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ABSTRACT

In this paper we investigate the convergence of the block Modified Accelerated Overrelaxation (MAOR) iterative method, when applied to the nonsingular linear system $Ax = b$, where A is a generalized consistently ordered (GCO) $(q,\ p-q)$-matrix. By mainly using the theory of block p-cyclic matrices, of positive matrices, and of regular splittings sufficient conditions for the convergence of the block MAOR and related methods are obtained. In this way known results are extended and improved and new ones are derived.

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1. INTRODUCTION AND PRELIMINARIES

In this paper we are concerned with the Modified Accelerated Overrelaxation (MAOR) iterative method for the solution of the nonsingular linear system

$$Ax = b \quad ,$$

where $A \in \mathbb{C}^{n,n}$ and $b \in \mathbb{C}^n$. It is assumed that when A is partitioned into a $p \times p$ block form it is as follows

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 & A_{1,s+1} & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 & 0 & A_{2,s+2} & \cdots & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_{qp} \\ A_{q+1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{q+2,2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & 0 & \cdots & A_{ps} & 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \quad (1.2)$$

with the diagonal blocks A_{jj}, $j = 1(1)p$, square and nonsingular and q relatively prime to p ($gcd(p, q) = 1$), where $p = s + q$. As is known the matrix A in (1.2) belongs to the class of p-cyclic matrices (see Varga [28]) or more precisely to that of generalized consistently ordered (GCO) $(q, p-q)$-matrices (see Young [30]).

Let $D := diag(A_{11}, A_{22}, \ldots, A_{pp})$, then the block Jacobi iteration matrix T^A, associated with A, has the form

$$T^A := I - D^{-1} A = \begin{bmatrix} 0_1 & 0 & 0 & \dots & 0 & T_{1,s+1} & 0 & \dots & 0 \\ 0 & 0_2 & 0 & \dots & 0 & 0 & T_{2,s+2} & \dots & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & T_{qp} \\ T_{q+1,1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & T_{q+2,2} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & 0 & \dots & T_{ps} & 0 & 0 & \dots & 0_p \end{bmatrix}, \quad (1.3)$$

where 0_j is the null matrix of the order of A_{jj} and $T_{jk} = - A_{jj}^{-1} A_{jk}$, $j = 1(1)p$, $k = 1(1)p$, $j \neq k$. Writing $A = D(I - L - U)$, with L and U strictly lower and strictly upper triangular matrices respectively, we have $T^A = L + U$. Let $\rho(\cdot)$ denote the spectral radius of a matrix and let $\overline{\mu} := \rho(|T^A|)$. In this paper, and unless otherwise specified, we shall be concerned with matrices A that belong to the class of matrices $\mathcal{B}$, where

$$\mathcal{B} := \left\{ \begin{array}{c} A \in \mathbb{C}^{n,n} \,/\, n \geq p \quad \text{arbitrary, } A \text{ is a block} \\ \text{GCO } (q,\ p-q)\text{–matrix with } \overline{\mu} := \rho(|T^A|) \end{array} \right\} \quad . \quad (1.4)$$

Very recently the new iterative method for the solution of linear systems, the Modified Accelerated Overrelaxation (MAOR), was introduced in [7]. The MAOR method for (1.1) is defined by

$$x^{(m+1)} = \mathcal{L}^A_{R,\Omega}\, x^{(m)} + c, \qquad m = 0, 1, 2, \ldots, \quad (1.5)$$

where

$$\begin{aligned} \mathcal{L}^A_{R,\Omega} &:= (I - RL)^{-1} \left[I - \Omega + (\Omega - R) L + \Omega U \right] \\ &= I - (I - RL)^{-1}\, \Omega D^{-1} A \end{aligned} \quad (1.6)$$

and

$$c := (I - RL)^{-1}\ \Omega D^{-1}\ b\quad . \tag{1.7}$$

In (1.6) – (1.7) the matrices R and Ω are defined as follows

$$\begin{aligned} R &:= \operatorname{diag}(r_1 I_1, r_2 I_2, \ldots, r_p I_P)\quad , \\ \Omega &:= \operatorname{diag}(\omega_1 I_1, \omega_2 I_2, \ldots, \omega_p I_p)\quad , \end{aligned} \tag{1.8}$$

where I_j is the identity matrix of the order of A_{jj} and r_j , ω_j, $j = 1(1)p$, are in general complex parameters with $\omega_j \neq 0$, $j = 1(1)p$. If $R = 0$, that is $r_j = 0$, $j = 1(1)p$, then (1.5) reduces to the Extrapolation Jacobi (EJ) method with p parameters ω_j where each one is associated with the corresponding jth row block of T^A (see e.g., [9]), while if $R = \Omega$, that is $r_j = \omega_j$, $j = 1(1)p$, it reduces to the Modified SOR(MSOR) method for (1.1). (See e.g., [3], [17], [27], [30], [10], [9] or [29].)

The purpose of this paper is to give sufficient conditions for the convergence or divergence of the block MAOR method and consequently of the methods which are derived from it. It is shown that the convergence results are applicable to the case where A is also an H-matrix. In general, several new results are obtained some of which extend and improve previously known ones.

2. CONVERGENCE OF THE BLOCK MAOR METHOD

We begin with the proof of two lemmas, the second one is a generalization of Lem.2.1 in [24], which are useful in the sequel.

