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STOCHASTIC ANALYSIS OF THE EFFECT OF RANDOM PERMEABILITY DISTRIBUTIONS ON CONFINE SEEPAGE

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Abstract

We describe two stochastic models for the determination of the statistical properties of the solution process. An error analysis of the models is considered which indicates the convergence of the overall procedure. Then a real case of a seepage encountered in an underground facility is treated based on the data provided by a field investigation. The outcome of the analysis is in concordance with the past engineering experience.

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It is generally recognized that within any homogeneous rock mass the physical properties exhibit a variability which must be considered in a design project.

This variability is due to different causes during the geologic formation of the strata, and is estimated by means of a site investigation. This will in turn provide the designer with the desired physical properties of the rock media at particular locations where the drillings are performed.

To handle the variability of the rock media an uncertainty factor is introduced in the simplified analytical model.

Two interrelated problems are then encountered:

A. How the above mentioned uncertainty of the physical properties will be inferred from the field information.

B. How to introduce the uncertainty of these properties in the already existing analytical model simulating the real world.

Both cases are treated by assuming that the rock properties $P_j(x,y,z)$ are spatial stochastic processes, as suggested by Cornel (1). Thus to a point in space (x,y,z) corresponds a merely probable value of P_j known also as statistical uncertainty which can be decreased at the expense of additional field informations.

The two problems are in the common practice solved independently and not coupled together to provide a consistent picture of the real phenomenon.

Usually the first problem is overcome by making the assumption that the statistical properties of the rock media are the same all over the region of interest or in other words that the process reflecting the randomness of the physical property is stationary. In that case there is no need of an inference model and one can proceed to solve the second problem by applying a first-order uncertainty analysis in connection with the already existing analytical model. This approach was adopted by several investigators in the field of continuum mechanics. B. Cambow (2) L. Esteva (3) and J. Padilla(6).

However, in rock mechanics the above procedure cannot be adopted and the first problem has to be solved exhaustively merely because of the nonstationarity of the physical properties.

In this case the rock volume is assumed as made up of a number of elementary volumes within each of which the physical property at hand if treated as stationary.

Moreover the properties assigned to each particular point in the two dimensional system must be inferred from the field data of a limited number of rock samples clearly then the need of an inference model is justified for:

1) The relatively large scale used to perform the analytical model.
2) The small amount of information of the rock media.

Krumbein (4) gives the fundamental techniques used in defining linear inference models. Such techniques are for example the method of least squares, fitting a polynomial in two variables, etc. All these techniques fail to provide an evaluation of how well the estimatron is performed, and present operational difficulties for the nonstationary case.

However, G. Matheron (5) proposed an inference scheme which is attractive for our case. Indeed with every obtained estimation of the property under consideration a parameter related with each estimation is indicating the performance level reached by the inference. This parameter is nothing else than the variance of the estimator. Moreover the nonstationarity can be treated as well as the stationarity case.

In the following a technique similar to the one suggested by G. Matheron is adopted to estimate the physical properties in the domaine of interest and to determine their spatial distribution. The results of the inference procedure are then used to treat the second problem.

In other words a coupling of the inference model and the analytical model, that is, a coupling of the data field investigation and the analytical simulation of the real phenomenon, is proposed. The analytical model is handled by using a finite element technique.

Implementation of the Inference model in a two-dimensional geometric space.

To determine the values of a rock property $z(x,y)$ a number of measurements are made on rock samples from boreholes. The set of points where observations are made is the set index β and Z_β represents the measured value of the random rock property at point β. Then the estimation $\hat{Z}(x_o,y_o)$ at any particular point (x_o,y_o) in the media is evaluated based on the given Z_β values.

A simple way that one can imagine to perform this estimation is to define $\hat{Z}$ in terms of the known values Z_β according to a linear combination as follows:

$$\hat{z}(x_o,y_o) = \sum_{\beta=1}^{\beta=n} b_\beta z_\beta$$

Where β and the known data points and b_β are the unknown weight coefficients to be determined by the inference model which is based on an optimal scheme of the random variable $z(x,y)$.

