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M. Moaveni
Detroit Edison Company

R. Cohen
Purdue University

J. F. Hamilton
Purdue University

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THE PREDICTION OF DYNAMIC STRAIN IN LEAF-TYPE COMPRESSOR VALVES WITH VARIABLE MASS AND STIFFNESS

Dr. M. Moaveni, Senior Research Engineer
Engineering Research Department, Detroit Edison, Detroit, Michigan

Dr. R. Cohen, Professor of Mechanical Engineering
and Director of the Ray W. Herrick Laboratories,
Purdue University, West Lafayette, Indiana

Dr. J. F. Hamilton, Professor of Mechanical Engineering
Ray W. Herrick Laboratories, Purdue University, West Lafayette, Indiana

INTRODUCTION

The continued operation of a compressor with automatic valves is highly dependent on the lift of its suction and discharge valves. The fatigue life of the valves is governed by the maximum level of cyclic stress in the valves during compressor operation.

The usual practice for the determination of valve stress is to experimentally measure the stress level for a given valve design under given operating pressures. Prediction of the valve stress if the valve design, such as the valve stop depth, or the operating pressures are changed involves remeasurement of the valve strain under the new conditions.

This paper discusses the development of a method for utilizing one set of valve strain measurements to:

1. Determine the maximum valve stress and its position on the valve, and
2. Predict the value of maximum valve stress for a change in valve stop position and/or operating pressures.

Valve strain energy analysis is used to estimate the valve response before and after the valve hits the stop. The dynamic strain modes of the valve are used to characterize the valve dynamics.

I. GENERAL THEORY -- DYNAMICS OF NON-UNIFORM WIDTH BEAMS

The differential equation of motion for the transverse vibration of beams in bending is given by

\[ \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] + m(x) \frac{\partial^2 y(x,t)}{\partial t^2} = P(x,t) \]  

(1)

Where x is the position coordinate along the length of the beam, y(x,t) is the absolute displacement at any point x and P(x,t) is the distributed load per unit length.

Using the approach of modal expansion, the beam motion may be expressed by the superposition of its natural modes of vibration

\[ y(x,t) = \sum_{n=1}^{\infty} \phi_n(x)Q_n(t) \]  

(2)

where \( \phi_n(x) \) are the beam's eigenfunctions (mode shapes) and \( Q_n(t) \) are the participation factors or time functions which depend on the forcing function involved. The axial strain due to bending is obtained from the relation

\[ \varepsilon(x,t) = h \frac{\partial^2 y(x,t)}{\partial x^2} = h \frac{\omega^2}{2} \int_{n=1}^{\infty} \frac{\partial^2 \phi_n(x)}{\partial x^2} Q_n(t) \]  

(3)

or

\[ \varepsilon(x,t) = h \sum_{n=1}^{\infty} \lambda_n^2 \psi_n(x)Q_n(t) = h \frac{\omega}{\lambda^2} \psi_n(x)Q_n(t) \]  

(4)

where h is the thickness of the beam and

\[ \psi_n(x) = \frac{d^2 \phi_n(x)}{dx^2} \left( \frac{1}{\lambda_n^2} \right) \]  

(5)

are nondimensional functions which represent the shape of the strain modes, and where

\[ \lambda_n^2 = \omega_n \frac{A(x)\rho}{EI(x)} \]  

(6)

is a convenient normalizing factor which increases with frequency and has the dimension (1/in)^2, and \( \rho \) is the mass density of the beam material.

For the case of a beam with uniform cross section both the displacement modes, \( \phi_n(x) \), and the strain modes, \( \psi_n(x) \), may be obtained from the exact functional relations [1,5], but for the beams with non-uniform cross sections, these functional relations are not usually known.

Researchers in the past [2,3,4] have tried to measure these modes in the laboratory or use equivalent uniform shapes for the shape of their valves. However, experimental evaluation of these modes is time consuming and most often does not have the high degree of accuracy which is required for a
successful stress prediction. In this work a mathematical model was selected so that the displacement and strain modes of any leaf-type valve (modeled as beams) can be evaluated analytically. The model is general and it represents the dynamics of continuous beams with variable mass and stiffness. In the model the differential equations of motion are converted into integral equations and these equations are solved by numerical integration on a digital computer.