Lemma 2.1: Let $B, \Gamma \in \mathbb{C}^{n,n}$ such that

$$\begin{aligned} B &= \operatorname{diag}(\beta_1 I_1, \beta_2 I_2, \ldots, \beta_p I_p)\quad , \\ \Gamma &= \operatorname{diag}(\gamma_1 I_1, \gamma_2 I_2, \ldots, \gamma_p I_p)\quad . \end{aligned} \tag{2.1}$$

Then the eigenvalues ξ of the matrix

$$BL + \Gamma U \tag{2.2}$$

are given by

$$\xi = \left[\prod_{j=1}^{q} \gamma_j \prod_{j=q+1}^{p} \beta_j \right]^{1/p} \mu \quad , \tag{2.3}$$

where $\mu \in \sigma(T^A)$ ($\sigma(\cdot)$ denotes the spectrum of a matrix) with T^A, L and U being defined in Section 1.

Proof: From the relationship

$$(BL + \Gamma U)^p = \left[\prod_{j=1}^{q} \gamma_j \prod_{j=q+1}^{p} \beta_j \right] (T^A)^p \tag{2.4}$$

and the p-cyclic nature of T^A the proof follows.□

Lemma 2.2: If B, Γ are given by (2.1) with $\beta_j, \ \gamma_j \geq 0, \ j = 1(1)p$, $\Delta = \mathrm{diag}(\delta_1 I_1, \delta_2 I_2, ..., \delta_p I_p)$ with $\delta_j > 0$, $j = 1(1)p$, and

$$\left[\prod_{j=1}^{q} \gamma_j \prod_{j=q+1}^{p} \beta_j \right] \overline{\mu}^p < \prod_{j=1}^{p} \delta_j \quad , \tag{2.5}$$

where $\overline{\mu} = \rho(|T^A|) = \rho(|L| + |U|)$, then the matrix

$$\hat{A} = \Delta - B|L| - \Gamma|U| \tag{2.6}$$

satisfies $\hat{A}^{-1} \geq 0$.

Proof: Let the matrix Q be defined by

$$Q := \Delta^{-1} B|L| + \Delta^{-1} \Gamma|U| \geq 0 \quad . \tag{2.7}$$

Then by Lem. 2.1 and (2.5) we have

$$\rho(Q) = \left[\left[\prod_{j=1}^{q} \gamma_j \prod_{j=q+1}^{p} \beta_j \right] \left[\prod_{j=1}^{p} \delta_j \right]^{-1} \right]^{1/p} \overline{\mu} < 1 \quad . \tag{2.8}$$

Because of (2.7) and (2.8), from Thm 3.8 in [28, p. 83] we obtain that $I - Q$ is nonsingular and $(I - Q)^{-1} \geq 0$. Thus, $\hat{A} = \Delta - \Delta Q = \Delta(I - Q)$ and $\hat{A}^{-1} = (I - Q)^{-1} \Delta^{-1} \geq 0$. □

The following theorem gives sufficient conditions for the convergence of the block MAOR method (1.5).

Theorem 2.1: If the acceleration and relaxation parameters r_j and ω_j, $j = 1(1)p$, respectively, of the MAOR method (1.5) satisfy

$$|1 - \omega_j| < 1, \quad j = 1(1)p \quad , \tag{2.9}$$

and

$$\left[\prod_{j=1}^{q} |\omega_j| \prod_{j=q+1}^{p} (|r_j| + |\omega_j - r_j|) \right] \bar{\mu}^p < \prod_{j=1}^{p} (1 - |1 - \omega_j|) \quad , \tag{2.10}$$

then the MAOR method converges ($\rho(\mathcal{L}^A_{R,\Omega}) < 1$).

Proof: Let

$$\begin{aligned} M &= I - RL, & N &= I - \Omega + (\Omega - R)L + \Omega U, \\ \tilde{M} &= I - |R|\ |L|, & \tilde{N} &= |I - \Omega| + |\Omega - R|\ |L| + |\Omega|\ |U|, \\ \tilde{A} &= \tilde{M} - \tilde{N} \end{aligned} \tag{2.11}$$

If we set $\tilde{\mathcal{L}}^A_{R,\Omega} = \tilde{M}^{-1} \tilde{N}$, then

$$\begin{aligned} 0 \ \leq\ & |\mathcal{L}^A_{R,\Omega}| = |M^{-1} N| \leq |(I - RL)^{-1}|\ |I - \Omega + (\Omega - R)L + \Omega U| \\ \leq\ & (I - |R|\ |L|)^{-1} (|I - \Omega| + |\Omega - R|\ |L| + |\Omega|\ |U|) = \tilde{\mathcal{L}}^A_{R,\Omega} \ , \end{aligned}$$

implying that

$$\rho(\mathcal{L}^A_{R,\Omega}) \leq \rho(\tilde{\mathcal{L}}^A_{R,\Omega}) \quad . \tag{2.12}$$

Since

$$\tilde{A} = \tilde{M} - \tilde{N} = (I - |I - \Omega|) - (|R| + |\Omega - R|)|L| - |\Omega|\ |U|$$

and (2.9), (2.10) hold, then by Lem. 2.2 we have $\tilde{A}^{-1} \geq 0$. Moreover since $\tilde{M}^{-1} \geq 0$ and $\tilde{N} \geq 0$, $\tilde{M} - \tilde{N}$ is a regular splitting of $\tilde{A}$ (see e.g., [28] or [2]) and therefore $\rho(\tilde{\mathcal{L}}^{A}_{R,\Omega}) < 1$. Consequently, from (2.12), $\rho(\mathcal{L}^{A}_{R,\Omega}) < 1$. □

Corollary 2.1: If the extrapolation (resp. relaxation) parameters ω_j, $j = 1(1)p$, of the EJ (resp. MSOR) method satisfy

$$|1 - \omega_j| < 1, \quad j = 1(1)p \quad , \tag{2.13}$$

and

$$\left[\prod_{j=1}^{p} |\omega_j| \right] \bar{\mu}^p < \prod_{j=1}^{p} (1 - |1 - \omega_j|) \quad , \tag{2.14}$$

then the EJ (resp. MSOR) method converges.

