z(x,y)] - E[\hat{z}(x,y)] = 0$.

Using the assumptions introduced in section (5.3.1) we obtain

$$E[(z-\hat{z})^2] = C(z,z) + \sum_{\alpha} b_{\alpha} \sum_{\beta} b_{\beta}[C(z_{\alpha},Z_{\beta}] - 2 \sum_{\alpha} b_{\alpha} C(z,z_{\alpha}) \quad (A.2)$$

and

$$E[z] - E[\hat{z}] = \sum_{\ell} a_{\ell} f^{\ell} - \sum a_{\ell}[\sum_{\alpha} b^{\alpha} f^{\ell}_{\alpha}] =$$

$$= \sum_{\ell} a_{\ell}[f^{\ell} - \sum_{\alpha} b^{\alpha} f^{\ell}_{\alpha}] = 0; \quad (A.3)$$

$$\alpha = 1,\dots,n \qquad \ell = 1,\dots,k$$

The minimization of the variance will be obtained using the method of Lagrange multipliers as follows.

The Lagrange function being

$$L = C(z,z) - 2\sum_{\alpha} b_{\alpha} C(z,z_{\alpha}) + \sum_{\alpha,\beta} b^{\alpha}b^{\beta} C(z_{\alpha},Z_{\beta}) -$$

$$- \sum_{\ell} (\mu_{\ell})[f^{\ell} - \sum_{\alpha} b^{\alpha} f^{\ell}_{\alpha}]; \quad (A.4)$$

$$\alpha,\beta = 1,\dots,n \qquad \ell = 1,\dots,k$$

The conditions to obtain the minimum are

$$\frac{\partial L}{\partial b^{\alpha}} = 0 \quad \text{for all } b\text{ 's} \quad (A.5)$$

$$\frac{\partial L}{\partial m_{\ell}} = 0 \quad \text{for all } \mu\text{ 's}$$

The unknowns being b 's and μ 's we obtain a linear system of α+k equations.

The differentiation of L with respect to b^α and M_ℓ gives:

First with respect to b^α.

$$-2\ C(z,z_\alpha) + \sum_\beta b^\beta\ C(z_\alpha,z_\beta) + \sum_\ell \mu_\ell\ f_\alpha^\ell = 0 \qquad \forall\ \alpha = 1,n \qquad (A.7)$$

Second with respect to μ_ℓ.

$$-f^\ell + \sum_\alpha b^\alpha\ f_\alpha^\ell = 0 \qquad \forall\ \ell = 1,k \qquad (A.8)$$

The system then can be written as

$$\sum_\beta b^\beta\ C(z_\alpha,z_\beta) + \sum_\ell \mu_\ell\ f_\alpha^\ell = 2\ C(z_\alpha,z)$$

$$\sum_\alpha b^\alpha\ f_\alpha^\ell = f^\ell \qquad (A.9)$$

The covariances $c(z_\alpha,z_\beta)$ and $c(z_\alpha,z)$ are obtained from the following relations:

$$C(z_\alpha,z) = -\gamma(Fz_\alpha - Fz) \text{ and } C(z_\alpha,z_\beta) = -\gamma(Fz_\alpha - Fz_\beta)$$

Then the linear system of equations becomes

$$\sum_\beta b^\beta\ \gamma(z_\alpha - z_\beta) + \sum_\ell \mu_\ell\ f_\alpha^\ell = 2\ \gamma(z_\alpha - z)$$

$$\sum_\beta b^\beta\ f^\ell(z_\beta) = f^\ell(z) \qquad (A.10)$$

where the b's and μ's are the unknown quantities. Therefore, solving this system the estimator of the variable is defined by:

