1975

The Hodie Method: A Brief Introduction with Summary of Computational Properties

Robert E. Lynch
Purdue University, rel@cs.purdue.edu

John R. Rice
Purdue University, jrr@cs.purdue.edu

Report Number:
75-170

THE HODIE METHOD

A Brief Introduction with Summary of Computational Properties

Robert E. Lynch and John R. Rice

Department of Computer Science
Purdue University
Lafayette, Indiana 47907

CSD-TR 170

November 18, 1975

ABSTRACT

This note describes the new HODIE method for obtaining high order difference schemes for solving differential equations. Technical details and proofs are omitted. The HODIE method is useful for boundary value problems or initial value problems with 1, 2 or 3 space variables. This note illustrates the method for second order elliptic boundary value problems. An extensive table of operation counts and storage requirement estimates are given for the HODIE and competing methods. These indicate the potential superiority of the HODIE method, a superiority which grows rapidly with increasing dimension or accuracy requirements.

THE HODIE METHOD

Robert E. Lynch and John R. Rice

1. General description. The acronym HODIE is derived from the descriptive phrase "High Order Difference approximation via Identity Expansion."

We consider the problem

$$(1) \qquad Lu = au_{xx} + bu_{xy} + cu_{yy} + du_x + cu_y + fu = g$$

where the coefficients are defined and sufficiently smooth on some domain. The operator is second order and, in general, the type of the operator is immaterial. The method also applies to ordinary differential equations as well as parabolic equations and, in general, to m-th order equations in n independent variables.

We use two sets of points, D (for differences) and H (for high order) and the HODIE approximation V is found from

$$(2) \qquad L_D V = I_H g$$

where

$$L_D V = \sum_{q \in D} Q(q)V(q) \ , \ I_H g = \sum_{r \in H} R(r)g(r)$$

We see that L_D is a finite difference approximation to L on the D points and I_H is a discretization of the identity operator on the H points. The operators L_D and I_H are determined by the linear equations derived from

(3) $$L_D p = I_H(Lp) \quad , \quad p \epsilon P_m$$

where P_m is the space of polynomials of degree at most m. The HODIE method can be viewed as a hybrid method: a combination of a finite difference operator, L_D, and an expansion of the identity operator, I, as I_H. The first involves the expansion of L in terms of polynomials and the second involves the expansion of g in terms of elements of LP_m.

There are two stages to the solution of a problem:

A. The determination of the coefficients of L_D and I_H in the HODIE equation (2) by solving (3). The equations (3) form a small linear system; the number of unknowns is one less than the total number of D and H points (one is used for normalization). Except for constant coefficients, one of these systems must be solved for each mesh point. In the case of a rectangular two dimensional domain with $h = 1/(N+1)$, this results in N^2 systems. The dimension of P_m is $k = (m+1) * (m+2)/2$ and a system (3) with k equations gives a $O(h^m)$ HODIE method. If there are r points in D and s points in H ($r+s = k+1$), then one can choose the basis of P_m so that the matrix of (3) is reducible and has the form $\begin{pmatrix} D_1 & E \\ 0 & H_1 \end{pmatrix}$ where D_1 is r-by-r and H_1 is s-by-s. Furthermore, D_1 is a very simple constant matrix, independent of the mesh points for natural ways of choosing the points D.

B. The solution of the HODIE equation (2). This is a large finite difference system, with as many equations as mesh points and the band width is independent of the order of the scheme.

2. SPECIFIC EXAMPLE: $4^{\underline{th}}$ ORDER HODIE WITH 9-POINT DIFFERENCES

The two point configurations are

x	x	x		o	o	o
x	x	x		o	o	o
x	x	x			o	
D-points (9 point stencil)				H-points		

These 16 points are used with the 15 basic functions of P_4:

Subset 1: 1 $\quad x$ $\quad y$ $\quad x^2-h^2$ $\quad y^2-h^2$ $\quad xy$ $\quad y(x^2-h^2)$ $\quad x(y^2-h^2)$ $\quad (x^2-h^2)(y^2-h^2)$

Subset 2: $x(x^2-h^2)$ $\quad yx(x^2-h^2)$ $\quad y(y^2-h^2)$ $\quad xy(y^2-h^2)$ $\quad x^2(x^2-h^2)$ $\quad y^2(y^2-h^2)$

The first 9 (Subset 1) are associated with D, the remaining 6 (Subset 2) with H. Note that those of Subset 2 are identically zero on D. The matrix for the L_D coefficients are:

$$
\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -h & 0 & h & 0 & -h & -h & h & h \\
0 & 0 & -h & 0 & h & h & -h & -h & h \\
-h^2 & 0 & -h^2 & 0 & -h^2 & 0 & 0 & 0 & 0 \\
-h^2 & -h^2 & 0 & -h^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -h^2 & h^2 & -h^2 & h^2 \\
0 & 0 & -h^3 & 0 & h^3 & 0 & 0 & 0 & 0 \\
0 & h^3 & 0 & -h^3 & 0 & 0 & 0 & 0 & 0 \\
h^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
$$

and the right hand side is

$$I_H(L(1),\ L(x),\ L(y),---,\ L[(x^2-h^2)(y^2-h^2)])$$

where (x,y) are H-points. After one value of I_H is normalized to be unity, the coefficient matrix for the I_H coefficients is 6 x 6 with little special structure.

