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Theoretical Limits of Stone Skipping

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Abstract

A simplified geometric model captures essential dynamics of stone skipping and allows easy computation of realistic stone trajectories under a variety of conditions. The model describes idealized collisions of flat, spin-stabilized stones with water, in which the water exerts sufficient reactive force on the stone to completely stall motion normal to the stone's bottom surface. Motion parallel to the stone's bottom surface remains unchanged. The necessary computations, including vector forces and accelerations, work and energy, algebra and trigonometry, are understandable by first year students of classical Newtonian physics. Comparison of this highly simplified theory with suitable experiments provides an interesting organizing theme for project based learning.

Introduction

As a boy I spent many hours skipping stones across the surface of the water on the shore of Lake Michigan. On a calm day with few waves, I learned to select a flat, smooth stone, about 5 to 10 centimeters in diameter and about one half to one centimeter thick, to bend down low, and with a side-arm motion fling it low and nearly flat across the water, giving the stone a spin with an extra finger flick in order to stabilize it in flight. Amazingly, with good technique, the stone would not sink on contact with the water, but would bounce or skip up in the air one or more times, with generally shorter and shorter distances between skips, before finally sinking to the bottom after a long and glorious run. The challenge was to get as many skips or as much distance as possible. Zero skips was a bust. One or two skips was disappointing, three or four skips was mediocre, five or six skips was satisfying, and six or more skips was exhilarating. The angle of the stone with respect to the surface of the water seemed to be critical. A large angle approaching 45 degrees produced one large jump and perhaps one or two more after that. A smaller angle produced more skips and a longer run. However, too small an angle would cause immediate sinking. Fine tuning the skill of stone skipping was a captivating pastime. Today stone skipping has become both a recreational and a competitive sport (www.stoneskipping.com).

The objectives of this paper are to explore the underlying physics of stone skipping, to explain why stones skip, to derive equations for the trajectory of an idealized skipping stone through the air, and in particular, to specify the number of skips and the total distance of travel of the stone, including especially the theoretical upper limits of performance. How many skips can one possibly get? This problem can provide an entertaining exercise for students to consolidate knowledge of first year physics without requiring advanced mathematics or a detailed description of the fluid flow around the colliding stone in three dimensions.

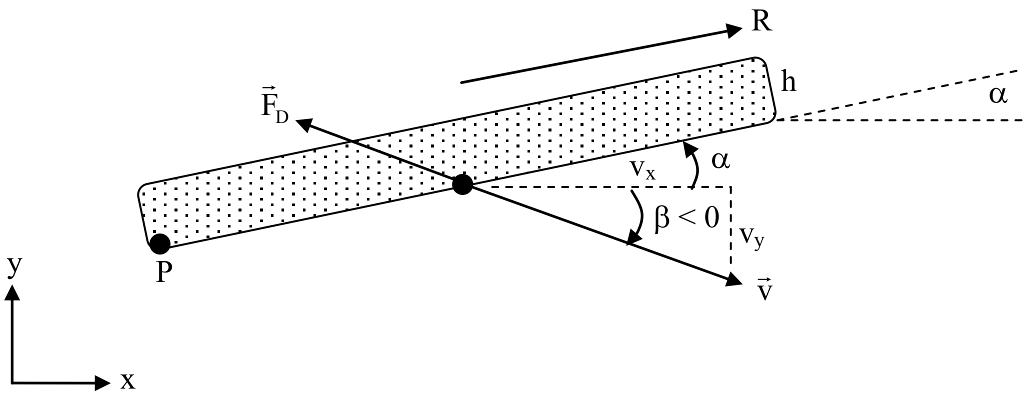


Figure 1. An idealized, spin stabilized skipping stone in flight. The radius of the stone is R, and the thickness is h. The velocity of the stone in air is $\vec{v}$. The tilt angle of the stone from the horizontal is α. The flight path angle is $\beta = \tan^{-1}(v_y/v_x)$, with $\beta < 0$ for a falling stone and $\beta > 0$ for a rising stone. Air resistance creates drag force, $\vec{F}_D$, which on average opposes forward motion.

Methods

An idealized skipping stone

Figure 1 is a sketch of an idealized flat stone, the angle of which with respect to the horizontal, α, is stabilized by rotational spin—the gyroscopic effect—and so is considered constant. The thickness of the stone is denoted h. The flat surface area of the stone is denoted $A = \pi R^2$ for a hockey puck shaped stone of radius R. If ρ_s is the mass density of the stone, then the mass of the stone is $m_s = \rho_s \pi R^2 h$.

The stone moves with instantaneous horizontal and vertical velocity coordinates v_x and v_y in a two-dimensional, x-y coordinate system. The flight path angle with respect to the horizon is $\beta = \tan^{-1}(v_y/v_x)$. A negative value of v_y , or a negative value of β , indicates that the stone is falling downward toward the water under the acceleration of gravity, g. A positive value of v_y or a positive value of β indicates that the stone is rebounding upward toward the sky. At time t = 0 the stone is launched over a flat surface of water from vertical height, y_0 , with initial horizontal velocity, v_{x0} , and initial vertical velocity, v_{y0}. Typically, $v_{y0} = 0$, and $\beta_0 = 0$. That is, the stone is launched horizontally.

Consider a point, P, at the trailing bottom edge of the stone. For simplicity, let the trajectory of P as a function of x, y, and time, t , represent the position of the stone in space and time. When y > 0 the stone is considered to be in the air, and when y < 0 the stone is considered to be in the water. In flight, ignoring air resistance, the acceleration of the stone in the x-direction, $a_x = 0$, and the acceleration in the y-direction, $a_y = -g$. If aerodynamic drag forces, $\vec{F}_D$, are included, the stone of mass m_s experiences additional vector drag acceleration, $\vec{a}_D = \vec{F}_D / m_s$, in a direction that opposes its forward motion. Including air resistance, the acceleration of the stone in the x-direction is $a_x = 0 - \frac{|\vec{F}_D|}{m_s}\cos\beta$, and the acceleration in the y-direction is $a_y = -g - \frac{|\vec{F}_D|}{m_s}\sin\beta$. When the stone is rising, we have $\beta > 0$, and drag acceleration is downward, toward the water. When the stone is falling we have $\beta < 0$, and drag acceleration is upward, toward the sky. (Here and in what follows the superscript arrow such as in $\vec{F}$ indicates a vector quantity, and $|\vec{F}|$ indicates the scalar magnitude of the vector.)

The crux of the stone skipping problem is to characterize the change in velocity of the stone after it hits the surface of the water. Reynolds numbers for this scenario of stone-water collision are Re ~ 10^5 (Bocquet, 2003; Rosellini, 2005), so that viscous forces can be neglected, and reactive inertial forces dominate. The following treatment gives expressions for the vector change in velocity of the stone, $\Delta\vec{v}_s$, with each skip for an idealized subset of all possible collisions in which the motion of the stone normal to its flat bottom surface is stalled by reactive forces before water overtops the stone, causing it to sink. Here this model is referred to as the "skipmax" model. For these conditions one can explore the factors governing the number of skips and the length of the run to generate hypotheses about how recreational and competitive throwers might improve their performance.