Proof: It follows by Thm. 2.1 for $R = 0$ (resp. $R = \Omega$). □

Remark: Thm 3.1 of [24] concerning the AOR method for (1.1) is obtained from Thm 2.1 in the special case $R = rI$ and $\Omega = \omega I$. □

A careful examination of the relationships (2.9) and (2.10) leads to the following theorem.

Theorem 2.2: Let (2.9) hold. Then a necessary condition for (2.10) to hold is $\bar{\mu} < 1$.

Proof: From (2.10) we have

$$\bar{\mu}^p < \prod_{j=1}^{q} \left[\frac{1 - |1 - \omega_j|}{|\omega_j|} \right] \prod_{j=q+1}^{p} \left[\frac{1 - |1 - \omega_j|}{|r_j| + |\omega_j - r_j|} \right] \quad . \tag{2.15}$$

On the other hand (2.9) imply

$$0 < 1 - |1 - \omega_j| \leq |1 - (1 - \omega_j)| = |\omega_j|, \quad j = 1(1)p \quad . \tag{2.16}$$

Moreover,

$$|\omega_j| = |r_j + (\omega_j - r_j)| \le |r_j| + |\omega_j - r_j|, \quad j = q+1(1)p \quad . \tag{2.17}$$

So, by virtue of (2.16) and (2.17), (2.15) gives $\overline{\mu}^p < 1$, that is $\overline{\mu} < 1$.

Remark: In view of Thm 2.2 in the remaining of this section we assume that $A \in \mathcal{B}$ satisfies also the assumption $\overline{\mu} < 1$. As we shall see in the end of this section (Thm 2.4) this assumption is satisfied in the case where A belongs to the class of H-matrices. □

If, now, we begin with (2.9)–(2.10), consider that $r_j, \omega_j \in I\!R$, $j = 1(1)p$, and at the same time strengthen the assumption (2.10), or equivalently (2.15), by requiring $\overline{\mu}$ to be strictly less than each of the p fractions in the right hand side of (2.15), then we can end up with the following statement. □

Theorem 2.3: Under the assumption $\overline{\mu} < 1$, with $r_j, \omega_j \in I\!R$, $j = 1(1)p$, the two sets of conditions in (2.18) and (2.19) below are equivalent.

$$\begin{aligned} &\text{i)} \quad \overline{\mu} < \frac{1 - |1-\omega_j|}{|\omega_j|}, \quad j = 1(1)q \quad , \\ &\text{ii)} \quad \overline{\mu} < \frac{1 - |1-\omega_j|}{|r_j| + |\omega_j - r_j|}, \quad j = q+1(1)p \quad . \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\text{i)} \quad 0 < \omega_j < \frac{2}{1+\overline{\mu}} \ (\le 2), \quad j = 1(1)p, \\ &\text{ii)} \quad \frac{\omega_j\overline{\mu} - (1 - |1-\omega_j|)}{2\overline{\mu}} < r_j < \frac{\omega_j\overline{\mu} + (1 - |1-\omega_j|)}{2\overline{\mu}}, \quad j = q+1(1)p. \end{aligned} \tag{2.19}$$

Furthermore, if either (2.18) or (2.19) hold, then the MAOR method converges.

Proof: From each of the p conditions in (2.18) we readily see that $1 - |1-\omega_j| > 0$, or equivalently, $0 < \omega_j < 2$, $j = 1(1)p$. By considering the two cases $0 < \omega_j \le 1$ and $1 < \omega_j < 2$, $j = 1(1)q$, having in mind the assumption $\overline{\mu} < 1$, it is found out that (2.18i) are equivalent to

$$0 < \omega_j < \frac{2}{1+\overline{\mu}} \ (\le 2), \quad j = 1(1)q \quad . \tag{2.20}$$

To derive relationships equivalent to those in (2.18ii) we distinguish three cases: a) $r_j \le 0$, b) $0 < r_j < \omega_j$, and c) $\omega_j \le r_j$, $j = q + 1(1)p$. In case (a), (2.18ii) give $\bar{\mu} < \dfrac{1 - |1 - \omega_j|}{\omega_j - 2r_j}$ and because $\omega_j - 2r_j > 0$ it is implied that

$$\frac{\omega_j\bar{\mu} - (1 - |1 - \omega_j|)}{2\bar{\mu}} < r_j \ (\le 0), \qquad j = q{+}1(1)p \quad . \tag{2.21}$$

Since $0 < \omega_j < 2$, $j = q + 1(1)p$, and the left hand side in (2.21) must be negative we obtain

$$0 < \omega_j < \frac{2}{1 + \bar{\mu}} \ (\le 2), \qquad j = q{+}1(1)p \quad , \tag{2.22}$$

