1975

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Report Number:
75-140

Buchi, J. R. and Owens, T. M., "Skolem Rings and Their Varieties" (1975). *Department of Computer Science Technical Reports*. Paper 89.
<https://docs.lib.purdue.edu/cstech/89>

SKOLEM RINGS and THEIR VARIETIES

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CSD TR 140

This work was supported by NSF Grant No. GJ 40891

INTRODUCTION

Skolem semi-lattices $\langle A, \vee, 0 \rangle$ may be defined as those which admit a binary operation $+$ (the symmetric pseudo-complement), whereby $x + y$ is the smallest element u , $x \vee u = y \vee u$. Define $x * y = (x \vee y) + y$, and $x \cdot y = (x * y) \vee (y * x)$. One can recover $\vee$ from $+$, $\cdot$ by: $x \vee y = (x + y) + (x \cdot y)$. So the structures $\langle A, +, \vee, 0 \rangle$ and $\langle A, +, \cdot, 0 \rangle$ are equivalent up to equational translation. We call these structures Skolem rings.

The lattice of closed sets of a topological space and the algebra of formulas of intuitionistic propositional calculus provide the best known examples of Skolem lattices $\mathcal{A} = \langle A, +, \vee, \wedge, 0 \rangle$ (see Curry [4], Rasiowa [4]). More natural still is the notion of a Skolem semi-lattice. We add a further reason for studying these structures; Theorem 1.1. The semi-lattice $\mathcal{A} = \langle A, \vee, 0 \rangle$ admits a nice representation $\text{Idl } \mathcal{A} \cong \text{Cgr } \mathcal{B}$ ($\text{Idl } \mathcal{A}$ is the ideal lattice of $\mathcal{A}$, $\text{Cgr } \mathcal{B}$ is the congruence lattice of an algebra $\mathcal{B}$), just in case $\mathcal{A}$ is Skolem. In this case the representation $U \mapsto (U)$ is given by $x \equiv y(U) \Leftrightarrow x + y \in U$, and the representing algebra $\mathcal{B}$ may be taken as the Skolem ring $\langle A, +, \vee, 0 \rangle$. Here "nice" means that 1) the class of 0 modulo $\equiv(U)$ is just the ideal U , and 2) $\equiv(U)$ is a congruence of $\vee$.

The representation problem will be discussed in section 1. As a bonus to this work an equational axiom system for Skolem rings is developed and Theorem 1.2: (Monteiro [9]) the class of all Skolem rings forms a variety. The laws for semi-lattices plus and the laws $x \leq x + 0$, $x + y \leq (x+z) \vee (y+z)$, and $(x \vee y) + y \leq x$ are an equational axiom system.

In section 3 we will clarify somewhat the relationship of Skolem rings to Topology. The semi-lattice of closed sets of a topological space is easily seen to be Skolem, here the $+$ operation is obtained by defining $A + B$ as the closure of the symmetric set difference $(A-B) \cup (B-A)$. We will prove in theorem 3.1 that every Skolem ring may be embedded as a subalgebra of the Skolem ring of closed sets of a topological space. The notion of a regular closed set has a natural analogue in Skolem rings, that of a regular element. We have the following counterpart to the Topological result; theorem 3.2: The regular elements of a Skolem ring $\mathcal{A}$ form a subalgebra of $\mathcal{A}$ and a Boolean ring.

The remainder of this paper is concerned with the subvariety lattice, or dually the lattice of equational extensions of Skolem rings. Theorem 4.1 asserts that those Skolem rings in which the $\cdot$ operation is a meet may be characterized as subdirect products of chains. This class of Skolem rings then forms a subvariety which Whalen and Nemitz [13] have shown is characterized by the associative

law for $\cdot$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. A characterization of the subdirectly irreducible Skolem rings developed in section 2 is used in theorem 5.1 to show that every finitely generated Skolem ring is finite. This result is in contrast to the situation for Skolem lattices. We conclude by showing that the Variety lattice of Skolem rings is completely distributive (theorem 5.2) and has the power of the continuum (theorem 5.3).

1.1. Discrete Systems

The closely related notions of a closure space, and a complete lattice are found throughout mathematics. The two spaces which most frequently arise are characterized by their $\mathbf{V}$ -bases. Let S be a closure space (complete lattice), call $a \in S$ strongly- $\mathbf{v}$ -irreducible if whenever $a \in \text{cl}\{b_1, \dots, b_n\}$ ($a \leq b_1 \vee \dots \vee b_n$), there is $i \leq n$ and $a \in \text{cl}\{b_i\}$ ($a \leq b_i$); a is called strongly-directed- $\mathbf{V}$ -irreducible (or compact) if whenever $a \in \text{cl}B$, $B \subseteq S$ ($a \leq \mathbf{V}B$), there are $b_1, \dots, b_n \in B$ and $a \in \text{cl}\{b_1, \dots, b_n\}$ ($a \leq b_1 \vee \dots \vee b_n$). Closure spaces and complete lattices which have $\mathbf{V}$ -bases of strongly- $\mathbf{v}$ -irreducible elements usually occur in continuous mathematics and are called topological. Those which have $\mathbf{V}$ -bases of strongly-directed- $\mathbf{v}$ -irreducible (compact) elements are most often met with in finitary mathematics, Logic, Geometry, Algebra, etc., and are called discrete.

As examples of the discrete complete lattices we mention $\text{Cgr } \mathcal{A}$ (the congruence lattice of the algebra $\mathcal{A}$), $\text{Sba } \mathcal{A}$ (the subalgebra lattice of $\mathcal{A}$), and $\text{Idl } \mathcal{A}$ (the ideal lattice of a semi-lattice $\mathcal{A} = \langle A, \vee, 0 \rangle$). Of course these systems may equally well be described by closure spaces. In each case the set of compact elements, $\text{Cgr}_0 \mathcal{A}$, $\text{Sba}_0 \mathcal{A}$, $\text{Idl}_0 \mathcal{A}$,

may be described as the finitely generated congruences, subalgebras, and ideals respectively.

These examples suggest a number of ways of representing a discrete complete lattice L , namely:

a) As all closed sets of a discrete space

$\mathcal{S} = \langle S, \text{Cl} \rangle$: Let $S = L_0$ = all compact elements of L , for $X \subseteq S$ define,

$$\text{cl } X = \{u \in S; u \leq \bigvee X\} = \bigcup \{\text{cl}\{x_1, \dots, x_n\}; x_1, \dots, x_n \in X\}$$

b) As all ideals of a semi-lattice $\mathcal{A} = \langle A, \vee, 0 \rangle$:

(Nachbin [12]). Let $A = L_0$, let $\vee$ be the join in L , 0 be the least element in L . Then $L \cong \text{Idl } \mathcal{A}$ where $\mathcal{A} = \langle A, \vee, 0 \rangle$.

c) As all subalgebras of an algebra $\mathcal{B}$: (Büchi [3]).

By Nachbin [12] we may assume $L = \text{Idl } \mathcal{A}$, $\mathcal{A} = \langle A, \vee, 0 \rangle$ a semi-lattice. For $a \in A$ define

$$f_a(x) = \begin{cases} x & \text{if } a \not\leq x \\ a & \text{if } a \leq x \end{cases}$$

Let $\mathcal{B} = \langle A, \vee, 0, \dots, f_a, \dots \rangle_{a \in A}$. Then $L \cong \text{Sba } \mathcal{B}$.

d) As all congruences of an algebra $\mathcal{B}$: (Grätzer-Schmidt [6]). This remarkable result involves a rather intricate construction. We will only mention several

interesting features of this construction here. Starting from a semi-lattice $\mathcal{A} = \langle A, \vee, 0 \rangle$ an isomorphism $u \mapsto \equiv(u)$ is constructed which takes $\mathcal{A}$ onto $\text{Cgr}_0 \mathcal{B}$. The algebra $\mathcal{B}$ is constructed in such a way that $\text{Cgr}_0 \mathcal{B} = \text{Pgr} \mathcal{B}$, i.e. finitely generated congruences are principal, and also every principal congruence $\text{Cg}_{\mathcal{B}}\{xy\}$ (the least congruence identifying x with y) is of the form $\text{Cg}_{\mathcal{B}}\{z0\}$ for a fixed element $0 \in B$.

Following from this construction we define a representation $\text{Idl } \mathcal{A} \cong \text{Cgr } \mathcal{B}$ to be nice if it has the above features of the Grätzer-Schmidt construction: that is if $u \mapsto \equiv(u)$ is a map of $\mathcal{A}$ onto $\text{Cgr } \mathcal{B}$ satisfying,

- 1) $u \leq v \implies \equiv(u) \subseteq \equiv(v) \quad \text{for } u, v \in A$
- 2) $(\forall u)_A (\exists x)_B \quad \equiv(u) = \text{Cg}\{x0\} \text{ for fixed } 0 \in B$
- 3) $(\forall x_1, y_1, x_2, y_2)_B (\exists u)_A \quad \text{Cg}\{x_1 y_1, x_2 y_2\} = \equiv(u) .$

It is immediate from 3 that nice representations satisfy,

- 3a) $(\forall x)_B (\exists u)_A \quad \equiv(u) = \text{Cg}\{x0\}$
- 3b) $(\forall xy)_B (\exists z)_B \quad \text{Cg}\{xy\} = \text{Cg}\{z0\}$
- 3c) $(\forall xy)_B (\exists z)_B \quad \text{Cg}\{x0, y0\} = \text{Cg}\{z0\} .$

Notice that 3a) together with 2) give an onto map $x \mapsto \bar{x}$ of B onto A by the rule,

$$\text{Cg}\{x_0\} = \equiv(\bar{x}).$$

Furthermore 3b) and 3c) show that operations $+$ and v may be defined on B (not necessarily in a unique way) so that,

$$\text{Cg}\{xy\} = \text{Cg}\{(x+y)_0\}$$

and

$$\text{Cg}\{x_0, y_0\} = \text{Cg}\{(xvy)_0\}.$$

Suppose now that $u \mapsto \equiv(u)$ is a nice representation of $\mathcal{A}$ onto $\text{Cgr } \mathcal{B}$. Let $u = \bar{y}$ for $y \in B$, then

$$x \equiv 0(u) \equiv x \equiv 0 \text{ (Cg}\{y_0\})$$

$$\equiv \text{Cg}\{x_0\} \subseteq \text{Cg}\{y_0\}$$

$$\equiv \equiv(\bar{x}) \subseteq \equiv(\bar{y})$$

$$\equiv \bar{x} \leq \bar{y} = u$$

and thus for the congruence class of 0 modulo u , $0^{\equiv(u)}$, we have

$$4) \quad 0^{\equiv(u)} = \{x; \bar{x} \in \text{id}\{u\}\}$$

where $\text{id}\{u\}$ denotes the least ideal in $\mathcal{A}$ containing u .
As the map $x \mapsto \bar{x}$ is onto,

$$\begin{aligned}
 5) \quad 0^{\equiv}(u) = 0^{\equiv}(v) \quad .\supset. \quad (\bar{x} \leq u \quad .\equiv. \quad \bar{x} \leq v) \\
 \quad .\supset. \quad u = v \quad .
 \end{aligned}$$

Also,

$$\begin{aligned}
 \equiv(xvy)^{-} &= \text{Cg}\{(xvy)0\} \\
 &= \text{Cg}\{x0\} \vee \text{Cg}\{y0\} \\
 &= \equiv(\bar{x}) \vee \equiv(\bar{y}) \\
 &= \equiv(\bar{x} \vee \bar{y}) \quad \text{by } 1
 \end{aligned}$$

and thus,

$$6) \quad (xvy)^{-} = \bar{x} \vee \bar{y} \quad .$$

Finally,

$$\begin{aligned}
 x \equiv y(u) \quad .\equiv. \quad \text{Cg}\{xy\} \subseteq \equiv(u) \\
 \quad .\equiv. \quad x + y \equiv 0 \quad (u)
 \end{aligned}$$

and thus by 4).

$$7) \quad x \equiv y(u) \quad .\equiv. \quad (x+y)^{-} \leq u.$$

Formulas 4) and 5) shows that nice representations are of the kind usually associated with ring theory, i.e., $\text{id}\{u\} \rightarrow 0 \equiv(u)$. 6) indicates the parallels between the $\vee$ operator on B and the $\vee$ on $\mathcal{A}$, while 7) states that $\equiv(u)$ is uniquely determined by the $+$ and $-$.