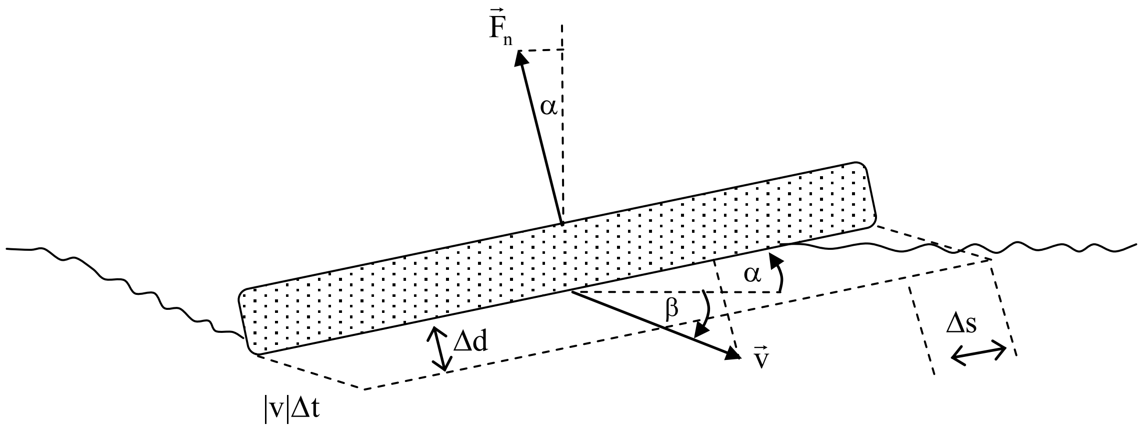


Figure 2. An idealized, spin stabilized skipping stone in the water. Here the stone is moving forward and downward into the water at velocity, $\vec{v}$. The reactive inertial force, $\vec{F}_n$, acting on the bottom surface of the stone pushes upward with force $\vec{F}_n$ cos(α) and backward with force $\vec{F}_n$ sin(α). The stone does work on the water when moving normal to its bottom surface through distance, Δd. A free slip condition at the water-stone boundary means that no work is done by movement over the orthogonal distance, Δs.

An idealized collision

To model the interaction of the stone with the water during successive skips, one can imagine the work done on the water by the stone, the equal and opposite work done on the stone by the water, and in turn, the change in velocity of the stone caused by the collision—at least for idealized "skipmax" cases. This approach captures the essence of stone skipping in a way that permits calculation of the trajectory of the stone over the course of multiple skips.

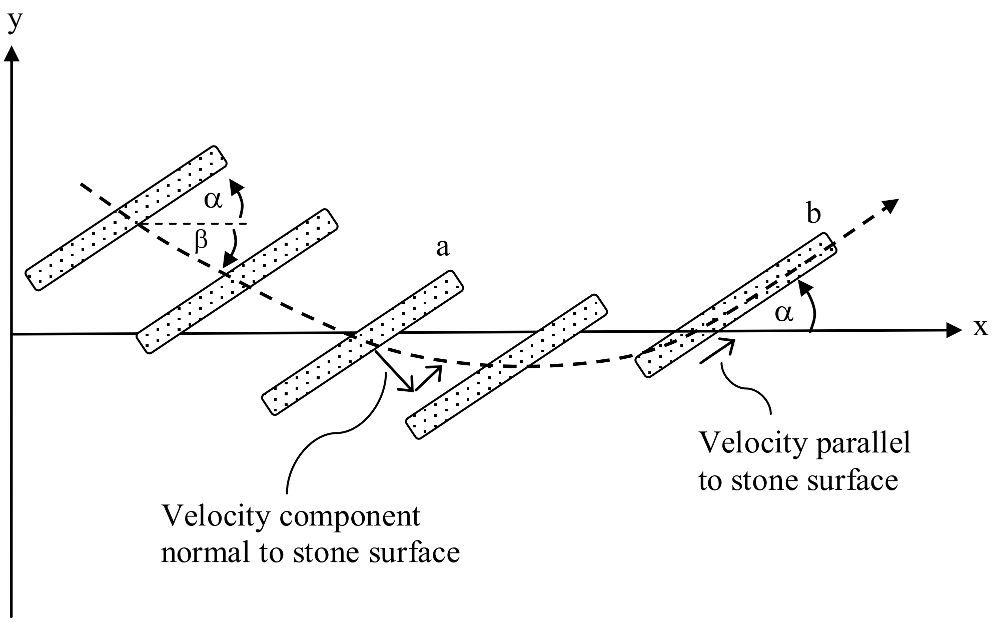


Figure 3. An idealized collision model, in which a spin-stabilized, flat stone does work to push water ahead of it during collision. In early positions (a) reactive force slows the stone in the direction normal to its surface, changing its trajectory until the flight path becomes parallel to the stone's tilt at angle, α, after which no more work is done (b). The stone exits the water at angle, α.

Figure 3 illustrates the flight path of an idealized skipping stone colliding with the surface of the water at spin-stabilized angle, α . The vertical scale is expanded to show detail. The water is regarded as an ideal fluid to allow frictionless slipping between continuous layers of water and also at the fluid-solid boundary. Hence, no work is done as the stone moves parallel to its flat bottom surface through distance Δs in Figure 2. Work is done transferring energy from the stone to the water only as the stone moves perpendicular to its bottom surface through distance Δd.

By Newton's first law, the reactive force on the stone is equal in magnitude and opposite in direction from the force that the stone exerts on the water to do work. By Newton's second law the product of the average reactive force and the brief time interval, Δt , of the collision equals the mass of the stone multiplied by the change in velocity of the stone: $\vec{F}\Delta t = m_s \Delta \vec{v}_s$. Hence, both the reactive force and the change in velocity of the stone point in the direction normal to the bottom surface of the stone. As long as the stone has sufficient kinetic energy to move water, and water does not overtop the stone, this effect will change the stone's trajectory until the flight path becomes parallel to the stone's surface at angle, α , after which time no more work is done (Figure 3). Thus, the reactive force reduces the velocity component normal to the stone toward zero, leaving only the velocity component parallel to the surface of the stone. Then the stone exits the water at angle, α, or very nearly α, as long as water does not overtop the stone, here temporarily ignoring gravity during the collision.

(a)

(b)

(c)

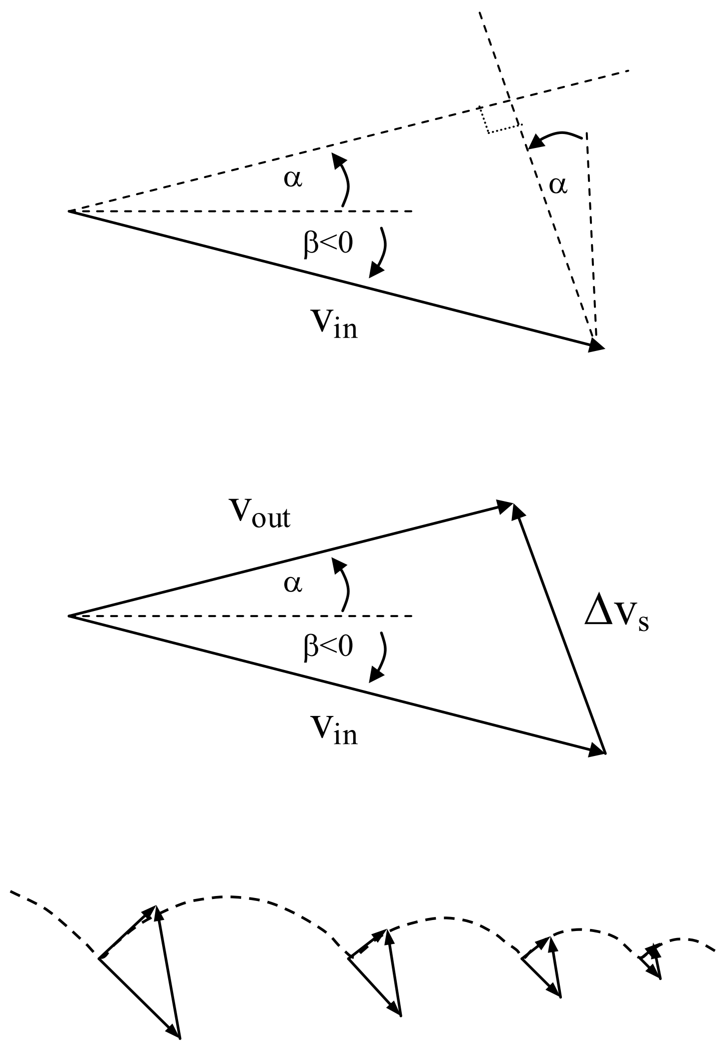


Figure 4. Vector addition $\vec{v}_{in} + \Delta\vec{v}_s = \vec{v}_{out}$ ***for computing outbound velocity of a skipping stone (a) and (b). Serial application of the vector addition rule to reconstruct airborne segments of the stone's trajectory (c).***