In case (b), we simply have $\bar{\mu} < \dfrac{1 - |1 - \omega_j|}{\omega_j}$ leading to (2.22) again. In case (c), it is $\bar{\mu} < \dfrac{1 - |1 - \omega_j|}{2r_j - \omega_j}$ or

$$(\omega_j \le)\ r_j < \frac{\omega_j\bar{\mu} + (1 - |1 - \omega_j|)}{2\bar{\mu}}, \qquad j = q + 1(1)p \quad . \tag{2.23}$$

From the fact that the right hand side of (2.23) must be strictly greater than ω_j, (2.22) follows. Hence the equivalent to (2.18ii) relationships are those in (2.22) together with all possible values for r_j obtained in the three cases just examined. These values, however, give the intervals for r_j, $j = q + 1(1)p$, in (2.19 ii). Noting that (2.20) and (2.22) give (2.19i) concludes the proof of the first part. To prove that the MAOR converges, we simply note that the right hand sides of (2.18) must be positive, which directly give (2.9), and that if we multiply all inequalities in (2.18) by members we obtain (2.10). Consequently, by Thm 2.1, the proof follows. □

Corollary 2.2: If $\bar{\mu} < 1$ and $0 < \omega_j < \dfrac{2}{1 + \bar{\mu}}$, $j = 1(1)p$, then the EJ and the MSOR methods for (1.1) converge. □

Corollary 2.3: If $\bar{\mu} < 1$, $0 < \omega < \dfrac{2}{1 + \bar{\mu}}$, and

$\dfrac{\omega\bar{\mu}-(1-|1-\omega|)}{2\bar{\mu}} < r < \dfrac{\omega\bar{\mu}+(1-|1-\omega|)}{2\bar{\mu}}$, then the AOR method for (1.1) converges. □

Remark: The results in Thms 2.1 and 2.3 are new and the ones in the former case are obviously stronger than those in the latter. This is not only because in Thm 2.1 complex parameters r_j and ω_j are considered but also because the domain of convergence defined by Thm 2.1 is larger than the one defined in Thm 2.3. However, even Thm 2.3 gives larger regions of convergence than previously known ones. For example, consider the MSOR method, for $p = 2$, for which it is known (see [15–16]) that in the real (ω_1, ω_2)-plane the region of convergence is the open quadrilateral R_1 whose vertices are the points $(0, 0)$, $(1, \bar{\mu})$, $\left(\frac{2}{1+\bar{\mu}}, \frac{2}{1+\bar{\mu}}\right)$, $\left(1, \frac{2}{1+\bar{\mu}}\right)$ (Fig. 1). Thm 2.3 gives as the region of convergence the open square R_2 with vertices $(0, 0)$, $\left(\frac{2}{1+\bar{\mu}}, 0\right)$, $\left(\frac{2}{1+\bar{\mu}}, \frac{2}{1+\bar{\mu}}\right)$, $\left(0, \frac{2}{1+\bar{\mu}}\right)$ (Fig. 2), while Thm 2.1 gives the open pentagon R_3, bounded by the straight lines $\omega_1 = 0$, $\omega_1 = \frac{2}{1+\bar{\mu}^2}$, $\omega_2 = 0$, $\omega_2 = \frac{2}{1+\bar{\mu}^2}$ and the hyperbola $(1-\bar{\mu}^2)\,\omega_1 \quad \omega_2 - 2\omega_1 - 2\omega_2 + 4 = 0$, with vertices $(0, 0)$, $\left(\frac{2}{1+\bar{\mu}^2}, 0\right)$, $\left(\frac{2}{1+\bar{\mu}^2}, 1\right)$, $\left(1, \frac{2}{1+\bar{\mu}^2}\right)$, $\left(0, \frac{2}{1+\bar{\mu}^2}\right)$ (Fig. 3). From the illustrative Figures 1–3 we have that $R_1 \subset R_2 \subset R_3$. It is interesting to note that as $\bar{\mu}$ tends to zero R_1 tends to the parallelogram with vertices (0, 0), (1, 0), (2, 2), (1, 2), while both R_2 and R_3 tend to the square with vertices (0, 0) (2, 0), (2, 2), (0, 2). Hence, there holds

$$\tilde{R}_1 := \lim_{\bar{\mu}\to 0} R_1 \subset \lim_{\bar{\mu}\to 0} R_2 = \lim_{\bar{\mu}\to 0} R_3 =: \tilde{R}_{2,3}$$

and the area of $\tilde{R}_1$ is half the area of $\tilde{R}_{2,3}$. Also, as $\bar{\mu}$ tends to one R_1 tends to an empty region (!), more specifically, to the open double line segment $\hat{R}_1$ with end-points (0, 0), (1, 1), while R_2 and R_3 tend to the unit square $\hat{R}_{2,3}$ $\left[(0, 0), (1, 0), (1, 1), (0, 1)\right]$. Obviously

$$\hat{R}_1 := \lim_{\bar{\mu}\to 1} R_1 = \phi(!) \subset \lim_{\bar{\mu}\to 1} R_2 = \lim_{\bar{\mu}\to 1} R_3 =: \hat{R}_{2,3} \quad . \quad \square$$

The theory developed so far applies also to a matrix $A \in \mathcal{B}$ in case A is a nonsingular H-matrix since then $\overline{\mu} = \rho(|T^A|) < 1$. It is reminded that $A = (\alpha_{ij}) \in \mathbb{C}^{n,n}$ is an H-matrix if its comparison matrix $\widetilde{\mathcal{M}}(A) = (m_{ij})$, with $m_{ii} = |a_{ii}|$, $m_{ij} = -|a_{ij}|$, $i, j = 1(1)n, j \neq i$, is a nonsingular M-matrix (see e.g., [2]). In fact $\overline{\mu} < 1$ holds for any nonsingular H-matrix in $p \times p$ block form not necessarily a p-cyclic one. We show it in Thm 2.4 after we state and prove the two lemmas below.

