The Hodie Method: A Brief Introduction with Summary of Computational Properties

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THE HODIE METHOD

A Brief Introduction with Summary of Computational Properties

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ABSTRACT

This note describes the new HODIE method for obtaining high order
difference schemes for solving differential equations. Technical details
and proofs are omitted. The HODIE method is useful for boundary value
problems or initial value problems with 1, 2 or 3 space variables. This
note illustrates the method for second order elliptic boundary value pro­
blems. An extensive table of operation counts and storage requirement
estimates are given for the HODIE and competing methods. These indicate
the potential superiority of the HODIE method, a superiority which grows
rapidly with increasing dimension or accuracy requirements.
1. General description. The acronym HODIE is derived from the descriptive phrase "High Order Difference approximation via Identity Expansion."

We consider the problem

\begin{equation}
Lu = au_{xx} + bu_{xy} + cu_{yy} + du_x + cu_y + f = g
\end{equation}

where the coefficients are defined and sufficiently smooth on some domain. The operator is second order and, in general, the type of the operator is immaterial. The method also applies to ordinary differential equations as well as parabolic equations and, in general, to m-th order equations in n independent variables.

We use two sets of points, D (for differences) and H (for high order) and the HODIE approximation \( V \) is found from

\begin{equation}
L_D V = I_H g
\end{equation}

where

\[
L_D V = \sum_{q \in D} Q(q) V(q), \quad I_H g = \sum_{r \in H} R(r) g(r)
\]

We see that \( L_D \) is a finite difference approximation to \( L \) on the D points and \( I_H \) is a discretization of the identity operator on the H points. The operators \( L_D \) and \( I_H \) are determined by the linear equations derived from
where $P_m$ is the space of polynomials of degree at most $m$. The HODIE method can be viewed as a hybrid method: a combination of a finite difference operator, $L_D$, and an expansion of the identity operator, $I$, as $I_H$. The first involves the expansion of $L$ in terms of polynomials and the second involves the expansion of $g$ in terms of elements of $LP_m$.

There are two stages to the solution of a problem:

A. The determination of the coefficients of $L_D$ and $I_H$ in the HODIE equation (2) by solving (3). The equations (3) form a small linear system; the number of unknowns is one less than the total number of $D$ and $H$ points (one is used for normalization). Except for constant coefficients, one of these systems must be solved for each mesh point. In the case of a rectangular two dimensional domain with $h = 1/(N+1)$, this results in $N^2$ systems. The dimension of $P_m$ is $k = (m+1) \cdot (m+2)/2$ and a system (3) with $k$ equations gives a $O(h^m)$ HODIE method. If there are $r$ points in $D$ and $s$ points in $H$ ($r+s = k+1$), then one can choose the basis of $P_m$ so that the matrix of (3) is reducible and has the form where $D_1$ is $r$-by-$r$ and $H_1$ is $s$-by-$s$. Furthermore, $D_1$ is a very simple constant matrix, independent of the mesh points for natural ways of choosing the points $D$.

B. The solution of the HODIE equation (2). This is a large finite difference system, with as many equations as mesh points and the band width is independent of the order of the scheme.
2. **SPECIFIC EXAMPLE: 4th ORDER HODIE WITH 9-POINT DIFFERENCES**

The two point configurations are

\[
\begin{align*}
&x \quad x \quad x \\
&x \quad x \quad x \\
&x \quad x \quad x
\end{align*}
\]

D-points (9 point stencil) H-points

These 16 points are used with the 15 basic functions of \( P_4 \):

Subset 1: \( 1 \ x \ y \ x^2 \ y^2 \ xy \ y(x^2) \ x(y^2) \ (x^2-h^2) (y^2-h^2) \)

Subset 2: \( x(x^2-h^2) \ yx(x^2-h^2) \ y(y^2-h^2) \ xy(y^2-h^2) \ x^2(x^2-h^2) \ y^2(y^2-h^2) \)

The first 9 (Subset 1) are associated with \( D \), the remaining 6 (Subset 2) with \( H \).