1.2. Fission-free representation, Skolem semi-lattices.

Let $u \rightarrow \equiv(u)$ be a nice representation of $\mathcal{A} = \langle A, \vee, 0 \rangle$ as $\text{Cgr}_0 \mathcal{B}$, $\mathcal{B} = \langle B, F \rangle$. Define:

$$x \sim y \equiv \text{Cg}\{x0\} = \text{Cg}\{y0\}$$

$$\equiv. \equiv(\bar{x}) = \equiv(\bar{y})$$

$$\equiv. \bar{x} = \bar{y}.$$

If $\sim$ is not the equality on B , we say that the representation uses fission.

If no fission is used in the representations, we have the fission-free condition,

$$\text{Cg}\{x0\} = \text{Cg}\{y0\} \supset x = y$$

that is,

$$\bar{x} = \bar{y} \supset x = y.$$

The representation given by the Grätzer-Schmidt construction, being very general, will not usually be the best representation of $\mathcal{A}$. In particular, their construction always involves infinite fission. That this can be improved can be seen in,

The Boolean Case: Let $\mathcal{B} = \langle A, +, \cdot, 0 \rangle$ be a Boolean ring, let $\vee$ be the join in $\mathcal{B}$. We call $\mathcal{A} = \langle A, \vee, 0 \rangle$ a Boolean semi-lattice, and it is well-known that $\text{Idl } \mathcal{A} = \text{Cgr } \mathcal{B} = \text{Cgr} \langle A, \vee, + \rangle$. Furthermore this representation is given by

$$x \equiv y(u) \iff x + y \leq u$$

is nice, and without fission.

There are many features of the Boolean case that are common to all nice, fission-free representations. Namely, the $\bar{}$ operation on B is uniquely defined, and hence so are the $+$ and $\vee$, moreover the map $x \mapsto \bar{x}$ is one to one from B onto A and we may therefore assume $A = B$, $x = \bar{x}$. So, for the nice fission-free representation we have,

1. $u \leq v \iff \equiv(u) \leq \equiv(v)$, hence congruences on $\mathcal{B}$ preserve the $\vee$ on $\mathcal{A}$, ie., $\text{Cgr} \langle A, F \rangle = \text{Cgr} \langle A, \vee, F \rangle$.
2. $\text{Cg}\{x0\} = \equiv(x)$.
3. $\text{Cg}\{xy\} = \text{Cg}\{(x + y)0\}$, hence $+$ is uniquely defined.

4. $Cg\{x0, y0\} = Cg\{(xvy)0\}$, hence v is uniquely defined, moreover,

$$Cg\{x0, y0\} = Cg\{x0\} \cup Cg\{y0\}$$

$$= \equiv(x) \cup \equiv(y) \quad \text{by 2}$$

$$= \equiv(xvy) \quad \text{by 1}$$

$$= Cg\{(xvy)^{\sim}\} \quad \text{and } v = v \text{ by 2}$$

5. $x \equiv 0(u) \Leftrightarrow x \leq u$, ie., $0^{\equiv}(u) = id\{u\}$

6. $0^{\equiv}(u) = 0^{\equiv}(v) \Rightarrow u = v$

7. $x \equiv y(u) \Leftrightarrow x + y \leq u$, hence $\equiv(u)$ is completely determined by $+$.

Using these properties, we may prove the following laws for the $+$ and $\vee$,

- i) $x \leq x + 0$ this follows by 7,5
- ii) $x + y \leq (x + z) \vee (y + z)$ as $Cg\{xy\} \subseteq Cg\{xz, yz\}$
- iii) $(xvy) + y \leq x$ as $Cg\{(xvy)y\} \subseteq Cg\{x0\}$.

Definition 1.1: $\mathcal{A} = \langle A, \vee, 0 \rangle$ is a Skolem semi-lattice if $\mathcal{A}$ admits a symmetric pseudo-complement $+$. That is if there is a binary operation $+$ on $\mathcal{A}$ such that $x + y =$ smallest u , $x \vee u = y \vee u$.

Lemma 1.1: Let $\mathcal{A} = \langle A, \vee, 0 \rangle$ be a semi-lattice, and let $+$ be an operation on $\mathcal{A}$ satisfying i), ii), iii) above, then for all $x, y, u \in A$,

1. $x + x = 0$
2. $x + y = y + x$
3. $x + 0 = x$
4. $x \vee u = y \vee u \implies x + y \leq u$

Proof:

1. $x + x = (x \vee 0) + x \leq 0$ by ii)
2. $x + y \leq (x + x) \vee (y + x) = (y + x)$ by 1.

The result follows by symmetry.

3. $x + 0 = (x \vee 0) + 0 \leq x$ by iii), and $x + 0 = x$ by i).
4. $x + 0 \leq (x + y) \vee (y + 0)$ by ii) and $x \leq (x + y) \vee y$ by 3). Hence $x \vee (x + y) \leq y \vee (x + y)$ and $x \vee (x + y) = y \vee (x + y)$ by symmetry. Conversely suppose $x \vee u = y \vee u$, then let $z = (x \vee u) = (y \vee u)$ in ii) to get,
 $x + y \leq (x + (x \vee u)) \vee (y + (y \vee u)) = ((x \vee u) + x) \vee ((y \vee u) + y)$
 by 2, and using iii) we get $x + y \leq u$. Q.E.D.

By Lemma 1.1, a semi-lattice $\mathcal{A}$ has a nice, fission-free representation as $\text{Cgr}_0 \mathcal{A}$ only if it is Skolem. In fact every Skolem semi-lattice has such a representation as we see in the following,

Lemma 1.2: Every Skolem semi-lattice $\mathcal{A}$ admits a nice representation without fission. This representation is given by $x \equiv y (u) .\equiv. x + y \leq u$ and $\mathcal{D} = \langle A, \vee, + \rangle$ will do.

Proof: Consider a Skolem semi-lattice $\mathcal{A}$ with symmetric pseudo-complement $+$, let $\mathcal{D} = \langle A, \vee, + \rangle$. Then

$$x \equiv y .\supset. x + x \equiv x + y$$

$$.\supset. 0 \equiv x + y$$

and

$$x + y \equiv 0 .\supset. x \vee (x + y) \equiv x \ \& \ y \vee (x + y) \equiv y$$

$$.\supset. x \vee y \equiv x \ \& \ x \vee y \equiv y$$

$$.\supset. x \equiv y .$$

Thus congruences on $\mathcal{D}$ are completely determined by the congruence classes of 0 . And as $x \equiv 0 \ \& \ y \equiv 0 .\supset. x \vee y \equiv 0$, and $x \equiv 0 \ \& \ y \leq x .\supset. y = (y \vee 0) \equiv (y \vee x) = x \equiv 0$, the congruence classes of 0 form ideals. The required map $u \mapsto \equiv(u)$ from $\mathcal{A}$ to $\text{Cgr}_0 \mathcal{D}$ is then given by

$$x \equiv y (u) .\equiv. x + y \leq u .\equiv. x + y \in \text{id}\{u\}.$$

Q.E.D.

Lemmas 1.1 and 1.2 are combined in the following theorems.

Theorem 1.1: Skolem semi-lattices $\mathcal{A}$ and only such admit nice representations without fission as $\text{Cgr}_0 \mathcal{A}$. The representation is given by

$$x \equiv y (u) \text{ .}\equiv\text{. } x + y \leq u$$

$$\text{.}\equiv\text{. } x \vee u = y \vee u$$

and $\mathcal{D} = \langle A, \vee, + \rangle$ will do.

Theorem 1.2: All Skolem semi-lattices $\langle A, +, \vee, 0 \rangle$ form an equational class (variety). An equational axiom system is given by the equations which make $\langle A, \vee, 0 \rangle$ a semi-lattice with zero, together with the axioms.

$$\text{i)} \quad x \leq x + 0$$

$$\text{ii)} \quad x + y \leq (x + z) \vee (y + z)$$

$$\text{iii)} \quad (x \vee y) + y \leq x$$

2.1. Relative Pseudo-complementation.

Skolem Rings may be defined alternatively as semi-lattices with a binary operation $/$, the relative pseudo-complement, which satisfies the rule

$$y \vee u \geq x \quad .\equiv. \quad u \geq x/y.$$

Then as

$$y \vee u = x \vee u \quad .\equiv. \quad y \vee u \geq x \ \& \ x \vee u \geq y$$

$$.\equiv. \quad u \geq x/y \ \& \ u \geq y/x$$

$$.\equiv. \quad u \geq (x/y) \vee (y/x).$$

The symmetric pseudo-complement $+$ may be defined by

$$a) \quad x + y = (x/y) \vee (y/x).$$

Similarly for a Skolem Ring $\mathcal{A} = \langle A, +, \vee, 0 \rangle$,

$$y \vee u \geq x \quad .\equiv. \quad y \vee u = x \vee y \vee u$$

$$.\equiv. \quad u \geq (x \vee y) + y$$

and thus the relative pseudo-complement $/$ is defined as

$$b) \quad x/y = (x \vee y) + y.$$

The following lemma gives a list of useful equations for Skolem Rings involving the $/$.

Lemma 2.1: Let $\mathcal{S}$ be a Skolem Ring, let $/$ be defined as in b). Then,

1. $x \leq y \vee u \quad .\equiv. \quad u \leq x/y$
2. $x \vee (y/x) = x \vee y$
3. $x \leq y \quad .\equiv. \quad x/y = 0$
4. $x/y \leq x$
5. $x \geq y \quad .\supset. \quad x/z \geq y/z \quad \& \quad z/x \leq z/y$
6. $x/(x/y) \leq y$
7. $(x \vee y)/z = (x/z) \vee (y/z)$
8. $(x/y)/z = x/(y \vee z) = (x/z)/y.$

Proof: 1) has already been shown, and 2) through 6) are obvious from 1). 7) is obtained by 1) as follows,

$$\begin{aligned}
 u \geq (x \vee y)/z & .\equiv. z \vee u \geq x \vee y \\
 & .\equiv. z \vee u \geq x \quad \& \quad z \vee u \geq y \\
 & .\equiv. u \geq x/z \quad \& \quad u \geq y/z \\
 & .\equiv. u \geq (x/z) \vee (y/z)
 \end{aligned}$$

8) follows similarly using the fact that $z \vee u \geq x/y \equiv y \vee z \vee u \geq x$.

Q.E.D.

2.2. Distributive Semi-lattices.

Finite distributive lattices form the simplest examples of Skolem lattices and semi-lattices. More generally complete lattices in which finite joins distribute over complete meets, i.e., in which $a \vee \bigwedge_{i \in I} \{b_i\} = \bigwedge_{i \in I} \{a \vee b_i\}$, are Skolem and in this case $a/b = \bigwedge \{c; b \vee c \geq a\}$. Conversely it is not hard to show that every Skolem lattice is distributive, and that a complete lattice is Skolem only if it is $\vee$ - $\bigwedge$ distributive in the above sense.

Definition 2.1 below extends to the notion of distributivity to semi-lattices. As we will show in Lemma 2.4, every Skolem semi-lattice is distributive in this sense.

Definition 2.1: A semi-lattice $\mathcal{A}$ is distributive if for any $x, y, z \in A$, whenever $z \leq x \vee y$, there are $u, v \in A$, $u \leq x$, $v \leq y$, and $z = u \vee v$.