Lemma 2.3: Any submatrix $\widetilde{A}$ which is obtained from a nonsingular H-matrix A by deleting any number of rows and the corresponding columns is a nonsingular H-matrix.

Proof: It follows from the fact that $\widetilde{\mathcal{M}}(A)$ is a nonsingular M-matrix and so is $\widetilde{\mathcal{M}}(A) = \mathcal{M}(\widetilde{A})$ (see e.g., [28, Thm 3.12 p. 85]). □

Lemma 2.4: For any nonsingular H-matrix A, there holds

$$|A^{-1}| \le \mathcal{M}^{-1}(A) \quad . \tag{2.24}$$

Proof: (2.24) is readily obtained if A is written as $A = D(I - B)$, where $D = \operatorname{diag}(a_{11}, a_{22}, ..., a_{nn})$. Then, because of $\rho(B) \le \rho(|B|) < 1$, it will be

$$|A^{-1}| = |(I-B)^{-1}| \ |D^{-1}| \le (I - |B|)^{-1} \ |D|^{-1} = \mathcal{M}^{-1}(A) \ . \square$$

Consider any matrix $A \in \mathbb{C}^{n,n}$ partitioned in a $p \times p$ block form and let T_p^A and T^A denote the point and the block Jacobi matrices, respectively, associated with A (provided they exist). Based on the previous definitions, lemmas and notations we can prove.

Theorem 2.4: Let $A \in \mathbb{C}^{n,n}$ be a nonsingular H-matrix partitioned in a $p \times p$ block form. There hold

$$\rho(T^A) \overset{(1)}{\le} \rho(|T^A|) =: \overline{\mu} \overset{(2)}{\le} \rho(T^{\mathcal{M}(A)}) \overset{(3)}{\le} \rho(T_p^{\mathcal{M}(A)}) \overset{(4)}{=} \rho(|T_p^A|) \overset{(5)}{<} 1 \quad . \tag{2.25}$$

Proof: It should be pointed out that some of the relationships (relns) in (2.25) are well-known while others are pretty obvious. For example relns (1) and (4). Reln (5) holds because A is a nonsingular H-matrix. To show the validity of reln (3) we consider two different splittings of the nonsingular M-matrix $\mathcal{M}(A) = M_p - N_p = M_b - N_b$, where $M_p = diag(|a_{11}|, |a_{22}|, ..., |a_{nn}|)$ and $M_b = diag(\mathcal{M}(A_{11}), \mathcal{M}(A_{22}), ..., \mathcal{M}(A_{pp}))$, corresponding to the point and the block partitioning of A, respectively. It is $M_p^{-1} \geq 0$, $M_b^{-1} \geq 0$, with the latter holding because M_b is the direct sum of nonsingular M-matrices, by Lem. 2.3 and Thm 3.12 of [28, p. 85], and $N_p \geq N_b \geq 0$. Observing now that $T^{\mathcal{M}(A)}$ and $T_p^{\mathcal{M}(A)}$ are the iteration matrices associated with the previous two regular splittings it implies that reln (3) holds [2, Cor. 5.7, p. 183]. Finally, to prove reln (2) it is sufficient and necessary to show that

$$|A_{ii}^{-1} A_{ij}| \leq \mathcal{M}^{-1}(A_{ii})\ |A_{ij}|, \quad i,\ j = 1(1)p, \quad j \neq i\ . \tag{2.26}$$

Since $|A_{ii}^{-1} A_{ij}| \leq |A_{ii}^{-1}|\ |A_{ij}|$ for all indices $i,\ j$ in (2.26), it suffices to have

$$|A_{ii}^{-1}| \leq \mathcal{M}^{-1}(A_{ii}), \quad i = 1(1)p\ . \tag{2.27}$$

By Lem. 2.3 A_{ii} is a nonsingular H-matrix and by Lem. 2.4 (2.27) hold true and so are (2.26) and reln (3), which concludes the proof. □

3. DIVERGENCE REGIONS OF THE MAOR ITERATION MATRIX.

We begin this section with the statement and proof of a weaker form of (2.10) of Thm 2.1. This may enable us to use the eigenvalue functional equation and obtain regions of divergence of the MAOR matrix.

Lemma 3.1: If the acceleration and relaxation parameters r_j and ω_j, $j = 1(1)p$, respectively of the MAOR method (1.5) satisfy the assumptions of Thm 2.1

$$|1 - \omega_j| < 1, \quad j = 1(1)p\ , \tag{3.1}$$

and (2.10), then (the MAOR method converges and)

$$\left[\prod_{j=1}^{q} (1 + |1 - \omega_j|) \prod_{j=q+1}^{p} |2r_j - \omega_j| \right] \bar{\mu}^p < \prod_{j=1}^{p} |2 - \omega_j| \tag{3.2}$$

holds.

