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A COLLOCATION METHOD FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract. An interpolation scheme based on piecewise cubic polynomials with Gaussian points as interpolation points is analyzed and applied to the solution of Fredholm equations of the second kind

Introduction. We consider an interpolation scheme based on piecewise polynomials with continuous first derivatives and the Gaussian points as interpolation points. A collocation method based on this interpolation scheme is applied to one and two dimensional Fredholm equations of the second kind.

This scheme has been applied to a collocation method by De Boor and Swartz [2] and Houstis [9] for the numerical solution of ordinary differential equations. Also, Douglas and Dupont [6], [7], [8] and Houstis [10], [11] have studied a collocation method for partial differential equations based on the above scheme. A survey of numerical methods for the solution of Fredholm integral equations of the second kind is given by Atkison [1]

In part I we present the formulation and error analysis of the interpolation scheme. In part II we apply this scheme to the solution of Fredholm integral equations of the second kind and give an experimental comparison with Nyström's method.

I. Piecewise Cubic Hermite Interpolation at the Gaussian points

1. One-dimensional interpolation scheme. Let $\Delta=(x_i)_1^{N+1}$ be a partition of $I\equiv[a,b], h_i\equiv|x_{i+1}-x_i|, I_i\equiv[x_i,x_{i+1}]$ and $h=\max h_i$. Throughout this report we denote by P_3 the set of polynomials of degree less than 4, and $P_{3,\Delta}$ the set of functions that reduce to polynomials of degree less than 4 in each subinterval $[x_i,x_{i+1}]$. Also we denote by H_Δ the (2N+2)-dimensional vector space of all continuously differentiable piecewise cubic polynomials with respect to Δ.

The Gaussian points in the subinterval $[x_j, x_{j+1}]$ are

$$(1.1) \qquad \xi_{2j+i} \equiv \frac{x_j+x_{j+1}}{2} + \frac{(-1)^i}{\sqrt{3}} \frac{h_j}{2} \qquad i = 1,2.$$

Let $E(I)$ be the space of real-valued functions defined on I. We introduce an interpolation operator

$$Q_N : \quad E(I) \to H_\Delta .$$

such that

(1.2) $(Q_N f)(\sigma_\ell) = f(\sigma_\ell)$, $\ell=1,\dots,2N+2$,

where $\sigma_1=a$, $\sigma_\ell=\xi_{2j+i}$, $j=1,\dots,N, i=1,2$, $\sigma_{2N+2}=b$.

This interpolation scheme is well defined. In fact, if $h(x)\varepsilon H_\Delta$ also interpolates f as above, then $e(x)\equiv Q_N f(x)-h(x)$ is a cubic polynomial on $[x_i,x_{i+1}]$, $0\le i\le N$ and $e(\sigma_i) = 0$, $1\le i\le 2N+2$. We show that $e(x)$ is identically zero in $[x_i,x_{i+1}]$. If this is not so, then without loss of generality we may assume that $e(x)\neq 0$ for all $x\varepsilon[x_1,x_2]$. Rolle's Theorem implies that $e(x_2)D_x e(x_2)>0$. Similarly, $D_x e$ restricted in $[x_2,x_3]$ has roots in (x_2,σ_4), (σ_4,σ_5). Thus, $e(x_3)D_x e(x_3)>0$. By induction $e(x_{N+1})D_x e(x_{N+1})>0$ contradicting the relation $e(x_{N+1})=0$. This proves that $e(x)\equiv 0$ in I.

2. Two-dimensional interpolation scheme. In this section we introduce a two-dimensional analogue of the interpolation scheme of the previous section. Let $\Delta_y = (y_j)_1^{M+1}$ be a partition of $[c,d]$, $J\equiv[c,d]$, $k_j\equiv|y_{j+1}-y_j|$, $J_j\equiv[y_j,y_{j+1}]$ and $k\equiv\max k_j$. Also, we denote by $\rho\equiv\Delta x\Delta y$ a partition of $[a,b]x[c,d]$ and by H_ρ the vector space of all piecewise bicubic polynomials $p(x,y)$ with respect to ρ, such that $D_x^\ell D_y^\eta p(x,y)$ is continuous on $[a,b]x[c,d]$ for all $0\le\ell,\eta\le 1$.