$$\hat{Z}(x,y) = \sum b^\beta\ Z_\beta \qquad (A.11)$$

and the variance of the estimate is

$$\sigma_{\hat{z}}^2 = \sum_\beta b^\beta\ \gamma(z_\beta - z) + \sum_\ell \mu_\ell\ f^\ell(z) \qquad (A.12)$$

```
      PROGRAM INFMOD (INPUT,OUTPUT,PLOT,TAPE1)
C
C     ------------------------------------------------------------------
C
C                                                     VERSION  DN1677
C
C     THIS PROGRAM  PERFORMS  AN  INFERENCE  MODEL  ANALYSIS
C     BASED  ON A  MOVING  AVERAGE  TECHNIQUE
C
C     INPUT  DATA
C
C         DELTA    = MESH  INTERVAL
C         NPL      = 1      PLOT  REQUIRED
C                  = 0      WITHOUT PLOT
C         W        = INITIAL  PARAMETER  OF  VARIOGRAM
C         CO       = INITIAL  DISCONTINUITY
C
C
C         NPGIV    = TOTAL  NUMBER  OF  GIVEN  POINTS
C         X(I),Y(I)= COORDINATES  OF  GIVEN  POINT (I)
C         Z(I)     = SPECIFIED   PHYSICAL PARAMETER AT LOCATION (I)
C
C     OUTPUT  OF  PROGRAM
C
C         AT  LOCATIONS SPECIFIED BY THE NODES OF THE GIVEN
C         MESH,  THE FIRST AND SECOND STATISTICAL MOMENTS
C         OF THE PHYSICAL PARAMETER UNDER INVESTIGATION
C         ARE ESTIMATED
C     --------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
      COMMON /B1/ W,CO,N,NNN
      COMMON /BBB/ BB(20,20),CV(20,20)
C
C     READ  INPUT  DATA
C     ==========================================
C
      READ 110, TESTS
  101 CONTINUE
      READ 114, DELTA,NPL
      READ 115, W,CO
      PRINT 111
C
      READ 113, NPGIV
      IF (NPGIV.EQ.0) GO TO 109
      DO 102 I=1,NPGIV
C
C
         READ 116, IDNC,X(I),Y(I)
  102 CONTINUE
C
      DO 103 I=1,NPGIV
  103 READ 116, IDN,Z(I)
C
      DO 104 I=1,NPGIV
  104 PRINT 115, X(I),Y(I),Z(I)
C
      PRINT 111
C
      CALL GRAPH (1,NPGIV,0,X,Y)
C
C     ==========================================
C
C     INITIALIZE
C     ==========================================
C
      DO 105 I=1,160
```

```
          NUM(I)=0
          NUMTRI(I,1)=0
          NUMTRI(I,2)=0
          NG(I)=0
          ND(I)=0
          NPOINTE(I)=0
          NUMEL(I)=0
  105 CONTINUE
      NPLA(1)=0
      NPLA(2)=0
      DO 106 J=1,32
          NHEIG(J)=0
  106 DIST(J)=0.
      DO 107 I=1,4
          XENT(I)=0.
          YENT(I)=0.
      DO 107 J=1,8
          NTR(I,J)=0
          NHEIG1(J)=0
          DIST1(J)=0.
  107 CONTINUE
C
C
C     DETERMINATION  OF  THE  MESH
C     ==========================================
C
C
      XMAX=X(1)
      XMIN=XMAX
      YMAX=Y(1)
      YMIN=YMAX
      DO 108 I=1,NPGIV
         X1=X(I)
         Y1=Y(I)
         IF (X1.LE.XMIN) XMIN=X1
         IF (Y1.LE.YMIN) YMIN=Y1
         IF (X1.GE.XMAX) XMAX=X1
         IF (Y1.GE.YMAX) YMAX=Y1
  108 CONTINUE
C
      PRINT 112, XMIN,XMAX,YMIN,YMAX
C
C     -------------------------------------------------------------
C
      CALL OPER (XAH)
C
C     -------------------------------------------------------------
C
C
C     IF (NPL. EQ. 0 ) GO TO  999
C     -   ------------------------------------------------
C     CALL PLT ( XAH,XMAX,XMIN,NINTX,NINTY )
C     --------------------------------------------------
C
      GO TO 101
C
C
  109 STOP
C
  110 FORMAT (A3)
  111 FORMAT (1H1)
  112 FORMAT (/2X,8F10.4)
  113 FORMAT (8I10)
  114 FORMAT (F10.0,I2)
  115 FORMAT (2F10.3,F15.4)
  116 FORMAT (I2,2F10.3)
C
      END
      SUBROUTINE OPER (XAH)
C
C     -------------------------------------------------------------
```

```
C
 101    CONTINUE
 102 CONTINUE
C
      RETURN
C
 103 FORMAT (5X,   7H   X    ,2X,   7H   Y    ,2X,   1H*,2X, 10HESTIMATED ,2
    1X, 10HESTIMATED ,2X,   1H*,2X, 12HCOEFFICIENT )
 104 FORMAT (5X,   7H        ,2X,   7H        ,2X,   1H*,2X, 10HMEAN      ,2
    1X, 10HVARIANCE  ,2X,   1H*,2X, 12HOF VARIATION)
 105 FORMAT (5X, 61H==================================================
    1===========,//)
 106 FORMAT (5X,F7.3,2X,F7.3,2X,  1H*,2X,F10.3,2X,F10.3,2X,  1H*,2X,F12
    1.3)
C
      END