**Strain Modes**

Substitution of Equation (2) into Equation (1) results in

\[ \frac{d^2}{dx^2} \left[ EI(x) \sum_{n=1}^{\infty} \frac{d^2 \phi_n(x)}{dx^2} \right] + m(x) \left[ \sum_{n=1}^{\infty} \frac{d^2 \phi_n(x)}{dx^2} \right] = P(x,t) \]

For free vibration

\[ P(x,t) = 0 \]

and

\[ y(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \sin \omega_n t \]

substituting in Equation (7), yields

\[ \frac{d^2}{dx^2} \left[ EI(x) \sum_{n=1}^{\infty} \frac{d^2 \phi_n(x)}{dx^2} \right] = m(x) \sum_{n=1}^{\infty} \phi_n(x) \omega_n^2 \]

integrating Equation (9) two times with respect to \( x \), yields

\[ EI(x) \sum_{n=1}^{\infty} \frac{d^2 \phi_n(x)}{dx^2} = \int \int m(x) \sum_{n=1}^{\infty} \phi_n(x) \omega_n^2 dx \]

Dividing by \( EI(x) \) and noting that \( \omega^2_n \) is a constant, with two more integrations the final form of Equation (10) may be written as

\[ \phi_n(x) = \int \int \frac{1}{EI(x)} \int \int m(x) \phi_n(x) dx \]

This equation indicates that for a continuous system there are infinite numbers of modes that will satisfy this equation, each one corresponding to a particular frequency. Equation (11) may be solved on a digital computer to obtain as many modes and frequencies desired for any compressor leaf-type valve with variable mass and stiffness. For the details of the process involved and comparison of the experimental and analytical results on displacement and strain modes, see published information in references [6,7].

**Time Functions**

The time functions or the factors \( Q_n(t) \) in Equation (2) determine how much each vibration mode contributes to the valve motion at any particular moment in time. For accuracy in the stress prediction, it is also important that the values of the time functions be known to the highest possible accuracy.

Through a general derivation, it can be shown [7] that the relation between these factors and the expressions which contain the involved forcing function and the natural modes of vibration may be written as

\[ Q_n(t) + \omega_n^2 Q_n(t) = \frac{\int P(x,t) dx}{\int m(x) \phi_n^2(x) dx} \]

The right hand side of this equation involves the integration of the product of the displacement modes, \( \phi_n(x) \), and the forcing function, \( P(x,t) \). Unfortunately, in most cases adequate mathematical expressions for the forcing functions in the compressor are not available because the equations of the gas flow around the valve and through the valve ports and their interaction with the valve dynamics is not known. To overcome the complexity of Equation (12), the following alternative approach was considered for the evaluation of these functions.

The approach used here calls for the direct recording of the strain time history of \( N \) strain gages applied to the valve. The number of gages \( N \) applied depends upon the decision or assumption of the number of modes necessary to describe the valve response such that the predicted results would be within a reasonable accuracy. In this study from the analysis of the strain signal, Figure 3, a three mode approximation to the valve dynamics was considered to be sufficient. However, after obtaining the three participation factors and comparing their magnitudes, it was found that the contribution of the third mode was negligible. For a value of \( N=3 \), the expanded form of Equation (4) becomes

\[
\begin{bmatrix}
\phi_K(t) \\
\phi_L(t) \\
\phi_M(t)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\psi_{1K} & \psi_{2K} & \psi_{3K} \\
\psi_{1L} & \psi_{2L} & \psi_{3L} \\
\psi_{1M} & \psi_{2M} & \psi_{3M}
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\]

where the number 1, 2, and 3 indicate the first, second, and third strain gages as this location will be predicted. The location of the highest strain on the valve thus does not have to be known or investigated by the application of other strain gages as this location will be predicted when the entire strain distribution is obtained by
substituting the results of Equation (13) into Equation (4).