As shown in Figure 4, for this skipmax scenario, we know that the outbound velocity vector, $\vec{v}_{out}$, must be approximately at angle α with respect to the horizontal. The reactive forces, and the consequent change in velocity vector, $\Delta\vec{v}_s$, must be normal or perpendicular to the surface of the stone, which is at angle, α, from the vertical. These two constraints define a right triangle for vector addition, $\vec{v}_{in} + \Delta\vec{v}_s = \vec{v}_{out}$, which determines the direction and magnitude of the outbound velocity of the stone.

By deduction from Figure 4, for total angle, $-\beta + \alpha$, at water entry in an idealized collision, we must have the magnitude of the outbound velocity vector

$$|\vec{v}_{out}| = |\vec{v}_{in}| \cos(-\beta + \alpha) \quad (1)$$

and the direction of the outbound velocity vector at angle, α, with respect to the horizon.

Note also by deduction from Figure 4 both the magnitude of the normal force on the stone and the magnitude of the consequent change in velocity, $|\Delta\vec{v}_s| = |\vec{v}_{in}| \sin(-\beta + \alpha)$, are proportional to the sine of the total angle, just as is observed experimentally (Rosellini, 2005, p.6). For shallow tilt angles, α, the dominant component of the normal force, $|\vec{F}_n| \cos\alpha$, is a vertical lifting force. It is this force that causes the stone to skip!

The skipmax assumption that the course correction by reactive forces is complete before the water overtops the stone, makes sense, especially for larger horizontal speeds, v_x. In such cases a relatively large volume of water is swept out by the bottom of the stone in a relatively short time interval, during which the reactive force can lift the stone before the elevated leading edge has time to submerge. Thinner, flatter stones are more likely to conform to the skipmax assumption.

Thus for a proper skipmax bounce, ignoring gravity during the collision, it is easy to specify approximately the initial conditions for the next skip of the stone, beginning at zero height and having velocity components

$$v_{xout} = |\vec{v}_{in}| \cos(-\beta + \alpha) \cos\alpha, \quad (2)$$

and

$$v_{yout} = |\vec{v}_{in}| \cos(-\beta + \alpha) \sin\alpha. \quad (3)$$

Including gravity during the collision

Equation (3) gives the vertical velocity component computed for a scenario in which the effect of gravity is ignored when the stone is in the water. To obtain a rough, zerothorder correction for the action of gravity, one can assume the typical maximal depth of point, P, at the trailing edge of the stone in the water is $R\sin(\alpha)$. As shown in Appendix 1 the vertical velocity, corrected for the energy required to lift the stone a small, constant distance, $R\sin(\alpha)$, is

$$\hat{v}_{yout} = \sqrt{v_{yout}^2 - 2gR\sin(\alpha)} = v_{yout}\sqrt{1 - \frac{2gR\sin(\alpha)}{v_{yout}^2}} \; . \tag{4}$$

If the argument of the square root in Equation (4) is less than zero, the stone sinks. In this way after a given collision, the positive horizontal and vertical exit velocity components at the water's surface ($y = 0$) may be computed. These values are then taken as initial conditions for the next flight, beginning at $y_0 = 0$. This process is repeated until the stone is unable to bounce.

Stopping criteria

At this juncture it is useful to define a minimal legal height, y_{legal}, for a skip. Surely, if the stone rises only a few micrometers above the water's surface, it would not be called a "skip" by most observers. An end-of-run condition in which the stone appears to slide forward a short distance with its trailing edge just below or just above the surface of the water is common in practical experience. During such rapid, close, low altitude skips or "pitty-pats" the stone appears to glide across the surface of the water, rather than making individual points of contact that can be counted easily (Kennedy, 2014). For purposes of quantitative analysis, let us define a minimum legal bounce height, $0 \le y_{legal} \le 1$ cm, such that if the stone bounces to a maximum height less than y_{legal}, it is not counted or scored. In this case the combined end-of-run stopping criteria are that if

$$1 - \frac{2g\left(R\sin(\alpha) + y_{legal}\right)}{v_{yout}^2} < 0 \quad \text{or} \quad v_{xout} < 0 \tag{5}$$

then stop. For the next skip after an impact to be counted, the stone must have both a positive horizontal speed and enough residual vertical kinetic energy to climb to a height of at least y_{legal}. The factor, y_{legal}, becomes especially important in evaluating high velocity, championship throws.

The forgoing treatment does not give the exact trajectory of the stone while it is in the water. However, as noted by Rosellini et al. (2005) and as known to practical stone skippers, the collisions with the water are very brief (~ 70 msec). The stone spends the overwhelming majority of the time in the air. Hence, we can still plot a reasonable approximation for the airborne trajectory for the stone at heights $y > 0$, as will now be described.

Numerical computation of stone trajectories in the air

The horizontal and vertical components of stone acceleration in air can be integrated numerically using the simple Euler method, implemented, for example, in Visual Basic code within an Excel spreadsheet on an ordinary laptop computer. Typical initial conditions are described in Results, Table 1. Specifically, given the horizontal and vertical accelerations of the stone in the air, a_x and a_y, double integration is performed as follows for each successive time increment, Δt :

$$v_x(t+\Delta t) \cong v_x(t) + a_x \Delta t \text{, and } v_y(t+\Delta t) \cong v_y(t) + a_y \Delta t . \qquad (6)$$

In turn,

$$x(t+\Delta t) \cong x(t) + v_x \Delta t \text{, and } y(t+\Delta t) \cong y(t) + v_y \Delta t . \qquad (7)$$

Given initial conditions at $t = 0$, specifically the initial height, y_0, and initial velocity components v_{x0} and v_{y0}, as the stone leaves the hand of the thrower, one can trace the evolution of the variables x and y as a function of time in a "marching solution", for any particular initial launch position until touchdown, when the stone returns to level $y = 0$. A subroutine can be created to perform this calculation for the initial throw ($y_0 > 0$, e.g. 50 cm) and also for subsequent skips ($y_0 = 0$). Stability and accuracy of numerical integration are ensured by using a sufficiently small value of Δt, such as 10 microseconds. Increasing or decreasing Δt without effect on the results confirms that a sufficiently small value was chosen for Δt.

Stone accelerations in air ignoring aerodynamic drag or air resistance are simply $a_x = 0$ and $a_y = -g$, the acceleration of gravity at the Earth's surface. To incorporate air resistance, which becomes important for higher initial throw velocities, one may include the aerodynamic drag force on the stone, which has direction opposite the stone's velocity and magnitude $|F_D| = \frac{1}{2}\rho_{air} C_D A |\vec{v}|^2$, where ρ_{air} is the mass density of air (0.00122 g/cm^3), constant, C_D, is a dimensionless drag coefficient or shape factor, typically ranging between 0 and 2, and A is the reference surface area, taken here as πR^2 for simplicity, and $\vec{v}$ is the forward velocity. Based on works of Hoerner (1965) the shape factor for a stone in flight can be estimated to be $C_D \approx 0.5$. For simplicity this constant value is used here, although there is probably some complex dependence on stone tilt, α, and on flight path angle, β, as well as on stone thickness, h.