Lemma 2.3 below gives evidence that definition 1 correctly extends the notion of a distributive lattice.

Lemma 2.3: The semi-lattice $\mathcal{A}$ is distributive if and only if $\text{Idl } \mathcal{A}$ is a distributive lattice. $\langle A, \vee, \wedge, 0 \rangle$ is a distributive lattice if and only if the semi-lattice $\langle A, \vee, 0 \rangle$ is distributive.

Proof: Suppose $\mathcal{A}$ is a distributive semi-lattice, let $I, J, K \in \text{Idl } \mathcal{A}$, and let $a \in (I \vee K) \wedge (I \vee J)$. Then there are $i_1, i_2 \in I$, $k \in K$, $j \in J$ and $a \leq i_1 \vee k$, $i_2 \vee j$. As $\mathcal{A}$ is distributive, there are $i' \leq i_1$, $k' \leq k$, and $a = i' \vee k'$. Thus $k' \in K$ and as $k' \leq a \leq j$, $k' \in J$, and $k' \in (J \wedge K)$. Then $a \in I \vee (J \wedge K)$ and $(I \vee J) \wedge (I \vee K) \leq I \vee (J \wedge K)$. The reverse inequality is obvious.

Conversely if $\text{Idl } \mathcal{A}$ is distributive, and $a \leq b \vee c$ then $\text{id}\{a\} = \text{id}\{a\} \wedge (\text{id}\{b\} \vee \text{id}\{c\}) = (\text{id}\{a\} \wedge \text{id}\{b\}) \vee (\text{id}\{a\} \wedge \text{id}\{c\})$, and there are $b' \leq a, b$, $c' \leq a, c$ and $a \leq b' \vee c'$. As $b', c' \leq a$ we have $a = b' \vee c'$.

The second statement follows from the first, and the well-known fact that $\langle A, \vee, \wedge, 0 \rangle$ is a distributive lattice if and only if $\text{Idl} \langle A, \vee, \wedge, 0 \rangle \cong \text{Idl} \langle A, \vee, 0 \rangle$ is distributive.

Q.E.D.

Lemma 2.4: Let $\mathcal{A} = \langle A, +, \vee, 0 \rangle$ be a Skolem semi-lattice then the semi-lattice $\langle A, \vee, 0 \rangle$ is distributive.

Proof: By Lemma 2.1,

$$z \leq x \vee y \quad \supset \quad z/x \leq y \quad \& \quad z \leq x \vee (z/x)$$

$$\quad \supset \quad z/(z/x) \leq x \quad \& \quad z \leq (z/(z/x)) \vee (z/x)$$

as z/x , $z/(z/x) \leq z$, we have $z = u \vee v$ for $u = z/(z/x) \leq x$ and $v = (z/x) \leq y$.

Q.E.D.

Corollary: If $\mathcal{A}$ is a Skolem Ring, then $\text{Idl } \mathcal{A}$ is distributive.

2.3. Homomorphisms, Subalgebras, $+$ - Process.

Theorem 1.1 gives a strong grip on the congruence relations and hence the homomorphisms of Skolem semi-lattices. If $\mathcal{A}$ and $\mathcal{B}$ are Skolem semi-lattices and $\phi: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$, then $U = \{a \in A; \phi(a) = 0\}$ is an ideal on $\mathcal{A}$ and

$$a) \quad \phi(x) = \phi(y) \equiv \phi(x) + \phi(y) = 0 \equiv \phi(x + y) = 0 \equiv$$

$$x + y \in U.$$

Conversely every ideal U on $\mathcal{A}$ determines a homomorphism ϕ of $\mathcal{A}$ by the rule $\phi(x) = \phi(y) \equiv x + y \in U$. Let $\mathcal{A}/U$ denote the homomorphic image of $\mathcal{A}$ determined by $U \in \text{Idl } \mathcal{A}$, then every homomorphic image of $\mathcal{A}$ is of the form $\mathcal{A}/U$ for some ideal U .

Lemma 2.5: Let $\mathcal{A}$ be a Skolem semi-lattice, then $\text{Cgr } \mathcal{A} \cong \text{Idl } \mathcal{A}$, with the isomorphism given by a).

For Skolem semi-lattice $\mathcal{A}$ and fixed $u \in A$, the set $\{a/u: a \in A\}$ forms a subalgebra of $\mathcal{A}$, as by Lemma 2.1 $(a/u) \vee (b/u) = (a \vee b)/u$ and $(a/u)/(b/u) = a/(u \vee (b/u)) = a/(b \vee u) = (a/b)/u$. We denote this subalgebra of $\mathcal{A}$ by $\mathcal{A}_u$. Then $\mathcal{A}_u \cong A/\text{id}\{u\}$. To see this, note that $x/(x/u) \leq u$ and

$(x/u)/x = 0 \leq u$ and thus $x + (x/u) \leq u$. The map $x \mapsto x/u$ is then a homomorphism of $\mathcal{A}_u$ induced by $\text{id}\{u\}$ and $\mathcal{A}_u \cong \mathcal{A}/\text{id}\{u\}$. We have shown

Lemma 2.6: Every homomorphic image of a Skolem ring $\mathcal{A}$ induced by a principal ideal $\text{id}\{u\}$, is isomorphic to the subalgebra $\mathcal{A}_u$ of $\mathcal{A}$. If every ideal in $\mathcal{A}$ is principal, in particular if $\mathcal{A}$ is finite, every homomorphic image of $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{A}$.

In addition to the formation of subalgebras, homomorphic images, and direct products, there is another important method of constructing Skolem semi-lattices which is often useful. Let $\mathcal{A}$ and $\mathcal{B}$ be Skolem semi-lattices, $\mathcal{A} = \langle A, +_A, \vee_A, 0_A \rangle$, $\mathcal{B} = \langle B, +_B, \vee_B, 0_B \rangle$, and assume $A \cap B = \emptyset$. Let $A' = A - \{0_A\}$ and define a new Skolem semi-lattice $\mathcal{A} + \mathcal{B}$ on $A' \cup B$ as follows:

$$a_1 \vee a_2 = a_2 \vee a_1 = \begin{cases} a_1 \vee_A a_2 & \text{if } a_1, a_2 \in A' \\ a_1 \vee_B a_2 & \text{if } a_1, a_2 \in B \\ a_1 & \text{if } a_1 \in A', a_2 \in B \end{cases}$$

$$a_1 + a_2 = a_2 + a_1 = \begin{cases} 0_B & \text{if } a_1 = a_2 \\ a_1 +_A a_2 & \text{if } a_1, a_2 \in A', a_1 \neq a_2 \\ a_1 +_B a_2 & \text{if } a_1, a_2 \in B \\ a_1 & \text{if } a_1 \in A', a_2 \in B \end{cases}$$

It is easy to verify that $\mathcal{A}^+ \mathcal{B} = \langle A' \cup B, +, \vee, 0_B \rangle$ is a Skolem semi-lattice. In the following example $\mathcal{A}$ is a four element Skolem semi-lattice, and $\mathcal{B}$ is the (unique) two element Skolem semi-lattice.

Figure 2.1

An example $\mathcal{A}^+ \mathcal{B}$, $\mathcal{B}^+ \mathcal{A}$

When $\mathcal{B}$ is the two element Skolem semi-lattice we will usually write $\mathcal{A}^+$ in place of $\mathcal{A}^+ \mathcal{B}$.

We will write $\mathcal{A} \subseteq_s \mathcal{B}$ if $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}$. The following Lemma is easily shown.

Lemma 2.7: Let $\mathcal{A}, \mathcal{B}$ be Skolem semi-lattices, then $\mathcal{A}^+ \mathcal{B}$ is a Skolem semi-lattice. The following statements are true,

- i) $\mathcal{A} \subseteq_s \mathcal{A}^+ \mathcal{B}$
- ii) $\mathcal{A} \subseteq_s \mathcal{C}, \mathcal{B} \subseteq_s \mathcal{D} \Rightarrow \mathcal{A}^+ \mathcal{B} \subseteq_s \mathcal{C}^+ \mathcal{D}$
- iii) $\mathcal{B} \in \text{Idl } \mathcal{A}^+ \mathcal{B}$
- iv) $\mathcal{A} \cong \mathcal{A}^+ \mathcal{B} / \mathcal{B}$

Definition 2.2: An ideal U of a Skolem semi-lattice $\mathcal{A}$ is called relatively maximal if there is an $a \in \mathcal{A}, a \notin U$, and for all $V \in \text{Idl } \mathcal{A}, U \subseteq V$ and $U \neq V$ imply $a \in V$.

The Relatively maximal ideals of $\mathcal{A}$ are thus the completely $\cap$ -irreducible members of $\text{Idl } \mathcal{A}$, and by Lemma 2.5, correspond to the $\cap$ -irreducible members of $\text{Cgr } \mathcal{A}$. It follows that

Lemma 2.8: $\mathcal{A}/U$ is a subdirectly irreducible factor of $\mathcal{A}$ precisely when U is relatively maximal. $\mathcal{A} \cong \mathcal{A}/\text{id}\{0\}$ is subdirectly irreducible just in case there is a smallest non-zero element in $\mathcal{A}$, i.e., when $\mathcal{A} \cong \mathcal{B}^+$ for $\mathcal{B} \subseteq_s \mathcal{A}$.

3.1. Topological Skolem Semi-lattices

Topological lattices, those which have a $\bigwedge$ -base of strongly- $\vee$ -irreducible elements, are easily seen to be $\vee$ - $\bigwedge$ distributive. They provide the most important examples of Skolem lattices and semi-lattices. In particular for Cld S , the Skolem semi-lattice of closed sets of a topological space S , the $+$ operation may be defined as the closure of the symmetric set difference, i.e., $A + B = \bigcap \{C : (A-B) \subseteq C \text{ \& } (B-A) \subseteq C\}$. It is natural to look to these topological structures, Cld S , for counter examples to possible Skolem equations. As we will see in Theorem 3.1, it is sufficient in such questions to confine our attention to those structures. In fact every Skolem semi-lattice may be embedded as a sub-algebra of Cld S , for a suitable space S . To see this, let $\mathcal{A} = \langle A, +, \vee, 0 \rangle$ be a Skolem semi-lattice and let $\mathcal{D}$ be an arbitrary dense set of proper ideals on $\mathcal{A}$ (i.e., $\mathcal{D} \subseteq \text{Idl } \mathcal{A}$ and for $a, b \in A$, $a \not\leq b$, - there is $I \in \mathcal{D}$, $b \in I$ and $a \notin I$). For $a \in A$, define $P_a = \{I \in \mathcal{D}; a \notin I\}$. Then

$$\begin{aligned} P_a \cup P_b &= \{I \in \mathcal{D}; a \notin I \text{ or } b \notin I\} \\ &= \{I \in \mathcal{D}; a \vee b \notin I\} \\ &= P_{a \vee b} \end{aligned}$$

The set $\{P_a; a \in A\}$ then forms a $\cap$ -base for a topology S on $\mathcal{D}$. As $\mathcal{D}$ is dense, $a \neq b \Rightarrow P_a \neq P_b$. The map $a \mapsto P_a$ is then an embedding of A into $\text{Cld}(S)$ which preserves the $\vee$. We have only to check now that $P_a + P_b = P_{a+b}$. By definition,

$$P_a + P_b = \cap \{P_c : P_a - P_b \subseteq P_c \text{ \& } P_b - P_a \subseteq P_c\}.$$

Now

$$\begin{aligned} P_a - P_b \subseteq P_c &\equiv (VI)_{\mathcal{D}}(a \notin I \text{ \& } b \in I \Rightarrow c \notin I) \\ &\equiv (VI)_{\mathcal{D}}(b \in I \text{ \& } c \in I \Rightarrow a \in I) \\ &\equiv (VI)_{\mathcal{D}}(b \vee c \in I \Rightarrow a \in I) \\ &\equiv a \leq b \vee c, \text{ as } \mathcal{D} \text{ is dense.} \end{aligned}$$

Similarly $P_b - P_a \subseteq P_c \equiv b \leq a \vee c$. And

$$\begin{aligned} P_a + P_b \subseteq P_c &\equiv a \vee c = b \vee c \\ &\equiv a + b \leq c \\ &\equiv P_{a+b} \subseteq P_c \end{aligned}$$

we have $P_a + P_b = P_{a+b}$. This proves

Theorem 3.1: Every Skolem semi-lattice may be embedded as a subalgebra of $\text{Cld}(S)$, for some topological space S .