Proof: Since (3.1) coincide with (2.9) it suffices to prove the validity of (3.2) under the assumption that (2.10) holds. For this we shall show that

$$\frac{1-|1-\omega_j|}{|\omega_j|} \le \frac{|2-\omega_j|}{1+|1-\omega_j|}, \quad j=1(1)q \quad , \tag{3.3}$$

and that

$$\frac{1-|1-\omega_j|}{|r_j|+|\omega_j-r_j|} \le \frac{|2-\omega_j|}{|2r_j-\omega_j|}, \quad j=q+1(1)p \quad . \tag{3.4}$$

In view of (3.1), (3.3) are equivalent to

$$[1-(1-\omega_j)(1-\overline{\omega}_j)]^2 \le \omega_j\,\overline{\omega}_j\,(2-\omega_j)(2-\overline{\omega}_j), \quad j=1(1)q \quad , \tag{3.5}$$

where $\overline{\omega}_j$ stands for the conjugate of ω_j. After some simple algebra we obtain

$$(Im\ \omega_j)^2 \ge 0, \quad j=1(1)q \quad , \tag{3.6}$$

which are always true. By observing that

$$0 < 1-|1-\omega_j| \le |1+(1-\omega_j)| = |2-\omega_j|, \quad j=q+1(1)p \tag{3.7}$$

and that

$$|r_j|+|\omega_j-r_j| \ge |r_j-(\omega_j-r_j)| = |2r_j-\omega_j|, \quad j=q+1(1)p \quad , \tag{3.8}$$

(3.4) are shown to hold which concludes the proof of the present lemma. □

Remark: It is noted that equality in (3.6) holds if and only if $\omega_j \in I\!R$, $j=1(1)q$. For equality in (3.7) we can obtain again as before $\omega_j \in I\!R$, $j=q+1(1)p$. Using the last conclusion we find out that equality in (3.8) holds if and only if $r_j \in I\!R$ and either $r_j \ge \omega_j$ or $r_j \le 0$, $j=q+1(1)p$. Consequently, for $\omega_j \in I\!R$, $j=1(1)p$, and r_j such that

$r_j \geq \omega_j$ or $r_j \leq 0$, $j = q + 1(1)p$, then (3.2) of Lem. 3.1 is equivalent to (2.10) of Thm 2.1. □

Recall now the eigenvalue functional equation which connects the eigenvalues μ of the Jacobi matrix T^A in (1.3) and λ of the MAOR matrix $\mathcal{L}^A_{R,\Omega}$ in (1.6) for any GCO $(q,\ p-q)$-matrix A in (1.2) (see [7]), that is

$$\prod_{j=1}^{p} (\lambda + \omega_j - 1) = \prod_{j=1}^{q} \omega_j \prod_{j=q+1}^{p} (\omega_j - r_j + r_j\,\lambda)\mu^p \quad . \tag{3.9}$$

As is known, if λ and μ are any two numbers satisfying (3.9) and

$$\omega_j - r_j + r_j\,\lambda \neq 0, \qquad j = q + 1(1)p \quad , \tag{3.10}$$

then $\mu \in \sigma(T^A)$ if and only if $\lambda \in \sigma(\mathcal{L}^A_{R,\Omega})$. It is noted that (3.10) always hold for the EJ matrix, while (3.10) becomes simply $\lambda \neq 0$ for the MSOR matrix. We also notice that when (3.9) holds sufficient conditions for (3.10) to hold are

$$\omega_j \neq r_j\,\omega_k, \qquad j = q + 1(1)p, \qquad k = 1(1)p \quad . \tag{3.11}$$

This is readily seen because the value of λ for which one of (3.10) becomes zero is $\lambda = 1 - \dfrac{\omega_j}{r_j}$ for some $j \in \{q+1,\ q+2,\ ...,\ p\}$. However, this value must make one of the factors of the left hand side of (3.9) vanish. This gives that $\omega_j = r_j\omega_k$ must hold for some $k \in \{1,\ 2,\ ...,\ p\}$.

Based on (3.9) and (3.11) we can prove.

Theorem 3.1: Let that the two sets of acceleration r_j, $j = 1(1)p$, and relaxation ω_j $(\neq 0)$, $j = 1(1)p$, parameters are real and satisfy (3.11). If one of the following conditions:

$$\begin{aligned} &i)\ \ \omega_j < 0 \quad \text{or} \quad \omega_j \geq 2, \quad j = 1(1)p, \\ &ii)\ \ \bar{\mu} \geq 1 \quad \text{and} \quad \prod_{j=1}^{p} \omega_j > 0 \\ &iii)\ \ \prod_{j=1}^{p} (2 - \omega_j) \leq \prod_{j=1}^{q} \omega_j \prod_{j=q+1}^{p} (2r_j - \omega_j)\, \bar{\mu}^p \end{aligned} \tag{3.12}$$

holds, then

$$\sup_{A \in \mathcal{B}} \{\rho(\mathcal{L}^A_{R,\Omega})\} \geq 1 \quad . \tag{3.13}$$

Proof: i) Let $0 \in \sigma(T^A)$, which is always possible in case at least two of the diagonal blocks of A are of different orders or all the diagonal blocks are of the same order and at least one of the off-diagonal non-identically zero blocks is singular. Then, from (3.9), and in view of (3.11), $\lambda = 1 - \omega_k \in \sigma(\mathcal{L}^A_{R,\Omega})$ for at least one $k = 1(1)p$. Thus $|1 - \omega_k| \geq 1$ implying $\rho(\mathcal{L}^A_{R,\Omega}) \geq 1$ and vice versa.