      SUBROUTINE PLT (XAH,XMAX,XMIN,NINTX,NINTY)
C
C     *******************************************
C
C     THIS  SUBROUTINE  PLOTS  THE  ESTIMATED  MEAN  VALUES
C
C
      DIMENSION XAH(20), YAH(20), WAH(300,4), DAH(20,20,2), TL(8)
      COMMON /BBB/ BB(20,20),CV(20,20)
C
C
C     PLOTTING       THE  COMPUTED  VALUES
C     ========================================
C
      MAXDIM=300
      DO 101 I=1,NINTY
      DO 101 J=1,NINTX
         DAH(I,J,1)=BB(I,J)
         DAH(I,J,2)=CV(I,J)
 101 CONTINUE
C
      XLNTH=6.
      YLNTH=4.
C
C     DELTAX = XMAX / XLNTH
C
      DELTAX=2.
      N1=NINTX
      NFNS=NINTY
C
      CALL PLOTS
C
      DO 104 L=1,2
         NNG=0
         READ 105, TL
         YMAH=0.
         DELTAY=2.
C
         DO 103 I=1,NFNS
            DO 102 J=1,NINTX
               YAH(J)=DAH(I,J,L)
 102        YAH(J)=DAH(I,J,L)
C
C     -----------------------------------------------------------------
C
            CALL HIDE (XAH,YAH,WAH(1,1),WAH(1,2),WAH(1,3),WAH(1,4),NNG,M
     1      AXDIM,N1,NFNS,TL,XLNTH,YLNTH,XMIN,DELTAX,YMAH,DELTAY)
C
C     -----------------------------------------------------------------
C
 103     CONTINUE
         CALL PLOT (14.,-2.,-3)
 104 CONTINUE
      CALL PLOT (0,0,999)
      RETURN
```

```
C
  105 FORMAT (8A10)
C
      END
      SUBROUTINE NUMB
C
C     ----------------------------------------------------------------
C     NUMBERING  OF  GIVEN  POINTS
C     ----------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      NINTX=(XMAX-XMIN)/DELTA+1
      XMAX=XMIN+NINTX*DELTA
      XMAX1=XMAX-XMIN
      NNDX=NINTX+1
      NINTY=(YMAX-YMIN)/DELTA+1
      YMAX=YMIN+NINTY*DELTA
      YMAX1=YMAX-YMIN
      NNDY=NINTY+1
C
C     PRINT 100, NINTX, NINTY
C     100  FORMAT( /2X,10I10)
C
      L4=4*NINTX
      LY=NINTY-1
      DO 104 I=1,NPGIV
         XI=X(I)-XMIN
         YI=Y(I)-YMIN
         IX=XI/DELTA
         IY=YI/DELTA
         IF (XI.LT.XMAX1) GO TO 101
         NUM(I)=(IY+1-IY/NINTY)*L4-1
         GO TO 103
  101    IF (YI.LT.YMAX1) GO TO 102
         NUM(I)=L4*LY+4*IX+3
         GO TO 103
  102    XN=XI-IX*DELTA
         YN=YI-IY*DELTA
         I1=1
         IF (XN.GT.YN) I1=2
         I2=0
         IF (YN.GT.(DELTA-XN)) I2=-5
         NUM(I)=IY*L4+4*IX+IABS(I1+I2)
  103    CONTINUE
  104 CONTINUE
      RETURN
C
      END
      SUBROUTINE SORT
C
C     ----------------------------------------------------------------
C     SORTING  THE  POINTS  IN  INCREASING  ORDER
C     ----------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
C
      JM=1
      N=NPGIV-1
      DO 105 I=1,N
         IA=I+1
         LI=I
         IF (NUM(IA).GE.NUM(LI)) GO TO 103
  101    CONTINUE
         IF (NG(LI).EQ.0) GO TO 102
```

```
      L1=LI
      LI=NG(LI)
      IF (NUM(IA).LE.NUM(LI)) GO TO 101
      NG(IA)=NG(L1)
      ND(IA)=ND(L1)
      NG(L1)=IA
      ND(LI)=IA
      GO TO 105
102   JM=IA
      NG(LI)=JM
      ND(IA)=LI
      GO TO 105
103   IF (ND(LI).EQ.0) GO TO 104
      L1=LI
      LI=ND(LI)
      IF ((NUM(IA)).GE.(NUM(LI))) GO TO 103
      ND(IA)=ND(L1)
      NG(IA)=NG(LI)
      ND(L1)=IA
      NG(LI)=IA
      GO TO 105
104   ND(LI)=IA
      NG(IA)=LI
  105 CONTINUE
      DO 106 K=1,NPGIV
         NUMTRI(K)=NUM(JM)
         NUMTRI(K,2)=JM
         JM=ND(JM)
  106 CONTINUE
      RETURN
C
      END
      SUBROUTINE INITL
C
C     ------------------------------------------------------------------
C     EVALUATION  OF  THE  PLACE  OCCUPIED  BY  THE  FIRST  NODE
C     OF  A  LINE  INSIDE  THE  VECTOR  NUMTRI
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      IND=1
      IND1=1
      L1=L4
      DO 104 I=1,NINTY
         IF (NUMTRI(IND).GT.L1) GO TO 103
         NPOINTE(I)=IND
         DO 102 J=1,1000
            IND=IND+1
            IF (IND.LE.NPGIV) GO TO 101
            K=I+1
            GO TO 105
101         CONTINUE
            IF (NUMTRI(IND).GT.L1) GO TO 103
102      CONTINUE
103      L1=L1+L4
  104 CONTINUE
      GO TO 107