Corollary: Every Skolem semi-lattice has an embedding into a Skolem lattice which preserves the $+$ and $\vee$.

3.2. Regular elements

A regular closed set in topology is defined as a set which is the closure of its interior. In order to understand the appropriate analogue for Skolem semi-lattices, we will first study the regular closed sets of the topology S of Theorem 3.1.

Define $\bar{P}_a = \{I \in \mathcal{D}; a \in I\}$, the set $\{\bar{P}_a; a \in A\}$ is then a $\cup$ -base for the open sets of S . For $T \in S$, let T^0 denote the interior of T , i.e., $T^0 = \cup\{\bar{P}_b; \bar{P}_b \subseteq T\}$. Then for $a \in A$, $P_a^0 = \cup\{\bar{P}_b; \bar{P}_b \subseteq P_a\}$ and

$$\bar{P}_b \subseteq P_a \equiv (VI)_{\mathcal{D}}(b \in I \supset a \notin I)$$

$$\equiv (VI)_{\mathcal{D}}(b \vee a \notin I)$$

$$\equiv (Vc)_A c \leq b \vee a$$

where the last equivalence follows since $\mathcal{D}$ is dense. Thus $\bar{P}_b \subseteq P_a$, and there are non-trivial regular closed sets, just in case there is a largest element $1 \in A$, and $b \vee a = 1$. And,

$$\bar{P}_b \subseteq P_a \equiv b \vee a = 1$$

$$\equiv 1/a \leq b$$

$$\equiv \bar{P}_b \subseteq \bar{P}_{1/a}$$

$$\equiv \bar{P}_{1/a} = P_a^0$$

In this case P_a is regular closed if and only if
 $P_a = \bigcap \{P_b; P_a^0 \subseteq P_b\} = \bigcap \{P_b; \bar{P}_{1/a} \subseteq P_b\}$, i.e. if and only if

$$\bar{P}_{1/a} \subseteq P_b \Leftrightarrow P_a \subseteq P_b$$

and this last statement is equivalent to,

$$b \vee (1/a) = 1 \Leftrightarrow a \leq b$$

$$\Leftrightarrow a \leq 1/(1/a)$$

As $1/(1/a) \leq a$ by Lemma 2.1, we have P_a is regular closed just in case $a = 1/(1/a)$. If for some element $b \in \mathcal{A}$, $a = 1/b$, then it is easy to show that $1/(1/a) = 1/(1/(1/b)) = 1/b = a$. This leads to the following definition.

Definition 3.1: In a Skolem semi-lattice $\mathcal{A}$ with identity 1 an element $a \in \mathcal{A}$ is regular if there is $b \in \mathcal{A}$, and $a = 1/b$.

Lemma 3.1. In a Skolem semi-lattice with identity, the regular elements form a subalgebra and a Boolean ring.

Proof: This follows from theorem 3.1; and the well known fact that the regular closed sets form a subalgebra of $\text{Cld}(S)$ and a Boolean ring.

We will see now that definition 3.1 may be modified to extend the concept of a regular element to Skolem semi-lattices without a 1. The new definition should satisfy

two conditions: 1) it should agree with definition 3.1 on Skolem semi-lattices with identity, and 2) the regular elements of an arbitrary Skolem semi-lattice should form a subalgebra and a Boolean ring. With this in mind we make the following definition.

Definition 3.2: An element a of a Skolem semi-lattice $\mathcal{A}$ is regular if there is an ideal $I_a \in \text{Idl } \mathcal{A}$ and $\text{id}\{a\}$ is the least ideal such that $A = I_a \vee \text{id}\{a\}$.

It is easy to verify that Definition 3.1 and 3.2 agree for Skolem semi-lattices with identity. Before we show that definition 3.2 meets condition 2), we will give an equivalent formulation of regularity.

Lemma 3.2: a is a regular element of a Skolem semi-lattice $\mathcal{A}$ if and only if $a = a/(b/a)$ for all $b \in A$.

Proof: Let a be a regular element of a Skolem semi-lattice $\mathcal{A}$. Then

$$b \in A = I_a \vee \text{id}\{a\} \quad \therefore (\exists i)_{I_a} \quad b \leq i \vee a$$

$$\therefore (b/a) \leq i \in I_a$$

and thus $b/a \in I_a$ for all $b \in A$. Then $b/a \in I_a$ and $a \leq (b/a) \vee (a/(b/a)) \therefore A = I_a \vee \text{id}\{a\} = I_a \vee \text{id}\{a/(b/a)\}$. As a is regular, $a \leq a/(b/a)$, and $a \geq a/(b/a)$ by Lemma 2.1, thus for all $b \in A$, $a = a/(b/a)$.

Conversely suppose $a = a/(b/a)$ for all $b \in A$, and let $I = \text{id}\{(b/a); b \in A\}$. It is clear that $A = I \vee \text{id}\{a\}$. Now suppose $A = I \vee J$ for some $J \in \text{Idl } \mathcal{A}$, then $a \in I \vee J \Rightarrow a \leq i \vee c$ for $i \in I, c \in J$, and thus $a \leq (c/a) \vee c$ for $c/a \in I$. Then $a = a/(c/a) \leq c$, and $\text{id}\{a\} \subseteq J$.
Q.E.D.

This new formulation is useful in proving the following Lemma.

Lemma 3.3: The regular elements of a Skolem semi-lattice $\mathcal{A}$ form a subalgebra of $\mathcal{A}$.

Proof: Let a, c be regular in $\mathcal{A}$. By Lemmas 3.2 and 2.1 it is sufficient to show for $b \in A$, $(a \vee c)/(b/(a \vee c)) \geq a \vee c$ and $(a/c)/(b/(a/c)) \geq a/c$.

Using Lemma 2.1,

$$(a \vee c)/(b/(a \vee c)) = a/(b/(a \vee c)) \vee c/(b/(a \vee c))$$

$$\geq a/(b/a) \vee c/(b/c)$$

$$= a \vee c \text{ as } a, c \text{ are regular.}$$

Again by Lemma 2.1,

$$(a/c)/(b/(a/c)) = (a/(b/(a/c)))/c$$

$$\geq (a/(b/a))/c$$

$$= a/c \text{ by regularity of } a.$$

Q.E.D.

Let $\text{Reg } \mathcal{A}$ denote the subalgebra of regular elements of $\mathcal{A}$. Let $a, b, c \in \text{Reg } \mathcal{A}$, and suppose $c \leq a, b$. Then $c \leq b \Rightarrow a/c \geq a/b$ and $(a/c) \vee (a/b) \geq (a/b) \vee (a/(a/b)) = a \geq c$, thus $(a/(a/b)) \geq c/(a/c) = c$ as c is regular. As $(a/(a/b)) \leq a, b$ by Lemma 2.1, it follows that $a/(a/b)$ is the meet $a \cdot b$ of a, b in $\text{Reg } \mathcal{A}$, and $\text{Reg } \mathcal{A}$ is a lattice. Moreover for $a, b \in \text{Reg } \mathcal{A}$, $a \cdot (b/a) = a/(a/(b/a)) = a/a = 0$, and $a \vee (b/a) = a \vee b$. Thus (b/a) is a complement for a in the interval $(a \vee b, 0)$. It follows that $\text{Reg } \mathcal{A}$ is generalized Boolean algebra, and thus a Boolean ring. Combined with Lemma 3.3, this proves

Theorem 3.2: The regular elements of a Skolem semi-lattice $\mathcal{A}$ form a subalgebra of $\mathcal{A}$ and a Boolean ring.

Corollary: A Boolean ring may be defined as a Skolem semi-lattice which satisfies the additional axiom $x/(y/x) = x$.

It is easy to check that $\text{Reg}(\mathcal{A}^+) \cong \text{Reg}(\mathcal{A})$, $\text{Reg}(\mathcal{A}) \subseteq_s \mathcal{A} \subseteq_s \mathcal{A}^+$. In general $\text{Reg}(\mathcal{A})$ need not be the largest boolean subalgebra of $\mathcal{A}$.

4.1 Skolem Rings

We have seen in Section 1 how the classical representation of Boolean semi-lattices $\langle A, +, \vee, 0 \rangle$ naturally extends to Skolem semi-lattices. The Boolean structure is usually presented and is more familiar in the ring form $\langle A, +, \cdot, 0 \rangle$ where

$$(a) \quad x \cdot y = (x \vee y) + (x + y),$$

the two structures being equivalent up to equational transformation. We will therefore try to introduce a corresponding $\cdot$ operation also for the more general Skolem semi-lattice; of course the two operations should coincide in the Boolean case. It turns out that (a) is not the appropriate definition, we define

$$x * y = ((x \vee y) + x) + y = y / (y/x)$$

(b)

$$x \cdot y = (x * y) \vee (y * x).$$

Definition 4.1: Let $\mathcal{A} = \langle A, +, \vee, 0 \rangle$ be a Skolem semi-lattice and let $\cdot$ be defined by (b). Then the structure $\langle A, +, \cdot, 0 \rangle$ is called a Skolem ring.

A Skolem ring is not generally a ring in the usual algebraic sense. The term is used here to indicate the parallels with Boolean rings.

Lemma 4.1: The following hold in any Skolem ring;

1. $x \cdot y = y \cdot x \leq x$
2. $x \leq y \Rightarrow x \cdot y = x$
3. $x/y \leq (x/y)/(y/x)$
4. $(x/y) \cdot (y/x) = 0$
5. $(x+y) \vee (x \cdot y) = x \vee y$
6. $x \cdot (y \vee z) \leq x \cdot y \vee x \cdot z$

Proof:

1. Using Lemma 2.1 $x/(x/y) \leq x$ and $y/(y/x) \leq x$, thus $x \cdot y = (x/(x/y)) \vee (y/(y/x)) \leq x$, $x \cdot y = y \cdot x$ follows by symmetry.

2. $x \leq y \Rightarrow x/y = 0$ Lemma 2.1
 $\Rightarrow x/(x/y) = x$
 $\Rightarrow x \leq (x/(x/y)) \vee (y/(y/x)) = x \cdot y$
 $\Rightarrow x = x \cdot y$ by 1).

3. It suffices to show $x \leq y \vee ((x/y)/(y/x))$. By Lemma 2.1, $y/x \leq y$ and thus $(x/y)/y \leq (x/y)/(y/x)$. Then $x \leq y \vee (x/y) = y \vee ((x/y)/y) \leq y \vee ((x/y)/(y/x))$.

4. By 3) and Lemma 2.1, $(x/y)/((x/y)/(y/x)) = 0$ and $(y/x)/((y/x)/(x/y)) = 0$, thus

$$\begin{aligned}(x/y) \cdot (y/x) &= (x/y)/((x/y)/(y/x)) \vee (y/x)/((y/x)/(x/y)) \\ &= 0.\end{aligned}$$

5. As $x+y = (y/x) \vee (x/y)$ we have

$$\begin{aligned}(x+y) \vee (x \cdot y) &= ((y/x) \vee (x/y)) \vee ((x/(x/y)) \vee (y/(y/x))) \\ &= ((y/x) \vee (y/(y/x))) \vee ((x/y) \vee (x/(x/y))) \\ &= y \vee x.\end{aligned}$$

6. By Lemma 2.1, $x/(x/(y \vee z)) \leq x/(x/y)$ and

$$\begin{aligned}(y \vee z)/(y \vee z)/x &= y/((y \vee z)/x) \vee z/((y \vee z)/x) \\ &\leq y/(y/x) \vee z/(z/x)\end{aligned}$$

thus,

$$\begin{aligned}x \cdot (y \vee z) &= x/(x/(y \vee z)) \vee (y \vee z)/((y \vee z)/x) \\ &\leq y/(y/x) \vee z/(z/x) \vee x/(x/y) \\ &\leq (y/(y/x) \vee x/(x/y)) \vee (z/(z/x) \vee x/(x/z)) \\ &= (x \cdot y) \vee (x \cdot z)\end{aligned}$$

Q.E.D.