Note that those of Subset 2 are identically zero on \( D \). The matrix for the \( L_D \) coefficients are:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -h & 0 & h & 0 & -h & -h & h & h & h \\
0 & 0 & -h & 0 & h & h & -h & -h & h & h \\
-h^2 & 0 & -h^2 & 0 & -h^2 & 0 & 0 & 0 & 0 & 0 \\
-h^2 & -h^2 & 0 & -h^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -h^2 & h^2 & -h^2 & h^2 & h^2 \\
0 & 0 & -h^3 & 0 & h^3 & 0 & 0 & 0 & 0 & 0 \\
0 & h^3 & 0 & -h^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and the right hand side is

\[ I_H (L(1), L(x), L(y), \ldots, L[(x^2-h^2)(y^2-h^2)]) \]
where \((x,y)\) are \(H\)-points. After one value of \(I_H\) is normalized to be unity, the coefficient matrix for the \(I_H\) coefficients is \(6 \times 6\) with little special structure.

We present an operations count (multiplications only) for solving (2) and (3) on the unit square. We ignore the \(N^2\) evaluations of the coefficients \(a(x,y)\) etc. that any method must make. We have assumed that \(b(x,y) \equiv 0\) in this example and later analysis.

A. Solution for the \(L_D\) and \(I_H\) operators (Eq. 3)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Multiplies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Formation of (I_H) equations</td>
<td>(98N^2)</td>
</tr>
<tr>
<td>(ii)</td>
<td>Solution of the (I_H) equations</td>
<td>(121N^2)</td>
</tr>
<tr>
<td>(iii)</td>
<td>Evaluation of right side of (L_D) equations</td>
<td>(122N^2)</td>
</tr>
<tr>
<td>(iv)</td>
<td>Solution of (L_D) equations (they are very simple)</td>
<td>(13N^2)</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>(254N^2)</td>
</tr>
</tbody>
</table>

B. Solution of the Difference equations (Eq. 2)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Multiplies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Evaluation of right side</td>
<td>(9N^2)</td>
</tr>
<tr>
<td>(ii)'</td>
<td>Solution by profile Gauss elimination</td>
<td>(2N^4)</td>
</tr>
<tr>
<td>(iii)&quot;</td>
<td>Solution by SOR type method (if applicable)</td>
<td>(9N^3)</td>
</tr>
</tbody>
</table>

The total multiplications is \(2N^4 + 363N^2\) or \(9N^3 + 363N^2\). Note that the HODIE method has determined \(16N^2\) unknowns.

The most efficient previously known fourth order method is collocation with Hermite cubics. For the same fineness of discretization there are \((N+1)^2\) elements and \(16(N+1)^2\) unknowns. The multiplication count for this collocation is \(64(N+1)^4 + 320(N+1)^2\) or \(16(N+1)^3 + 320(N+1)^2\) (Note that the count given in the Prenter and Russell preprint are wrong), depending on whether profile Gauss or SOR (if applicable) is used.
It is clear that the 4th order HODIE method involves substantially fewer multiplications and we also note that it requires $2N^3$ (for Gauss) or $9N^2$ (for SOR) words of memory compared to $16(N+1)^3$ or $16(N+1)^2$ for collocation. This comparison, of course, assumes that these two methods produce exactly the same accuracy whereas one might actually be much more accurate than the other for the same $h$ values. Only computational experiments will tell.

3. FURTHER OBSERVATIONS ABOUT THE HODIE METHOD

A. If $L$ has constant coefficients then the $36N^2$ term becomes $9N^2$ and if, in addition, $g(x,y) = 0$ then this term disappears. Similar savings occur for collocation.

B. Technical problems occur if the null space of $L$ contains $P_m$ (e.g. for the Laplacian) and one cannot raise the order above six except by using a non polynomial basis. This limitation does not seem to be of practical importance.

C. The HODIE difference equations have the same structure as the ordinary finite difference equations as $h \to 0$. Thus, choices of $D$ where SOR, etc. is known to be effective allow SOR to be applied to HODIE methods independent of the order (provided $h$ is not too big). For example, the 5-point star when $b(x,y) = 0$ ($D$ contains 5, instead of 9 points as in Section 2).