  105 CONTINUE
      DO 106 I=K,NINTY
         NPOINTE(I)=0
  106 CONTINUE
  107 NPOINTE(NNDY)=99999
      RETURN
C
      END
      SUBROUTINE SEARCH (XT,YT,NX,NY)
```

```
C
C     ------------------------------------------------------------------
C     SEARCH  OF  THE  EIGT  GIVEN  POINTS  AROUND  THE  EXAMINED NODE
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPGINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,KS,NBEG
      COMMON /B3/ NKRIG,NADR
C
      LNX=NINTX-NX
      KNY=NINTY-NY
      DO 101 I=1,4
      DO 101 J=1,8
         NTR(I,J)=0
  101 CONTINUE
      IF ((NX.GE.NNDX).OR.(NY.GE.NNDY)) GO TO 102
C
C     ------------------------------------------------------------------
C
      CALL SECT1 (NX,NY)
C
C     ------------------------------------------------------------------
C
      CALL SECT2 (NX,NY)
C
C     ------------------------------------------------------------------
C
  102 CONTINUE
      IF ((NX.LE.0).OR.(NY.GE.NNDY)) GO TO 103
C
C     ------------------------------------------------------------------
C
      CALL SECT3 (NX,NY)
C
C     ------------------------------------------------------------------
C
      CALL SECT4 (NX,NY)
C
C     ------------------------------------------------------------------
C
  103 CONTINUE
      IF ((NX.LE.0).OR.(NY.LE.0)) GO TO 104
C
C     ------------------------------------------------------------------
C
      CALL SECT5 (NX,NY)
C
C     ------------------------------------------------------------------
C
      CALL SECT6 (NX,NY)
C
C     ------------------------------------------------------------------
C
  104 CONTINUE
      IF ((NY.LE.0).OR.(NX.GE.NNDX)) GO TO 105
C
C     ------------------------------------------------------------------
C
      CALL SECT7 (NX,NY)
C
C     ------------------------------------------------------------------
C
      CALL SECT8 (NX,NY)
C
C     ------------------------------------------------------------------
C
  105 CONTINUE
      NKRIG=0
```

```
      NADR=0
C
C     ------------------------------------------------------------------
C
      CALL EIGT (XT,YT,0)
C
C     ------------------------------------------------------------------
C
      RETURN
C
      END
      SUBROUTINE SECT1 (NX,NY)
C
C     ------------------------------------------------------------------
C     SECTOR  FROM  0.  TO  PI /4
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      INUM=2
      NTROUV=0
      NLIGN=NTROUV
      NCOL=NLIGN
      NUMEL(1)=L4*NY+4*NX+2
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)+1
C
C     ------------------------------------------------------------------
C
      CALL LOOK1 (INUM,1,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NCOL=NCOL+1
      IF (NCOL.GE.LNX) RETURN
      NLIGN=0
      NC4=4*NCOL
      NELEM4=NELEM+NC4-L4-1
      DO 103 I=1,NCOL
         INUM=4*I
         NUMEL(INUM-3)=NELEM4+L4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)+1
         NUMEL(INUM-1)=NUMEL(INUM-2)+1
         NUMEL(INUM)=NUMEL(INUM-1)+1
         NLIGN=NLIGN+1
         IF (NLIGN.GE.KNY) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-4)+L4
      GO TO 101
C
C     ------------------------------------------------------------------
C
  104 CALL LOOK1 (INUM,1,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NCOL.LT.LNX) GO TO 102
      RETURN
C
      END
      SUBROUTINE SECT4 (NX,NY)
C
C     ------------------------------------------------------------------
```

```
C       SECTOR  FROM  3 PI/4  TO  PI
C       ---------------------------------------------------------------
C
        COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
       1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
       2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
        COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
       11(8),LNX,KNY,K8,NBEQ
C
        INUM=2
        NTROUV=0
        NLIGN=NTROUV
        NCOL=NLIGN