Lemma 4.2: Let $\mathcal{A} = \langle \Lambda, +, \vee, 0 \rangle$ be a Skolem semi-lattice and let $\cdot$ be defined by (b). Then $\vee$ can be defined in the Skolem ring $\langle \Lambda, +, \cdot, 0 \rangle$ by

$$(c) \quad x \vee y = (x+y) + (x \cdot y).$$

The notions of a Skolem semi-lattice and a Skolem ring are equivalent.

Proof: The second statement follows directly from (c). To prove (c) let $u = (x+y) + (x \cdot y)$; as $x+y, x \cdot y \leq x \vee y$ it is clear that $u \leq x \vee y$. It suffices to show $x, y \leq u$. By Lemma 2.1 and 4.1

$$(x+y) \vee u = (x \cdot y) \vee u = (x+y) \vee (x \cdot y) = x \vee y$$

and thus $x \leq (x+y) \vee u, (x \cdot y) \vee u$. Using the distributivity of $\vee$ we may write $x = z_1 \vee u_1 = z_2 \vee u_2$ for $z_1 \leq x+y, z_2 \leq x \cdot y, u_1, u_2 \leq u$. Again by distributivity $z_1 \leq z_2 \vee u_2$ implies $z_1 = z \vee u'$ for $z \leq z_2, u' \leq u_2$. Let $v_1 = u_1 \vee u'$, then $x = z \vee v_1$ for $z \leq x+y, x \cdot y$ and $v_1 \leq u$. Similarly $y = w \vee v_2$ for $w \leq x+y, x \cdot y$ and $v_2 \leq u$. As $w \leq x \cdot y \leq x$, $x = (z \vee w) \vee v_1$ and similarly $y = (z \vee w) \vee v_2$. Then, $(x+y) \leq v_1 \vee v_2$ and thus $(z \vee w) \leq (x+y) \leq v_1 \vee v_2 \leq u$. As $z, w, v_1, v_2 \leq u$, we have $x, y \leq u$.

Q.E.D.

As Skolem semi-lattices and Skolem rings are equivalent up to equational translation, we will use the two terms interchangeably.

4.2. Correct Skolem Rings

The $\cdot$ operation in a Boolean ring $\mathcal{A}$ is the meet operation of the lattice, that is $x \cdot y = x \wedge y =$ the greatest

lower bound of x, y in $\mathcal{A}$. This contrasts with the general Skolem case where the meet $x \wedge y$ may not exist, or if it exists may not be equationally definable. As we have seen the $\cdot$ operation for Skolem rings was derived by modifying the Boolean definition, and as formulas (1) and (2) of Lemma 4.1 show, it retains some of the characteristics of a meet. The question arises: for which Skolem rings is the $\cdot$ operation a meet? More generally: for which Skolem rings is a meet equationally definable?

Whalen and Nemitz [13] have shown that the $\cdot$ operation is the meet precisely when it satisfies the associative law $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. This result is included in Theorem 4.1 below. It is easily seen that these "associative" structures are the only ones in which the meet is equationally definable.

Linear ordered sets with smallest element 0 are the simplest examples of Skolem rings in which $x \cdot y = x \wedge y$. We call these structures chains and use C_n , $n \in \omega$ to denote the n -element chain.

Definition 4.2: Let $U \in \text{Idl } \mathcal{A}$, for Skolem ring $\mathcal{A}$. U is said to be prime if $x \cdot y \in U$ implies $x \in U$ or $y \in U$. x and y are said to be 0-divisors if $x, y \neq 0$ and $x \cdot y = 0$.

Lemma 4.3: Let P be an ideal in the Skolem ring $\mathcal{A}$. Then

P is prime $\equiv \mathcal{A}/P$ is a chain.

Proof: Suppose P is prime, let $x, y \in A$, and ϕ be the homomorphism $\phi: A \rightarrow \mathcal{A}/P$. As $(x/y) \cdot (y/x) = 0 \in P$ by 4. of

Lemma 4.4, either $x/y \in P$ or $y/x \in P$. Suppose $x/y \in P$ then $0 = \phi(x/y) = \phi(x)/\phi(y)$ and thus $\phi(x) \leq \phi(y)$, and $\mathcal{A}/P$ is a chain.

Conversely suppose $\mathcal{A}/P$ is a chain, $x, y \in A$, $x \cdot y \in P$. As $\mathcal{A}/P$ is a chain we may assume $\phi(x) \geq \phi(y)$. Then by Lemma 4.1, $\phi(y) = \phi(x) \cdot \phi(y) = \phi(x \cdot y) = 0$, and thus $y \in P$.
Q.E.D.

Corollary: Let $\mathcal{A}$ be a Skolem ring. Then,

$\mathcal{A}$ has no 0-divisors $\Leftrightarrow \mathcal{A}$ is a chain.

If the $\cdot$ operation is a meet in some Skolem ring $\mathcal{A}$, then by Lemmas 2.3 and 2.4 $\langle A, \vee, \cdot, 0 \rangle$ is a distributive lattice and thus every relatively maximal ideal is prime. The proof goes as follows: Let P be relatively maximal in $\mathcal{A}$, say $a \notin P$ and for all $b \in P$, $a \in P \vee \text{id}\{b\}$. Let $b, c \notin P$, then $a \in P \vee \text{id}\{b\}$, $P \vee \text{id}\{c\}$ and thus $a = p_1 \vee b' = p_2 \vee c'$ for $p_1, p_2 \in P$, $b' \leq b$, $c' \leq c$ by distributivity. Then as $\cdot$ is the $\wedge$, by Lemmas 2.3, and 2.4,

$$\begin{aligned}
 (*) \quad a &= a \cdot a = (p_1 \vee b') \cdot (p_2 \vee c') \\
 &= (p_1 \cdot p_2 \vee p_1 \cdot c' \vee p_2 \cdot b') \vee b' \cdot c'
 \end{aligned}$$

thus $a \in P \vee \text{id}\{b' \cdot c'\}$ and,

$$(**) \quad b' \leq b, c' \leq c \Rightarrow b' \cdot c' \leq b \cdot c$$

then $a \in P \vee id\{b \cdot c\}$, $b \cdot c \notin P$ and P is prime.

Then $x \cdot y = x \wedge y$ in $\mathcal{A}$ implies that every relatively maximal ideal is Prime, which implies by Lemma 4.2, that every subdirectly irreducible factor of $\mathcal{A}$ is a chain, and $\mathcal{A}$ is a subdirect product of chains. Conversely as $x \cdot y$ is the meet of x, y in every chain, and as the statement " $\cdot$ is meet" is equational (e.g. replace $\wedge$ with $\cdot$ in the usual lattice equations) we have the following:

Lemma 4.4: For a Skolem ring $\mathcal{A}$, the following statements are equivalent:

- i) the $\cdot$ operation is the meet in $\mathcal{A}$
- ii) $\mathcal{A}$ is a subdirect product of chains.

Remark 1: A closer look at the proof just given for Lemma 4.4 shows that only two properties need be assumed for the $\cdot$ operation on $\mathcal{A}$ in order for the relatively maximal ideals of $\mathcal{A}$ to be prime. Namely at (*) it is enough to assume the distributive law

- (1) $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$, and at (**) it is sufficient that
- (2) $x \leq y \Rightarrow x \cdot z \leq y \cdot z$.

In fact it is easily seen that (1) is equivalent with (2), and both are equivalent to the associative law (3) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. This is summarized in the following theorem.

Theorem 4.1: The following are equivalent statements for a Skolem ring $\mathcal{A} = \langle A, +, \cdot, 0 \rangle$.

- a) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- b) $x \leq y \Rightarrow x \cdot z \leq y \cdot z$
- c) $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
- d) every relatively maximal ideal of $\mathcal{A}$ is prime.
- e) $\mathcal{A}$ is a subdirect product of chains.
- f) $\cdot$ is a meet operation on $\mathcal{A}$.

Proof: (b) $\leftrightarrow$ (c) is trivial, (d) $\leftrightarrow$ (e) $\leftrightarrow$ (f) by Lemmas 4.3 and 4.4. (f) $\rightarrow$ (a) is trivial. We will prove (a) $\rightarrow$ (b) and (c) $\rightarrow$ (d).

(a) $\rightarrow$ (b): Using 2 of Lemma 4.1, we have,

$$x \leq y \Rightarrow x \cdot y = x$$

$$\Rightarrow (x \cdot y) \cdot z = x \cdot z$$

$$\Rightarrow (x \cdot y) \cdot (z \cdot z) = x \cdot z$$

by (a) and 1 of Lemma 4.1 we may rewrite this last expression

as $(x \cdot z) \cdot (y \cdot z) = x \cdot z$ and conclude again by 2 of Lemma 4.1 that $x \cdot z \leq y \cdot z$.

(c) $\rightarrow$ (d): Using the fact that c) $\rightarrow$ b) we may assume

$$x \leq y \Rightarrow x \cdot z \leq y \cdot z \quad \text{and} \quad x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$$

and conclude by applying Remark 1 to the proof of Lemma 4.3.

Q.E.D.

Definition 4.3: We will call a Skolem ring satisfying any of the equivalent statements of Theorem 4.1 a Correct Skolem ring.

Theorem 4.1 may be read as asserting that the Variety of correct Skolem rings is identical with the Variety generated by chains. As every n -element subset of a chain $\mathcal{L}$ which includes the 0 of $\mathcal{L}$ forms a subalgebra isomorphic to C_n , an equation fails in $\mathcal{L}$ only if it fails in some finite subchain of $\mathcal{L}$. We may conclude,

Corollary 1: (Whalen, Nemitz [13]). The Variety of Correct Skolem rings is the Variety generated by the finite chains.

Moreover as Whalen and Nemitz point out, every subdirectly irreducible Skolem ring $\mathcal{A}$ which is not a chain must have a subalgebra of the form in Figure 4.1.

Figure 4.1

An example of a non-correct Skolem Ring

To see this, let $x, y \in \mathcal{A}$ such that $x \not\leq y$ and $y \not\leq x$, then for $a = (x/y)$, $b = (y/x)$ and u the smallest non-zero element in $\mathcal{A}$. The set $\{(a \vee b), a, b, u, 0\}$ is easily seen to be a subalgebra of $\mathcal{A}$. Note that in this case $a \cdot b = 0$ and hence $\cdot$ is not the meet. Further the four elements $a, b, (a \vee b), 0$ form a subalgebra of $\mathcal{A}$ which does not include $a \wedge b = u$, and thus the meet is not equationally definable in $\mathcal{A}$. We have the following corollary to Theorem 4.1,

Corollary 2: The Variety generated by the finite chains is the only Variety of Skolem rings for which a meet may be equationally defined, and in this case $x \wedge y = x \cdot y$.

5.1. The Subvariety Lattice of Skolem Rings.

Let A denote the class of all algebras of a given similarity type, and $\mathcal{S}$ the class of all identities or equations of that type. For $K \subseteq A$, $M \subseteq \mathcal{S}$, define $K^\tau \subseteq \mathcal{S}$, $M^\mu \subseteq A$ by: K^τ is the set of all identities which hold for every algebra in K , M^μ is the set of all algebras which satisfy the identities in M . This describes a Galois connection (see Birkhoff [2]), between two important closure spaces, the one over universal algebras, the other over equational theories.