The Gaussian points τ in the subinterval $[y_i, y_{i+1}]$ are

$$\tau_{2i+j} \equiv \frac{y_i + y_{i+1}}{2} + \frac{(-1)^j}{\sqrt{3}} \frac{k_i}{2}, \quad j=1,2 .$$

A two-dimensional interpolation operator is defined as the tensor product

$$Q_\rho \equiv Q_N \otimes Q_M = Q_N Q_M$$

3. Error analysis. In this section, we establish a priori bounds for the interpolation scheme introduced in section 2 for a uniform partition Δ of $[0,1]$ with mesh length $h = N^{-1}$.

Let

$$\phi(x) = \begin{cases} (1-x)^2(1+2x) & 0 \le x \le 1 \\ (1+x)^2(1-2x) & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi(x) = \begin{cases} x(1-x)^2 & 0 \le x \le 1 \\ x(1+x)^2 & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases} .$$

The basis functions of the C^1 cubic piecewise polynomial space H_Δ are defined by

$$B_{2i}(x) = \phi\left(\frac{x-x_i}{h}\right), \quad 1 \le i \le N+1 ,$$

$$B_{2i-1}(x) = h\psi\left(\frac{x-x_i}{h}\right), \quad 1 \le i \le N+1 .$$

For later use, we define the Gramian matrix

$$G_N \equiv \left[B_i(\sigma_j) \;\; ; \; i,j=1,\ldots,2N+2 \right]$$

of the interpolation operator Q_N. Using the $(2N+2)\times(2N+2)$ matrix

$$H_N \equiv \begin{bmatrix} 1 & & & & & \\ & h & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & 1 & \\ & & & & & h \end{bmatrix},$$

we find that

$$H_N^{-1}\,G_N \equiv \left[\begin{array}{c|c|c|cccc} \begin{matrix}1\\0\end{matrix} & A & & & & & \\ \hline & B & A & & & 0 & \\ \hline & & & \cdot & & & \\ & & & & \cdot & & \\ & 0 & & & & B & A \\ \hline & & & & & & B \quad \begin{matrix}1\\0\end{matrix} \end{array}\right]$$

where

$$A \equiv \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad B \equiv \begin{bmatrix} \beta & \alpha \\ -\delta & -\gamma \end{bmatrix}$$

and

$$\alpha = \frac{9+4\sqrt{3}}{18}, \quad \beta = \frac{9-4\sqrt{3}}{18}, \quad \gamma = \frac{3+\sqrt{3}}{36}, \quad \delta = \frac{3-\sqrt{3}}{36}$$

We will also use the matrix

$$T \equiv BA^{-1} = \begin{bmatrix} -7 & 48 \\ 1 & -7 \end{bmatrix}$$

It is easy to see that for all integers n, $(T^{\circ} \equiv I)$,

$$T^{n} = \begin{bmatrix} a_n & 48c_n \\ c_n & a_n \end{bmatrix}$$

where

$$a_{n+1} = -7a_n + 48c_n \ , \ c_{n+1} = a_n - 7c_n \ .$$

More generally, from $T^{s+t} = T^s T^t$ we get

$$a_{s\pm t} = a_s a_t \pm 48 c_s c_t \ , \ c_{s\pm t} = c_s a_t \pm a_s c_t$$

(3.1)

$$c_s a_t = \frac{1}{2} (c_{s+t} + c_{s-t})$$

$$c_s c_t = \frac{1}{96} (a_{s+t} - a_{s-t})$$

$$a_s a_t = \frac{1}{2} (a_{s+t} + a_{s-t})$$

$$a_{-\ell} = a_\ell \ , \ c_{-\ell} = -c_\ell \ .$$

Let $\lambda_n \equiv |a_n/c_n| = -a_n/c_n$. Since $\det(T^n) = 1$, we can easily show that λ_n is decreasing with n and for all n