ii) Let $\mu^p = \overline{\mu}^p \in \sigma((T^A)^p)$, which is possible in case e.g., A is an M-matrix. Then any λ which satisfies

$$P(\lambda\,; r_j,\, \omega_j,\, \overline{\mu}) = 0 \quad , \tag{3.14}$$

where

$$P(\lambda\,; r_j,\, \omega_j,\, \mu) := \prod_{j=1}^{p} (\lambda + \omega_j - 1) - \prod_{j=1}^{q} \omega_j \prod_{j=q+1}^{p} (\omega_j - r_j + r_j\,\lambda)\mu^p \quad , \tag{3.15}$$

is an eigenvalue of $\mathcal{L}^A_{R,\Omega}$. Observe now that if (3.12ii) holds, it is

$$P(1\,; r_j,\, \omega_j,\, \overline{\mu}) = \prod_{j=1}^{p} \omega_j (1 - \overline{\mu}^p) \leq 0 \quad ,$$

which combined with the fact that $P(\lambda\,; r_j,\, \omega_j,\, \overline{\mu}) \geq 0$ for λ sufficiently large implies that there exists a $\lambda^* \geq 1$ such that $P(\lambda^*\,; r_j,\, \omega_j,\, \overline{\mu}) = 0$. Thus, $\lambda^* \in \sigma(\mathcal{L}^A_{R,\Omega})$. Hence $\rho(\mathcal{L}^A_{R,\Omega}) \geq 1$.

iii) Let $\mu^p = (-1)^q\, \overline{\mu}^p \in \sigma((T^A)^p)$, a case which is possible if e.g., A_{jj}, $j = 1(1)p$, are M-matrices, $A_{j,p-q+j} \geq 0$, $j = 1(1)q$, while $A_{q+j,j} \leq 0$, $j = 1(1)p - q$. Then, any $\lambda = -\nu$ satisfying (3.14) will also satisfy

$$Q(\nu\,; r_j,\, \omega_j,\, (-1)^{q/p}\, \overline{\mu}) = 0 \quad , \tag{3.16}$$

where

$$Q(\nu\,;\,r_j,\,\omega_j,\,\mu) := \prod_{j=1}^{p}(\nu-\omega_j+1) - (-1)^q \prod_{j=1}^{q}\omega_j \prod_{j=q+1}^{p}(r_j-\omega_j+r_j\,\nu)\,\mu^p \tag{3.17}$$

and will be an eigenvalue of $\mathcal{L}^{A}_{R,\Omega}$. By a similar argument as in (ii) previously it can be proved using (3.12iii) that there exists a $\nu^* \geq 1$ satisfying (3.16) and therefore a $(-1 \geq)\ \lambda^* = -\nu^* \in \sigma(\mathcal{L}^{A}_{R,\Omega})$. So, $\rho(\mathcal{L}^{A}_{R,\Omega}) \geq 1$ follows. This concludes the proof of the theorem. □

Remark: Thm 3.1 is an extention, in one direction, of the basic Thm 3.1 of [25] which concerns the scalar case $R = rI \neq 0$ and $\Omega = \omega I$. □

4. CONCLUDING REMARKS AND DISCUSSION

As has already been seen the results of this paper and in particular those in Section 2 extend and generalize other known ones. Our effort in Section 2 was to establish sufficient conditions for the convergence of the MAOR method. To make these conditions as strict as possible which will enable us to determine the precise domain of convergence of the MAOR method, as this was done for the SSOR method by Neumaier and Varga [19] for the entire class of H-matrices and by Hadjidimos and Neumann [5] for each class of GCO $(q,\ p-q)$-matrices, seems to be a complicated problem. This can be realized from Thm 3.1, when domains of divergence were obtained. However, we would like to point out that the conditions we considered in Thm 3.1 may be relaxed if one considers particular methods as e.g., the EJ and/or the MSOR ones or if one restricts oneself to subclasses of the class of matrices $\mathcal{B}$ as e.g., the one where all diagonal blocks of A are square and the non-identically zero blocks of T^A are nonsingular. For the latter a deeper analysis of (3.9) in view of (3.10) is needed. An investigation along the lines of filling up the gap between the convergence and divergence domains of the MAOR method is being made.

A very interesting and attractive problem is that of deriving "optimal" or "good" values for the parameters involved so that convergence of the MAOR method is achieved in an "optimal" sense. In the general case a solution to this problem does not seem to be achieved in a straightforward manner. For this one should bear in mind the kind of difficulties one should overcome in the determination of the optimal parameters when only two real ones are involved. As for example, in the scalar 2-cyclic AOR method (e.g., [26], [20], [1], [23], [18], [9], etc. the scalar 3-cyclic AOR [22] or even in the 2-cyclic MSOR method (e.g., [17], [27], [10], [9], [8], [11], [29]), where in the most cases "optimal" parameters are obtained based on previous works on 2- and k-step iterative methods (see e.g., [13], [14], [12], [21], etc.). The problem of the determination of optimal parameters for the MAOR method in cases of both theoretical and practical interest is also being investigated.