The acceleration due to air resistance has average direction opposite the forward velocity and has magnitude given by the drag force divided by stone mass or

$$|a_D| = \frac{|F_D|}{m_s} = \frac{\frac{1}{2}\rho_{air} C_D A |v|^2}{\rho_s A h} = \frac{1}{2}\frac{\rho_{air}}{\rho_s}\frac{C_D}{h}|\vec{v}|^2. \tag{8}$$

With reference to Figure 1, drag acceleration on the stone in flight has horizontal and vertical components

$$a_{Dx} = -\frac{1}{2}\frac{\rho_{air}}{\rho_s}\frac{C_D}{h}|\vec{v}|^2 \cos\beta \tag{9a}$$

and

$$a_{Dy} = -\frac{1}{2}\frac{\rho_{air}}{\rho_s}\frac{C_D}{h}|v|^2 \sin\beta\,. \tag{9b}$$

Then, accounting for air resistance, stone acceleration components in air become

$$a_x = a_{Dx} \tag{10a}$$

and

$$a_y = -g + a_{Dy}. \tag{10b}$$

Numerical computations of changes in stone velocity with each skip

After either the initial throw or a skip, as soon as the computed height of the stone above the water, y(t), becomes less than zero, the subsequent change in velocity of the stone caused by collision with the water is easily computed using Equations (2) and (4) subject to the stopping criteria of Equation (5). In this way after a given collision, horizontal and vertical exit velocity components at the water's surface (y = 0) are computed, and these values are then taken as initial conditions for the next flight, beginning at $y_0 = 0$. In plotting the trajectory of the stone, the unknown horizontal distance that the stone travels in the water from its point of entry is estimated roughly to be its diameter, or 2R. Then the next airborne trajectory is computed using the rebound velocity components as inputs. This process is repeated until a stopping criterion is met.

Results

Standard model

Table 1: Standard model

Variable	Value	Units	Definition
h	1	cm	Stone thickness
R	4	cm	Stone radius
rhos	2.5	grams/cm^3	Stone mass density
rhow	1.0	grams/cm^3	Water mass density
rhoair	0.00122	grams/cm^3	Air mass density
y0	50	cm	Launch height
ylegal	0 to 1	cm	Minimum legal skip height for counting
vx0	1000	cm/sec	Horizontal launch velocity
vy0	0	cm/sec	Vertical launch velocity (positive = up)
alpha	0.3	radians	Stone surface angle with horizon
	17	degrees	
dt	0.00001	sec	Time step for numerical integration

Table 1 shows standard model parameters for the idealized skipping stone. Stone mass density is similar to that of typical rocks and dense concrete (www.simetric.co.uk). The launch height represents a child's sidearm throw. The launch angle is horizontal and the launch speed is based on a little league baseball change-up pitch speed of 40 miles per hour = 18 m/sec (www.efastball.com) and values near 10 m/sec in prior work (Bocquet, 2003). The angle of the flat stone with respect to the horizon is 0.3 radians (17 degrees) in keeping with common experience.

Typical stone trajectories

Figure 5(a) shows stone trajectories for the standard model with and without the presence of air resistance. The solid curve (Plus drag) includes aerodynamic drag on the stone during its flight through the air. The dashed curve (Minus drag) shows the calculated trajectory if air resistance is eliminated by setting the density of air equal to zero. The effect of drag is important even for moderately successful throws and becomes even greater for championship level throws of much higher launch velocity. The Plus Drag trajectory represents a good throw with multiple skips. The horizontal axis ranges from 0 to 2000 cm. The vertical axis ranges from zero to 75 cm to show detail of the skips. The apparent skip heights and water entry and exit angles in the figure are correspondingly exaggerated. For this standard model, including drag, there are 9 skips and the flight distance is 13.71 meters when the minimum legal skip height, $y_{legal} = 0$. When y_{legal} is increased to between 0.2 cm and 0.5 cm, there are only 8 skips, and the flight distance is only 13.57 meters. The last very low altitude rebound is not counted. Realistic stone skipping behavior can be predicted by the simple underlying physics that is embodied in the skipmax model.

Figure 5(b) shows the stone trajectory for the otherwise standard model in Figure 5(a) with reduced stone tilt angle $\alpha = 14$ degrees (0.25 radians). There are 13 skips and a flight distance of 16.00 meters. The last three skips are very low in height. If y_{legal} is increased to 0.5 cm, then only 10 skips are counted over 15.55 meters. This feature of hard-to-judge terminal "pitty-pat" is characteristic of low tilt angle throws.

Figure 5(c) shows the stone trajectory for the otherwise standard model with increased tilt angle $\alpha = 23$ degrees (0.4 radians) at launch. There are 5 skips and a flight distance of 10.6 meters. The trajectory shows physically realistic skipping behavior. In particular, the initial large hop near 500 cm in Figure 8(c) is typical of excessive tilt angles, α, that tend to reduce the total number of skips.