In the first space, the map $K \rightarrow K^{\tau\mu}$ defines the closure operator. Closed sets, those of the form $K^{\tau\mu}$ for $K \subseteq A$, are called Varieties. $K^{\tau\mu}$ is called the variety generated by K , denoted $V(K)$. If $K = \{\mathcal{A}\}$ for an algebra $\mathcal{A} \in A$, we will write $V(\mathcal{A})$ in place of $V(\{\mathcal{A}\})$. By a well-known result of Birkhoff[1], $V(K) = HSP(K)$ where H, S, P denote the closure respectively under the taking of homomorphisms, subalgebras and direct products. If K_1, K_2 are varieties, $K_1 \subseteq K_2$ then K_1 is said to be a subvariety of K_2 . The subvarieties of a given variety K , ordered by inclusion, thus form a complete lattice $VL(K)$ called the subvariety lattice of K . $Th(K)$, the complete lattice of equational

theories extending K^T is defined dually over $\mathcal{S}$. $\text{Th}(K)$ and $\text{VL}(K)$ are dually isomorphic by the Galois connection.

The class of all Skolem Rings, (S.R.), forms a Variety by theorem 1.2, in this section we will give a description of $\text{VL}(\text{S.R.})$ and dually $\text{Th}(\text{S.R.})$. We will use the fact that varieties of the form $V(\mathcal{A})$ for $\mathcal{A}$ a subdirectly irreducible Skolem ring form a $\vee$ -base for $\text{VL}(\text{S.R.})$.

Let $\mathcal{L}_n$ denote the n -element chain and $\mathcal{D}$ the Skolem Ring of Figure 4.1. The one element algebra $\mathcal{L}_1$ is a homomorphic image of every Skolem Ring and thus $V(\mathcal{L}_1)$ is the zero of the lattice $\text{VL}(\text{S.R.})$: it corresponds to the largest theory in $\text{Th}(\text{S.R.})$, the inconsistent theory $x = y$. As every non-trivial subdirectly irreducible Skolem ring has $\mathcal{L}_2$ as a subalgebra, $V(\mathcal{L}_2)$ is the smallest non-trivial subvariety, and $\mathcal{L}_2^T$ is the maximally consistent or complete theory in $\text{Th}(\text{S.R.})$. Let $\mathcal{A}$ be a subdirectly irreducible Skolem ring, and let u be the smallest non-zero element in $\mathcal{A}$. If $\mathcal{A}$ is not $\mathcal{L}_2$, then there is $a \in \mathcal{A}$, $a > u$ and the set $\{a, u, 0\}$ forms a subalgebra of $\mathcal{A}$ isomorphic to $\mathcal{L}_3$. As the Boolean identity of theorem 3.2 $x = x/(y/x)$ holds in $\mathcal{L}_2$ but fails in $\mathcal{L}_3$, $V(\mathcal{L}_3)$ properly contains $V(\mathcal{L}_2)$. Similarly it may be shown that every variety which properly contains $V(\mathcal{L}_3)$ contains either $V(\mathcal{L}_4)$ or $V(\mathcal{D})$. The base of the lattice $\text{VL}(\text{S.R.})$ thus looks like,

Figure 5.1
Base of $VL(S.R.)$

It follows further that $V(L_2)$ is the variety of all Boolean rings, and hence Stones' result: every Boolean ring is a subdirect product of L_2 . Any identity such as $x = x/(y/x)$ which holds in L_2 but fails in L_3 together with the Skolem ring identities of theorem 1.2, will serve as an equational set of axioms for L_2^r .

The Whalen-Nemitz result in section 4 shows that the variety of correct Skolem rings is the join of varieties generated by finite Skolem rings. It is not difficult to show that this is also the case for the variety of all Skolem rings. This follows from a result due to McKinsey and Tarski [9,10], using theorem 3.1. To see this suppose $\phi(x_1, \dots, x_n)$ is an equation which fails in some Skolem ring $\mathcal{A} = \langle A, +, \vee, 0 \rangle$. We may assume by theorem 3.1 that A is a distributive lattice. Say ϕ fails for $x_1 = a_1, \dots, x_n = a_n$, for $a_i \in A$. Let B' be the finite subset of A containing $a_1, \dots, a_n$ and all subterms of the expression $\phi(a_1, \dots, a_n)$.

in A . Let B be the finite distributive sublattice of A generated by B' and let $\mathcal{D} = \langle B, +', \vee, 0 \rangle$ be the Skolem ring on B . For $a, b \in B$, if $a + b \leq B$ then $a +' b = \bigwedge \{c \in B, a \vee c = b \vee c\} = a + b$. Hence $\phi(x_1, \dots, x_n)$ fails in $\mathcal{D}$ for $x_i = a_i$.

Theorem 5.1 below shows that in fact the varieties of the form $V(\mathcal{A})$ for $\mathcal{A}$ finite and subdirectly irreducible form a V -base for $VL(S.R.)$. This follows from the more general fact that the finitely generated Skolem rings are finite. This result is rather surprising in view of the example given by McKinsey and Tarski [9,10] of an infinite Skolem lattice generated by only two elements.

We will let $\mathcal{F}_n$ denote the free Skolem ring generated by n elements. A simple argument shows that $\mathcal{F}_n$ is finite. To see this, let x, y be the two generators of $\mathcal{F}_2$, and let $\mathcal{A}^+$ be a subdirectly irreducible factor of $\mathcal{F}_2$. Let ϕ be the onto projection $\phi: \mathcal{F}_2 \rightarrow \mathcal{A}^+$, and let u be the smallest non-zero element of $\mathcal{A}^+$. As ϕ is onto $\mathcal{A}^+$, $\phi(x), \phi(y)$ generate $\mathcal{A}^+$ and as $u \notin \mathcal{A} \subseteq_s \mathcal{A}^+$, either $\phi(x) = u$ or $\phi(y) = u$. Say $\phi(x) = u$, then $\mathcal{A} \cong \mathcal{L}_3$ or $\mathcal{L}_2$ according as $\phi(y) > \phi(x)$ or $\phi(y) \leq \phi(x)$. In either case $\mathcal{A} \in V(\mathcal{L}_3)$.

Lemma 5.1: $\mathcal{F}_2 \in V(\mathcal{L}_3)$, $\mathcal{F}_2$ is finite.

Proof: The first statement has already been proven, the second statement follows from the first and following general fact,

Lemma 5.2: Let $\mathcal{A}, \mathcal{B}$ be algebras of the same type and $A \in V(\mathcal{B})$. If $\mathcal{B}$ is finite and $\mathcal{A}$ is finitely generated, then $\mathcal{A}$ is finite.

Proof: Suppose $\mathcal{A}$ is generated by n -elements, then as $\mathcal{A} \in \text{HSP}\{\mathcal{B}\}$ there is an $\mathcal{A}' \in \text{SP}\{\mathcal{B}\}$, and $\mathcal{A} \in \text{H}\{\mathcal{A}'\}$. We may assume $\mathcal{A}'$ is generated by n -elements, it suffices to show $\mathcal{A}'$ is finite. Suppose $\mathcal{A}'$ is generated by $\{a_1, \dots, a_n\}$ and $\mathcal{A}'$ is a subalgebra of $\prod_{i \in I} \mathcal{B}_i$ where $\mathcal{B}_i = \mathcal{B}$ for all i . Let $\phi_i: \mathcal{A}' \rightarrow \mathcal{B}_i$ be the i th projection function. We may define an equivalence relation θ on I as follows: $i \sim j(\theta)$ if and only if $\phi_i(a_k) = \phi_j(a_k)$ for all $k \leq n$. As $\mathcal{B}$ is finite, say $|\mathcal{B}| = m$, there are at most m^n different equivalence classes i^θ , $i \in I$. Pick a representative for each such equivalence class and let $I' \subseteq I$ be this set of representatives. Then clearly $\mathcal{A}'$ is isomorphic to a subalgebra of $\prod_{i \in I'} \mathcal{B}_i$, and I' and $\mathcal{B}$ are finite, $\mathcal{A}'$ and $\mathcal{A}$ are finite and $|A| \leq m \cdot m^n$.

Q.E.D.

It follows that $|\mathcal{F}_2| \leq 3 \cdot 3^2 = 27$. Direct computation shows that $\mathcal{F}_2 \cong \mathcal{L}_3 \times \mathcal{L}_3 \times \mathcal{L}_2$ and has 18 elements. This argument is extended for all $\mathcal{F}_n$, $n \in \omega$ in the following:

Theorem 5.1: $\mathcal{F}_{n+1} \in V(\mathcal{F}_n^+)$, $\mathcal{F}_{n+1}$ is finite for all $n \in \omega$.

Proof: The proof is by induction, the case $n = 1$ having been covered in Lemma 2.1, where $\mathcal{L}_3 \cong \mathcal{F}_1^+$. Assume for all $k < n$, $\mathcal{F}_{k+1} \in V(\mathcal{F}_k^+)$ and $\mathcal{F}_{k+1}$ is finite. Let $\mathcal{A}^+$ be a

subdirectly irreducible factor of $\mathcal{F}_{n+1}$ and $\phi: \mathcal{F}_{n+1} \rightarrow \mathcal{A}^+$ the onto projection. As ϕ is onto, $\mathcal{A}^+$ is generated by $\leq n+1$ elements, thus $\mathcal{A} \subseteq_s \mathcal{A}^+$ is generated by $\leq n$ elements and $\mathcal{A} \in H(\mathcal{F}_n)$. As $\mathcal{F}_n$ is finite by induction hypothesis, $\mathcal{A} \subseteq_s \mathcal{F}_n$ by lemma 2.6. Then $\mathcal{A}^+ \subseteq_s \mathcal{F}_n^+$ by lemma 2.7, and $\mathcal{F}_{n+1} \in V(\mathcal{F}_n^+)$. $\mathcal{F}_{n+1}$ is then finite by Lemma 5.2, and the induction hypothesis.

Q.E.D.

Corollary 1: Every finitely generated Skolem ring is finite.

Corollary 2: Varieties of the form $V(\mathcal{A})$ for $\mathcal{A}$ finite and subdirectly irreducible form a V-base for $VL(SR)$.

Proof of Corollary 2: If an equation ϕ fails in a variety K of Skolem Rings, it fails in some finitely generated and hence finite member of K .

Q.E.D.

As Lemma 5.3, below shows, varieties of the form $V(\mathcal{A})$, for $\mathcal{A}$ finite, are strongly-directed-V-irreducible in $VL(S.R.)$.

Lemma 5.3: Let K be a set of Skolem rings. Let $\mathcal{A} \in V(K)$. If $\mathcal{A}$ is finite, then there is a finite subset $K' \subseteq K$ and $\mathcal{A} \in V(K')$.

Proof: As $\mathcal{A} \in HSP(K)$, there are $K_i \in K$, $\mathcal{B} \subseteq_s \prod_{i \in I} K_i$, and $\phi: \mathcal{B} \rightarrow \mathcal{A}$ a homomorphism of $\mathcal{B}$ onto $\mathcal{A}$. Since $\mathcal{A}$ is finite we may assume $\mathcal{B}$ is finitely generated and thus by

theorem 5.1, $\mathcal{D}$ is finite. Let $B = \{b_1, \dots, b_n\}$ and for each $i < j \leq n$ pick $k_{ij} \in I$ such that $\pi_{k_{ij}}(b_i) \neq \pi_{k_{ij}}(b_j)$ where π_k denotes the k -th projection $\mathcal{D} \rightarrow K_k$. Let $I' = \{k_{ij} : i < j \leq n\} \subseteq I$ and let $\mathcal{D}_{I'}$ be the projection $\pi_{I'}$ of $\mathcal{D}$ onto $\prod_{i \in I'} K_i$. Then $\pi_{I'}$ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}_{I'}$ and $\mathcal{A} \in \text{HSP}\{K_i : i \in I'\}$.

Q.E.D.

