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Holographic currents and Chern-Simons terms

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Holographic currents and their associated Ward identities are derived in the framework of gravity/gauge duality. Holographic improvements of the energy-momentum tensor and R-symmetry current which are consistent with the Ward identities are displayed. The effects of specific string loop corrections to the bulk action are included as four derivative effective Lagrangian terms and their contributions to the trace and R-symmetry anomalies of the boundary theory are determined. As an example, the construction is applied to the $\mathcal{N} = 2$ conformal supergravity which is taken to be dual to a boundary $SU(N) \times SU(N)$, $\mathcal{N} = 1$ superconformal field theory.

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I. INTRODUCTION

Type IIB string theory on an $\mathrm{AdS}_5 \times Y_5$ background, where Y_5 is a Sasaki-Einstein manifold, for large radii and with a stack of N-$D3$-branes at the tip of the conical singularity is approximated by a $\mathcal{N} = 2$ conformal supergravity which, in turn, is dual to a $\mathcal{N} = 1$ superconformal field theory on the Minkowski space-time M_4 boundary of the AdS_5 space [1–3]. The values of the central charges of the superconformal field theory (SCFT), a and c, designate different boundary field theories dual to the bulk AdS supergravity [3]. Indeed, the energy-momentum tensor trace and the R-symmetry current anomalies calculated holographically in the supergravity theory and obtained in terms of the central charges in the boundary field theory have been shown to match in leading order [4], while imposing subleading order matching provides a measure of the string loop corrections to the supergravity action through the value of the coefficient governing the higher derivative action terms [5–10].

The trace anomaly of the boundary $\mathcal{N} = 1$ SCFT in the presence of background gravitational and $U(1)_R$ gauge fields and in the absence of sources related to any other global symmetries has the general form [3]

$$\theta^\mu{}_\mu = \frac{1}{16\pi^2}[cW^2 - aR^2_{\mathrm{GB}}] + \frac{c}{6\pi^2}F_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

where $W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + 1/(d-2)[g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}] - 1/(d-2)(d-1)(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma})R$ is the Weyl tensor in d-dimensional space-time with $R_{\mu\nu\rho\sigma}$ the Riemann tensor and R^2_{GB} the Einstein-Gauss-Bonnet invariant $R^2_{\mathrm{GB}} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ in all dimensions. $F_{\mu\nu}$ is the $U(1)_R$ gauge field strength tensor. Expanding the gravitational contributions to the anomaly yields

$$\theta^\mu{}_\mu = \frac{1}{16\pi^2}\Big[2(2a - c)R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}(c - 3a)R^2 + (c - a)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big] + \frac{c}{6\pi^2}F_{\mu\nu}F^{\mu\nu}. \qquad (2)$$

For an R current defined so that it lies in the same $\mathcal{N} = 1$ supersymmetry (SUSY) supercurrent multiplet as the energy-momentum tensor and with the trace and R-current anomalies also appearing in their own SUSY anomaly multiplet, then the R-symmetry current anomaly is given in terms of the same a and c coefficients and has the form [3]

$$\partial_\mu(\sqrt{-g}R^\mu) = \frac{1}{24\pi^2}(c - a)\epsilon^{\mu\nu\rho\sigma}R^{\xi\zeta}{}_{\mu\nu}R_{\xi\zeta\rho\sigma} + \frac{1}{18\pi^2}(5a - 3c)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \qquad (3)$$

In order for these currents, along with the SUSY current, to lie in the same $\mathcal{N} = 1$ supermultiplet, they must have appropriate normalization and improvements. If the R charges are defined differently, say by a rescaling by r, $R \to rR$, then the contribution to the $F\tilde{F}$ term will be scaled as r^3 while the $R\tilde{R}$ contribution to the anomaly triangle will be scaled only linearly with r. Furthermore, if the overall normalization of the R current is also scaled so that $R^\mu \to \lambda R^\mu$, then the two terms on the right-hand side of the divergence Eq. (3) get modified by λr and λr^3 factors, respectively. These factors will be necessary in matching the holographically defined boundary currents to the conventional SCFT supercurrent components.

The field theory determination of these anomaly coefficients is [3]

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$$a = \frac{3}{32}(3\,\mathrm{Tr}R^3 - \mathrm{Tr}R), \qquad c = \frac{1}{32}(9\,\mathrm{Tr}R^3 - 5\,\mathrm{Tr}R), \quad (4)$$

where R is the R-symmetry charge of the fields and the trace is over all chiral fermion fields. Inverting these expressions yields the linear and cubic nature of their triangle contributions $\mathrm{Tr}[R] = -16(c-a)$ and $\mathrm{Tr}[R^3] = 16(5a-3c)/9$. The leading N^2 contribution to the anomaly coefficients implies that $c = a$ for the class of models considered here. Hence, the gravitational field contribution to the trace anomaly occurs only through the Ricci tensor and scalar curvature terms and not the Riemann tensor term, while there is no leading order gravitational contribution to the R-symmetry anomaly. The subleading large N contributions to both anomalies are reflected not only in the $(c-a) \neq 0$ difference of the central charges but also in the subleading corrections to each individually.