An alternate approach would be to estimate the mean value of $Z(x,y)$, namely $\bar{Z}(x,y)$ according to a linear combination.

$$\hat{Z} = \sum_{\beta=1}^{\beta=n} a_\beta \, Z_\beta$$

where β are the known data points as previously and a_β are the unknown weight coefficients.

The random variable $Z(x,y)$ is characterized spatially by the expression.

$$Z(x,y) = \bar{Z}(x,y) + FZ(x,y)$$

where $\bar{Z}(x,y)$ the mean and $FZ(x,y)$ the fluctuating term around the mean. Then the two overmentioned estimation procedures lead to two distinctive groups of assumptions characterizing the inference model.

The first group is given by

1) $E[Z(x,y)] = \bar{Z}(x,y)$

2) $E[Z(x_1,y_1)Z(x_2 y_2)] = \bar{Z}(x_1 y_1)\,\bar{Z}(x_2 y_2) + C(x_1, x_2)$

where $\bar{Z}(x,y)$ is the trend and $c(x_1, x_2)$ the covariance of the variable $Z(x,y)$. However these assumptions are in many cases dealing with rocks, too restrictive and need to be replaced by some more flexible assumptions.

This is realized by considering the rate of change of the random variable $Z(x,y)$ and will lead to the second group of assumptions. The second group indeed is defined by:

1) $E[Z(x_1 y_1) - Z(x_2 y_2)] = \bar{Z}(x_1 y_1) - \bar{Z}(x_2 y_2)$

2) $E[Z(x_1 y_1) - Z(x_2 y_2)]^2 = 2\,\gamma(x_1 x_2)$

where $\gamma(x_1 x_2)$ is the variogram

In both set nevertheless of assumptions there is a need to characterize the nature of the randomness of the variable $Z(x,y)$.

Therefore the following hypothesis concerning the randomness are added to each group. More specifically we assume:

First locally at a point (x,y) the mean $\bar{Z}(x,y)$ is approximated by known functions. Indeed,

$$\bar{Z}(x,y) = \sum_{i=1}^{u} a_i f^i(x,y)$$

a_i being some unknown weight coefficients and $f^i(x,y)$ the apriori known functions.

Secondly the covariances $c(x_i,x_2)$ are computed based on field measurements (See Padilla) and can be represented as approximated functions of the form $c(x_1,x_2) = k\, e^{-\alpha r}$

where r = distance between x_1 and x_2, α and k some fitting parameters.

At that point following the general trend of thought the two groups of assumptions correspond two inference models with two different goals reached in each case.

In the first model the goal will be to make the best estimation for the mean $\bar{Z}(x,y)$ while in the second model the best estimation is required for the random variable $Z(x,y)$ itself.

Both models lead to the problem of identifying the best estimators among all possible functions satisfying the hypothesis covering the randomness of the rock media.

These goals are achieved in each case by optimizing the expression of the variance using the method of lagrangian multipliers under the constraints imposed by the assumptions concerning the first moments.

In appendix (1) the computations give the following results:

FOR MODEL 1 concerning the mean $\bar{Z}(x,y)$

The method of Lagragian multipliers leads to a set of $n+k$ equations with $n+k$ unknowns namely the weight coefficients:

$$\sum_{\beta} a_{\ell}^{\beta} \; c_{\alpha\beta} - \sum_{s} \mu_{s\ell} \, f_{\alpha}^{s} = 0 \qquad \alpha,\beta = 1, \ldots n$$

$$\sum a_{\ell}^{\beta} \, f_{\alpha}^{s} - \delta_{\ell}^{s} = 0 \qquad \ell = 1, \ldots k$$

where a_{ℓ}^{β} the unknown weight coefficients, $\mu_{s\ell}$ the Lograngian multipliers and δ_{ℓ}^{s} the kronecker delta