We present an operations count (multiplications only) for solving (2) and (3) on the unit square. We ignore the N^2 evaluations of the coefficients a(x,y) etc. that any method must make. We have assumed that $b(x,y) \equiv 0$ in this example and later analysis.

A. Solution for the L_D and I_H operators (Eq. 3)		Multiplies
(i)	Formation of I_H equations	$98N^2$
(ii)	Solution of the I_H equations	$121N^2$
(iii)	Evaluation of right side of L_D equations	$122N^2$
(iv)	Solution of L_D equations (they are very simple)	$13N^2$
	Total	$254N^2$

B. Solution of the Difference equations (Eq. 2)		
(i)	Evaluation of right side	$9N^2$
(ii)'	Solution by profile Gauss elimination	$2N^4$
(iii)"	Solution by SOR type method (if applicable)	$9N^3$

The total multiplications is $2N^4 + 363N^2$ or $9N^3 + 363N^2$. Note that the HODIE method has determined $16N^2$ unknowns.

The most efficient previously known fourth order method is collocation with Hermite cubics. For the same fineness of discretization there are $(N+1)^2$ elements and $16(N+1)^2$ unknowns. The multiplication count for this collocation is $64(N+1)^4 + 320(N+1)^2$ or $16(N+1)^3 + 320(N+1)^2$ (Note that the count given in the Prenter and Russell preprint are wrong), depending on whether profile Gauss or SOR (if applicable) is used.

It is clear that this $4^{\underline{th}}$ order HODIE method involves substantially fewer multiplications and we also note that it requires $2N^3$ (for Gauss) or $9N^2$ (for SOR) words of memory compared to $16(N+1)^3$ or $16(N+1)^2$ for collocation. This comparison, of course, assumes that these two methods produce exactly the same accuracy whereas one might actually be much more accurate than the other for the same h values. Only computational experiments will tell.

3. FURTHER OBSERVATIONS ABOUT THE HODIE METHOD

A. If L has constant coefficients then the $363N^2$ term becomes $9N^2$ and if, in addition, $g(x,y) \equiv 0$ then this term disappears. Similar savings occur for collocation.

B. Technical problems occur if the null space of L contains P_m (e.g. for the Laplacian) and one cannot raise the order above six except by using a non polynomial basis. This limitation does not seem to be of practical importance.

C. The HODIE difference equations have the same structure as the ordinary finite difference equations as $h \rightarrow 0$. Thus, choices of D where SOR, etc. is known to be effective allow SOR to be applied to HODIE methods independent of the order (provided h is not too big). For example, the 5-point star when $b(x,y) \equiv 0$. (D contains 5, instead of 9 points as in Section 2).

D. There are certain special points (Gauss type, such as used in collocation) which give higher accuracy. The study of the exploitation of these points is not complete except for the one dimensional case.

E. The general form of the HODIE matrix of (2) and (3) may be viewed as

$$\begin{pmatrix} A & 0 & 0 \\ 0 & D & E \\ 0 & 0 & H \end{pmatrix}$$

where

$H \quad = \quad \mathrm{Diag} \quad (H_1, H_2, ----, \; H_{N^2})$

$H_i \quad = \quad (r-1) \times (r-1)$ general matrix

and

$D \quad = \quad \mathrm{Diag} \quad (D_1, D_2 \; ----, \; D_{N^2})$

$D_i \quad = \quad s \times s$ simple constant matrix.

The matrix A is the usual finite difference type matrix whose coefficients come from the solution of the D equations and whose corresponding right sides come from the H equations.

F. HODIE is a generalization of the Mehrstellenverfahren method as described by Collatz. But, our formulation differs in several ways from other such schemes discussed in the literature. It also has various connections with other methods, in particular, it is closely related to finite element methods such as least squares and collocation.

G. If there is a single point for the operator I_H, then the scheme reduces to a usual difference approximation derived by, say, the use of divided central differences; typically such a scheme has $O(h)$ or $O(h^2)$ accuracy. If such a scheme converges, then so do higher order schemes.

H. Boundary conditions and curved boundaries do not introduce difficulties as they do in usual finite difference formulations, including Mehrstellenverfahren schemes.