$$\sqrt{48} < \lambda_n \leqq 7, \; \lambda_1 = 7$$

(3.2) $$c_n = (-1)^{n+1}|c_n|, \; a_n = (-1)^n|a_n|$$

$$|a_{n+1}|>|a_n| \;, \; |c_{n+1}|>|c_n|$$

Since

$$|a_n| = \frac{1}{2}(|c_{n+1}|-|c_{n-1}|), |c_n| = \frac{1}{96}(|a_{n+1}|-|a_n|)$$

we also have

(3.3)
$$\sum_{\ell=q}^{p} |a_\ell| = \frac{1}{2}(|c_{p+1}|+|c_p|-|c_q|-|c_{q-1}|)$$

$$\sum_{\ell=q}^{p} |c_\ell| = \frac{1}{96}(|a_{p+1}|+|a_p|-|a_q|-|a_{q-1}|) \;.$$

We introduce a (2N+2)x(2N+2) matrix R in partition form

$$\left[\begin{array}{cc|c|cc}
r_{11} & r_{12} & \cdots & r_{1,2N+1} & r_{1,2N+2} \\
\hline
\multicolumn{2}{c|}{R_{11}} & \cdots & \multicolumn{2}{c}{R_{1,N+1}} \\
\multicolumn{2}{c|}{\vdots} & & \multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c|}{R_{N,1}} & \cdots & \multicolumn{2}{c}{R_{N,N+1}} \\
\hline
r_{2N+2,1} & r_{2N+2,2} & \cdots & r_{2N+2,2N+1} & r_{2N+2,2N+2}
\end{array}\right]$$

where the first and last rows are defined as

$$[r_{1,2j-1},\ r_{1,2j}] \equiv \frac{(-1)^{j+1}}{c_N} [c_{N-j+1}\ a_{N-j+1}]$$

$$[r_{2N+2,2j-1},\ r_{2N+2,2j}] \equiv \frac{(-1)^{N-j}}{c_N} [-c_{j-1}\ a_{j-1}] \qquad j=1,\ldots,N+1$$

while the 2x2 matrices $R_{n,m}$ are defined as

$$R_{n,m} \equiv A^{-1} [(-T)^{n-1} Z_m + \sigma_{n,m} (-T)^{n-m}], \quad n=1,\ldots,N \ , \quad m=1,\ldots,N+1$$

with

$$Z_1 \equiv \begin{bmatrix} 0 & \lambda_N \\ 0 & 1 \end{bmatrix}, \quad Z_m \equiv \frac{(-1)^m}{c_N} \begin{bmatrix} c_{N-m+1} & a_{N-m+1} \\ 0 & 0 \end{bmatrix} \quad m=2,\ldots,N+1$$

and

$$\sigma_{n,m} = \begin{cases} 1 & \text{if } 2 \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

The reason for introducing the matrix R becomes apparent by the following Lemma, which shows that we have explicitly constructed the inverse of the matrix $H_N^{-1} G_N$

Lemma 3.1. The matrix $H_N^{-1} G_N$ is invertible and its inverse is the matrix R.

Proof: Let $S \equiv R(H_N^{-1} G_N)$. It is enough to show that S=I. We partition S into blocks:

$$S = \begin{bmatrix} s_{11} & \tau_{11} & \cdots & \tau_{1N} & s_{1,2N+2} \\ \omega_{11} & S_{11} & \cdots & S_{1N} & \omega_{1,2N+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_{N1} & S_{N1} & & S_{NN} & \omega_{N,2N+2} \\ s_{2N+2,1} & \tau_{2N+2,1} & \cdots & \tau_{2N+2,N} & s_{2N+2,2N+2} \end{bmatrix}$$

where each s_{ij} is 1x1, S_{ij} is 2x2, ω_{ij} is 2x1 and τ_{ij} is 1x2.