The variance of the estimation is given by

$$E[\bar{z}(x,y)]^2 = \sum_{\ell} \mu_{\ell} \, f^{\ell}(x,y)$$

FOR MODEL 2 concerning the random variable $Z(x,y)$

Similarly the following system is obtained (Appendix (1))

$$\sum_{\beta}^{n} b^{\beta} \, \gamma_{\alpha\beta} + \sum_{\ell=1}^{k} \mu_{\ell} \, f_{\alpha}^{\ell} = \gamma(x_{\alpha}, x) \qquad \alpha,\beta = 1, \ldots n$$

$$\sum_{\alpha} b^{\alpha} f_{\alpha}^{\ell} = f^{\ell}(x) \qquad \ell = 1, \ldots k$$

$$\sum_{\alpha} b^{\alpha} = 1$$

where b^{β} the unknown weight coefficients and μ_{ℓ} the Lagrongian multipliers.

The variance of estimation being:

$$E[z - z^*]^2 = \sum_{\alpha=1}^{n} b^{\alpha} \, \gamma(x_{\alpha}\, x) + \sum_{\ell}^{k} \mu_{\ell} \, f_{\alpha}^{\ell}$$

UNCERTAINTY ANALYSIS OF THE ANALYTICAL MODEL

As mentioned previously the analytical model is treated using the finite element technique, which provides us the transfer mechanism between a set of inputs $\{S\}$ and set of outputs generally unknown $\{u\}$.

Then the general solutions is given by the following relation in matrix form

$$\{u\} = [k]^{-1} \{S\}$$

[k] is generally known as transfer matrix and is defined function of the random variables $Z_1, Z_2, \ldots Z_n$.

$$[k] = f(z_1, z_2 \ldots z_n)$$

Applying now the first order uncertainty analysis as described by Papoulis ( ) the following moments are obtained:

FIRST MOMENT

$$E\left[\{u(z_1(x,y), z_2(x,y))\}\right] \simeq \{u(\bar{z}_1(x,y), \bar{z}_2(x,y)\} + \frac{1}{2}\left[\sigma^2_{z_1(x,y)} \frac{\partial^2\{u(\bar{z}_1(x,y)\, \bar{z}_2(x,y)\}}{\partial z_1^2(x,y)}\right.$$

The second part of the second member can be neglected being a very small quantity

SECOND MOMENT

$$E\left[\{u(z_1(x,y), z_2(x,y))\}\right]^2 \cong \sigma^2_{z_1(x,y)} \left(\frac{\partial\{u(\bar{z}_1(x,y), \bar{z}_2(x,y)\}}{\partial z_1(x,y)}\right)^2 + \sigma^2_{z_2}\left(\frac{\partial\{u(\bar{z}_1\, \bar{z}_2)\}}{\partial z_2}\right)^2 +$$

$$+ 2\, \frac{\partial\{u(\bar{z}_1, \bar{z}_2\}}{\partial z_1}\, \frac{\partial\{u(\bar{z}_1, \bar{z}_2)}{\partial z_2}\, cov(z_1, z_2)$$

Then the partial derivatives are the solutions of the following system:

$$[k(z_1\, z_2)]\, \frac{\partial\{u(z_1\, z_2)\}}{\partial z_i} = \frac{\partial\{s\}}{\partial z_i} - \frac{\partial[k(z_1\, z_2)]}{\partial z_i}\{u\} \quad i = 1,2$$

The solution will be obtained using the classical finite element methodology.

Coupling of the Inference Model and the Analytical Uncertainty Model

In this procedure the statistical properties of the solution process are determined by the previously given expressions where the moving average or spatial mean $\bar{Z}_1(x,y)$, $\bar{Z}_2(x,y)$ and the variances $\sigma_{Z_1}(x,y)$, $\sigma_{Z_2}(x,y)$ are provided by the inference model. By substituting these quantities in the statistical relations of the dependent random variable {W}, its coefficient of variation is defined. This in turn is an essential statistical quantity used to evaluate the performance of the analytical model.

The convergence of the overall procedure is considered in Appendix 2 and checked through several examples. In general the results are improved both in the analytical and statistical sense when the mesh becomes denser.

Algorithm Description

The geometric domain under investigation is divided using a rectangular mesh common for the Inference Model and the analytical model (Finite element mesh).