On the supergravity side, the leading order contributions to the anomalies are given by the 2 derivative terms in the bosonic part of the supersymmetric action

$$\Gamma_{\mathrm{Leading}} = \int d^5x\, \det E\Big[\frac{1}{2\kappa}R + \Lambda - \frac{Z_A}{4}F_{MN}F^{MN} + CE^{MNRST}F_{MN}F_{RS}A_T\Big], \quad (5)$$

where $\det E E^{MNRST} = \epsilon^{MNRST}$ and E^A_M is the gravitational fünfbein. The AdS_5 gravitational constant is $2\kappa = 16\pi G_5$ and the cosmological constant is $\Lambda = -6/\kappa R_5^2$ with R_5 the AdS_5 radius of curvature. The $U(1)_R$ field strength tensor $F_{MN} = \partial_M A_N - \partial_N A_M$ while the rescaled gauge field wave function normalization factor Z_A is determined by matching the holographic energy-momentum tensor anomaly to that of the boundary SCFT as given in Eq. (2). The $\mathcal{N} = 2$ SUSY determines the Chern-Simons coefficient (see the Appendix)

$$C = \frac{(Z_A 2\kappa)^{3/2}}{24\kappa\sqrt{3}}. \quad (6)$$

The leading order trace anomaly was found in Ref. [4] in the absence of the $U(1)_R$ field strength. After its inclusion, the trace anomaly is

$$\theta^\mu{}_\mu = \left(\frac{1}{8}\frac{R_5^3}{\kappa}\right)\Big[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big] + \left(\frac{R_5 Z_A}{4}\right)F_{\mu\nu}F^{\mu\nu}. \quad (7)$$

Comparing this to Eq. (2) yields the leading order results that $a = c$ and identifies the gravitational constant as $R_5^3/\kappa = a/\pi^2$. Requiring consistency for the F^2 contribution yields that the wave function normalization factor satisfies $Z_A R_5 = 2a/3\pi^2$. The direct variation of the action with respect to R-symmetry transformations, $\delta_R A_M = \partial_M \Omega$, yields the divergence of the holographic R current in leading order as

$$\delta_R \Gamma_{\mathrm{Leading}} = -C\int_{M_4} d^4x\,\Omega(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \quad (8)$$

AdS/CFT duality implies that the holographic R-current anomaly coefficient C is

$$\lambda r^3 C = \frac{1}{18\pi^2}(5a - 3c) = \frac{a}{9\pi^2}. \quad (9)$$

Inserting the definition of C from Eq. (6) in terms of a and c requires the normalization of the charges and current to obey $\lambda r^3 = 3$ in order for the R current to belong to the $\mathcal{N} = 1$ supercurrent multiplet.

The subleading contributions to the anomalies arise due to higher order 4 derivative terms in the action coming from string loop effects along with subleading corrections to the coefficients of the 2 derivative terms [5–10]. The R-symmetry anomaly arises directly from the variation of the five-dimensional mixed $U(1)_R$ gauge field and gravitational field Chern-Simons term as

$$\Gamma_{RRA} = -D\int d^5x\, \det E E^{MNRST} R^X{}_{YMN} R^Y{}_{XRS} A_T, \quad (10)$$

where $D \equiv -\gamma R_5^3/3$ and γ sets the scale of the SUSY completion of the 4 derivative contributions to the action. In the notation of Ref. [10] $16\pi G_5\gamma = c_2/192$. The $U(1)_R$ transformation, $\delta_R A_T = \partial_T \Omega$, yields the gravitational contribution to the R-symmetry anomaly as

$$\delta_R \Gamma_{RRA} = -D\int_{M_4} d^4x\,\Omega(x)\epsilon^{\mu\nu\rho\sigma}R^{\xi\zeta}{}_{\mu\nu}R_{\xi\zeta\rho\sigma}. \quad (11)$$

Exploiting AdS/CFT duality, the boundary SCFT result for the gravitational contribution to the R anomaly is such that the holographic R current and R charges must include the rescaling by λ and r in order for the R current to fit in the supercurrent with the holographic energy-momentum tensor, so that

$$\lambda r D = \frac{1}{24\pi^2}(c - a). \quad (12)$$

As in leading order case, the 2-derivative pure $U(1)_R$ gauge Chern-Simons term is responsible for the gauge field contribution to the R anomaly. With an AdS action term of the form [11,12],

$$\Gamma_{FFA} = C\int d^5x\, \det E E^{MNRST} F_{MN}F_{RS}A_T, \quad (13)$$

where C is fixed by supersymmetry [10] through subleading order (see the Appendix) to be $C = 2R_5^3[1 + 12\gamma(2\kappa)/R_5^2]/27(2\kappa)$. As previously, its contribution to the R anomaly is

$$\delta_R \Gamma_{FFA} = -C\int_{M_4} d^4x\,\Omega(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \quad (14)$$

AdS/CFT duality implies that the R-anomaly coefficient C is

$$\lambda r^3 C = \frac{1}{18\pi^2}(5a - 3c). \tag{15}$$

Note that $(c - a)/a = 8D/(3r^2C + 12D) = 8D/3r^2C$ to subleading order.