Performing the multiplication of the matrices R and $H_N^{-1} G_N$ we obtain

$s_{11} = r_{11} = 1$

$$\begin{aligned} \tau_{ij} = [s_{1,2j} \;\; s_{1,2j+1}] &= [r_{1,2j-1} \;\; r_{1,2j}]\, A + [r_{1,2j-1} \;\; r_{1,2j+2}]\, B \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1} \;\; a_{N-j+1}] - [c_{N-j} a_{N-j}]\, T\}A \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1} \;\; a_{N-j+1}] - [c_{N-j+1} \;\; a_{N-j+1}]\}A \\ &= [0, \; 0] \end{aligned}$$

and

$$s_{1,2N+2} = r_{1,2N+1} = 0 \quad .$$

Similarly

$$\omega_{i,1} = \omega_{i,2N+2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \tau_{2N+2,j} = [0 \;\; 0], \; i,j = 1,\ldots,N$$

and

$$s_{2N+2,1} = 0 \quad , \; s_{2N+2,2N+2} = 1.$$

For the square blocks $S_{n,m}$ we find

$$S_{n,m} = R_{n,m} A + R_{n,m+1} B$$

$$= A^{-1}(-T)^{n-1}\{Z_m + Z_{m+1} T + (\sigma_{n,m} - \sigma_{n,m+1})(-T)^{1-m}\} A$$

From the definition of Z_m and T we obtain $Z_m + Z_{m+1} T = \delta_1^m I.$ Then from the definition of $\sigma_{n,m}$ we get

$$S_{n,m} = \delta_n^m I.$$

This concludes the proof of Lemma 3.1.

Lemma 3.2. If G_N is the Grammian of the interpolation operator Q_N then

(3.4) $\quad ||(H_N^{-1} G_N)^{-1}||_\infty < 100$

for all $N \geq 2$.

Proof. Let

$$||R||_\ell \equiv \Sigma_{m=1}^{2N+2} |r_{\ell m}| \quad .$$

From the definition of R and relations (3.1), (3.2), (3.3), we obtain

$$||R||_1 \equiv \Sigma_{j=1}^{N}(|r_{1,2j+1}|+|r_{1,2j}|)$$

$$= \frac{1}{|c_N|}\Sigma_{j=1}^{N}(|c_{N-j+1}|+|a_{N-j+1}|)$$

$$= \frac{1}{|c_N|}\Sigma_{\ell=1}^{N}(|c_\ell|+|a_\ell|)$$

$$\leq \frac{7}{12}\frac{|a_N|}{|c_N|}+\frac{9}{2}+\frac{5}{15}\frac{|a_1|}{|c_N|}-\frac{7}{2}\frac{|c_1|}{|c_N|}$$

$$\leq 23/2$$

It is easy to see that $||R||_{2N+2} = ||R||_1$. For the remaining rows we use (3.1) (3.2) to get that for $2 \leq m \leq n$

$$AR_{n,m}=(-T)^{n-1}\, Z_m + \sigma_{n,m}\, (-T)^{n-m}$$

$$= \frac{1}{2|c_N|}\begin{bmatrix} -|c_{N-n-m+2}|+|c_{N-n+m}| & |a_{N-n-m+2}|+|a_{N-n+m}| \\ \frac{1}{48}(-|a_{N-n-m+2}|+|a_{N-n+m}|) & |c_{N-n-m+2}|+|c_{N-n+m}| \end{bmatrix}$$

while for $n < m$

$$AR_{n,m} = \frac{1}{2|c_N|}\begin{bmatrix} -|c_{N+n-m}|+|c_{N-n-m+2}| & |a_{N+n-m}|+|a_{N-n-m+2}| \\ \\ \frac{1}{48}(|a_{N+n-m}|-|a_{N-n-m+2}|) & |c_{N+n-m}|-|c_{N-n-m+2}| \end{bmatrix}.$$

Finally, for $m=1$

$$AR_{n,1} = \frac{1}{|c_N|}\begin{bmatrix} 0 & |a_{N-n+1}| \\ 0 & |c_{N-n+1}| \end{bmatrix}$$