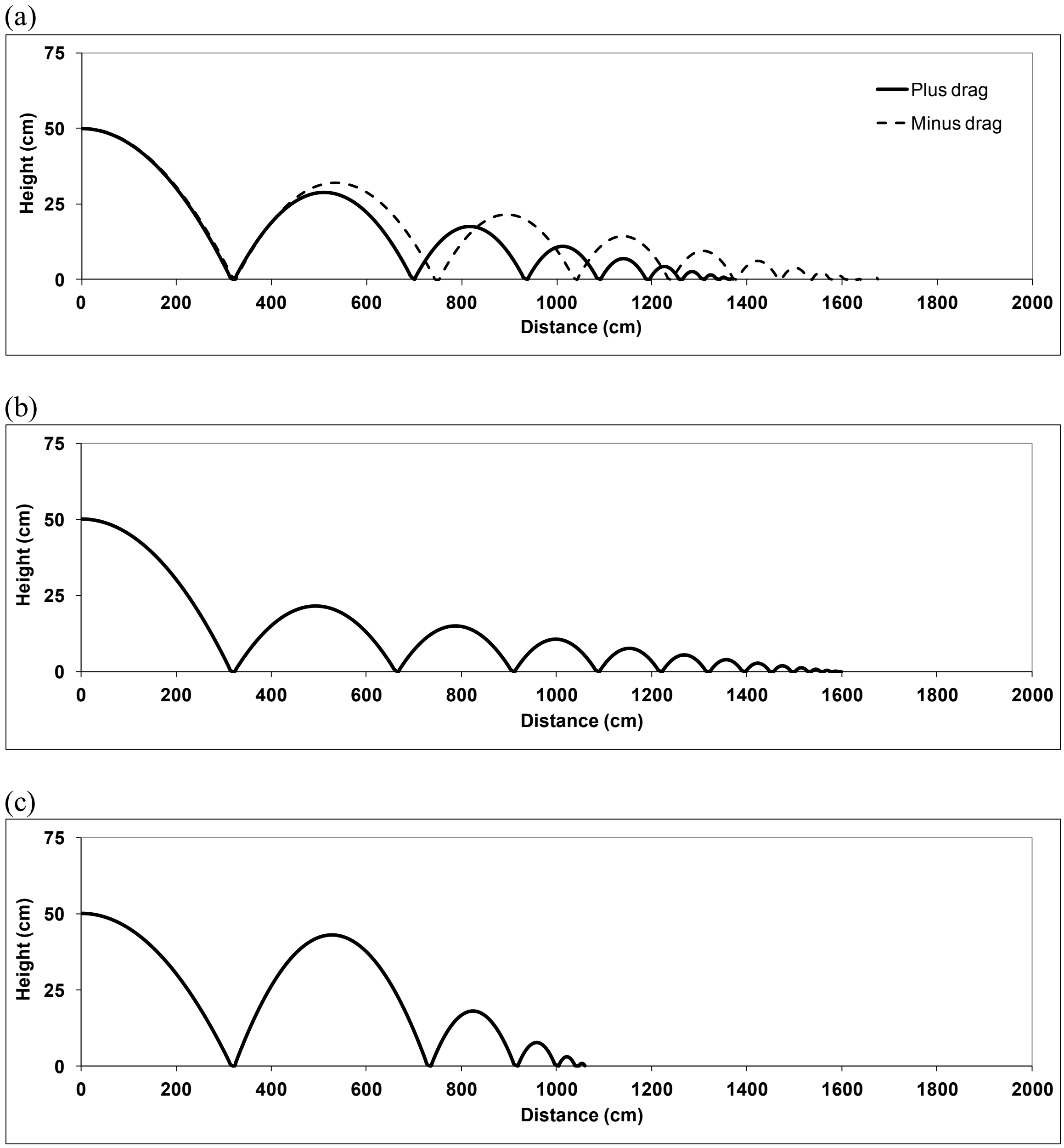


Figure 5. Typical stone trajectories for the standard model. Note the difference in horizontal and vertical length scales, which exaggerates apparent skip height. (a) Plus Drag trajectory computed for air density = 0.00122 g/ml. Minus Drag trajectory computed for air density 0. Tilt angle = 17 degrees (0.3 radians). Minimum legal skip height, $y_{legal} = 0$. Other model parameters are those listed in Table 1. (b) tilt angle $\alpha = 14$ degrees (0.25 radians). (c) tilt angle $\alpha = 23$ degrees (0.4 radians).

Stone tilt angle

A more complete study of the effects of changing the tilt angle, α , of the stone with respect to the horizon is shown in Figure 6.

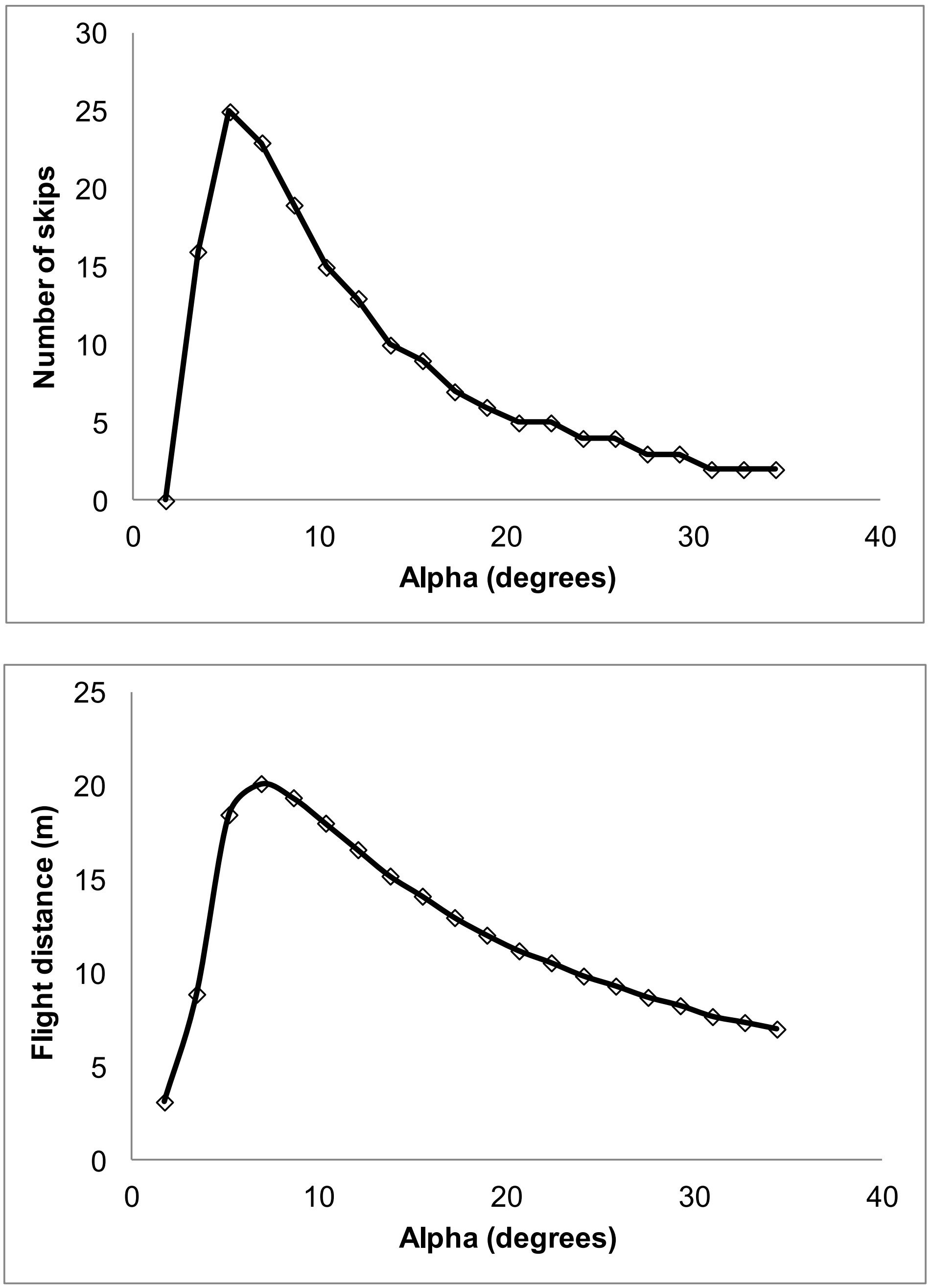


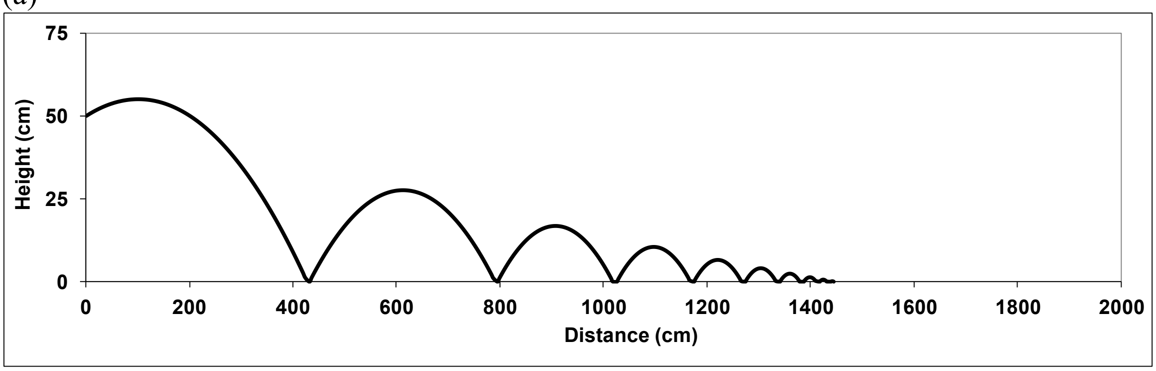
Figure 6. Effects of stone tilt angle, α , on figures of merit for stone skipping. For these plots a minimum legal skip height of 1.0 cm was required for counting.