On the other hand, the SUSY completion of the mixed Chern-Simons term requires many more 4 derivative terms to be added to the supergravity action [5–10] albeit with relative coefficients determined by the supergravity transformations and overall coefficient fixed by γ [10]. In the SUSY case discussed here, the Weyl tensor squared term has been shown [10] to be the source of the corresponding subleading pure gravitational corrections to the trace anomaly. The two independent gravitational tensor couplings to the $U(1)_R$ field strength bilinear along with the three field strength covariant derivative bilinears provide the subleading field strength squared corrections as shown in the Appendix for arbitrary coefficients. (Indeed, the coupling of the Weyl tensor to the field strength, $F_{MN}F_{RS}W^{MNRS}$, does not contribute to the anomaly while the remaining couplings to the Ricci tensor and scalar curvature do contribute.) For the conformal supergravity values of the coefficients [10], this action reads

$$\Gamma_{\text{Subleading}} = \gamma \int d^5x \det E\Big[W_{MNRS}W^{MNRS} + \frac{R_5}{3}E^{MNRST}R^X{}_{YMN}R^Y{}_{XRS}A_T + \frac{2}{9}R_5^2 F_{MN}F^{MN}R - \frac{32}{9}R_5^2 F_{MR}F_N{}^R R^{MN} - \frac{16}{3}R_5^2 F_{MN}\nabla^N\nabla_R F^{MR} - \frac{8}{3}R_5^2\nabla^M F^{NR}\nabla_M F_{NR} - \frac{8}{3}R_5^2\nabla^M F^{NR}\nabla_N F_{MR} + \text{SUSY Completion}\Big]. \tag{16}$$

The remaining SUSY completion terms do not contribute to either anomaly. The holographic trace anomaly in the general non-SUSY case is considered in the Appendix. Specializing the results of the Appendix to the $\mathcal{N} = 2$ conformal supergravity case, the holographic trace anomaly has the form

$$\theta^\mu{}_\mu = \left(-\frac{1}{24}\frac{R_5^3}{\kappa} - \frac{1}{3}R_5\gamma\right)R^2 + \left(\frac{1}{8}\frac{R_5^3}{\kappa} + 2R_5\gamma\right)R_{\mu\nu}R^{\mu\nu} - R_5\gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \left(\frac{R_5 Z_A}{4} - \frac{160}{9}R_5\gamma\right)F_{\mu\nu}F^{\mu\nu}. \tag{17}$$

Using AdS/CFT duality to match the trace anomaly in the SCFT form, Eq. (2), to the holographic trace anomaly, Eq. (17), leads to the values for the gravitational constant κ and the scale of the subleading corrections γ

$$\frac{R_5^3}{\kappa} = \frac{a}{\pi^2} \qquad \gamma R_5 = -\frac{(c-a)}{16\pi^2}. \tag{18}$$

Consistency also requires the value of the coefficient of the Riemann tensor squared to be given as $(c - a)/16\pi^2$, in agreement with the above results. In addition, it is in agreement with the conformal supergravity symmetry constraints relating the subleading gravitational contribution to the trace anomaly to just the Weyl tensor squared as given in Eq. (16). The $U(1)_R$ field strength contribution to the anomaly involves only the coefficient c. This part of the trace anomaly fixes the wave function renormalization of the $U(1)_R$ gauge field to be

$$Z_A R_5 = \frac{2}{3}\frac{a}{\pi^2} + \frac{10}{9}\frac{(c-a)}{\pi^2}. \tag{19}$$

The $\mathcal{N} = 2$ supergravity constraints require the supersymmetric values of C and D to be given by

$$C = \frac{2R_5^3}{27(2\kappa)}\left[1 + 12\frac{\gamma(2\kappa)}{R_5^2}\right] = \frac{1}{3}\frac{(5a-3c)}{18\pi^2}, \qquad D = -\frac{\gamma R_5}{3} = \frac{1}{2}\frac{(c-a)}{24\pi^2}, \tag{20}$$

where the second equality employs the relationship of the supergravity coefficients to a and c from the trace anomaly above. Turning to the R-symmetry AdS/CFT anomaly matching Eqs. (12) and (15), the Chern-Simons coefficients imply the need to normalize the R charges and current so that $r = \sqrt{3/2}$ and $\lambda = \sqrt{8/3}$ for the holographic R current to belong to the supercurrent.