Using again the relations (3.1) through (3.3) we now find

$$\sum_{m=1}^{N} ||AR_{n,m}||_1 \leq$$

$$\frac{1}{2}\left[2\frac{|a_{N-n+1}|}{|c_N|} + \frac{1}{96}\left(\frac{|a_{N-n+1}|}{|c_N|} + \frac{|a_{N-n}|}{|c_N|} + \frac{|a_N|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|}\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{|c_{N-n}|}{|c_N|} + 9 + \frac{|c_{N-1}|}{|c_N|} + \frac{|a_N|}{|c_N|}\right)\right] \leq \frac{35}{3}.$$

and

$$\sum_{m=1}^{N} ||AR_{n,m}||_2 \leqq$$

$$\frac{1}{2}\left[2\,\frac{|c_{N-n+1}|}{|c_N|} + \frac{1}{96}\left(\frac{|a_{N-n}|}{|c_N|} + 9\,\frac{|a_N|}{|c_N|} + 48 + \frac{|c_{N-n+1}|}{|c_N|}\right.\right.$$

$$\left.\left. + \frac{|c_{N-n}|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|} + \frac{|a_n|}{|c_N|} + \frac{|c_{N-1}|}{|c_N|}\right)\right] \leqq 2.$$

By definition now, we have for $\ell=1,2$

$$||R||_{2n+\ell} = \sum_{m=1}^{N} ||A^{-1}AR_{n,m}||_\ell \leq \sum_{m=1}^{N} ||A^{-1}||_\infty ||AR_{n,m}||_\ell$$

while

$$||A^{-1}||_\infty = \frac{7\sqrt{3}+9}{4}$$

Thus, for the norm $||R||_\infty = \max_i ||R||_i$ the following bound holds

$$||R||_\infty = ||(H^{-1}G_N)^{-1}||_\infty < 100.$$

This concludes the proof of Lemma 3.2.

Remark. As the proof of Lemma 3.2 suggests the bound (3.4) can be improved. Our conjecture is that a more careful analysis will show that the norm $||(H_N^{-1}G_N)^{-1}||_\infty$ is decreasing in N, that

$$\lim_{N\to\infty}||(H_N^{-1}G_N)^{-1}||_\infty = \frac{69-29\sqrt{3}}{2}$$

and that for all $N \geq 2$

$$\frac{69-29\sqrt{3}}{2} \leq ||(H_N^{-1}G_N)^{-1}||_\infty \leq ||H_2^{-1}G_2||_\infty = \frac{33\sqrt{3}+9}{7}$$

Numerical experiments confirm this conjecture.

Lemma 3.3. Let Q_N be the interpolation operator defined by (1.2). Then Q_N is bounded in the L_∞-norm.

Proof. We can express the $Q_N f$ interpolant as

$$(Q_N f)(x) = \sum_{i=1}^{N+1} \{ a_i B_{2i}(x) + b_i B_{2i-1}(x)/h \}$$

where B_i's are the previously defined basis functions of H_Δ. Let $\underline{f}$ denote the column vector $\{f(\sigma_j)\}_{j=1}^{2N+2}$ and set

$$L_N = (H_N^{-1} G_N)^T .$$

Then,

$$\underline{f} = L_N \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \end{bmatrix} .$$

In Lemma 3.2 we prove that

$$\| L_N^{-1} \|_\infty < 100$$

therefore

$$\max_i (\max(|a_i|, |b_i|)) \le 100\|f\|_{L_\infty} .$$

For all $x \in [x_i, x_{i+1}]$, $1 \le i \le N$, we have

$$(Q_N f)(x) = a_i B_{2i}(x) + b_i B_{2i-1}(x)/h + a_{i+1} B_{2i+2}(x) + b_{i+1} B_{2i+1}(x)/h$$

and since B_{2i}, B_{2i-1}/h are bounded by unity we obtain

$$(3.5) \qquad \|Q_N f\|_{L_\infty} \le 400\|f\|_{L_\infty} .$$

Theorem 3.1. If $f \in W^{s,\infty}(I)$, $s = 0,1,2,3,$ or 4, then

(i) $Q_N f \to f$, as $N \to \infty$

and

(ii) for the interpolation error we have

$$\|Q_N f - f\|_{L_\infty} \le ch^s \|D^s f\|_{L_\infty}$$

where c is independent of h.