        NUMEL(1)=L4*NY+4*NX-3
        NELEM=NUMEL(1)
  101   NUMEL(INUM)=NUMEL(INUM-1)+1
C
C       ---------------------------------------------------------------
C
        CALL LOOK1 (INUM,4,NLIGN,NY,NTROUV,1)
C
C       ---------------------------------------------------------------
C
        IF (NTROUV.EQ.4) RETURN
  102   CONTINUE
        NCOL=NCOL+1
        IF (NCOL.GE.NX) RETURN
        NLIGN=0
        NC4=4*NCOL
        NELEM4=NELEM-L4-NC4
        DO 103 I=1,NCOL
           INUM=4*I
           NUMEL(INUM-3)=NELEM4+L4*I
           NUMEL(INUM-2)=NUMEL(INUM-3)+1
           NUMEL(INUM-1)=NUMEL(INUM-2)+1
           NUMEL(INUM)=NUMEL(INUM-1)+1
           NLIGN=NLIGN+1
           IF (NLIGN.GE.KNY) GO TO 104
  103   CONTINUE
        INUM=INUM+2
        NUMEL(INUM-1)=NUMEL(INUM-4)+L4-1
        GO TO 101
C
C       ---------------------------------------------------------------
C
  104   CALL LOOK1 (INUM,4,NLIGN,NY,NTROUV,1)
C
C       ---------------------------------------------------------------
C
        IF (NTROUV.EQ.4) RETURN
        IF (NCOL.LT.NX) GO TO 102
        RETURN
C
        END
        SUBROUTINE SECT5 (NX,NY)
C
C       ---------------------------------------------------------------
C       SECTOR  FROM  PI  TO  5  PI/4 .
C       ---------------------------------------------------------------
C
        COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
       1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
       2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
        COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
       11(8),LNX,KNY,K8,NBEQ
C
        INUM=2
        NTROUV=0
        NLIGN=NTROUV
        NCOL=NLIGN
        NUMEL(1)=L4*NY-L4+4*NX
```

```
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)-3
C
C     ----------------------------------------------------------------
C
      CALL LOOK1 (INUM,5,NLIGN,NY,NTROUV,-1)
C
C     ----------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NCOL=NCOL+1
      IF (NCOL.GE.NX) RETURN
      NLIGN=0
      NC4=4*NCOL
      NELEM4=NELEM-NC4+L4
      DO 103 I=1,NCOL
         INUM=4*I
         NUMEL(INUM-3)=NELEM4-L4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)-1
         NUMEL(INUM-1)=NUMEL(INUM-2)-1
         NUMEL(INUM)=NUMEL(INUM-1)-1
         NLIGN=NLIGN+1
         IF (NLIGN.GE.NY) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-5)-L4
      GO TO 101
C
C     ----------------------------------------------------------------
C
  104 CALL LOOK1 (INUM,5,NLIGN,NY,NTROUV,-1)
C
C     ----------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NCOL.LT.NX) GO TO 102
      RETURN
C
      END
      SUBROUTINE SECT8 (NX,NY)
C
C     ----------------------------------------------------------------
C     SECTOR  FROM  7 PI/4  TO 2 PI
C     ----------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,MENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      INUM=2
      NTROUV=0
      NLIGN=NTROUV
      NCOL=NLIGN
      NUMEL(1)=L4*NY-L4+4*NX+4
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)-1
C
C     ----------------------------------------------------------------
C
      CALL LOOK1 (INUM,8,NLIGN,NY,NTROUV,-1)
C
C     ----------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NCOL=NCOL+1
      IF (NCOL.GE.LNX) RETURN
      NLIGN=0
```

```
      NC4=4*NCOL
      NELEM4=NC4+NELEM+L4
      DO 103 I=1,NCOL
         INUM=4*I
         NUMEL(INUM-3)=NELEM4-L4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)-1
         NUMEL(INUM-1)=NUMEL(INUM-2)-1
         NUMEL(INUM)=NUMEL(INUM-1)-1
         NLIGN=NLIGN+1
         IF (NLIGN.GE.NY) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-4)-L4+1
      GO TO 101
C
C     ------------------------------------------------------------------
C
  104 CALL LOOK1 (INUM,8,NLIGN,NY,NTROUV,-1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NCOL.LT.LNX) GO TO 102
      RETURN
C
      END
      SUBROUTINE SECT2 (NX,NY)