D. There are certain special points (Gauss type, such as used in collocation) which give higher accuracy. The study of the exploitation of these points is not complete except for the one dimensional case.
E. The general form of the HODIE matrix of (2) and (3) may be viewed as:

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & D & E \\
0 & 0 & H \\
\end{pmatrix}
\]

where

\[
H = \text{Diag} \left( H_1, H_2, \ldots, H_{n_2} \right)
\]

\[
H_i = (r-1) \times (r-1) \text{ general matrix}
\]

and

\[
D = \text{Diag} \left( D_1, D_2, \ldots, D_{n_2} \right)
\]

\[
D_i = s \times s \text{ simple constant matrix}.
\]

The matrix A is the usual finite difference type matrix whose coefficients come from the solution of the D equations and whose corresponding right sides come from the H equations.

F. HODIE is a generalization of the Mehrstellenverfahren method as described by Collatz. But, our formulation differs in several ways from other such schemes discussed in the literature. It also has various connections with other methods, in particular, it is closely related to finite element methods such as least squares and collocation.

G. If there is a single point for the operator \( I_{H} \), then the scheme reduces to a usual difference approximation derived by, say, the use of divided central differences; typically such a scheme has \( O(h) \) or \( O(h^2) \) accuracy. If such a scheme converges, then so do higher order schemes.
H. Boundary conditions and curved boundaries do not introduce difficulties as they do in usual finite difference formulations, including Mehrstellenverfahren schemes.

I. The nested dissection technique can be applied to the difference equations generated. For two dimensional problems this gives a multiplication count of the same order as SOR with a direct method.

4. MULTIPLICATION COUNT AND STORAGE COMPARISONS OF METHODS

We consider three methods: \( m^{\text{th}} \) order HODIE, \( m^{\text{th}} \) order to collocation and "equivalent" classical second order finite differences. We implicitly assume that the first two give the same accuracy for the same \( h \) and that the equivalent second order finite difference gives the same accuracy using a mesh with \( N^{m/2} \) points in each variable. The tables present selected combinations of the following possibilities:

- **Methods**: \( 4^{\text{th}} \) and \( 6^{\text{th}} \) order HODIE, collocation and "equivalent" finite differences.

- **Operators**: Variable coefficients, constant coefficients, homogeneous constant coefficients. We always have the \( u_{xy} \) term missing for the counts made.

- **Dimensions**: 2 and 3

- **Solution Methods**: Profile Gauss elimination, SOR and related schemes.

The particular HODIE methods involve a 9-point finite difference approximations. For many of the cases we do not know whether SOR and related iterative schemes are actually applicable, thus these entries must be considered as tentative. We always assume that \( N \) or \( N+1 \) iterations are sufficient for convergence. All table entries are in thousands of multiplications or words of storage.
Fourth Order, Variable Coef. in 2-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10/.25</td>
<td>177/7</td>
<td>788/31</td>
<td>10/.22</td>
<td>15/2</td>
<td>85/3</td>
</tr>
<tr>
<td>10</td>
<td>56/2</td>
<td>1900/40</td>
<td>200000/2000</td>
<td>45/.9</td>
<td>124/8</td>
<td>5100/50</td>
</tr>
<tr>
<td>20</td>
<td>465/16</td>
<td>25000/295</td>
<td>2^+7/64000</td>
<td>217/3.6</td>
<td>735/28</td>
<td>320000/800</td>
</tr>
<tr>
<td>50</td>
<td>13000/250</td>
<td>866000/4000</td>
<td>3^+9/1.5^+7</td>
<td>2000/32</td>
<td>9300/170</td>
<td>8^+7/31000</td>
</tr>
<tr>
<td>100</td>
<td>203000/2000</td>
<td>1.3^+7/33000</td>
<td>2^+13/2^+9</td>
<td>12600/90</td>
<td>69000/250</td>
<td>5^+9/50000</td>
</tr>
</tbody>
</table>

Table 1. Comparison of multiplication and storage counts for fourth order methods for two dimensional, variable coefficients problems. The formulas used for this table are (in the order used):

- \(2N^4 + 363N^2 / 2N^3, 128(N+1)^4 + 320(N+1)^2 / 32(N+1)^3, 2N^8 + 11N^4 / 2N^6, \)
- \(9N^3 + 363N^2 / 9N^2, 64(N+1)^3 + 320(N+1)^2 / 64(N+1)^2, 5N^6 + 11N^4 / 5N^4.\)