Proof. Let $\partial_H f$ be the Hermite interpolant of f, defined by interpolation of f and its first derivative at the nodes of the partition Δ. From the triangle inequality we find

$$(3.6) \quad \|f - Q_N f\|_{L_\infty} \le (1 + \|Q_N\|)\|f - \partial_H f\|_{L_\infty} .$$

Moreover, for the Hermite interpolation error, it is known [3, p. 236]

$$(3.7) \quad \|f - \partial_H f\|_{L_\infty} \le ch^s \|D^s f\|_{L_\infty}, \quad s > 0.$$

From (3.5), (3.6), (3.7) and Lemma 3.3, we now get

$$||f-Q_N f||_{L_\infty} \leq ch^s ||D^s f||_{L_\infty} .$$

This proves conclusions (i) and (ii).

Theorem 3.2. If $f \in W^{s,\infty}(\Omega)$, $s = 1,2,3,4$, and $\Omega = [0,1]^2$, then

$$(3.8) \quad ||f-Q_\rho f||_{L_\infty} \leq c(h^s ||D_x^s f||_{L_\infty} + h^p k^q ||D_x^p D_y^q f||_{L_\infty} + k^s ||D_y^s f||_{L_\infty})$$

$$\leq c\, \rho^s ||f||_{W^{s,\infty}} \quad , \quad \text{where } s = p + q$$

and p,q are Integers.

Proof. From the triangle inequality we have

$$||f-Q_\rho f||_{L_\infty} \leq ||f-Q_N f||_{L_\infty} + ||Q_N(f-Q_M f)||_{L_\infty}$$

$$\leq ||f-Q_N f||_{L_\infty} + ||Q_N(f-Q_M f) - (f-Q_M f)||_{L_\infty} + ||f-Q_M f||_{L_\infty}$$

Using the results of Theorem 3.1, we now get

$$||f-Q_\rho f||_{L_\infty} \leq h^s ||D_x^s f||_{L_\infty} + h^p ||D_x^p (f-Q_M f)||_{L_\infty} + k^s ||D_y^s f||_{L_\infty}$$

$$\leq h^s ||D_x^s f||_{L_\infty} + h^p k^q ||D_x^p D_y^q f||_{L_\infty} + k^s ||D_y^s f||_{L_\infty}$$

$$\leq c\rho^s ||f||_{W^{s,\infty}}$$

which completes the proof of the Theorem.

II. Two-dimensional Collocation

4. Procedure and error estimation. In this section we consider the problem of approximating the solution of the integral equation

(4.1) $\Lambda u = u(P) - \lambda \int_\Omega k(P;Q)u(Q)dQ = f(P)$ where $\Omega = [0,1]^2$ and,

for brevity, $P=(x,y)$, $Q=(s,t)$ and $dQ=dsdt$.

$$Ku \equiv \int_\Omega k(P;Q)u(Q)dQ.$$

We seek an approximation $u_\rho \varepsilon H_\rho$ to u of the form

$$(4.2) \quad u_\rho(P) \equiv \sum_{i=1}^{2N+2} \sum_{j=1}^{2M+2} \alpha_{ij} \; B_i(x)B_j(y)$$

such that

$$(4.3) \quad (I-\lambda Q_\rho K)u_\rho = Q_\rho f$$

Theorem 4.1. If

A1: λ is not an eigenvalue of the kernel $k(P;Q)$

A2: The right side and the kernel of equation (4.1) were in $W^{s,\infty}(\Omega)$, $s = 1,2,3$, or 4.

Then

(i) for sufficiently small $|\rho|$ the collocation system (4.3) is uniquely solvable and

(ii) for the error of approximation we have

$$(4.4) \quad ||u-u_\rho||_{L_\infty} \le c||u-Q_\rho u||_{L_\infty}$$
$$\le c\rho^s||u||_{W^{s,\infty}} .$$

Proof By the definition of the operator K we obtain

(4.5) $||Ku-Q_\rho Ku||_{L_\infty}=||\int_\Omega (k(\cdot,\Omega)-Q_\rho k(\cdot,\Omega))ud\Omega||_{L_\infty}$ and since