The computations will be performed in each node using a number of known realizations of the random variable $Z(x,y)$. Therefore a zone of influence, characteristic of the media and depending on the covariance $c(x_1,x_2)$ is defined at each node of the mesh. In this zone eight given points are selected for efficiency and influence the computed estimation of the random variable $Z(x,y)$ at that particular node.

Indeed at every nodal point (x,y) a system of fourteen equations is solved and the estimator $\hat{Z}(x,y)$ computed according to the previously defined relations. The outcome of the procedure will of course depend on the number and closeness of the measured information of $Z(x,y)$ provided by the field investigation.

If the informations are not enough in number for all the domaine of interest then the estimator violates the original assumptions, but the variance on the other hand indicates the poor performance of the estimation and gives the exact location in which more informations are needed.

The flow chart in figure (1) gives the sequence in which the computations will be performed by program INFMODA.

Several examples were treated to test the program, the more significant being the following:

A square of 400m by 400m is examined and the random value $Z(x,y)$ is assumed to possess a realization lying on a portion of a sphere as shown in figure (2). The domaine is divided into squares of 25 x 25m having 289 nodes in which the computations are performed.

The apriori known function characterizing the behavior of the mean $\bar{Z}(x,y)$ was taken as a quadratic function of the form:

$$f(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$$

On the other hand the covariance was given by

$$C\left((x_1 y_1),(x_2 y_2)\right) = e^{-r}$$

where $r = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

The results are given in table (1) and are conform with what was expected. An interesting point to be mentioned, is that the computed variances are more sensitive to the location of the given information than the gradient of the mean $\frac{\partial \bar{Z}(x,y)}{\partial x}$, $\frac{\partial \bar{Z}(x,y)}{\partial y}$

The outcome of the Inference model is directly introduced at the nodes of the triangular element used in the finite element procedure. Then the uncertainty analysis is conducted in a conventional way.

NUMBER OF GIVEN INFORMAT.	TYPE OF MODEL	MAX ERROR	EXPECTED VALUE	MAX VARIANCE	MAX COEFFICIENT OF VARIATION
9 points	MODEL 1				
	MODEL 2	0.049	3.45	1.08	0.30
25 points	MODEL 1				
	MODEL 2	0.0047	3.61	0.569	0.21
81 points	MODEL 1				
	MODEL 2	0.0007	3.62	0.249	0.14

TABLE 1.