As an example, taking the above mentioned Sasaki-Einstein manifold to be $Y_5 = T^{1,1}$, these anomaly matching results can be applied to the large N strong 't Hooft coupling superconformal $SU(N) \times SU(N)$ SUSY gauge theory with a pair of bi-fundamental $(N, \bar{N})$ chiral matter fields A_1 and A_2 and a pair of anti–bi-fundamental $(\bar{N}, N)$ chiral matter fields B_1 and B_2 [12–14]. The central charges calculated in the boundary field theory determine the two supergravity parameters related to the gravitational constant κ and γ which sets the scale for the four derivative terms. The A and B chiral superfields have $R = 1/2$, so the $\tilde{A}$ and $\tilde{B}$ matter fermions have $R = -1/2$. The $2(N^2 - 1)$ gluinos have $R = 1$. Hence, the boundary SCFT has

$$a = \frac{27}{64}N^2 - \frac{3}{8}, \qquad c = \frac{27}{64}N^2 - \frac{1}{4}, \tag{21}$$

with $(c - a) = 1/8$. Thus the gravitational constant is seen to be

$$\frac{R_5^3}{\kappa} = \frac{27}{64\pi^2}N^2\left(1 - \frac{8}{9N^2}\right), \tag{22}$$

and the wave function renormalization Z_A is found to be

$$Z_A R_5 = \frac{9}{32\pi^2} N^2 \left(1 - \frac{32}{81N^2}\right), \tag{23}$$

while the coefficients of the 4 derivative terms are given by γ and are found to be subleading

$$\gamma R_5 = -\frac{1}{128\pi^2}. \tag{24}$$

The anomaly matching results aside, the purpose of this paper is to construct the energy-momentum tensor and R-symmetry current [15] in the case that the subleading mixed Chern-Simons term is added to the leading order supergravity action. This truncated 4 derivative action will provide the leading gravitational contributions to each anomaly (leading N^2 contributions with $c = a$ in the trace anomaly case and subleading N^0 contributions with $c \neq a$ in the R-anomaly case). The pure $U(1)_R$ gauge field contributions are leading order N^2 terms for each anomaly. In Sec. II, the near-boundary field equations are solved and the abbreviated action is holographically renormalized through the addition of boundary counterterms [16–18]. In Sec. III the Brown-York energy-momentum tensor [19] and R-symmetry current are constructed via the boundary source variational principle for the action [4,20]. The covariant divergence of the energy-momentum tensor has the Brown-York form and yields the diffeomorphism invariance of the renormalized action Ward identity [21], thus providing its interpretation as a boundary energy-momentum tensor. Improvements to both currents are constructed consistent with the Ward identities and trace and R-symmetry anomalies. They are shown to follow from the variation of boundary action improvement terms.

II. HOLOGRAPHIC SUPERGRAVITY ACTION

The bosonic part of the bulk AdS_5 $\mathcal{N} = 2$ conformal supergravity action including the single 4 derivative mixed Chern-Simons term is given by

$$\begin{aligned}\Gamma = \int d^5x \det E \Big[&\frac{1}{2\kappa} R + \Lambda - \frac{Z_A}{4} F_{MN} F^{MN} \\ &+ C E^{MNRST} F_{MN} F_{RS} A_T \\ &- D E^{MNRST} R^X{}_{YMN} R^Y{}_{XRS} A_T \Big] \\ &+ \frac{1}{\kappa} \int_{\rho=\epsilon} d^4x \sqrt{-\gamma} K(\gamma) + \Gamma_{\text{Counterterms}},\end{aligned} \tag{25}$$

where

$$ds^2 = \frac{R_5^2}{\rho} \left[g_{\mu\nu}(x,\rho) dx^\mu dx^\nu - \frac{d\rho^2}{4\rho} \right] \tag{26}$$

is the Fefferman-Graham metric for the (asymptotic) AdS_5 space of radius R_5. Here $\gamma_{\mu\nu} = (R_5^2/\rho) g_{\mu\nu}(x,\rho)|_{\rho=\epsilon}$ is the induced metric on the fixed $\rho = \epsilon$ hypersurface homeomorphic to the M_4 boundary. The extrinsic curvature on that surface is given by the gradient of the normal vector there yielding $K_{\mu\nu} = (\rho/R_5) \partial \gamma_{\mu\nu}/\partial\rho|_{\rho=\epsilon}$. The counterterm boundary action $\Gamma_{\text{Counter-terms}} = \int_{\rho=\epsilon} d^4x \sqrt{-\gamma} B(\gamma)$ is determined through holographic renormalization and normalization conditions as discussed at the end of this section. The Riemann tensor is given in terms of the Christoffel symbol Γ^R_{MN} as

$$R^R{}_{SMN} = \partial_N \Gamma^R_{SM} - \partial_M \Gamma^R_{SN} + \Gamma^L_{SM} \Gamma^R_{LN} - \Gamma^L_{SN} \Gamma^R_{LM}, \tag{27}$$