$||k(\cdot,\Omega)-Q_\rho k(\cdot,\Omega)||_{L_\infty}\to o$ as $|\rho|\to o$

it follows from (4.5) that

(4.6) $$||K-Q_\rho K||\to o \qquad \text{as} |\rho|\to o$$

In fact, for any $\varepsilon>o$ there exists $\delta_o(\varepsilon)$ such that for $|\rho|<\delta_o(\varepsilon)$ the inequality $||K-Q_\rho K||<\varepsilon$ holds. But in this case, on the sphere $||u||_{L_\infty}=1$ we have

$$||(I-\lambda Q_\rho K)u||_{L_\infty}\geq ||(I-\lambda K)u||_{L_\infty}-|\lambda|\ ||(K-Q_\rho K)\ u||_{L_\infty}$$
$$\geq \alpha-|\lambda|\varepsilon$$

where

$$o < \alpha \leq \inf_{||u||_{L_\infty}=1} ||(I-\lambda K)u||_{L_\infty}.$$

Consequently, for sufficiently small $|\rho|$, the relationship

(4.7) $$\inf_{||u||_{L_\infty}=1} ||(I-\lambda Q_\rho K)u||_{L_\infty}\geq \beta > o$$

holds, from which conlusion (i) follows.

Now from (4.3) it follows easily that

$$u-u_\rho=(I-\lambda Q_\rho)^{-1} \quad (u-Q_\rho u)$$

from which

(4.8) $$||u-u_\rho||_{L_\infty} \leq ||(I-\lambda Q_\rho K)^{-1}|| \quad ||u-Q_\rho u||_{L_\infty} \leq \frac{1}{\beta}\ ||u-Q_\rho u||_{L_\infty}$$

and combining this inequality with that of (3.8) we obtain (4.4). This concludes the proof of Theorem 4.1.

Finally, we remark that the above results also hold for one-dimensional integral equations of the second kind

5. Numerical results In this section we present some numerical results conserning the approximation of the solution of some one-dimensional integral equations taken from [2]. The numerical solutions are computed by one-dimensional analogue of the collocation scheme introduced in part II with a three-point Gaussian rule and by Nystrom's method with Simpson's numerical integration rule ([1]). The partition Δ used is uniform with mesh size $h=1/N$. The rate of convergence estimate $\log\left(\frac{\text{error for } h}{\text{error for } h/2}\right) / \log 2$ is also given.

The integral equation

$$u(s)-\lambda\int_a^b k(s,t)u(t)dt=f(s) \qquad a\leq s\leq b$$

is solved for various kernel functions k, right side functions f and parameters λ.

Case (i): $k(s,t)=\cos(\pi st)$, $0\leq s,t\leq 1$, $\lambda=1$.
The right side f is chosen so that

$$u(s)=e^{s}\cos(\pi s)$$

	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	4.55×10^{-2}		1.18×10^{-2}	
6	4.24×10^{-3}	3.4	5.85×10^{-4}	4.3
12	3.37×10^{-4}	3.7	3.47×10^{-5}	4.1
24	2.59×10^{-5}	3.7	2.14×10^{-6}	4.
48	1.75×10^{-6}	3.9	1.33×10^{-7}	4.

Case(ii): $k(s,t)=e^{\beta st}$, $o \leq s$, $t \leq 1$, $\lambda=1$. For the numerical example we pick f so that

$$u(s)=e^{\alpha s} \qquad \alpha=1, \qquad \beta=5,$$

	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	1.33×10^{-3}		1.60×10^{-1}	
6	2.17×10^{-5}	5.9	1.55×10^{-2}	3.37
12	2.72×10^{-7}	6.3	1.02×10^{-3}	3.90
24	1.1×10^{-8}	4.6	6.49×10^{-5}	3.97
48	8.57×10^{-10}	3.7	4.06×10^{-6}	4.00