II. EXPERIMENTAL MEASUREMENTS

The two compressors used for the experimental work of this study had considerable differences in their valve geometry, capacity and application. In general, they may be classified in the category of single acting, reciprocating, hermetic compressors using a halocarbon for a refrigerant. Figures 1 and 2 show the suction valve and valve plate assembly of the two compressors. These valves will be referred to as the AUIPl2 and 104 suction valves respectively.

Strain gages used in this study were of MA Series, miniature transducer gages, supplied by Micromeasurement Inc. These are self temperature compensation gages with highest level of accuracy and stability. The experience and techniques developed by the Ray W. Herrick Laboratories [3,5] were used for the successful application of these gages for measuring the dynamic strain of the valves during the actual operation of the compressor. The top trace of Figure 3 shows a typical strain-time history of the 104 suction valve.

For better understanding of the valve motion and analysis of strain signal, a variable reluctance proximity transducer was installed in the valve plate to measure the valve tip displacement [7]. The location of the transducer was primarily governed by where it could be conveniently installed without interrupting the normal gas flow. The bottom trace in Figure 3 is the output of this transducer which represents the motion near the tip of the 104 suction valve.

III. STRAIN SIGNAL ANALYSIS

As indicated by Equation (4) the strain is a function of the position x along the valve and the time t. At the moment when the tip of the valve first contacts the stop, t=t_c, the strain in the valve is called the contact strain. The moment of contact may be identified by noting the disturbance in the strain signal when valve strikes the stop. It was found that this disturbance was a reliable indicator for contact strain [3,7], and the use of other kinds of indicators, like an electrical contactor in the stop position of the valve was not necessary.

The level of the highest strain in the valve which occurs at t=t_m, soon after the valve strikes the stop, is defined here as the maximum strain. The moments of stop contact and maximum strain are shown in sequence in Figure 3.

In the method of modal analysis, consideration of the correct boundary conditions of the valve is of utmost importance for a successful prediction of strain and deflection along the valve. Experimental time histories indicate that the tip of the valve, immediately after striking the stop position rebounds from the stop and does not remain in contact with it when maximum strain occurs [7]. Thus, the clamped-free boundary conditions were used for both contact and maximum strain predictions along the valve. Furthermore, the change in strain which occurred in the valve during the time Δt=t_m-t_c was called the "rebound" strain. The relation between maximum, contact, and rebound strain may be written as

\[ S_{m} = S_{c} + S_{r} \]
\[ \varepsilon_m = \varepsilon_c + \varepsilon_r \]

Strain and deflection distribution along the valve for any moment in time during opening and closing of the valve, may be obtained from Equations (4) and (2) respectively. The two moments considered were the moment of valve tip contact with the stop, \( t = t_c \), and the moment of maximum strain, \( t = t_m \). The values for the strain modes, \( \varepsilon_0(x) \), and the displacement modes, \( \delta_n(x) \), used in the above equations, were the ones predicted by the mathematical model. The measured strain values at contact time and maximum time were substituted into the right side of Equation (13) to compute the time functions at \( t = t_c \) and \( t = t_m \) respectively. Results are shown in Figures 4 and 5.

**Figure 4. Theoretical Distribution of Dynamic Strain and Deflection at Contact Time, \( t = t_c \), for AUIP12 Valve**

![Figure 4](image)

**Figure 5. Theoretical Distribution of Dynamic Strain and Deflection at Maximum Time, \( t = t_m \), for AUIP12 Valve**

![Figure 5](image)

### IV. COMPRESSOR DESIGN CHANGES AND MAXIMUM STRAIN

#### Change in Stop Position

The depth of the stop position in the compressor block is an important factor to control the maximum strain in the valve. The main reason for providing a stop is to limit the maximum strain by restricting the valve motion. The values of the modal participation factors, \( Q_n(t) \), in Equations (2) and (4) are functions of time and they will change when contact and maximum times are changed, due to the changes in the stop position. To this end, the relations between these factors and the valve stop position had to be developed. For a two mode approximation to the valve response (found to be sufficient to predict stress along the valve) equation for the valve tip position \( \delta \) and the valve strain energy are

\[ \delta = \phi_1(t)Q_1(t) + \phi_2(t)Q_2(t) \]