FLOW CHART

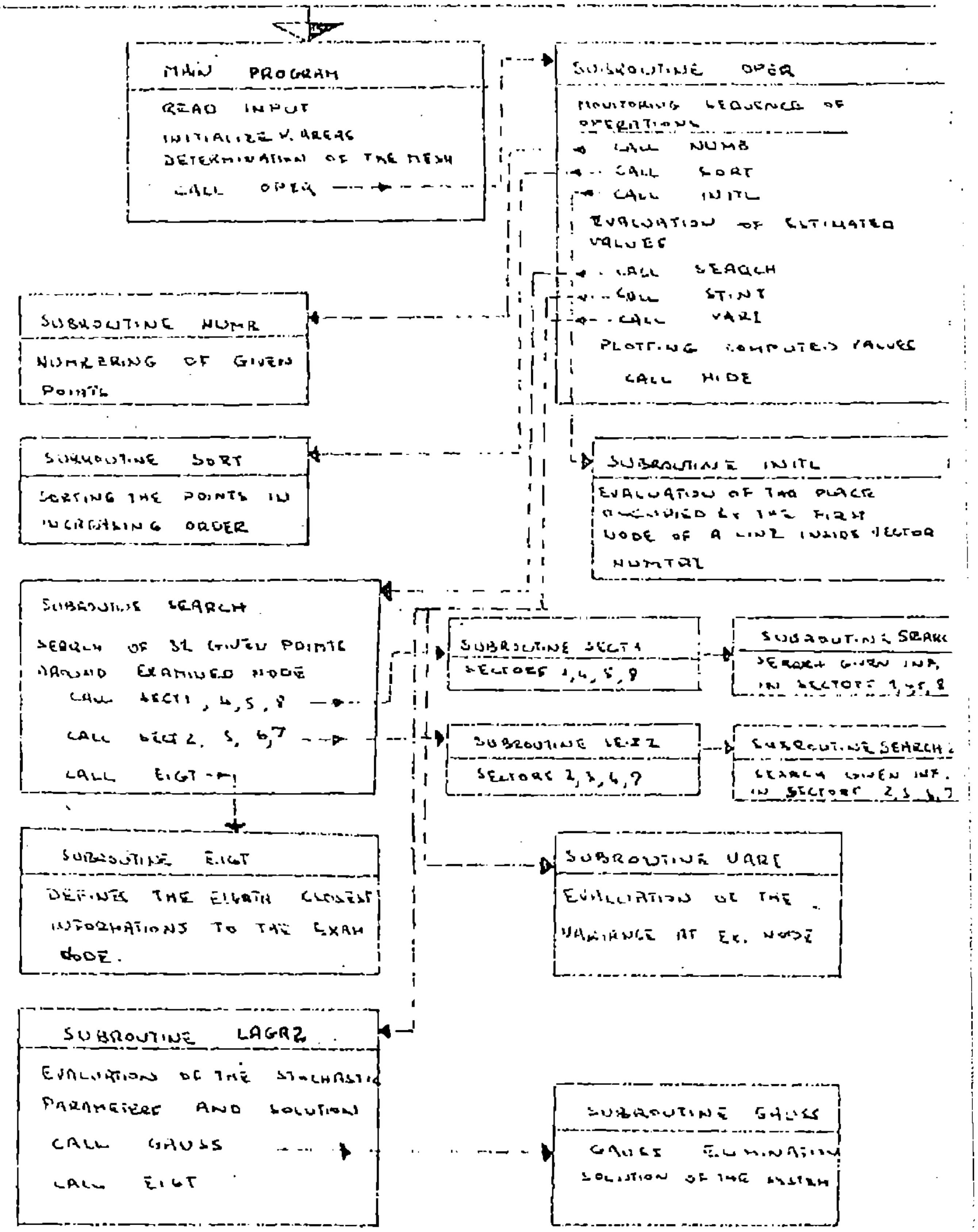