with $\Gamma^R_{MN} = \frac{1}{2} g^{RS} [\partial_N g_{SM} + \partial_M g_{SN} - \partial_S g_{MN}]$. The $U(1)$ R-symmetry gauge field A_M has field strength tensor $F_{MN} = \nabla_M A_N - \nabla_N A_M = \partial_M A_N - \partial_N A_M$. The world tensor $E^{MNRST} = \epsilon^{ABCDE} E_A^{-1M} E_B^{-1N} E_C^{-1R} E_D^{-1S} E_E^{-1T} = (1/\det E) \epsilon^{MNRST}$ with the fünfbein

$$E_M{}^A = \begin{pmatrix} \frac{R_5}{\sqrt{\rho}} e_\mu{}^a(x,\rho) & 0 \\ 0 & \frac{R_5}{2\rho} \end{pmatrix}_{MA}, \tag{28}$$

where $g_{\mu\nu}(x,\rho) = e_\mu{}^a(x,\rho) \eta_{ab} e_\nu{}^b(x,\rho)$ with $\eta_{ab} = (+,-,-,-)_{ab}$.

Varying the bulk action with fixed boundary conditions for the fields yields the field equations. The Einstein equation has the form

$$R_{MN} - \tfrac{1}{2} g_{MN} R = g_{MN} \kappa \Lambda - \kappa T_{MN}, \tag{29}$$

with the Ricci tensor defined by $R_{MN} \equiv R^R{}_{MNR}$ and the scalar curvature given by $R \equiv g^{MN} R_{MN}$. The bulk energy-momentum tensor T_{MN} is obtained as

$$T_{MN} = T^{\text{Maxwell}}_{MN} + \nabla_X \Theta^X_{MN}, \tag{30}$$

with the R-symmetry gauge field's contribution to the energy-momentum tensor given by the Maxwell symmetric form

$$T^{\text{Maxwell}}_{MN} = Z_A F_{MR} F^R{}_N - g_{MN} \left(-\frac{Z_A}{4} F_{RS} F^{RS} \right), \tag{31}$$

while the mixed gauge-gravity Chern-Simons contribution to the energy-momentum tensor is

$$\Theta^X_{MN} = 2 D E^{QRSTU} F_{TU} (g_{QM} R^X{}_{NRS} + g_{QN} R^X{}_{MRS}). \tag{32}$$

The covariant derivative is defined according to

$$\nabla_M T_N{}^R \equiv \partial_M T_N{}^R - \Gamma^P_{MN} T_P{}^R + \Gamma^R_{MP} T_N{}^P. \tag{33}$$

The Maxwell equation is generalized to include the Chern-Simons terms so that

$$\begin{aligned} Z_A \nabla_M F^{ML} &+ 3 C E^{MNRSL} F_{MN} F_{RS} \\ &= D E^{MNRSL} R^X{}_{YMN} R^Y{}_{XRS}. \end{aligned} \tag{34}$$

Using the relations $\nabla_M E_N{}^B = 0 = \nabla_M E^{PQRST}$, this can be written as

$$Z_A \partial_M(\det EF^{ML}) + 3C \det EE^{MNRSL} F_{MN} F_{RS} = D \det EE^{MNRSL} R^X{}_{YMN} R^Y{}_{XRS}. \tag{35}$$

Applying the Fefferman-Graham metric, the Einstein equations can be expanded to have the forms (here the covariant derivative is made utilizing $g_{\mu\nu}$)

$$\begin{aligned}\rho[&2g''_{\mu\nu} - 2g'_{\mu\lambda} g^{\lambda\rho} g'_{\rho\nu} + g'_{\mu\nu} g^{\rho\sigma} g'_{\rho\sigma}] \\ &+ R_{\mu\nu}(g) - 2g'_{\mu\nu} - g_{\mu\nu} g^{\rho\sigma} g'_{\rho\sigma} \\ &= -\kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \kappa g^{\rho\sigma} T_{\rho\sigma} \\ &- \rho g_{\mu\nu}\left[\frac{1}{3\rho^2}(6 + R_5^2 \kappa\Lambda) + \frac{1}{6\rho}\kappa g^{\rho\sigma} T_{\rho\sigma} + \frac{4}{3}\kappa T_{44}\right],\end{aligned} \tag{36}$$

$$\begin{aligned}g^{\rho\sigma} g''_{\rho\sigma} &- \frac{1}{2}(g^{\rho\sigma} g'_{\sigma\mu} g^{\mu\nu} g'_{\nu\rho}) \\ &= \frac{1}{3\rho^2}[6 + R_5^2 \kappa\Lambda] + \frac{1}{6\rho}\kappa g^{\mu\nu} T_{\mu\nu} + \frac{4}{3}\kappa T_{44},\end{aligned} \tag{37}$$