\[ V(t) = \frac{1}{2} \int_0^\delta \varepsilon(x,t)\left[\varepsilon''(x,t)Q_1(t) + \varepsilon''(x,t)Q_2(t)\right] dx \]

and the values of the time functions are found by solving these equations as

\[ Q_1(t) = \frac{1}{2A_2} \left[ -A_1 \delta \pm \sqrt{(A_1 \delta)^2 - 4A_2(a_2^2 - \frac{2V(t)}{E}2)} \right] \]

\[ Q_2(t) = \frac{1}{\phi_2(t)} \left[ \delta - \phi_1(t)Q_1(t) \right] \]

where the constants \( A_i \)'s are functions of valve geometry and boundary conditions [7].

From these equations the time factors \( Q_1(t) \) and \( Q_2(t) \) can be determined for any moment in time provided that the expression for the strain energy, \( V(t) \) and the valve tip deflection, \( \delta \), is known. In this analysis, \( \delta \) is regarded as an independent parameter and at contact time it corresponds to the stop depth. At this point it remains to find an expression for the strain energy in the valve as a function of time or valve tip deflection.

**Mathematical Expression for Valve Strain Energy**

The valve strain may be characterized by the amount of strain energy in the valve at any moment in time. This strain energy is given by

\[ V(t) = \frac{1}{2} \int_0^\delta \varepsilon(x,t)\left[\varepsilon''(x,t)\right]^2 dx \]

where \( \varepsilon(x) \) is the local bending stiffness along the valve. By substituting of Equation (2) and (4) into Equation (15) it can be shown [7] that the expression for the strain energy in the valve up to contact may be written as

\[ V(t) = \frac{E}{2} \left[ C_1 + C_2q_1(t)^2 + C_3q_2(t)^2 + \ldots \right] \]

where the constant \( C_i \)'s are functions of the valve response to gas forces, valve geometry and its boundary condition. Assuming that the ratios of the relative contribution of different modes to the valve response remain substantially constant as long as the initial conditions or valve boundary conditions are not changed; then the expression in the bracket in Equation (16) remains a constant [7] and the equation may be written as

\[ V(t) = \frac{E}{2} K_2 \delta^2 \]

The value of \( K_2 \) may be determined from Equation (17) if a known valve tip deflection and the corresponding valve strain energy is known. This information
can be obtained from the moment of valve tip contact with the stop for which the experimental strain data has been collected. Designating this stop depth \( \delta_0 \) (usually \( \delta_0 \) is the standard stop depth) and the corresponding contact strain energy \( V_0(t_c) \), then from Equation (17)

\[
K_2 = \frac{2V(t_c)}{E_0^2} \delta_0^2 \tag{18}
\]

and substituting back into Equation (17)

\[
\frac{2V(t)}{E} = \frac{2V_0(t)}{E_0^2} \delta_0^2 \tag{19}
\]

for any tip position up to contact with the stop. To evaluate the contact strain energy \( V_0(t_c) \), the values of strain distribution \( \varepsilon(x,t_c) \), shown in Figure 4 for the AUP12 valve, are substituted in Equation (15). The result is substituted in Equation (19) to obtain the valve strain energy as a function of tip deflection.

Approximate Mathematical Expressions For Maximum Energy

The maximum energy is defined to be the valve strain or potential energy at \( t=t_m \) when maximum strain occurs. At this moment the tip of the valve (\( \delta=\delta_m \)) is somewhere between the seat and stop depth and the rate or slope of the strain signal is zero, Figure 3. Thus, the kinetic energy for the moment of maximum strain is zero and all the energy in the valve is its strain or potential energy, \([5,7]\]. Figure 6 shows the valve deflection curve (No. 3) for the moment of maximum strain. The deflection curve No. 2, is for the same valve tip deflection, \( \delta=\delta_m \), using the precontact deflection curve.

\[
V(t_m) = V(t) \delta = \delta_m + \Delta V \tag{20}
\]

where \( \Delta V \) represents the net energy added to the valve by the gas forces during the time interval between deflection curves 2 and 3. Substituting the expressions given by Equation (19) for \( V(t) \), when \( \delta=\delta_m \) into Equation (20) results in:

\[
\frac{2V(t_m)}{E} = \frac{2V_0(t)}{E_0^2} \delta_0^2 + \frac{2\Delta V}{E} \tag{21}
\]