I. The nested disection technique can be applied tc the difference equations generated. For two dimensional problems this gives a multiplication count of the same order as SOR with a direct method.

4. MULTIPLICATION COUNT AND STORAGE COMPARISONS OF METHODS

We consider three methods: $m^{\underline{th}}$ order HODIE, $m^{\underline{th}}$ order to collocation and "equivalent" classical second order finite differences. We implicitly assume that the first two give the same accuracy for the same h and that the equivalent second order finite difference gives the same accuracy using a mesh with $N^{m/2}$ points in each variable. The tables present selected combinations of the following possibilities:

Methods: $4^{\underline{th}}$ and $6^{\underline{th}}$ order HODIE, collocation and "equivalent" finite differences.

Operators: Variable coefficients, constant coefficients, homogeneous constant coefficients. We always have the u_{xy} term missing for the counts made.

Dimensions: 2 and 3

Solution Methods: Profile Gauss elimination, SOR and related schemes.

The particular HODIE methods involve a 9-point finite difference approximations. For many of the cases we do not know whether SOR and related iterative schemes are actually applicable, thus these entries must be considered as tentative. We always assume that N or N+1 iterations are sufficient for convergence. All table entries are in thousands of multiplications or words of storage.

Fourth Order, Variable Coef. in 2-Dim.

N	Gauss Elimination HODIE	Gauss Elimination COLLOC.	Gauss Elimination EQUIV. FD	SOR Solution HODIE	SOR Solution COLLOC.	SOR Solution EQUIV. FD
5	10/.25	177/7	788/31	10/.22	15/2	85/3
10	56/2	1900/40	200000/2000	45/.9	124/8	5100/50
20	465/16	25000/295	2^{+7}/64000	217/3.6	735/28	320000/800
50	13000/250	866000/4000	$3^{+9}/1.5^{+7}$	2000/32	9300/170	8^{+7}/31000
100	203000/2000	1.3^{+7}/33000	$2^{+13}/2^{+9}$	12600/90	69000/250	5^{+9}/500000

Table 1. Comparison of multiplication and storage counts for fourth order methods for two dimensional, variable coefficients problems. The formulas used for this table are (in the order used):

$2N^4+363N^2/2N^3$, $128(N+1)^4 + 320(N+1)^2/32(N+1)^3$, $2N^8+11N^4/2N^6$,

$9N^3+363N^2/9N^2$, $64(N+1)^3 + 320(N+1)^2/64(N+1)^2$, $5N^6+11N^4/5N^4$

The format is (thousands of multiplications)/(thousands of storage words)

Fourth Order, Constant Coef. in 2-Dim.

N	Gauss Elimination HODIE	Gauss Elimination COLLOC.	Gauss Elimination EQUIV. FD	SOR Solution HODIE	SOR Solution COLLOC.	SOR Solution EQUIV. FD
5	1.5	166	781	1.3	14	78
10	21	1850	200000	10	85	5000
20	324	21000	2^{+7}	76	600	320000
50	12500	800000	3^{+9}	1150	8500	8^{+7}
100	200000	1.3^{+6}	2^{+13}	9100	66000	5^{+9}

Table 2. Comparison of multiplication counts for fourth order methods for two dimensional, constant coefficient problems. The formulas used for this table are (in the order used):

$2N^4+9N^2$, $128(N+1)^4$, $2N^8$, $9N^3+9N^2$, $64(N+1)^3$, $5N^6$

The storage counts are the same as in Table 1.

Fourth Order, Constant Coef. and Homogeneous in 2-Dim.

	Gauss Elimination			SOR Solution		
N	HODIE	COLLOC.	EQUIV. FD	HODIE	COLLOC.	EQUIV. FD
5	1.2	166	781	1.1	14	78
20	320	21000	2^{+7}	72	600	320000
100	200000	1.3^{+6}	2^{+13}	9000	66000	5^{+9}

Table 3. Comparison of multiplication counts for fourth order methods for two dimensional, constant coefficients and homogeneous problems. The formulas are:

$2N^4$, $128(N+1)^4$, $2N^8$, $9N^3$, $64(N+1)^3$, $5N^6$.

Sixth Order, Variable Coef. in 2-Dim.