$$\nabla_\mu(g^{\rho\sigma} g'_{\rho\sigma}) - \nabla^\rho g'_{\rho\mu} = 2\kappa T_{\mu 4}, \tag{38}$$

where the prime indicates differentiation with respect to ρ, so that, for example, $g'_{\mu\nu} \equiv \partial g_{\mu\nu}/\partial\rho$. Likewise the Maxwell equations have the detailed structure (here the permutation tensor $E^{\mu\nu\rho\sigma}(g) = \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$)

$$\begin{aligned}2R_5 Z_A \partial_\mu[\det e g^{\mu\nu} A'_\nu] &+ 3C \det e E^{\mu\nu\rho\sigma}(g) F_{\mu\nu}(A) F_{\rho\sigma}(A) \\ &- D \det e E^{\mu\nu\rho\sigma}(g) R^\kappa{}_{\lambda\mu\nu}(g) R^\lambda{}_{\kappa\rho\sigma}(g) \\ &= +4\rho D \det e E^{\mu\nu\rho\sigma}(g) R^{\lambda\kappa}{}_{\mu\nu}(g) g'_{\kappa\rho} g'_{\lambda\sigma} \\ &+ 2\rho D \det e E^{\mu\nu\rho\sigma}(g)(\nabla_\nu g'_{\mu\kappa} \\ &- \nabla_\mu g'_{\nu\kappa}) g^{\kappa\lambda}(\nabla_\sigma g'_{\lambda\rho} - \nabla_\rho g'_{\lambda\sigma}),\end{aligned} \tag{39}$$

$$\begin{aligned}\rho\frac{\partial}{\partial\rho}[Z_A \det e g^{\lambda\rho} A'_\rho] - \frac{1}{4}\partial_\mu[Z_A \det e g^{\mu\rho} g^{\lambda\sigma} F_{\rho\sigma}(A)] &= \frac{6\rho}{R_5} C \det e E^{\tau\rho\sigma\lambda}(g) A'_\tau F_{\rho\sigma}(A) \\ &- \frac{2\rho}{R_5} D \det e E^{\nu\rho\sigma\lambda}(g)\Big[g^{\mu\tau}\Big(-\frac{1}{\rho} g_{\mu\nu} - 2\rho g''_{\mu\nu} + \rho g^{\pi\kappa} g'_{\kappa\mu} g'_{\pi\nu}\Big)(\nabla_\sigma g'_{\tau\rho} - \nabla_\rho g'_{\tau\sigma}). \\ &+ \Big(\frac{1}{2} g^{\tau\pi}\nabla_\nu g'_{\pi\mu} - \Gamma'^{\tau}_{\mu\nu}(g)\Big)\Big(R^\mu{}_{\tau\rho\sigma}(g) + \frac{1}{\rho} g_{\tau\rho}\delta^\mu_\sigma - g'_{\tau\rho}\delta^\mu_\sigma - \frac{1}{\rho} g_{\tau\sigma}\delta^\mu_\rho + g'_{\tau\sigma}\delta^\mu_\rho \\ &- g_{\tau\rho} g^{\mu\pi} g'_{\pi\sigma} + \rho g'_{\tau\rho} g^{\mu\pi} g'_{\pi\sigma} + g_{\tau\sigma} g^{\mu\pi} g'_{\pi\rho} - \rho g'_{\tau\sigma} g^{\mu\pi} g'_{\pi\rho}\Big)\Big].\end{aligned} \tag{40}$$

The boundary currents have an expectation value in the presence of the background gravitational and $U(1)_R$ gauge fields given in terms of the asymptotic behavior of the bulk fields at the M_4 boundary of the AdS_5 space. Consequently, the field equations need only be solved close to the boundary at $\rho = 0$ in order to determine these one point functions. The $U(1)_R$ gauge field $A_\mu(x, \rho)$ has the asymptotic form in ρ close to the boundary given by

$$A_\mu(x, \rho) = A^{(0)}_\mu(x) + \rho A^{(2)}_\mu(x) + \rho \ln(\rho/R_5^2) B_\mu(x). \tag{41}$$

Likewise for $\rho \sim 0$, the gravitational field has the behavior

$$\begin{aligned}g_{\mu\nu}(x, \rho) &= g^{(0)}_{\mu\nu}(x) + \rho g^{(2)}_{\mu\nu}(x) + \rho^2 g^{(4)}_{\mu\nu}(x) \\ &+ \rho^2 \ln(\rho/R_5^2) h_{\mu\nu}(x).\end{aligned} \tag{42}$$