The amount of energy designated by \( \Delta V \) above, is called the rebound energy. This is the net energy added to the valve during the time interval \( \Delta t_m = t_m - t_{m-1} \) where \( t_{m-1} \) corresponds to the moment when the stop is at \( \delta=\delta_m \) before contact with the stop. In high speed compressors, this time interval is very short. In fact, the total time from \( t=0.0 \) up to \( t=t_m \) was about one millisecond in the AUP12 and 1.5 milliseconds in the 104 compressor; also, the rebound energy, and the assumed function was

\[
\delta = \delta_m + r_e \tag{22}
\]

The valve rebound was assumed to be a function of the rebound energy, and the assumed function was written as

\[
r_e = R_1 \Delta V \tag{23}
\]

where \( R_1 \) is a constant which can be evaluated from Equation (23) when the values of \( r_e \) and \( \Delta V \) are obtained for a set of compressor operating conditions as discussed earlier in this paper.

Total Tip Excursion

The total tip excursion is defined as the maximum valve tip deflection under a set of operating conditions with no stop in the compressor. This total excursion designated here by \( \delta_m \) is a function of valve and compressor design and operating condition. The operating condition determines the mass rate of flow which is primarily a function of suction or inlet pressure. Changes in discharge pressure do not have significant effect on the level of maximum strain which is proportional to valve deflection and thus they are neglected here \([7,5]\).
A study of Doige’s experimental data on the determination of valve contact with the stop and observation of the contactor signal in the 104 compressor, resulted in the derivation [7] of an expression for the total tip excursion as a function of suction pressure in the form of

$$\delta_t = \delta_{to} \left(\frac{P_s}{P_{so}}\right)^{1/4}$$

(24)

where $\delta_{to}$ and $P_{so}$ are the corresponding known valve tip contact and suction pressure respectively.

A knowledge of the total tip excursion is necessary for the determination of the maximum level of the energy stored or given to the valve by the gas forces up to the time $t=t_{c}$, under any operating condition. When the depth of the stop position is changed, it will finally reach a point where the valve tip does not contact any more. If the suction pressure is increased, the contact occurs again. Therefore, for any operating condition, i.e., suction pressure, there is a limit on the position of the stop depth for which the valve tip just contacts the stop and does not rebound. In this case $t_c=t_m$ and $\delta_m=\delta_t$. Thus Equations (19) and (21) become

$$2V(t) = \frac{2}{E} \frac{V_0(t)}{\delta_0^2} \delta^2 \left(\frac{P_s}{P_{so}}\right)^{1/2}, \quad t=t_c=t_m$$

(25)

This indicates that for the moment of $t=t_c=t_m$ the valve contact strain energy and maximum energy are equal and their values cannot exceed the limit given by Equation (25). This limit is a function of the compressor operating condition which is represented here by the suction pressure, $P_s$.

V. PREDICTION OF MAXIMUM STRAIN WHEN STOP POSITION IS CHANGED

The energy expressions and other corresponding relations which have been developed so far were applied to the 104 and AU1P12 suction valves with the only input being measured values of the contact and maximum strain from three gages on the valve and the value of a suction pressure $P_{so}$ with its corresponding tip excursion. Figure 7 shows the results of the application of the energy expressions to the AU1P12 valve. This figure represents the contact and maximum energy of the valve as a function of valve tip deflection. For the moment of contact, the tip deflection is the same as the stop depth, but for the moment of maximum strain, the relation between tip deflection and stop depth is determined from Equation (22). The level of the maximum total energy for the particular operating condition shown in this figure was obtained from Equation (25) with the values of $P_s=80$ psig and $\delta_t=0.305$ inches. Another way to obtain this level is to compute the total excursion for any operating condition from Equation (24) and locate it on the horizontal axis of Figure 7. The corresponding value of contact strain energy would be the desired level.