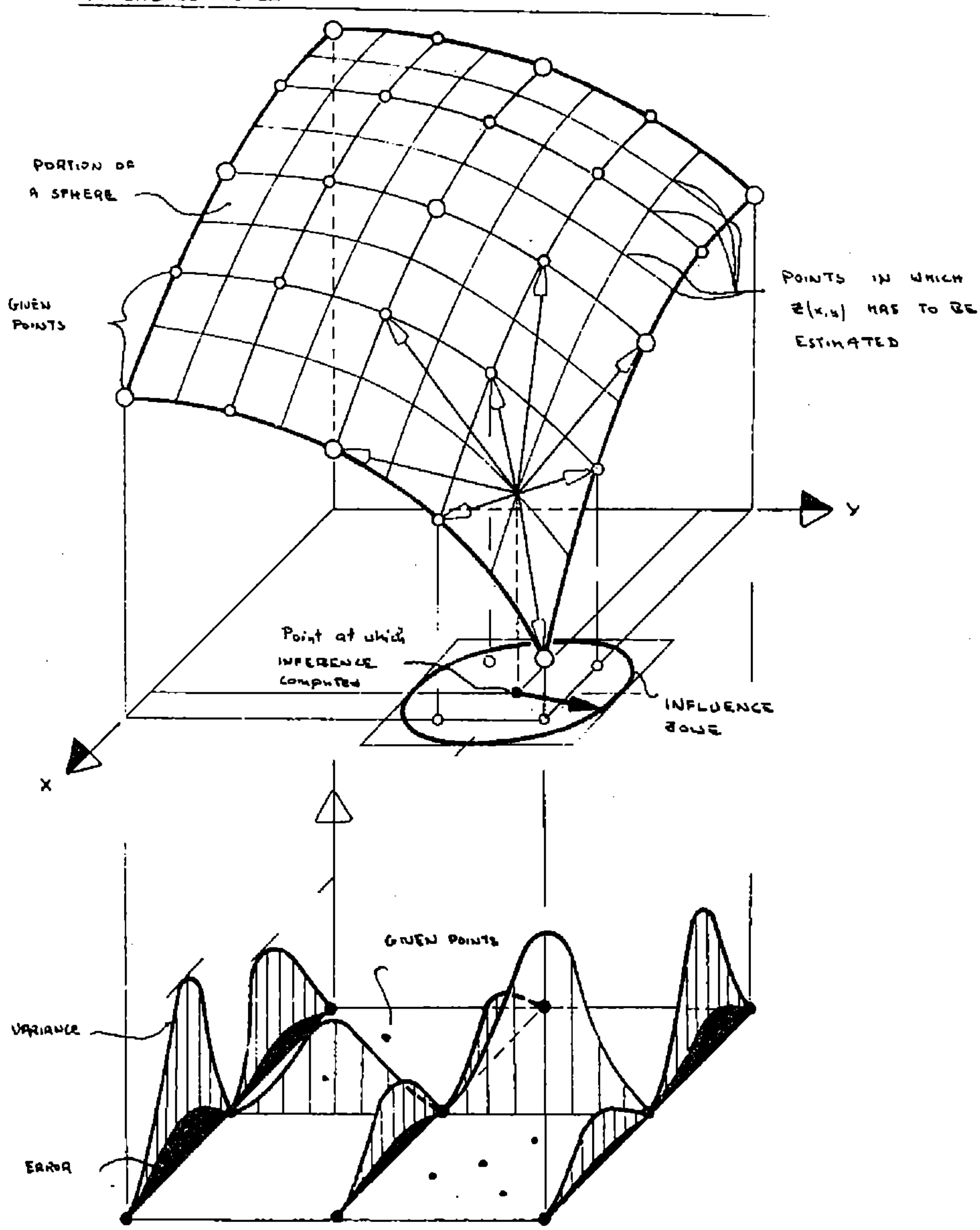
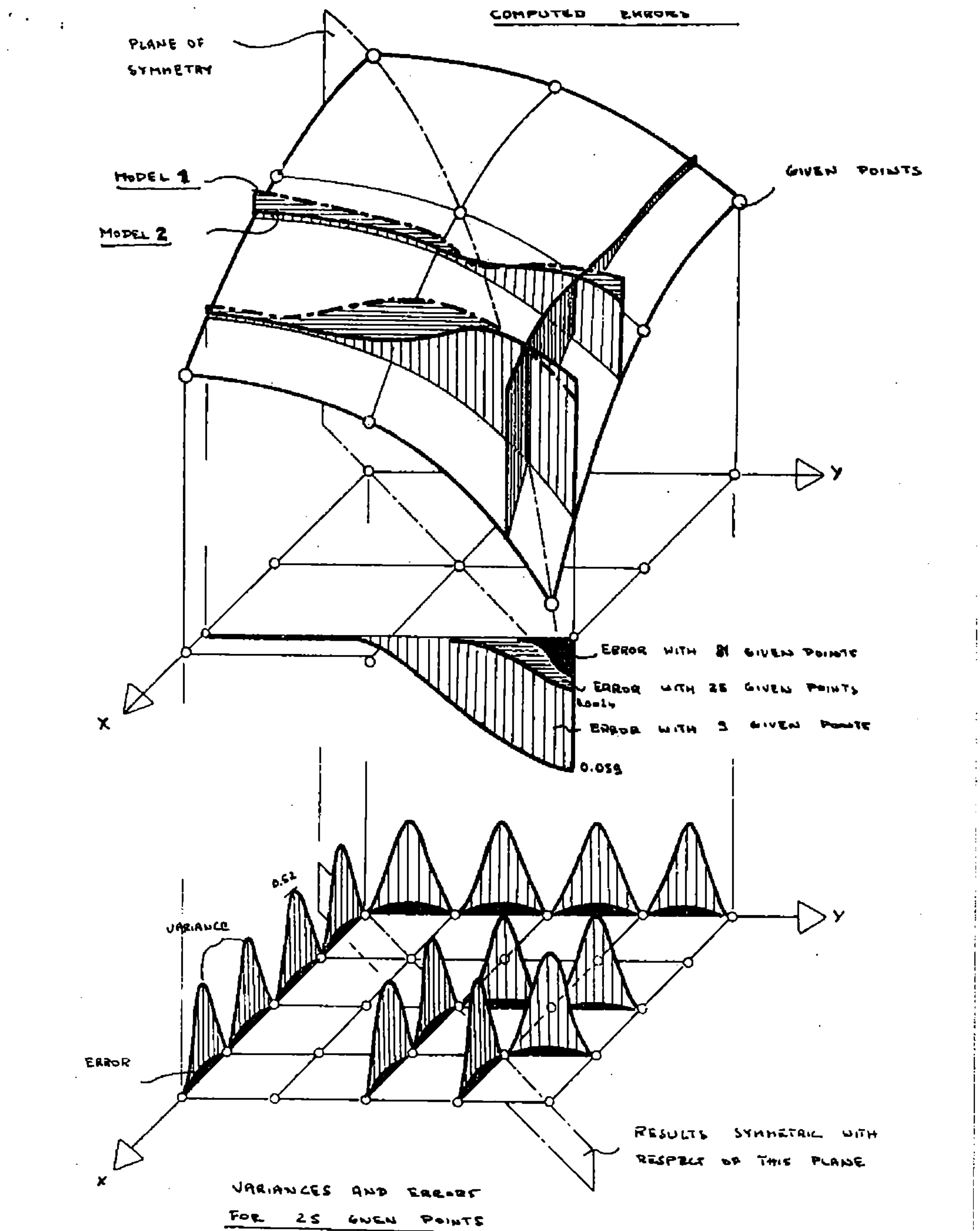