The format is (thousands of multiplications)/(thousands of storage words)

Fourth Order, Constant Coef. in 2-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.5</td>
<td>166</td>
<td>781</td>
<td>1.3</td>
<td>14</td>
<td>78</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>1850</td>
<td>200000</td>
<td>10</td>
<td>85</td>
<td>5000</td>
</tr>
<tr>
<td>20</td>
<td>324</td>
<td>21000</td>
<td>2^+7</td>
<td>76</td>
<td>600</td>
<td>320000</td>
</tr>
<tr>
<td>50</td>
<td>12500</td>
<td>800000</td>
<td>3^+9</td>
<td>1150</td>
<td>8500</td>
<td>8^+7</td>
</tr>
<tr>
<td>100</td>
<td>200000</td>
<td>1.3^+6</td>
<td>2^+13</td>
<td>9100</td>
<td>66000</td>
<td>5^+9</td>
</tr>
</tbody>
</table>

Table 2. Comparison of multiplication counts for fourth order methods for two dimensional, constant coefficient problems. The formulas used for this table are (in the order used):

- \(2N^4 + 9N^2, 128(N+1)^4, 2N^8, 9N^3 + 9N^2, 64(N+1)^3, 5N^6.\)

The storage counts are the same as in Table 1.
Fourth Order, Constant Coef. and Homogeneous in 2-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.2</td>
<td>166</td>
<td>781</td>
<td>1.1</td>
<td>14</td>
<td>78</td>
</tr>
<tr>
<td>20</td>
<td>320</td>
<td>21000</td>
<td>2^7</td>
<td>72</td>
<td>600</td>
<td>320000</td>
</tr>
<tr>
<td>100</td>
<td>200000</td>
<td>1.3^46</td>
<td>2^13</td>
<td>9000</td>
<td>66000</td>
<td>5^9</td>
</tr>
</tbody>
</table>

Table 3. Comparison of multiplication counts for fourth order methods for two dimensional, constant coefficients and homogeneous problems. The formulas are:

\[2N^4, 128(N+1)^4, 2N^3, 9N^3, 64(N+1)^3, 5N^6.\]

Sixth Order, Variable Coef. in 2-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV. FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>96/.25</td>
<td>2000/35</td>
<td>490000/3900</td>
<td>97/.22</td>
<td>289/26</td>
<td>9940/78</td>
</tr>
<tr>
<td>10</td>
<td>403/2</td>
<td>22000/200</td>
<td>2^9/2^6</td>
<td>392/9</td>
<td>1400/89</td>
<td>5^9/5000</td>
</tr>
<tr>
<td>20</td>
<td>1850/16</td>
<td>287000/20000</td>
<td>8^12/1^9</td>
<td>1600/4</td>
<td>8360/322</td>
<td>3^9/320000</td>
</tr>
<tr>
<td>50</td>
<td>22000/250</td>
<td>1^7/165000</td>
<td>5^17/4^12</td>
<td>11000/22</td>
<td>106000/1900</td>
<td>1^13/8^7</td>
</tr>
</tbody>
</table>

Table 4. Comparison of multiplication and storage counts for sixth order methods for two dimensional, variable coefficients problems. The super convergence phenomena of collocation and the use of special "Gauss-type" points for HODIE raises the effective order possible with the counts given. The formulas are:

\[2N^4 + 3830N^2/2N^3, 1458(N+1)^4 + 3650(N+1)^2/162(N+1)^3, 2N^{12} + 11N^6/2N^9\]
\[9N^3 + 3830N^2/9N^2, 729(N+1)^3 + 3650(N+1)^2/729(N+1)^2, 5N^9 + 11N^6/5N^6\]
Sixth Order, Constant Coef. in 2-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>Gauss Elimination</th>
<th>SOR Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HODIE</td>
<td>COLLOC.</td>
</tr>
<tr>
<td>5</td>
<td>1.7</td>
<td>1890</td>
</tr>
<tr>
<td>10</td>
<td>22</td>
<td>21300</td>
</tr>
<tr>
<td>20</td>
<td>328</td>
<td>284000</td>
</tr>
<tr>
<td>50</td>
<td>12600</td>
<td>1^{+7}</td>
</tr>
</tbody>
</table>