Case (iii): $k(s,t)= t-s$, $o \leq t$, $s \leq 1$, $\lambda=1$. Choose f so that

$$u(s)=s^{\alpha/2}, \quad \alpha=1,3,5,7,9.$$

α=1	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	9.79×10^{-3}	.5	4.32×10^{-3}	
6	6.92×10^{-3}	.5	1.52×10^{-3}	1.5
12	4.87×10^{-3}	.5	5.33×10^{-4}	1.5
24	3.43×10^{-3}	.5	1.88×10^{-4}	1.5
48	2.42×10^{-3}	.5	6.66×10^{-5}	1.5

α=3	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	4.21×10^{-4}		1.36×10^{-4}	
6	1.46×10^{-4}	1.5	2.30×10^{-5}	2.60
12	5.10×10^{-5}	1.5	3.95×10^{-6}	2.54
24	1.79×10^{-5}	1.5	6.90×10^{-7}	2.52
48	6.32×10^{-6}	1.5	1.17×10^{-7}	2.56

α=5	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	1.05×10^{-4}		1.03×10^{-3}	
6	1.82×10^{-5}	2.5	7.55×10^{-5}	3.77
12	3.21×10^{-6}	2.5	5.25×10^{-6}	3.85
24	5.64×10^{-7}	2.5	3.66×10^{-7}	3.84
48	9.95×10^{-8}	2.5	2.57×10^{-8}	3.83

α=7	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	1.61×10^{-4}		2.22×10^{-4}	
6	1.50×10^{-4}	3.5	1.37×10^{-5}	4.
12	1.24×10^{-6}	3.5	8.52×10^{-7}	4.
24	1.09×10^{-7}	3.5	5.31×10^{-8}	4.
48	9.74×10^{-9}	3.5	3.3×10^{-9}	4.

α=9	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	4.32×10^{-4}		4.56×10^{-4}	
6	4.66×10^{-5}	4.64	2.88×10^{-5}	3.98
12	6.97×10^{-7}	4.74	1.80×10^{-6}	4.
24	2.61×10^{-8}	4.74	1.12×10^{-7}	4.
48	8.75×10^{-10}	4.90	7.00×10^{-9}	4.

<u>Case(iv):</u>

$$\lambda = .3 \quad \text{and} \quad k(s,t)= \begin{cases} -s(1-t) & o \leq s \leq t < 1 \\ -t(1-s) & o \leq t \leq s < 1 \end{cases}$$

Choose f so that

$$u(s)=25\ s^5(1-s)$$

	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	4.75×10^{-2}		3.39×10^{-3}	
6	4.05×10^{-3}	3.6	5.69×10^{-4}	2.6
12	3.09×10^{-4}	3.7	1.44×10^{-4}	2.
24	2.49×10^{-5}	3.6	3.61×10^{-5}	2.
48	2.22×10^{-6}	3.5	9.01×10^{-6}	2.

Case (v): $k(s,t)=t-s \quad o \le s,\ t \le 1 \quad , \quad \lambda=1$

Choose f so that

$u(s)=s\ln(s)$

	COLLOCATION		NYSTROM	
N	Max. Error	Rate	Max. Error	Rate
3	$1.45 \quad 10^{-2}$		$1.73 \quad 10^{-3}$	
6	$7.24 \quad 10^{-3}$	1.	$4.22 \quad 10^{-4}$	2.
12	$3.63 \quad 10^{-3}$	1.	$1.04 \quad 10^{-4}$	2.
24	$1.86 \quad 10^{-3}$	1.	$2.50 \quad 10^{-5}$	2.
48	$9.09 \quad 10^{-4}$	1.	$6.45 \quad 10^{-6}$	1.95

The estimated rates of convergence in cases (i)-(v) are in good agreement with those suggested by Theorem 4.1 depending on the smoothness of the solution.

The above data indicate that the collocation method is faster than Nystrom for problems with smooth solutions and non-smooth kernels. The Nystrom's method runs faster in the cases of non-smooth solutions but we believe that collocation with non-uniform mesh will do equally well. It is worth noting that for both methods all the time is spent in solving the linear system.

The cases (i)-(v) were solved by the collocation method described in this paper with a two-point Gaussian rule. The results obtained are less accurate, but the computed rate estimates agree with the a priori estimate obtained in Theorem 4.1 as expected.

All the numerical experiments were carried out on a CDC 6500 in single precision.

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