MAIN PROGRAM
READ INPUT
INITIALIZE V. AREAS
DETERMINATION OF THE MESH
CALL OPER
SUBROUTINE OPER
MONITORING SEQUENCE OF OPERATIONS
CALL NUMB
CALL SORT
CALL INITL
EVALUATION OF ESTIMATED VALUES
CALL SEARCH
CALL STINT
CALL VARI
PLOTTING COMPUTED VALUES
CALL HIDE
SUBROUTINE NUMB
NUMBERING OF GIVEN POINTS
SUBROUTINE SORT
SORTING THE POINTS IN INCREASING ORDER
SUBROUTINE INITL
EVALUATION OF THE PLACE OCCUPIED BY THE FIRST NODE OF A LINE INSIDE VECTOR NUMTAL
SUBROUTINE SEARCH
SEARCH OF 32 GIVEN POINTS AROUND EXAMINED NODE
CALL SECT1, 4, 5, 8
CALL SECT2, 3, 6, 7
CALL EIGT
SUBROUTINE SECT1
SECTORS 1, 4, 5, 8
SUBROUTINE SEARC
SEARCH GIVEN INF. IN SECTORS 1, 4, 5, 8
SUBROUTINE SECT2
SECTORS 2, 3, 6, 7
SUBROUTINE SEARCH2
SEARCH GIVEN INF. IN SECTORS 2, 3, 6, 7
SUBROUTINE EIGT
DEFINES THE EIGHTH CLOSEST INFORMATIONS TO THE EXAM NODE.
SUBROUTINE VARI
EVALUATION OF THE VARIANCE AT EX. NODE
SUBROUTINE LAGR2
EVALUATION OF THE STOCHASTIC PARAMETERS AND SOLUTION
CALL GAUSS
CALL EIGT
SUBROUTINE GAUSS
GAUSS ELIMINATION
SOLUTION OF THE SYSTEM

ILLUSTRATION OF THE CONCEPTUAL BASIS CONCERNING THE INFERENCE MODEL :