	Gauss Elimination			SOR Solution		
N	HODIE	COLLOC.	EQUIV. FD	HODIE	COLLOC.	EQUIV. FD
5	96/.25	2000/35	490000/3900	97/.22	289/26	9940/78
10	403/2	22000/200	$2^{+9}/2^{+6}$	392/.9	1400/88	$5^{+6}/5000$
20	1850/16	287000/20000	$8^{+12}/1^{+9}$	1600/4	8360/322	$3^{+9}/320000$
50	22000/250	$1^{+7}/165000$	$5^{+17}/4^{+12}$	11000/22	106000/1900	$1^{+13}/8^{+7}$

Table 4. Comparison of multiplication and storage counts for sixth order methods for two dimensional, variable coefficients problems. The super convergence phenomena of collocation and the use of special "Gauss-type" points for HODIE raises the effective order possible with the counts given. The formulas are:

$2N^4+3830N^2/2N^3$, $1458(N+1)^4+3650(N+1)^2/162(N+1)^3$, $2N^{12}+11N^6/2N^9$

$9N^3+3830N^2/9N^2$, $729(N+1)^3+3650(N+1)^2/729(N+1)^2$, $5N^9+11N^6/5N^6$

Sixth Order, Constant Coef. in 2-Dim.

	Gauss Elimination			SOR Solution		
N	HODIE	COLLOC.	EQUIV FD	HODIE	COLLOC.	EQUIV FD
5	1.7	1890	488000	1.6	158	9770
10	22	21300	2^{+9}	11	970	5^{+6}
20	328	284000	8^{+12}	80	6750	3^{+9}
50	12600	1^{+7}	5^{+17}	1200	97000	1^{+13}

Table 5. Comparison of multiplication counts for sixth order methods for two dimensional, constant coefficients problems. The formulas are

$2N^4+19N^2$, $1458(N+1)^4$, $2N^{12}$, $9N^3+19N^2$, $729(N+1)^3$, $5N^9$

Fourth Order, Variable Coef. in 3-Dim.

	Gauss Elimination			SOR Solution		
N	HODIE	COLLOC.	EQUIV FD	HODIE	COLLOC.	EQUIV FD
5	406/6	287000/995	1^{+7}/19500	267/3.5	1500/111	2950/109
10	22000/200	2^{+7}/20600	$2^{+11}/2^{+7}$	2300/27	12800/680	714000/7000
20	5^{+6}/6400	2^{+9}/523000	$3^{+15}/2^{+10}$	20300/216	137000/4750	2^{+8}/448000
50	2^{+9}/625000	$9^{+11}/4^{+7}$	$1^{+21}/2^{+14}$	420000/3400	3^{+6}/68000	$3^{+11}/1^{+8}$

Table 6. Comparison of multiplication and storage counts for fourth order methods for three dimensional, variable coefficients problems. The formulas are

$2N^7+2000N^3/2N^5$, $1024(N+1)^7+4100(N+1)^3/128(N+1)^5$, $2N^{14}+14N^6/2N^{10}$

$27N^4+2000N^3/27N^3$, $512(N+1)^4+4100(N+1)^3/512(N+1)^3$, $7N^8 + 14N^6/7N^6$

Fourth Order, Constant Coef. in 3-Dim.

	Gauss Elimination			SOR Solution		
N	HODIE	COLLOC.	EQUIV FD	HODIE	COLLOC.	EQUIV FD
5	157	287000	1^{+7}	18	664	2700
10	20000	2^{+7}	2^{+11}	279	7500	700000
20	3^{+6}	2^{+9}	3^{+15}	4300	100000	2^{+8}
50	2^{+9}	9^{+11}	1^{+21}	68000	3^{+6}	3^{+11}

Table 7. Comparison of multiplication counts for fourth order methods for three dimensional constant coefficient problems. The formulas are:

$2N^7+9N^3$, $1024(N+1)^7$, $2N^{14}$, $27N^4+9N^3$, $512(N+1)^4$, $7N^8$

5. CONCLUSIONS

Any conclusions drawn from these estimates must be regarded as tentative. However, these tables indicate that the HODIE method has the potential to significantly increase the domain of solvable problems and to substantially cut the cost of solution of many current computations. Define Routine, Practical and Feasible computations as those with costs less than \$10, \$500 and \$10,000, respectively. Indicate a computation by the triple (accuracy, dimension, operator type) and then we can classify HODIE computations as follows for problems with smooth, well-behaved solutions:

	Using Gauss	Using SOR
Routine	(10^{-6},2D,var.coef.) (10^{-3},3D,var.coef.)	(10^{-7},2D,var.coef.) (10^{-4},3D, var.co
Practical	(10^{-10},2D, var. coef.) (10^{-4}, 3D, var. coef.)	(10^{-5},3D,var. coef.)
Feasible	(10^{-5},3D,var.coef.)	(10^{-7},3D,var.coef.)

These estimates also suggest that it is reasonable to solve time dependent problems with three space variables and obtain modest accuracy.

It is clear that a systematic study is needed to find those HODIE methods where SOR, etc. is effective. The consequences for different choices of the D and H points needs to be studied from many points of view.

There is a clear need to implement some versions of the HODIE method to test the validity of the conclusions suggested by this analysis.