These plots show a clear optimum near 8 degrees. However, there is a crash toward zero skips at shallower tilt angles, especially when a minimum legal skip height (here 1.0 cm) is required for counting. The sharp optimum tilt adds to the excitement of stone skipping as a competitive sport, requiring skill to find the best angle, but with danger of dramatic failure on the verge of dramatic success. The tilt effect also has implications for the assumption of spin stabilization of the stone in the idealized model presented here. For real stones the trailing edge of the stone will contact the water first, producing a torque on the stone tending to rotate the stone toward a flatter angle, α . However, if this effect is modest, the small reduction in α will still allow the stone to skip subsequently, as long as one begins sufficiently far from the steep slopes in Figure 6. That is, one can start deliberately with a steeper tilt angle and allow the stone to "flatten out" during successive skips.

Launch angle

Figure 7 illustrates the effects of launch angle, defined as the angle whose tangent is the ratio of vertical to horizontal launch speeds.

(a)



(b)

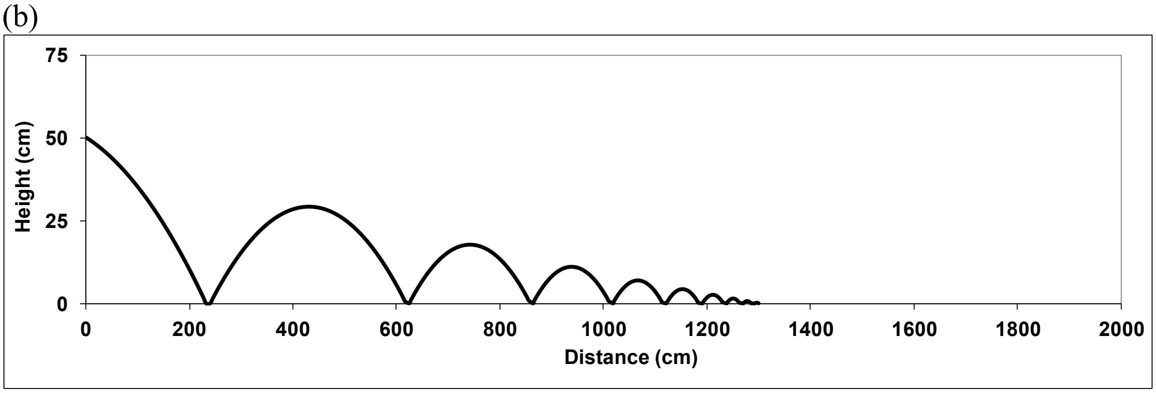
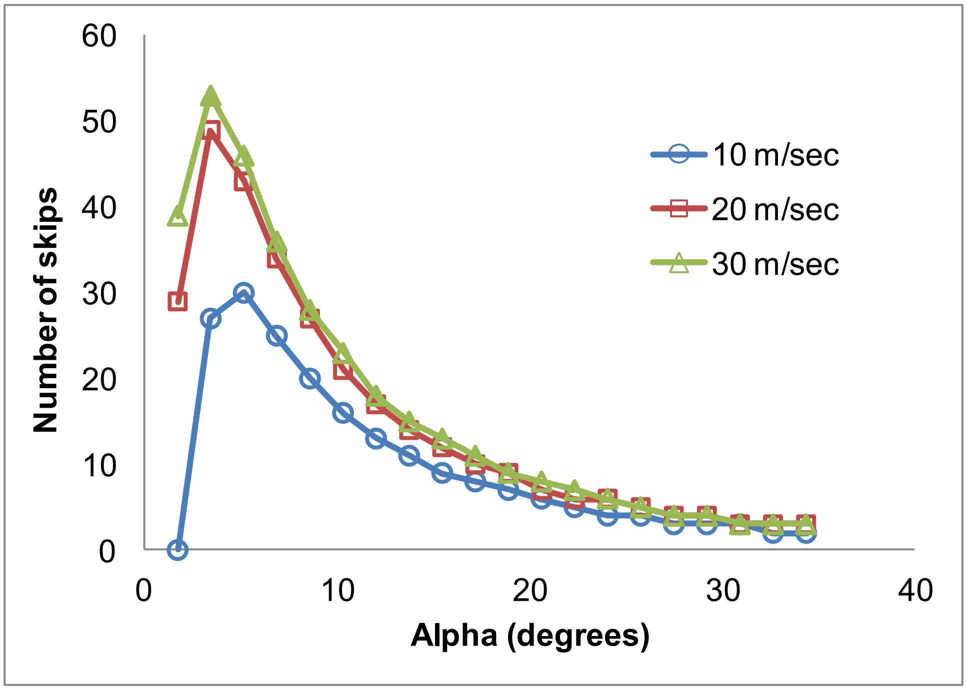


Figure 7. Otherwise standard model with slightly positive and negative launch angles. Minimum legal skip height, $y_{legal} = 0$. Note difference in horizontal and vertical length scales, which exaggerates apparent skip height.

In Figure 7(a) the initial vertical to horizontal velocity ratio is 100/1000 (+5.7 degrees elevation). In Figure 7(b) the initial vertical to horizontal velocity ratio is −100/1000 (−5.7 degrees elevation). The skipping pattern is similar for the two throws. With positive launch angle the point of first contact is farther from the launch site, and with negative elevation the point of first contact is closer to the launch site. In (a) there are 9 skips and the flight distance is 14.5 meters. In (b) there are also 9 skips, and the flight distance is 13.0 meters.

Launch velocity

Figure 8 shows the effects of increasing horizontal launch speed and stone tilt angle upon skipping performance with a minimum legal skip height of 0.5 cm. For reference a professional baseball player can throw balls approaching 100 miles/hour or 45 m/sec (www.efastball.com). A good sidearm fastball tends to be slower in the range of 80 miles/hour or 36 m/sec.



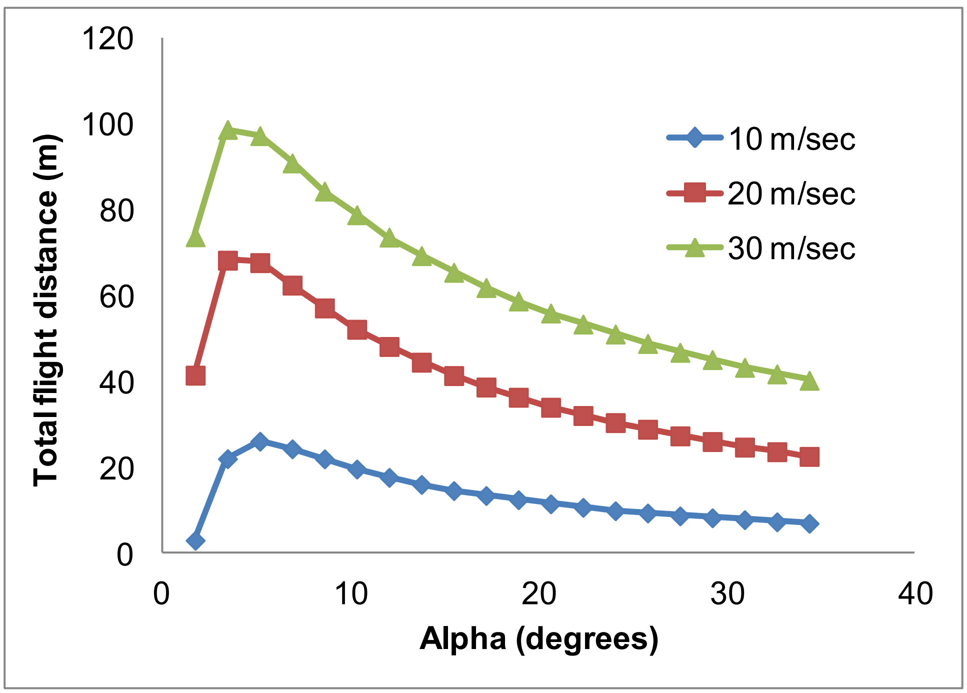
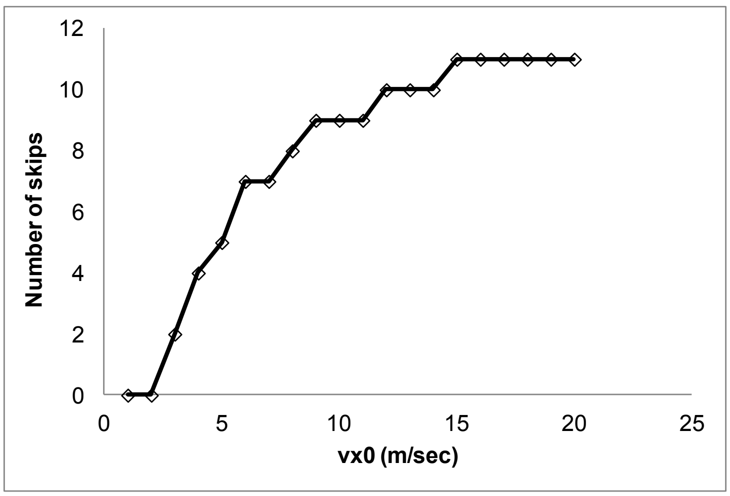