VARIANCES AND ERRORS FOR AN ASYMMETRIC CASE


COMPUTED ERRORS
PLANE OF SYMMETRY
MODEL 1
MODEL 2
GIVEN POINTS
Y
X
ERROR WITH 81 GIVEN POINTS
ERROR WITH 25 GIVEN POINTS
ERROR WITH 9 GIVEN POINTS
0.059
0.52
VARIANCE
ERROR
Y
X
RESULTS SYMMETRIC WITH RESPECT OF THIS PLANE
VARIANCES AND ERRORS FOR 25 GIVEN POINTS

APPENDIX I

The problem, therefore, consist to minimize $E[(z-\hat{z})^2]$ the variance with the constrain $E[z(x,y)] - E[\hat{z}(x,y)] = 0$.

In appendix three the above quantities are evaluated and was found to be:

$$E[(z-\hat{z})^2] = c(z,z) + \sum_\alpha b_\alpha \sum_\beta b_\beta [c(z_\alpha, z_\beta)] - 2 \sum_\alpha b_\alpha \, c(z, z_\alpha)$$

and

$$E[z] - E[\hat{z}] = \sum_k a_k f^k - \sum a_k [\sum_\alpha b^\alpha f_\alpha^k] =$$

$$= \sum_k a_k [f^k - \sum_\alpha b^\alpha f_\alpha^k] = 0$$

The minimization of the variance will be obtained using the method of Lagrangian multipliers as followed.

The Lagrangian function being

$$\mathcal{L} = c(z,z) - 2 \sum_\alpha b_\alpha c(z, z_\alpha) + \sum_{\alpha,\beta} b^\alpha b^\beta c(z_\alpha, z_\beta) -$$

$$- \sum_k (\mu_k) [f^k - \sum_\alpha b^\alpha f_\alpha^k]$$

The conditions to obtain the minimum are

$$\frac{\partial \mathcal{L}}{\partial b^\alpha} = 0 \qquad \text{for all } b^\alpha\text{'s}$$

$$\frac{\partial \mathcal{L}}{\partial m_k} = 0 \qquad \text{for all } m_k\text{'s}$$

The unknowns being b^α's and m_k's we obtain a linear system of $\alpha+k$ equations.

The differentiation of $\mathcal{L}$ with respect to b^α and m_k gives:

First with respect to b^α.

$$-2\, c(z, z_\alpha) + \sum_\beta b^\beta c(z_\alpha, z_\beta) + \sum_k m_k f_\alpha^k = 0 \qquad \forall\ \alpha = 1, n$$

Second with respect to m_k.

$$-f^k + \sum_\alpha b^\alpha f_\alpha^k = 0 \qquad \forall\ k = 1, p$$

The system then can be written as

$$\alpha = 1 \quad \sum_\beta b^\beta c(z_\alpha, z_\beta) + \sum_k m_k f_\alpha^k = 2\, c(z_\alpha, z)$$

$$\begin{matrix}\alpha = n \\ k = 1\end{matrix} \quad \sum_\alpha b^\alpha f_\alpha^k = f^k$$

The covariances $c(z_\alpha, z_\beta)$ and $c(z_\alpha, z)$ are obtained in appendix three and are given in the following relations:

$$c(z_\alpha, z) = \gamma(Fz_\alpha - Fz) \quad \text{and} \quad c(z_\alpha, z_\beta) = \gamma(Fz_\alpha - Fz_\beta)$$

Then the linear system of equations becomes

$$\sum_\beta^n b^\beta \gamma(z_\alpha - z_\beta) + \sum_k m_k f_\alpha^k = 2\, \gamma(z_\alpha - z)$$

$$\sum_\beta b^\beta f^k(z_\beta) = f^k(z)$$

where the b's and m's are the unknown quantities. Therefore, solving this system the estimator of the variable is defined by:

$$\hat{z}(x,y) = \sum_\beta^n b^\beta z_\beta$$ and the variance of the estimate is

$$\sigma_{\hat{z}}^2 \quad \sum b^\beta \gamma(z_\beta - z) + \sum_k m_k f^k(z)$$

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