C
C     ------------------------------------------------------------------
C     SECTOR  FROM  PI/4  TO  PI/2
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIGI(8),DIST
     1I(8),LNX,KNY,K8,NBEG
C
      INUM=2
      NTROUV=0
      NCOL=NTROUV
      NLIGN=NCOL
      NUMEL(1)=L4*NY+4*NX+1
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)+3
C
C     ------------------------------------------------------------------
C
      CALL LOOK2 (INUM,2,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NLIGN=NLIGN+1
      IF (NLIGN.GE.KNY) RETURN
      NCOL=1
      NELEM4=NELEM+L4*NLIGN-4
      DO 103 I=1,NLIGN
         INUM=4*I
         NUMEL(INUM-3)=NELEM4+4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)+1
         NUMEL(INUM-1)=NUMEL(INUM-2)+1
         NUMEL(INUM)=NUMEL(INUM-1)+1
         NCOL=NCOL+1
         IF (NCOL.GT.LNX) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-2)+1
      GO TO 101
C
```

```
C     ------------------------------------------------------------------
C
  104 CALL LOOK2 (INUM,2,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NLIGN.LT.KNY) GO TO 102
      RETURN
C
      END
      SUBROUTINE SECT3 (NX,NY)
C
C     ------------------------------------------------------------------
C     SECTOR  FROM  PI/2  TO  3 PI/4
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      INUM=2
      NTROUV=0
      NCOL=NTROUV
      NLIGN=NCOL
      NUMEL(1)=L4*NY+4*NX-1
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)+1
C
C     ------------------------------------------------------------------
C
      CALL LOOK2 (INUM,3,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NLIGN=NLIGN+1
      IF (NLIGN.GE.KNY) RETURN
      NCOL=1
      NLI4=L4*NLIGN
      NELEM4=NLI4+NELEM+4
      DO 103 I=1,NLIGN
         INUM=4*I
         NUMEL(INUM-3)=NELEM4-4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)-1
         NUMEL(INUM-1)=NUMEL(INUM-2)-1
         NUMEL(INUM)=NUMEL(INUM-1)+3
         NCOL=NCOL+1
         IF (NCOL.GT.NX) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-4)-3
      GO TO 101
C
C     ------------------------------------------------------------------
C
  104 CALL LOOK2 (INUM,3,NLIGN,NY,NTROUV,1)
C
C     ------------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NLIGN.LT.KNY) GO TO 102
      RETURN
C
      END
      SUBROUTINE SECT4 (NX,NY)
C
C     ------------------------------------------------------------------
```

```
C      SECTOR  FROM  5 PI/   TO  3 PI/2
C      ----------------------------------------------------------------
C
       COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
      1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
      2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
       COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
      11(8),LNX,KNY,K8,NBEQ
C
       INUM=2
       NTROUV=0
       NCOL=NTROUV
       NLIGN=NCOL
       NUMEL(1)=L4*NY-L4+4*NX-1
       NELEM=NUMEL(1)
   101 NUMEL(INUM)=NUMEL(INUM-1)-1
C
C      ----------------------------------------------------------------
C
       CALL LOOK2 (INUM,6,NLIGN,NY,NTROUV,-1)
C
C      ----------------------------------------------------------------
C
       IF (NTROUV.EQ.4) RETURN
   102 CONTINUE
       NLIGN=NLIGN+1
       IF (NLIGN.GE.NY) RETURN
       NCOL=1
       NLI4=NLIGN*L4
       NELEM4=NELEM-NLI4+4
       DO 103 I=1,NLIGN
           INUM=4*I
           NUMEL(INUM-3)=NELEM4-4*I
           NUMEL(INUM-2)=NUMEL(INUM-3)-1
           NUMEL(INUM-1)=NUMEL(INUM-2)-1
           NUMEL(INUM)=NUMEL(INUM-1)+3
           NCOL=NCOL+1
           IF (NCOL.GT.NX) GO TO 104
   103 CONTINUE
       INUM=INUM+2
       NUMEL(INUM-1)=NUMEL(INUM-4)-3
       GO TO 101
C
C      ----------------------------------------------------------------
C
   104 CALL LOOK2 (INUM,6,NLIGN,NY,NTROUV,-1)
C
C      ----------------------------------------------------------------
C
       IF (NTROUV.EQ.4) RETURN