Figure 8. Effects of horizontal launch speed and stone tilt angle on figures of merit for stone skipping. Here a minimum legal skip height of 0.5 cm was required.

Figure 9 illustrates the skip/no-skip threshold phenomenon in the launch velocity domain. Launch velocity must exceed a certain critical threshold before skipping occurs. Above the threshold, progressively increasing launch velocities result in a decelerating increase in the number of skips. Total flight distance increases more readily once skipping behavior occurs. Depending on the initial conditions, a stone has to have a minimum initial velocity in order to bounce. The calculated minimal skipping velocity near 2.5 m/sec agrees with experimental findings. Indeed, using the exact parameters for "Stone 1" tested by Rosellini et al. (2005) the skipmax model predicts a maximum horizontal launch velocity for zero skips of 2.6 m/sec. The experimentally observed maximum no-skip launch velocity was 2.7 m/sec. This agreement of theory and experiment is especially important, because it is at the low end of the range of launch velocities where the skipmax model is most likely to fail.



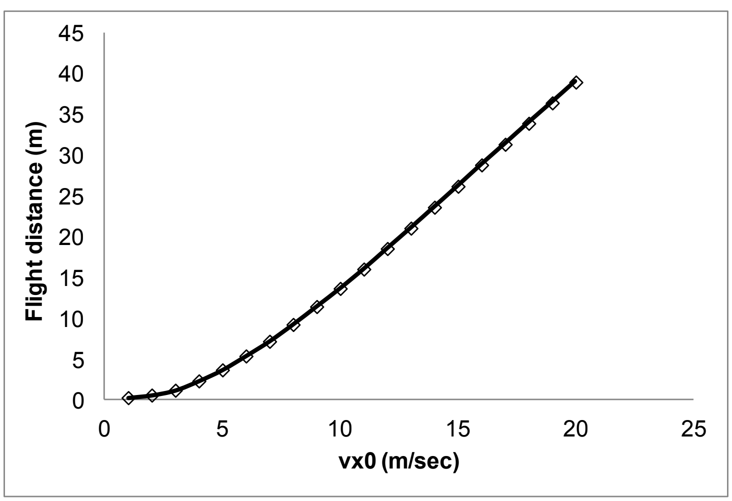


Figure 9. Number of skips and total flight distance for the standard model as a function of horizontal launch velocity, Vx0. Vertical launch velocity is zero. Alpha is 0.3 radians. Minimum legal skip height is zero cm.

Superior performance

Figure 10 illustrates the trajectory of an idealized stone with model parameters tuned to enhance skipping behavior: alpha 0.1 radians (5.7 degrees) and launch speed 36.0 m/sec. Zero minimum skip height was required. There are 67 skips over a distance of 114 meters. Note the extended distance scale. The pattern of a few early high skips, followed by a large number of low, quick skips is similar to that seen on recorded videos (search YouTube championship stone skipping).

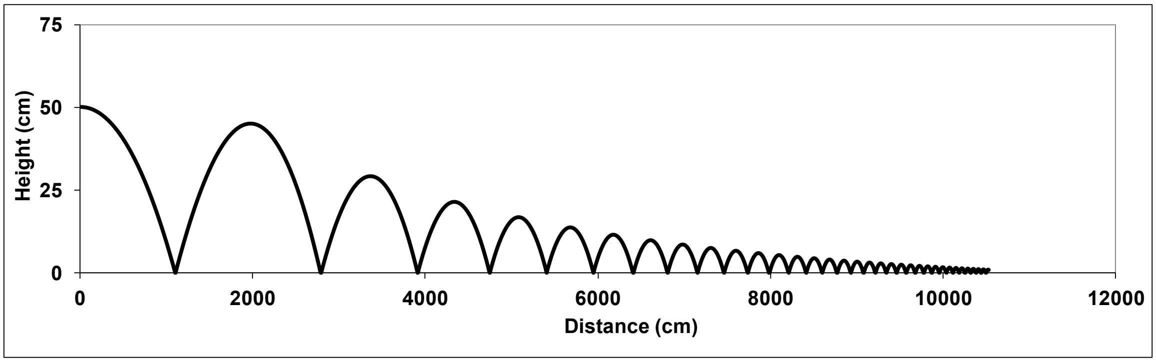


Figure 10. Stone skipping with tuned parameters: alpha 0.1, launch speed 3600 cm/sec. Note difference in horizontal and vertical distance scales, which distorts the apparent heights of the skips.

Figure 11 represents a simulation of a potential world record throw. Note the longer distance scale. The horizontal launch speed is 45 m/sec, and the tilt angle is 0.09 radian = 5.1 degrees. There are 78 skips over a distance of 135 meters. The world record set by Russell Byars in 2007 was 51 skips (BBC News, 2007). The world record set by Maxwell Steiner in 2014 was 65 skips (Truscott, 2014). Most recently Kurt Steiner achieved 88 skips (www.stoneskipping.com).

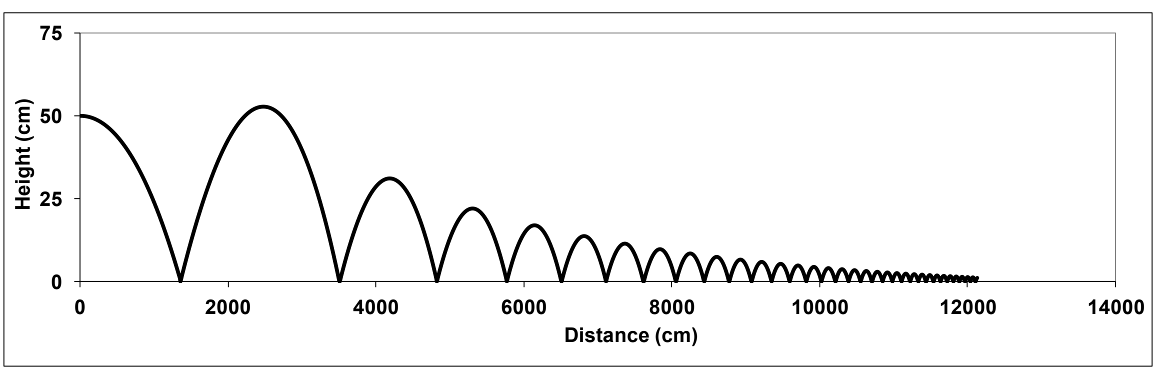


Figure 11. Stone skipping with tuned parameters for a near world record throw: Alpha 0.09 radian, launch speed 45 m/sec. No minimum skip height was required.