Substituting these expansions into the Maxwell equations results in their right-hand sides vanishing as $\rho \to 0$ while the fields B_μ and the covariant divergence of $A^{(2)}_\mu$ being determined in terms of $A^{(0)}_\mu$. The transverse part of $A^{(2)}_\mu$ is undetermined. It corresponds to the other linearly independent solution to the second order Dirichlet problem and it appears as the subleading asymptotic behavior of $A_\mu(x, \rho)$. A second boundary condition deeper into the bulk would be needed for its specification. Using $g^{(0)}$ to raise and lower indices, the $U(1)_R$ gauge field has the asymptotic solution

$$\begin{aligned}B^\mu(x) &= \tfrac{1}{4}\nabla^{(0)}_\nu F^{(0)\nu\mu} \\ 2R_5 Z_A \nabla^{(0)}_\mu A^{(2)\mu}(x) &= -3CE^{\mu\nu\rho\sigma}(g^{(0)}) F^{(0)}_{\mu\nu} F^{(0)}_{\rho\sigma} \\ &+ DE^{\mu\nu\rho\sigma}(g^{(0)}) R^{(0)\kappa}{}_{\lambda\mu\nu} R^{(0)\lambda}{}_{\kappa\rho\sigma}.\end{aligned} \tag{43}$$

In order to find the asymptotic solution to the Einstein equations, the bulk energy-momentum tensor must be expanded in terms of ρ where it is found from Eqs. (30)–(32) that

$$\begin{aligned}T_{\mu\nu}(x, \rho) &= \rho T^{(2)}_{\mu\nu}(x) + \cdots, \\ T_{44}(x, \rho) &= T^{(0)}_{44}(x) + \cdots, \\ T_{\mu 4}(x, \rho) &= \rho T^{(2)}_{\mu 4}(x) + \rho \ln(\rho/R_5^2)\tilde{T}_{\mu 4}(x) + \cdots,\end{aligned} \tag{44}$$

with

$$T^{(2)}_{\mu\nu}(x) = \frac{1}{R_5^2}\Big[Z_A F^{(0)}_{\mu\rho}F^{(0)\rho}{}_{\nu} + \frac{Z_A}{4}g^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma}F^{(0)\rho\sigma}\Big], \qquad T^{(0)}_{44}(x) = -\frac{Z_A}{16R_5^2}F^{(0)}_{\rho\sigma}F^{(0)\rho\sigma},$$
$$T^{(2)}_{\mu 4}(x) = -\frac{1}{R_5^2}Z_A F^{(0)}_{\mu\rho}g^{(0)\rho\sigma}[A^{(2)}_{\sigma} + B_{\sigma}] + \frac{1}{R_5^3}DE^{(0)\rho\sigma\pi\lambda}F^{(0)}_{\pi\lambda}[\nabla^{(0)}_{\rho}g^{(2)}_{\mu\sigma} - \nabla^{(0)}_{\sigma}g^{(2)}_{\mu\rho}]$$
$$- \frac{1}{R_5^3}DE^{(0)\rho\sigma\pi\lambda}\nabla^{(0)}_{\alpha}[F^{(0)}_{\pi\lambda}(R^{(0)\alpha}{}_{\mu\rho\sigma} - g^{(0)}_{\mu\rho}g^{(0)\alpha\xi}g^{(2)}_{\xi\sigma} + g^{(0)}_{\mu\sigma}g^{(0)\alpha\xi}g^{(2)}_{\xi\rho})],$$
$$\tilde{T}_{\mu 4}(x) = -\frac{1}{R_5^2}Z_A F^{(0)}_{\mu\rho}g^{(0)\rho\sigma}B_{\sigma}. \tag{45}$$

Note that $g^{(0)\rho\sigma}T^{(2)}_{\rho\sigma} = 0$ and $\nabla^{(0)\mu}T^{(2)}_{\mu\nu} = 4\tilde{T}_{\nu 4}$. It is useful to consider the Maxwell contribution to $T^{(2)}_{\mu 4}$ separately

$$T^{(2)\,\mathrm{Maxwell}}_{\mu 4} = -\frac{1}{R_5^2}Z_A F^{(0)}_{\mu\rho}g^{(0)\rho\sigma}[A^{(2)}_{\sigma} + B_{\sigma}], \tag{46}$$

and hence the mixed Chern-Simons contribution is just $T^{(2)\mathrm{D}}_{\mu 4} = T^{(2)}_{\mu 4} - T^{(2)\,\mathrm{Maxwell}}_{\mu 4}$.