![Figure 7](image-url)

**FIGURE 7.** Contact and Maximum Strain Energy as a Function of Tip Deflection for AU1P12 Valve. $(P_s, P_{so})=(80,300)$ psig

To predict the value of contact strain when stop depth is changed, the corresponding values of contact strain energy and tip deflection (Figure 7 or Equation 19) are substituted in Equation (14) to obtain the corresponding values of $Q_1(t)$ and $Q_2(t)$. These values are then used in Equations (2) and (4) to evaluate the contact strain and deflection distribution for the particular stop depth or tip deflection. In a similar manner the maximum strain and deflection distributions for the moment of $t=t_m$ are obtained; except that in this case the maximum energy curve in Figure 7 or Equation (21) is used. Figure 8 shows the predicted contact and maximum strain for the location $x=.031$ inches from the clamped end, as the stop depth is increased. Experimental data are also shown in the figure for the purpose of comparison. The dotted line shown in Figure 8 is the result of the prediction of maximum strain by Doige, using the theory he developed in his study [9]. Figures 9 and 10 show the entire strain and deflection distribution as stop depth is changed for the moments of $t=t_c$ and $t=t_m$ respectively. The increment in the stop depth was $\Delta s=.03$ inches. Figure 11 shows similar results for the 104 suction valve at $t=t_m$.

![Figure 8](image-url)

**FIGURE 8.** Theoretical Contact and Maximum Strain ($\varepsilon_c$ and $\varepsilon_m$) as a Function of Stop Depth Compared to Experimental Data, (Gage Location $x=.031$ in). AU1P12 Valve

- Experiment $\varepsilon_c$
- Experiment $\varepsilon_m$
- Theory (Model)
Changes In Compressor Operating Conditions

For the evaluation of maximum strain as a function of operating conditions, the following relations had to be found [7]:

a) The relation between the total tip excursion and suction pressure which was shown in Equation (24).
b) The relation between the rebound energy and suction pressure which was found to be approximately

\[ \frac{2}{E} \cdot \Delta V = \frac{2}{\bar{E}} \cdot \frac{\Delta V}{P_{80}} \cdot (P_S)^{1/2} \quad (26) \]

where \( \Delta V_0 \) and \( P_{80} \) correspond to the particular operating condition for which the strain data is obtained.
c) The approximate relation between the suction pressure and valve tip rebound which was written in the form of

\[ r_e = \frac{r_{80}}{(P_8)^{1/4}} \cdot (P_S)^{1/4} \quad (27) \]

where \( r_{80} \) is the corresponding valve tip rebound to the known suction pressure \( P_{80} \).

Equation (26) was applied to the AU1P12 valve to obtain the values of the rebound energy for different compressor operating conditions (suction pressure). These values were then used in Equation (21) to evaluate the maximum energy in the valve as a function of valve tip deflection or stop depth for any particular operating condition. Equations (24) and (25) were also used to determine the corresponding total tip excursion and the level of maximum total energy respectively. The results for two different operating conditions are shown in Figure 12.
These energy curves or their corresponding equations were then used, as described earlier in this paper, to predict the time factors and consequently the maximum strain and deflection distribution along the valve as a function of stop depth for a new compressor operating condition. A typical result is shown in Figure 13.

![Figure 13](image_url)

**FIGURE 13.** Results of Maximum Strain and Deflection Distribution as a Function of Valve Tip Deflection. AU1P12 Valve (P_s, P_d) = (30, 100) psig

**VI. SUMMARY**

The results of this paper may be used to predict the dynamic stress distribution along the leaf-type compressor valves when changes in compressor design, operating conditions or small changes in the valve geometry are made.

The mathematical model for nonuniform beams is a useful tool for predicting lower frequencies, strain and displacement modes of leaf-type compressor valves with variable mass and stiffness. By measuring the strain time histories on a few locations along the valve, the nominal strain and deflection distribution along the valve can be predicted using the approach of modal analysis and the techniques shown in this paper.

The contact and maximum energy in the valve as a function of compressor design changes and/or operating conditions can be evaluated using the results of the energy expressions and concepts discussed here. The knowledge of valve strain energy can be effectively used to predict valve maximum stress as a function of the above changes.

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**VIII. REFERENCES**


Note: $\phi''(x) = \frac{d^2 \phi(x)}{dx^2}$