       IF (NLIGN.LT.NY) GO TO 102
       RETURN
C
       END
       SUBROUTINE SECT7 (NX,NY)
C
C      ----------------------------------------------------------------
C      SECTOR  FROM  3 PI/2 TO  7 PI/4
C      ----------------------------------------------------------------
C
       COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
      1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
      2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
       COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
      11(8),LNX,KNY,K8,NBEQ
C
       INUM=2
       NTROUV=0
       NCOL=NTROUV
       NLIGN=NCOL
       NUMEL(1)=L4*NY-L4+4*NX+1
```

```
      NELEM=NUMEL(1)
  101 NUMEL(INUM)=NUMEL(INUM-1)+1
C
C     -----------------------------------------------------------------
C
      CALL LOOK2 (INUM,7,NLIGN,NY,NTROUV,-1)
C
C     -----------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
  102 CONTINUE
      NLIGN=NLIGN+1
      IF (NLIGN.GE.NY) RETURN
      NCOL=1
      NLI4=L4*NLIGN
      NELEM4=NELEM-NLI4-4
      DO 103 I=1,NLIGN
         INUM=4*I
         NUMEL(INUM-3)=NELEM4+4*I
         NUMEL(INUM-2)=NUMEL(INUM-3)+1
         NUMEL(INUM-1)=NUMEL(INUM-2)+1
         NUMEL(INUM)=NUMEL(INUM-1)+1
         NCOL=NCOL+1
         IF (NCOL.GT.LNX) GO TO 104
  103 CONTINUE
      INUM=INUM+2
      NUMEL(INUM-1)=NUMEL(INUM-2)+1
      GO TO 101
C
C     -----------------------------------------------------------------
C
  104 CALL LOOK2 (INUM,7,NLIGN,NY,NTROUV,-1)
C
C     -----------------------------------------------------------------
C
      IF (NTROUV.EQ.4) RETURN
      IF (NLIGN.LT.NY) GO TO 102
      RETURN
C
      END
      SUBROUTINE LOOK1 (INUM,NUMSEC,NLIGN,NY,NTROUV,KDIR)
C
C     -----------------------------------------------------------------
C     SEARCHING  THE  GIVEN  INFORMATION  IN  SECTORS  1,4,5,8
C     -----------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,KS,NBEQ
C
      KR=1
      NOLI=NY+(1-KDIR)/2
      IF (NLIGN.EQ.0) GO TO 107
      DO 106 I=1,NLIGN
         NOLI=NOLI+KDIR
         IF (NOLI.GE.NNDY) RETURN
         NUMP=NPOINTE(NOLI)
         IF (NUMP.NE.0) GO TO 101
         KR=KR+4
         GO TO 106
  101    NUMP4=NUMP+L4
         NUMPA=MIN0(NUMP4,NPGIV)
         JJ=0
         DO 105 J=1,100
            JJ=JJ+1
            IF (JJ.GE.5) GO TO 106
            DO 103 J1=NUMP,NUMPA
               IF (NUMTRI(J1,1)-NUMEL(KR)) 103,102,104
  102          NTROUV=NTROUV+1
```

```
                NTR(NTROUV,NUMSEC)=NUMTRI(J1,2)
                KR=KR-1
                JJ=JJ-1
                NUMP=J1+1
                IF (NTROUV.EQ.4) RETURN
                GO TO 105
  103       CONTINUE
  104       NUMP=NPOINTE(NOLI)
  105    KR=KR+1
  106 CONTINUE
      IF (KR.GT.INUM) RETURN
  107 NOLI=NY+KDIR*NLIGN+(1+KDIR)/2
      IF (NOLI.GE.NNDY) RETURN
      NUMP=NPOINTE(NOLI)
      IF (NUMP.EQ.0) RETURN
      NUMP4=NUMP+L4
      NUMPA=MIN0(NPGIV,NUMP4)
      II=0
      DO 111 I=1,100
         II=II+1
         IF (II.GE.3) GO TO 112
         DO 109 J=NUMP,NUMPA
            IF (NUMTRI(J,1)-NUMEL(KR)) 109,108,110
  108       NTROUV=NTROUV+1
            KR=KR-1
            II=II-1
            NUMP=J+1
            NTR(NTROUV,NUMSEC)=NUMTRI(J,2)
            IF (NTROUV.EQ.4) RETURN
            GO TO 111
  109    CONTINUE
  110    NUMP=NPOINTE(NOLI)
  111 KR=KR+1
  112 CONTINUE
      RETURN
C
      END
      SUBROUTINE LOOK2 (INUM,NUMSEC,NLIGN,NY,NTROUV,KDIR)
C
C     ------------------------------------------------------------------
C     SEARCHING  THE  GIVEN  INFORMATION  IN  SECTORS  2,3,6,7
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),ND(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,NINTX,NINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
C
      NOLI=NY+KDIR*NLIGN+(1+KDIR)/2
      IF (NOLI.GE.NNDY) RETURN
      NUMP=NPOINTE(NOLI)
      IF (NUMP.EQ.0) RETURN
      NUMP4=NUMP+L4
      NUMPA=MIN0(NUMP4,NPGIV)