For such low tilt angle, high launch velocity throws, many of the terminal skips become quite low in height and require video analysis to confirm. Figure 12 shows the effects of requiring a minimum legal skip height on the number of skips counted for the simulated world record throw in Figure 11. The world record may depend on whether skips of 1 millimeter or less in height are counted.

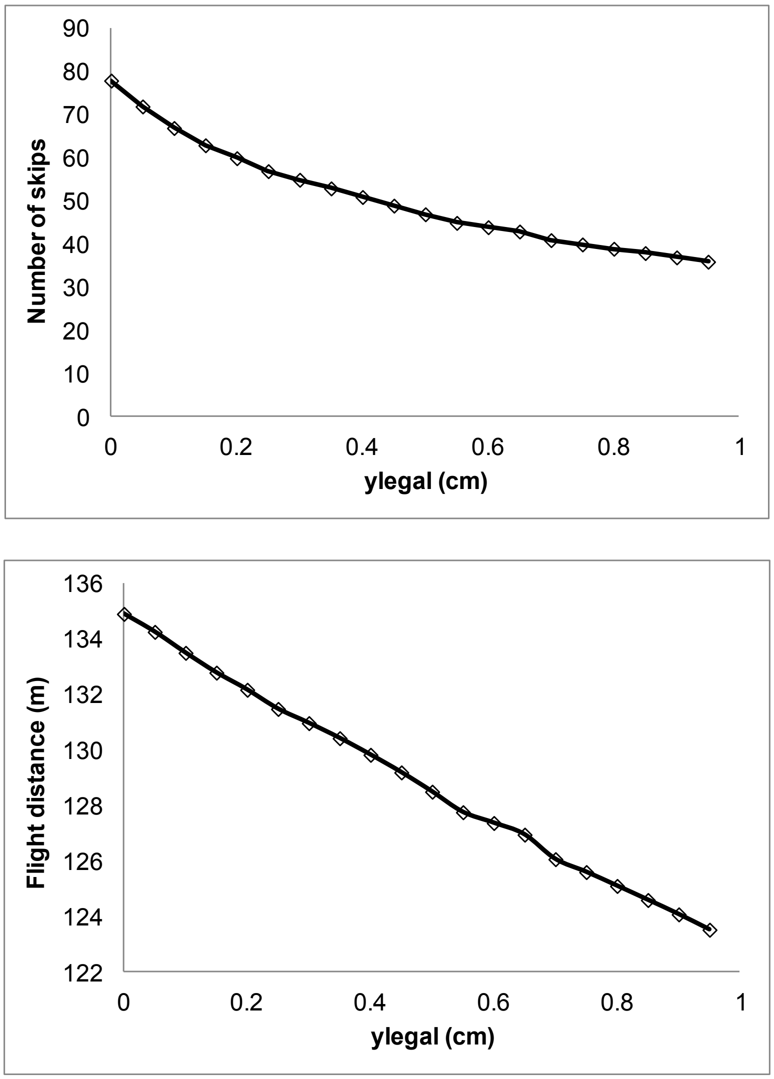


Figure 12. Effects of a required minimum skip height on figures of merit for the simulated world record throw in Figure 11.

Discussion

Skipping flat stones across the surface of a river, lake, or pond is a challenging and enjoyable pastime for children of all ages. Stones skip because collisions with the water stall velocity components normal to the tilted bottom surface of the stone, while preserving velocity components parallel to the bottom surface. Using elementary physics, based upon Newton's laws of motion, it is possible to estimate the trajectory of an idealized skipping stone using a simple computer program.

Students can practice math and science skills, as well as coding skills, by creating a mathematical model of stone skipping and then testing its predictions experimentally with actual stones and throws using ubiquitous video capture technology. The initial launch velocity, launch height, and stone angles can be determined from video frames recorded at about 30 frames per second. An alternative approach indoors would be to use a strobe light and long exposure times with an ordinary film camera. Stone weight, volume, and dimensions can be measured to specify remaining model parameters. A pendulum and a pre-calibrated blower can be used to measure the drag coefficient of a stone in the air.

There are many interesting follow-on experiments to be done to test and refine the skipmax model. Observations to compare with theoretical predictions might include the threshold launch speed for skipping, the spacing between skips, or initial rebound height as a function of throw parameters. On a small scale one could build a catapult to launch coins with spin stabilization toward a pan full of water.

The present geometric model includes several simplifications to make the problem more accessible to beginning level students. These include perfectly round stones, perfect spin stabilization, perfectly efficient stone-water collisions, a perfectly flat water surface, and absence of any friction between the stone and the water. The present model does not include all possible instances of "plunks" in which water overtops the stone on the first collision, since the model is silent regarding the exact depth of penetration of the stone into the water. Nevertheless, the skipmax model does predict realistic skipping behavior and stone trajectories, especially at higher forward speeds, without resort to three dimensional computational dynamics to characterize complex fluid-structure interactions. The model gives reasonable estimates of championship and world record throws at the high end of the performance spectrum, as well as realistic threshold launch velocities for skipping at the low end. Notably, the derived reactive normal force on the stone shows a dependence on $\sin(\theta = -\beta + \alpha)$, just as has been observed experimentally (Rosellini, 2005, p.6), rather than dependence on $\sin^2(\theta)$ or some other trigonometric function.

Previous investigators (Bocquet, 2003, Truscot, 2014) have characterized reactive inertial forces during stone-water collisions in a simple way by applying the aerodynamic drag equation for a compressible fluid such as air to the hydrodynamic problem of stone collisions with an incompressible fluid such as water, then substituting the density of water for that of air, and choosing a suitable drag coefficient, C_D. The drag equation is in essence a descriptive curve fit, with C_D being highly dependent on the shape and orientation of the solid body being tested. This approach essentially models water as a very heavy or very dense gas.

In the present analysis we do use the drag equation for estimating the effects of air resistance when the stone is in flight through the air. Air resistance, or aerodynamic drag, comes into play especially for high velocity throws at the championship level. However, water is regarded as an ideal, non-viscous, and incompressible liquid. This approach makes it easier to relate stone skipping phenomena to principles of classical Newtonian physics. It also allows computation of the flight trajectories of skipping stones on the basis of fundamental principles, predicts realistic stone trajectories, and may suggest insights for improved performance in the recreational and sometimes competitive sport of stone skipping.

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Appendix 1: correction for effects of gravity during stone-water collisions

It is possible to resolve kinetic energy into orthogonal, x-y, components thus

$$U_{total} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right) = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 = U_x + U_y\,.$$

Turning to the vertical component, U_y , the total energy of the stone emerging from the water surface, corrected for the effects of gravity, equals the total energy calculated in the absence of gravity minus the work done to lift the trailing edge of a stone of mass, m_s , vertically through a nominal distance, $R\sin(\alpha)$, from below the surface of the water to level $y = 0$. This amount of extra work is $m_s\, g\, R \sin(\alpha)$. In the absence of friction the vertical kinetic energy of the stone at the surface is

$$U_y = \frac{1}{2}m_s v_y^2 - m_s g R \sin(\alpha) = \frac{1}{2}m_s \hat{v}_y^2\,.$$

The corrected upward velocity component of the stone, $\hat{v}_y$, after having climbed out of the small potential energy well is

$$\hat{v}_y = \sqrt{v_y^2 - 2gR\sin(\alpha)}\,.$$

Here for simplicity the vertical distance, $R\sin(\alpha)$, is taken as a constant for all skips and represents half submersion of the bottom surface of the stone at the bottom of the potential energy well.