Closure Spaces $\mathcal{S}$ which are both discrete and topological are called singular. They have the property that for $B \subseteq \mathcal{S}$, $\text{cl } B = \bigvee_{b \in B} \text{cl}\{b\} = \bigcup_{b \in B} \text{cl}\{b\}$; that is, the closure of a set is the union of the closure of its point subsets. It follows that singular complete lattices (the lattices of closed sets of a singular space) are completely distributive. We will show by Lemma 5.3, and a result due to B. Jonsson [8], that $\text{VL}(\text{S.R.})$ is also topological and hence singular..

Lemma: (Jonsson [8]). Let K be a class of algebras of the same type such that $\text{Cgr}(\mathcal{D})$ is distributive for each $\mathcal{D} \in V(K)$. If $\mathcal{A}$ is subdirectly irreducible and $\mathcal{A} \in V(K)$, then $\mathcal{A} \in \text{HSP}_u(K)$ where P_u denotes closure under the taking of ultraproducts.

Corollary: Let $\mathcal{A}$, K be as above, if $K_1, K_2 \subseteq K$, and $\mathcal{A} \in V(K_1 \cup K_2)$, then $\mathcal{A} \in V(K_1)$ or $\mathcal{A} \in V(K_2)$.

As we have noted, for any variety K , varieties of the form $V(\mathcal{A})$ for $\mathcal{A}$ subdirectly irreducible form a V -base for $VL(K)$. If in addition K has the property that for every algebra $\mathcal{A} \in K$, $\text{Cgr}(\mathcal{A})$ is distributive, then $\mathcal{A}$ subdirectly irreducible implies $V(\mathcal{A})$ is strongly- $\vee$ -irreducible, and $VL(K)$ is topological. For in this case by the corollary to Jonsson's Lemma, if $\mathcal{A} \in (\mathcal{A}_1) \vee \dots \vee (\mathcal{A}_n)$ for $\mathcal{A}_1, \dots, \mathcal{A}_n \in K$, then there is $i \leq n$ and $\mathcal{A} \in \vee(\mathcal{A}_i)$.

Lemma 5.4. $VL(SR)$ is topological.

Proof: By Jonsson's Lemma, we have only to show that $\text{Cgr} \mathcal{A}$ is distributive for every Skolem ring $\mathcal{A}$, and this follows as $\text{Cgr} \mathcal{A} \cong \text{Idl} \mathcal{A}$ (theorem 1.1, lemma 2.5) and $\text{Idl}(\mathcal{A})$ is distributive (lemmas 2.3 and 2.4).

Combining Lemmas 5.3 and 5.4 we get the following

Theorem 5.2: $VL(S.R.)$ is singular. Varieties of the form $V(\mathcal{A})$ for $\mathcal{A}$ finite and subdirectly irreducible are both strongly- $\vee$ -irreducible and strongly-directed- $\vee$ -irreducible and form a $\vee$ -base for $VL(SR)$.

Corollary: $VL(S.R.)$ is completely distributive.

5.2. Non-finitely Based Theories of Skolem Rings.

Definition 5.1: An equational theory is finitely based if it is axiomatized by a finite number of identities.

Theorem 1.2 shows that the theory of Skolem rings is finitely based, and theorem 3.2 shows that $\mathcal{L}_2^\tau$ is finitely based. Similarly by picking identities $\phi_1, \phi_2 \in \mathcal{L}_3^\tau$ such that ϕ_1 fails in $\mathcal{L}_4$ and ϕ_2 fails in $\mathcal{D}$ we see by figure 4.1 that ϕ_1, ϕ_2 together with the Skolem ring identities of theorem 1.2 form a finite base for $\mathcal{L}_3^\tau$. This argument is easily generalized by means of theorems 4.2 & 4.4, to show that the theory of every finite Skolem ring is finitely based.

Definition 5.2: Let $K \in \text{VL}(\text{S.R.})$, we will call $\{\mathcal{A}_i : i \in I\}$ where each $\mathcal{A}_i$ is a finite subdirectly irreducible Skolem ring, a cover for K , if for each i , $K \vee V(\mathcal{A}_i)$ covers K in $\text{VL}(\text{S.R.})$, and if whenever $K' \supseteq K$, for $K' \in \text{VL}(\text{S.R.})$ either $K' = K$ or there is an $i \in I$ and $\mathcal{A}_i \in K'$. The set $\{\mathcal{A}_i : i \in I\}$ is a minimal cover for K if it is a cover for K , and for each proper subset $I' \subset I$, $\{\mathcal{A}_i : i \in I'\}$ is not a cover for K .

Lemma 5.5: Every variety K of Skolem rings has a minimal cover.

Proof: This follows directly from Theorem 5.1, Corollary 2. For each $K' \in \text{VL}(\text{S.R.})$ which properly contains K , there is a finite subdirectly irreducible $\mathcal{A} \in K'$, $\mathcal{A} \notin K$. The

minimal cover for K consists of all such $\mathcal{A}$ of minimal cardinality.

Q.E.D.

It is easily seen that for every variety K of Skolem rings which has a finite cover, K^τ is finitely based. For if $\{\mathcal{A}_i : i \leq n\}$ is a cover for K , we may pick an identity $\phi_i \in K^\tau$ for each $i \leq n$ such that $\phi_i \notin (\mathcal{A}_i)^\tau$, the set $\{\phi_i : i \leq n\}$ together with the Skolem ring identities of theorem 1.2 will form a finite generating set for K^τ .

Lemma 5.6: Let $\mathcal{A}$ be any finite Skolem ring, then $\mathcal{A}^\tau$ is finitely based.

Proof: We will show that $VL(\mathcal{A})$ has a finite cover. As $\mathcal{A}$ is finite, we may write $V(\mathcal{A}) = V(\{\mathcal{A}_i : i \leq n\})$ for some finite set of subdirectly irreducible Skolem rings $\mathcal{A}_i$. Let m be the maximum cardinality of the $\mathcal{A}_i$, the result will follow from theorem 5.1, by showing that each member $\mathcal{B}$ of a minimal cover for $V(\mathcal{A})$ is generated by $\leq m + 1$ elements. Let $\mathcal{B}$ be a member of a minimal cover for $V(\mathcal{A})$, and let $\mathcal{B}_1$ be a maximal proper subalgebra of $\mathcal{B}$. As $\mathcal{B}$ is finite, we have $|\mathcal{B}_1| < |\mathcal{B}|$ and thus $\mathcal{B} \notin HS(\mathcal{B}_1) = HSP_u(\mathcal{B}_1)$. By the Jonsson Lemma this gives $\mathcal{B} \notin V(\mathcal{A}) \vee V(\mathcal{B}_1)$ and thus $V(\mathcal{A}) = V(\mathcal{A}) \vee V(\mathcal{B}_1)$. Again by the Jonsson Lemma we have $\mathcal{B}_1 \in HSP_u(\mathcal{A}_i) = HS(\mathcal{A}_i)$ for some i and hence $|\mathcal{B}_1| \leq m$. It follows that $\mathcal{B}$ has $\leq m + 1$ generators.

Q.E.D.

Remark: It is easily seen that every theory of Skolem Rings generated by two identities $\phi_1: \sigma_1 = \tau_1$ and $\phi_2: \sigma_2 = \tau_2$ is equivalent to the theory generated by the single identity $\phi: (\sigma_1 + \tau_1) \vee (\sigma_2 + \tau_2) = 0$. Thus we have a result analogous to that of first-order sentences, namely every finitely based theory is one-based, i.e., is the theory generated by a single equation.

Definition 5.3: A set of identities $\{\phi_i: i \in I\}$ of the same type is independent if each distinct subset generates a distinct theory. That is if for $i \notin I' \subset I$, we have $\phi_i \notin \{\phi_j: j \in I'\}^{\mu\tau}$.

Lemma 5.7: The following are equivalent statements:

1. There exists a non-finitely based theory of Skolem rings.
2. There exists an infinite set of finite Skolem rings $\{\mathcal{A}_i: i \in \omega\}$ with the property that $\mathcal{A}_i \not\subseteq_s \mathcal{A}_j$ for $i \neq j$.
3. There is an infinite independent set $\{\phi_i: i \in \omega\}$ of Skolem ring identities.
4. There are $2^{\aleph_0}$ distinct theories of Skolem rings.

Proof:

1 $\rightarrow$ 2: Let $K \in \text{VL}(\text{S.R.})$ and K^τ be non-finitely based. Let $\{\mathcal{A}_i: i \in I\}$ be a minimal cover for K . As K^τ is non-finitely based, $|I| = \aleph_0$, and any countable subset of the $\mathcal{A}_i$ will do for 2.

2 $\rightarrow$ 3: Let $\{\mathcal{A}_i : i \in \omega\}$ be a countable set of finite Skolem rings with the property that $i \neq j \Rightarrow \mathcal{A}_i \not\subseteq_s \mathcal{A}_j$. Then for $i \neq j$, $\mathcal{A}_i \not\subseteq_s \mathcal{A}_j$ which implies by Lemma 2.6, $\mathcal{A}_i \notin HS(\mathcal{A}_j)$ and by Jonssons Lemma $\mathcal{A}_i \notin V(\mathcal{A}_j)$. Let $K_i = V(\{\mathcal{A}_j : j \neq i, j \in \omega\})$. It follows from Lemma 5.4, that $\mathcal{A}_i \notin K_i$. For each $i \in \omega$ we may pick $\phi_i \in K_i^T$, $\phi_i \notin \mathcal{A}_i^T$. The set $\{\phi_i : i \in \omega\}$ picked in this manner is clearly independent.

3 $\rightarrow$ 4, and 4 $\rightarrow$ 1 are clear.

Q.E.D.

In order to answer the question of the existence of a non-finitely based theory of Skolem rings, it seems easiest to look for an infinite set $\{\mathcal{A}_i : i \in \omega\}$ of Skolem rings as in 2 of Lemma 5.7. We will now exhibit such a set together with an infinite Skolem ring $\mathcal{A}_\omega^+$ whose theory is easily shown to be non-finitely based.

We will let $\mathcal{A}_i$, $i \in \omega$ denote the Skolem rings associated with the distributive lattices of figure 5.2. Each of the structures $\mathcal{A}_n$ may be described as being generated by two chains $a > b > b_2 > \dots > b_{2n} > d$ and $b_1 > b_3 > \dots > b_{2n+1} > c$ such that (1) a, b are incomparable with b_1 , (2) c is incomparable b_{2n}, d , (3) b_i is incomparable with b_{i+1} all $i \leq 2n$, (4) $b > b_3, b_{2n+1} > d$, and $b_i > b_{i+3}$ all $i \leq 2n - 2$, and (5) b_1 and b_{2n} are the only two elements of $\mathcal{A}_n$ incomparable with four distinct elements. Where by "x incomparable with y" we mean $x \not\leq y$ and $y \not\leq x$.

We will say that $\mathcal{A}_n$ has rank n . This construction may be extended to form similar Skolem rings with ranks of arbitrarily large cardinality. $\mathcal{A}_\omega$ will be used to denote such a structure with an $\omega + \omega^*$ sequence of b_i .

The important feature of this construction is the interlocking nature of the b_i -sequence. Namely $b_i > b_{i+3}$ but b_i incomparable with b_{i+1} . Let ϕ be an embedding of $\mathcal{A}_n$ into $\mathcal{A}_m$, $n, m \in \omega$. It follows that if $b_i \in \mathcal{A}_n$, $b'_j \in \mathcal{A}_m$ and $\phi(b_i) = b'_j$, then $\phi(b_{i+1}) = b'_{j+1}$. By (5) $\phi(b_1) = b'_1$ and $\phi(b_{2n}) = b'_{2m}$. Thus ϕ is an embedding of $\mathcal{A}_n$ into $\mathcal{A}_m$ if and only if $n = m$ and ϕ is an isomorphism of $\mathcal{A}_n$. We have

Lemma 5.8: For each of the Skolem rings $\mathcal{A}_i$, $i \in \omega$ described in Figure 4.2, we have $i \neq j \Rightarrow \mathcal{A}_i \not\subseteq_s \mathcal{A}_j$.