      IN=0
      DO 104 I=1,160
         IN=IN+1
         IF (IN.GT.INUM) RETURN
         DO 102 J=NUMP,NUMPA
            IF (NUMTRI(J,1)-NUMEL(IN)) 102,101,103
  101       NTROUV=NTROUV+1
            NTR(NTROUV,NUMSEC)=NUMTRI(J,2)
            IF (NTROUV.EQ.4) RETURN
            IN=IN-1
            NUMP=J+1
            GO TO 104
  102    CONTINUE
  103    NUMP=NPOINTE(NOLI)
  104 CONTINUE
      RETURN
```

```
C
      END
      SUBROUTINE EIGT (XT,YT,NFLAG)
C
C     ------------------------------------------------------------------
C     DEFINES THE EIGTH CLOSEST INFORMATIONS TO THE EXAM. NODE
C     ------------------------------------------------------------------
C
      COMMON X(300),Y(300),Z(300),NUM(300),NUMTRI(300,2),NG(300),NB(300)
     1,NPOINTE(300),NPLA(2),XENT(4),YENT(4),DELTA,NPGIV,XMIN,XMAX,YMIN,Y
     2MAX,MINTX,MINTY,NNDX,NNDY,L4,NENT(4,2)
      COMMON /SEC/ NUMEL(300),NTR(4,8),NHEIG(32),DIST(32),NHEIG1(8),DIST
     11(8),LNX,KNY,K8,NBEQ
      COMMON /B3/ NKRIG,NADR
      DIMENSION NHG(8)
C
C
      IF (NFLAG.NE.0) GO TO 110
      DO 101 I=1,32
         DIST(I)=0.
         NHEIG(I)=0
  101 CONTINUE
      K8=0
      DO 102 I=1,8
         NTRI=NTR(1,I)
         IF (NTRI.EQ.0) GO TO 102
         K8=K8+1
         NHEIG(K8)=NTRI
  102 CONTINUE
      DO 103 I=1,K8
         N8I=NHEIG(I)
         DIST(I)=SQRT((X(N8I)-XT)**2+(Y(N8I)-YT)**2)
  103 CONTINUE
      DO 107 J=2,4
         JN=0
         JJ=0
         DO 104 I=1,8
            NTRI=NTR(J,I)
            IF (NTRI.EQ.0) GO TO 104
            JN=JN+1
            NHG(JN)=NTRI
C
            JJ=JJ+1
            NHEIG1(JJ)=NTRI
            DIST1(JJ)=SQRT((X(NTRI)-XT)**2+(Y(NTRI)-YT)**2)
  104    CONTINUE
         JJ1=JJ-1
         DO 105 I=1,JJ1
            IP1=I+1
         DO 105 K=IP1,JJ
            IF (DIST1(K).GE.DIST1(I)) GO TO 105
            DIST1I=DIST1(I)
            NHEIG1I=NHEIG1(I)
            DIST1(I)=DIST1(K)
            NHEIG1(I)=NHEIG1(K)
            DIST1(K)=DIST1I
            NHEIG1(K)=NHEIG1I
  105    CONTINUE
         DO 106 I=1,JJ
            K8=K8+1
            NHEIG(K8)=NHEIG1(I)
            DIST(K8)=DIST1(I)
  106    CONTINUE
  107 CONTINUE
C
      DO 108 I=1,6
         NHEIG(8+I)=NHG(I)
  108 CONTINUE
      DO 109 I=9,14
         IF (NHEIG(I).GT.0) GO TO 109
         NHEIG(I)=NHEIG(I-8)
```

```
  109 CONTINUE
C
C
      RETURN
  110 CONTINUE
      NH=NHEIG(NADR)
      NHEIG(NADR)=NHEIG(NGIV)
      NHEIG(NGIV)=NH
      IF (NHEIG(NADR).NE.0) RETURN
      NADR=NADR-1
      IF (NADR.NE.0) GO TO 111
      NBEQ=K3
      RETURN
  111 NGIV=9
      GO TO 110
C
      END
      SUBROUTINE GAUSS (IC,N,C,B,V)
C
C     ------------------------------------------------------------
C     N  = NUMBER  OF  EQUATIONS
C     C  = MATRIX  OF  COEFFICIENTS
C     B  = CONSTANT  VECTOR
C     V  =  SOLUTION  VECTOR
C     ----------------------------------------
C
      DIMENSION C(IC,IC), B(IC), V(IC)
      DATA LEVEL/1/
C
C
      IF (LEVEL.EQ.1) GO TO 102
      DO 101 I=1,N
         PRINT 112, (C(I,J),J=1,N),B(I)
  101 CONTINUE
  102 CONTINUE
C
      LAST=N-1
      DO 108 I=1,LAST
         M=I
         ITEMP=I+1
         DO 103 J=ITEMP,N
C
C     FIND  THE  ELEMENT  IN  THE  I-TH  COLUMN  WITH  THE  MAXIM.
C     ABSOLUTE  VALUE  FOR  SCALING
C     ===================================
C
            IF (ABS(C(M,I)).GT.ABS(C(J,I))) GO TO 103
            M=J
  103    CONTINUE
         IF (M.EQ.I) GO TO 105
C
C     CHANGE  M-TH  ROW  TO I-TH  ROW  SO  THAT  MAX  ELEMENT  APPEARS
C     ON  THE  DIAGONAL
C     =============================
C
         DO 104 J=I,N
            TEMP=C(M,J)
            C(M,J)=C(I,J)
  104    C(I,J)=TEMP
C
C     CONSTANT  VECTOR  IS  CHANGED
C
         TEMP=B(M)
         B(M)=B(I)
         B(I)=TEMP
  105    DO 107 J=ITEMP,N
C
C     MULTIPLY  AND  SUBSTRACT  SO  THAT  ALL ELEMENTS  BELOW
C     THE  DIAGONAL  TERM  ARE  ZERO
C     ==========================================
C
```