Table 5. Comparison of multiplication counts for sixth order methods for twodimensional, constant coefficients problems. The formulas are

\[2N^4 + 19N^2, 1458(N+1)^4, 2N^{12}, 9N^3 + 19N^2, 729(N+1)^3, 5N^9\]

Fourth Order, Variable Coef. in 3-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>Gauss Elimination</th>
<th>SOR Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HODIE</td>
<td>COLLOC.</td>
</tr>
<tr>
<td>5</td>
<td>406/6</td>
<td>287000/995</td>
</tr>
<tr>
<td>10</td>
<td>22000/200</td>
<td>2^{+7}/20600</td>
</tr>
<tr>
<td>20</td>
<td>5^{+6}/6400</td>
<td>2^{+9}/523000</td>
</tr>
<tr>
<td>50</td>
<td>2^{+9}/625000</td>
<td>9^{+11}/4^{+7}</td>
</tr>
</tbody>
</table>

Table 6. Comparison of multiplication and storage counts for fourth order methods for three dimensional, variable coefficients problems. The formulas are

\[2N^7 + 2000N^3/2N^5, 1024(N+1)^7 + 4100(N+1)^3/128(N+1)^5, 2N^{14} + 14N^6/2N^{10},
27N^4 + 2000N^3/27N^3, 512(N+1)^4 + 4100(N+1)^3/512(N+1)^3, 7N^8 + 14N^6/7N^6\]
Fourth Order, Constant Coef. in 3-Dim.

<table>
<thead>
<tr>
<th>N</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV FD</th>
<th>HODIE</th>
<th>COLLOC.</th>
<th>EQUIV FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>157</td>
<td>287000</td>
<td>1 + 7</td>
<td>18</td>
<td>664</td>
<td>2700</td>
</tr>
<tr>
<td>10</td>
<td>20000</td>
<td>2 + 7</td>
<td>2 + 11</td>
<td>279</td>
<td>7500</td>
<td>700000</td>
</tr>
<tr>
<td>20</td>
<td>3 + 6</td>
<td>2 + 9</td>
<td>3 + 15</td>
<td>4300</td>
<td>100000</td>
<td>2 + 8</td>
</tr>
<tr>
<td>50</td>
<td>2 + 9</td>
<td>9 + 11</td>
<td>1 + 21</td>
<td>68000</td>
<td>3 + 6</td>
<td>3 + 11</td>
</tr>
</tbody>
</table>

Table 7. Comparison of multiplication counts for fourth order methods for three dimensional constant coefficient problems. The formulas are:

\[ 2N^7 + 9N^3, 1024(N+1)^7, 2N^{14}, 27N^4 + 9N^3, 512(N+1)^6, 7N^8 \]

5. CONCLUSIONS

Any conclusions drawn from these estimates must be regarded as tentative. However, these tables indicate that the HODIE method has the potential to significantly increase the domain of solvable problems and to substantially cut the cost of solution of many current computations. Define Routine, Practical and Feasible computations as those with costs less than $10, $500 and $10,000, respectively. Indicate a computation by the triple (accuracy, dimension, operator type) and then we can classify HODIE computations as follows for problems with smooth, well-behaved solutions:

Using Gauss  | Using SOR
-------------|------------
Routine      | (10^{-6}, 2D, var. coef.) (10^{-3}, 3D, var. coef.)  | (10^{-7}, 2D, var. coef.) (10^{-4}, 3D, var. coef.) |
Practical    | (10^{-10}, 2D, var. coef.) (10^{-4}, 3D, var. coef.) | (10^{-5}, 3D, var. coef.) |
Feasible     | (10^{-5}, 3D, var. coef.)                            | (10^{-7}, 3D, var. coef.) |

These estimates also suggest that it is reasonable to solve time dependent problems with three space variables and obtain modest accuracy.
It is clear that a systematic study is needed to find those HODIE methods where SOR, etc. is effective. The consequences for different choices of the D and H points needs to be studied from many points of view.

There is a clear need to implement some versions of the HODIE method to test the validity of the conclusions suggested by this analysis.