Lemma 5.9: Let $\mathcal{A}$ be any non-trivial ultraproduct of the set $\{\mathcal{A}_i^+ : i \in \omega\}$ of Lemma 5.8. Then $\mathcal{A}^\tau = (\mathcal{A}_\omega^+)^{\tau}$ and the equational theory of $\mathcal{A}_\omega^+$ is non-finitely based.

Proof: The features (1)-(5) of the $\mathcal{A}_i$ can be stated in sentences of first order, it follows that any ultraproduct of the $\mathcal{A}_i^+$ must be similarly constructed, with non-trivial ultraproducts having infinite (in fact uncountable) rank. It is clear that any two such Skolem rings of infinite rank have the same finite subalgebras and thus the same equational theory.

Figure 5.2
Skolem Rings $\mathcal{A}_n$ of Lemma 5.8

As no proper homomorphism of $\mathcal{A}_\omega$ will satisfy feature (5) of the construction, by the Jonsson lemma $\mathcal{A}_i^+ \notin V(\mathcal{A}_\omega^+)$. Now suppose $\mathcal{A}_\omega^\tau$ is finitely based, then by the remark following Lemma 5.6 $\mathcal{A}_\omega^\tau = \{\phi\}^{\mu\tau}$ for a single identity ϕ . As $\mathcal{A}_i^+ \notin V(\mathcal{A}_\omega^+)$ we have $\phi \notin (\mathcal{A}_i^+)^{\mu\tau}$ for all i . ϕ then fails in every ultraproduct of the $\mathcal{A}_i$, and thus $\phi \notin (\mathcal{A}_\omega^+)^{\mu\tau}$. It follows that $(\mathcal{A}_\omega^+)^{\mu\tau}$ is non-finitely based. Q.E.D.

Remark: If we consider $\mathcal{A}_\omega$ as a Skolem lattice, that is as an algebra $\langle A_\omega, +, \vee, \wedge, 0 \rangle$, any two elements $(b_i \wedge b_{i+1}), b_{i+2}$ generate an infinite subalgebra. This is in contrast to the situation for Skolem rings, Lemma 5.1.

5.3. Positive Implication Algebras, Skolem Lattices

Let $\mathcal{P}$ be a propositional calculus of the intuitionistic logic of Heyting and Brouwer (see Heyting [7]). Let P be the set of all formulas of $\mathcal{P}$. We may define a quasi-ordering $\leq$ on P by $A \leq B \equiv \vdash_{\mathcal{P}} A \rightarrow B$. Let $\&$ and $\vee$ be the conjunction and disjunction on $\mathcal{P}$, let $(\equiv)$ denote the equivalence relation on P induced by $\leq$, and $1 = [A \rightarrow A]^{(\equiv)}$, $0 = [\neg A \& A]^{(\equiv)}$. The structure $\langle P/(\equiv), \rightarrow, \vee, \&, 1 \rangle$ forms a Skolem lattice in the dual form, that is, $\vee$ and $\&$ are join and meet operations for the partial ordering $\leq$ on $P/(\equiv)$, and $(A \& B) \leq C \equiv \vdash_{\mathcal{P}} (A \& B) \rightarrow C \equiv \vdash_{\mathcal{P}} B \rightarrow (A \rightarrow C) \equiv B \leq (A \rightarrow C)$. Algebras of this form are also called relatively pseudo-complemented lattices (Birkhoff [2]). Relatively pseudo-complemented lattices with the zero element

0 are called Heyting algebras, pseudo-complemented algebras, and in their dual form Brouwerian algebras, (Rasiowa [14]).

The equational theory of implication, that is, the equational theory of the structures $\langle P/(\equiv), \rightarrow, 1 \rangle$ has also been studied; Diego [5], in particular, has proven a result for these structures corresponding to our theorem 5.1. Following Rasiowa we will call such algebras Positive implication algebras.

In this section we will compare the variety (and dually the theory) lattices of positive implication algebras (P.I.), Skolem rings (S.R.) and Skolem lattices (S.L.).

Recall from Lemma 2.1 that

- 1) $x/(y \vee z) = (x/y)/z$
- 2) $(x \vee y)/z = (x/z) \vee (y/z)$

we use these two facts to prove a result for Skolem rings analogous to the disjunctive normal form in propositional calculus.

Lemma 5.10: Let ϕ be any term in the Skolem ring operations $/, \vee, 0$. Then there are terms $\psi_1, \dots, \psi_n$ in $/, 0$ such that $\phi = \bigvee_{i=1}^n \psi_i$ is a Skolem ring identity.

Proof: By induction on the length of the expression ϕ . If ϕ is of the form $\phi_1 \vee \phi_2$ the result follows immediately from the induction assumption. Otherwise ϕ is of the form ϕ_1/ϕ_2 and we may assume there are terms ψ_i, ψ'_i such that

$\phi_1 = \psi_1 \vee \dots \vee \psi_n$ and $\phi_2 = \psi'_1 \vee \dots \vee \psi'_m$. Then

$\phi = \phi_1 / \phi_2 = (\psi_1 \vee \dots \vee \psi_n) / \phi_2 = (\psi_1 / \phi_2) \vee \dots \vee (\psi_n / \phi_2)$ using I).

Each $\psi_i / \phi_2 = (\dots (\psi_i / \psi'_1) / \psi'_2) / \dots / \psi'_m$ and

$$\phi = \bigvee_{i=1}^n (\dots (\psi_i / \psi'_1) / \psi'_2) / \dots / \psi'_m$$

Q.E.D.

Corollary 1: Let ϕ be any Skolem ring identity. Then there are identities $\psi_1, \dots, \psi_n$ in $/, 0$ such that the Skolem ring theory generated by ϕ is the Skolem ring theory generated by $\{\psi_1, \dots, \psi_n\}$.

Proof: ϕ is equivalent to an identity of the form $\tau = 0$.

By the Lemma, there are $\psi'_1, \dots, \psi'_n$ in $/, 0$ such that $\tau = \bigvee_{i=1}^n \psi'_i$, then $\tau = 0 \equiv (\bigvee_{i=1}^n \psi'_i) = 0 \equiv \psi'_i = 0$ all $i \leq n$, then let ψ_i be the identity $\psi'_i = 0$.

Corollary 2: Every Skolem ring theory can be axiomatized by identities in $/, 0$, together with the two identities 1) and 2) above.

Proof: The partial ordering on a Skolem ring is completely determined by $/, 0$, that is, $a \leq b \equiv a/b = 0$. It is easily verified that 1) and 2) suffice to make $\vee$ a join for this partial ordering, while $x \leq y \vee z \equiv x/(y \vee z) = 0 \equiv (x/y)/z = 0 \equiv (x/y) \leq z$.

Q.E.D.

Ideals on a positive implication algebra $\mathcal{A}$ are defined as subsets $I \subseteq A$ such that $a \in I, b/a \in I \Rightarrow b \in I$.

Ideals on Skolem lattices are defined in the usual way. It follows as in Lemma 2.5 that for P.I. structures or S.L. structures $\mathcal{A}$, $\text{Cgr}(\mathcal{A}) \cong \text{Idl}(\mathcal{A})$ and $\text{Idl}(\mathcal{A})$ is distributive. It follows from theorem 5.1 and Jonssons Lemma that $\text{V.L.}(\text{P.I.})$ (and $\text{th}(\text{P.I.})$) is completely distributive.

Each finite subdirectly irreducible Skolem ring $\mathcal{A} = \langle A, /, \vee, 0 \rangle$ has associated with it a unique finite, subdirectly irreducible P.I. algebra $\mathcal{A}' = \langle A, /, 0 \rangle$, and as $\mathcal{A}$ is finite, a unique finite, subdirectly irreducible Skolem lattice $\mathcal{A}^\wedge = \langle A, /, \vee, \wedge, 0 \rangle$, where $\wedge$ is the meet operation on $\mathcal{A}$. By corollary 2 above, $\mathcal{A}^\wedge$ is generated by $(\mathcal{A}')^\wedge$ together with the identities 1) and 2). To each variety $K \in \text{V.L.}(\text{P.I.})$ there corresponds a variety $K^\vee \in \text{V.L.}(\text{S.R.})$ given by the rule $\mathcal{A} \in K^\vee \Leftrightarrow \mathcal{A}' \in K$ where $\mathcal{A}$ is finite and subdirectly irreducible.

Lemma 5.11: The map $\phi: K \rightarrow K^\vee$ defines a homomorphism of $\text{V.L.}(\text{P.I.})$ onto $\text{V.L.}(\text{S.R.})$.

Proof: Let $K_1, K_2 \in \text{V.L.}(\text{P.I.})$. That $\phi(K_1 \cap K_2) = \phi(K_1) \cap \phi(K_2)$ is clear, we must show $\phi(K_1 \vee K_2) = \phi(K_1) \vee \phi(K_2)$. Let $\mathcal{A}$ be a finite subdirectly irreducible Skolem ring and suppose $\mathcal{A} \in \phi(K_1 \vee K_2)$, then the corresponding P.I. algebra $\mathcal{A}' \in K_1 \vee K_2$. As $\mathcal{A}$ is subdirectly irreducible, $\mathcal{A}'$ is also, hence by the distributivity of $\text{V.L.}(\text{P.I.})$ and the Jonsson Lemma $\mathcal{A}' \in K_1$ or $\mathcal{A}' \in K_2$ and $\mathcal{A} \in \phi(K_1)$ or $\mathcal{A} \in \phi(K_2)$. Then $\phi(K_1 \vee K_2) \subseteq \phi(K_1) \vee \phi(K_2)$. The reverse inclusion is obvious. That ϕ is onto follows by Corollary 2 above. Q.E.D.

For $K \in V.L.(S.R.)$ let $K' \in V.L.(P.I.)$ and $K^\wedge \in V.L.(S.L.)$ be defined as follows: K' is generated by the algebras $\mathcal{A}' = \langle A, /, 0 \rangle$ where $\mathcal{A}$ is finite, subdirectly irreducible, $\mathcal{A} \in K$, and $K^\wedge$ is similarly generated by the algebras $\mathcal{A}^\wedge$.

Lemma 5.12: The map $\phi: K \rightarrow K'$ defines an embedding of $V.L.(S.R.)$ as a complete sub-lattice of $V.L.(P.I.)$. The map $\psi: K \rightarrow K^\wedge$ defines an embedding of $V.L.(S.R.)$ as a complete sub-lattice of $V.L.(S.L.)$.

Proof: As $V.L.(P.I.)$ is distributive, ϕ is an embedding of $V.L.(S.R.)$ as a sub-lattice by theorem 5.1 and Jonssons Lemma. It is clear that ϕ preserves arbitrary meets, and Lemma 5.3 is used to show ϕ preserves arbitrary joins. The second statement is proved similarly. Q.E.D.

By theorem 5.2 for any independent set $\{\phi_i: i \in I\}$ of Skolem ring identities there are finite Skolem rings $\mathcal{A}_i$ such that $\phi_j \in \mathcal{A}_i^\tau \iff j \neq i$. Since each $\mathcal{A}_i$ is finite there is a $\wedge$ -operation on $\mathcal{A}_i$ and $\phi_j \in \langle \mathcal{A}_i, /, \vee, \wedge, 0 \rangle^\tau \iff j \neq i$. The identities are independent when considered as identities in S.L.

Theorem 5.3: There is a countable set of independent Skolem ring and Skolem lattice identities. There is a countable set of independent identities for positive implication algebras.

Proof: The first statement follows from lemmas 5.7 and 5.8 and the above remarks. The second statement follows by generalizing lemmas 5.7 and 5.8 to the P.I-algebras.

Q.E.D.

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