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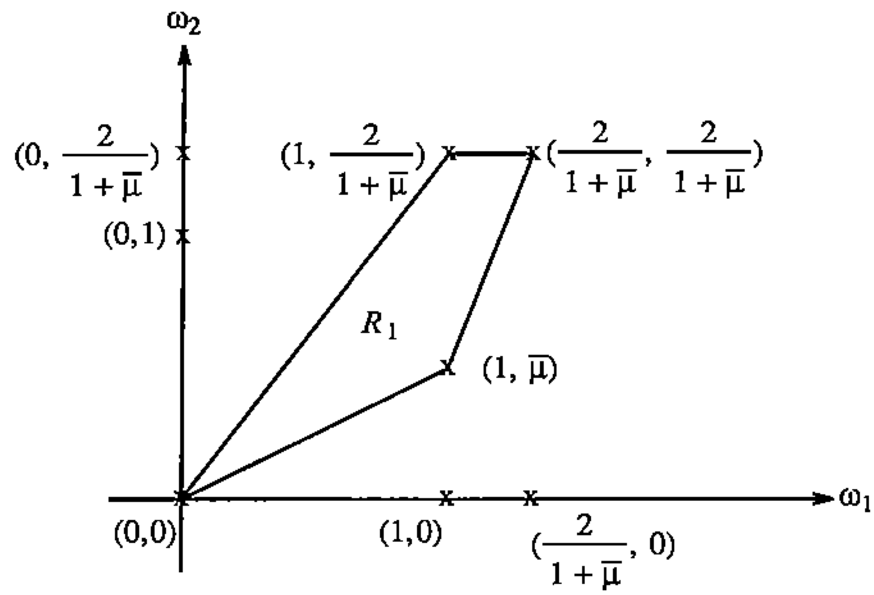
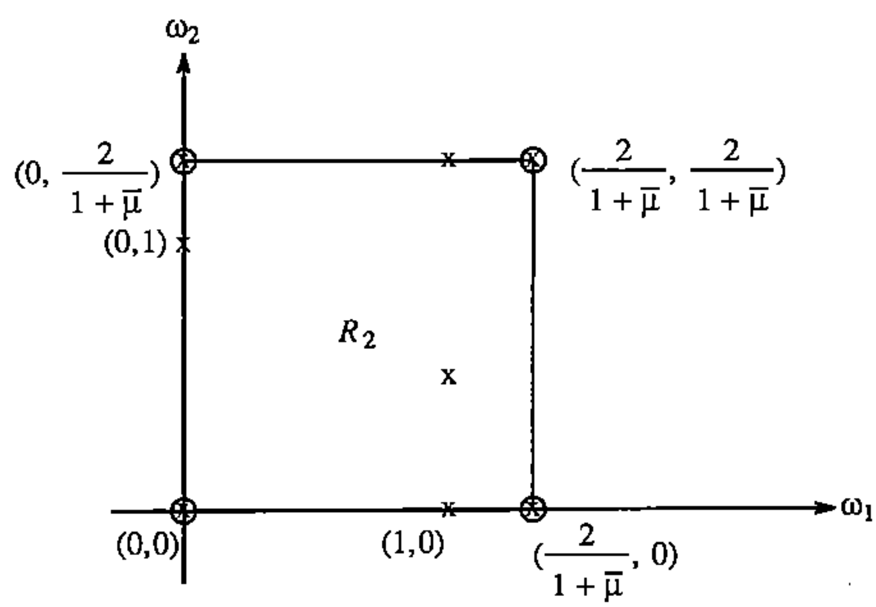


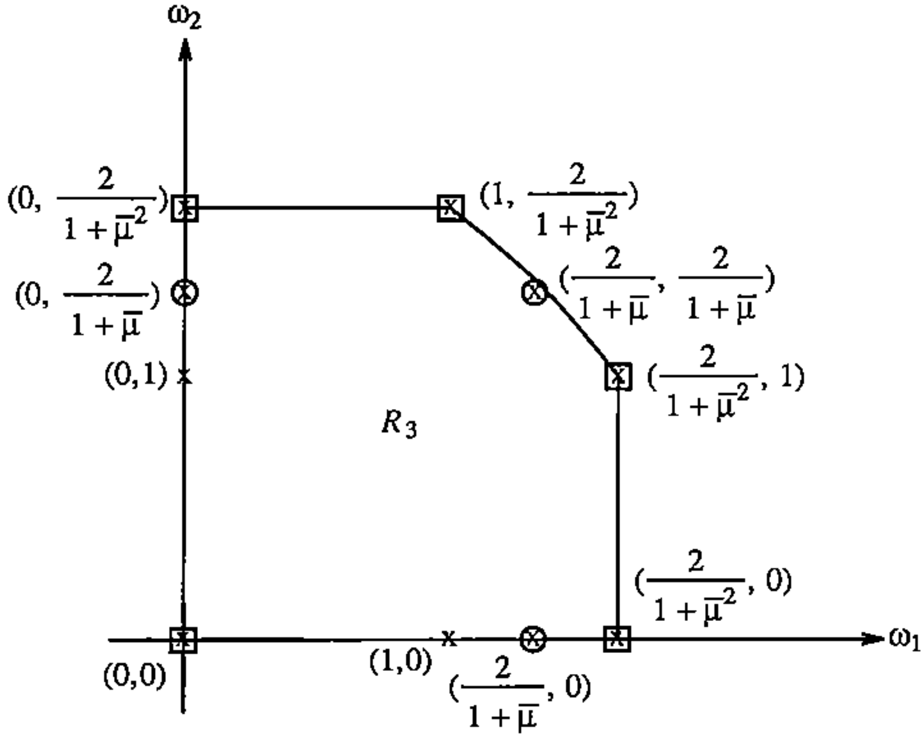
Region of Convergence R_1 ($\bar{\mu} = 0.5$)

Figure 1.



Region of Convergence R_2 ($\bar{\mu} = 0.5$)

Figure 2.



Region of Convergence R_3 ($\bar{\mu} = 0.5$)

Figure 3.