The metric coefficients $g^{(2)}$ and h can be determined from the field equation expansion in terms of the boundary metric $g^{(0)}$ and the boundary R-symmetry gauge field $A^{(0)}_{\mu}$ while only the trace and divergence of $g^{(4)}$ is determined in terms of $g^{(0)}$ and $A^{(0)}_{\mu}$ by the near-boundary expansion. The remaining components of $g^{(4)}$ being fixed by a needed second boundary condition deeper into the bulk for these second order differential equations. From Eq. (36) and the vanishing of the bulk energy-momentum tensor $T_{\mu\nu}$ as $\rho \sim 0$, it is found that $\kappa\Lambda R_5^2 = -6$ and

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\Big[R^{(0)}_{\mu\nu} - \frac{1}{6}g^{(0)}_{\mu\nu}R^{(0)}\Big], \tag{47}$$

along with $g^{(0)\mu\nu}g^{(2)}_{\mu\nu} = (1/6)R^{(0)}$. Expanding Eq. (37) immediately yields $g^{(0)\mu\nu}h_{\mu\nu} = 0$ and

$$g^{(0)\mu\nu}g^{(4)}_{\mu\nu} = \frac{1}{4}(g^{(0)\rho\sigma}g^{(2)}_{\sigma\lambda}g^{(0)\lambda\kappa}g^{(2)}_{\kappa\rho}) + \frac{2}{3}\kappa T^{(0)}_{44}$$
$$= \frac{1}{16}[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{2}{9}R^{(0)2}] - \frac{\kappa Z_A}{24R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}. \tag{48}$$

Returning to Eq. (36), $h_{\mu\nu}$ is obtained

$$h_{\mu\nu} = \frac{1}{2}g^{(2)}_{\mu\lambda}g^{(0)\lambda\rho}g^{(2)}_{\rho\nu} - \frac{1}{8}g^{(0)}_{\mu\nu}[g^{(0)\rho\sigma}g^{(2)}_{\sigma\lambda}g^{(0)\lambda\kappa}g^{(2)}_{\kappa\rho}] - \frac{1}{4}R^{(2)}_{\mu\nu} - \frac{1}{4}\kappa T^{(2)}_{\mu\nu} = \frac{1}{8}\Big[R^{(0)}_{\mu\rho}R^{(0)\rho}{}_{\nu} - \frac{1}{3}R^{(0)}R^{(0)}_{\mu\nu} - \frac{1}{36}g^{(0)}_{\mu\nu}R^{(0)2}\Big]$$
$$- \frac{1}{32}g^{(0)}_{\mu\nu}\Big[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{2}{9}R^{(0)2}\Big] - \frac{1}{8}\Big[R^{(0)}_{\mu\rho}R^{(0)\rho}{}_{\nu} - R^{(0)\rho\sigma}R^{(0)}_{\sigma\mu\nu\rho} + \frac{1}{6}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}R^{(0)} - \frac{1}{2}\nabla^{(0)}_{\rho}\nabla^{(0)\rho}R^{(0)}_{\mu\nu}$$
$$+ \frac{1}{12}g^{(0)}_{\mu\nu}\nabla^{(0)}_{\rho}\nabla^{(0)\rho}R^{(0)}\Big] - \frac{1}{4R_5^2}\kappa\Big[Z_A F^{(0)}_{\mu\rho}F^{(0)\rho}{}_{\nu} + \frac{Z_A}{4}g^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma}F^{(0)\rho\sigma}\Big], \tag{49}$$

where the Ricci tensor has been expanded close to the boundary as $R_{\mu\nu}(g) = R^{(0)}_{\mu\nu}(g^{(0)}) + \rho R^{(2)}_{\mu\nu}(g^{(0)}) + \cdots$ and likewise for the scalar curvature $R(g) = R^{(0)}(g^{(0)}) + \rho R^{(2)}(g^{(0)}) + \cdots$. The expansion coefficients are given by

$$R^{(2)}_{\mu\nu}(g^{(0)}) = \frac{1}{2}\Big[R^{(0)}_{\mu\rho}R^{(0)\rho}{}_{\nu} - R^{(0)\rho\sigma}R^{(0)}_{\sigma\mu\nu\rho} + \frac{1}{6}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}R^{(0)} - \frac{1}{2}\nabla^{(0)}_{\rho}\nabla^{(0)\rho}R^{(0)}_{\mu\nu} + \frac{1}{12}g^{(0)}_{\mu\nu}\nabla^{(0)}_{\rho}\nabla^{(0)\rho}R^{(0)}\Big]$$
$$R^{(2)}(g^{(0)}) = -\frac{1}{2}\Big[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{6}R^{(0)2}\Big]. \tag{50}$$

Note that $g^{(0)\mu\nu}h_{\mu\nu} = 0$ as found above. Finally, Eq. (38) yields the divergence of the coefficients $g^{(2)}$, h, and $g^{(4)}$

$$\nabla^{(0)\rho}g^{(2)}_{\rho\mu} = \frac{1}{6}\nabla^{(0)}_{\mu}R^{(0)},$$
$$\nabla^{(0)\rho}h_{\rho\mu} = -\kappa\tilde{T}_{\mu 4} = \frac{\kappa Z_A}{R_5^2}F^{(0)}_{\mu\rho}g^{(0)\rho\sigma}B_{\sigma} = -\frac{1}{4}\nabla^{(0)\rho}T^{(2)}_{\rho\mu}$$
$$\nabla^{(0)\rho}g^{(4)}_{\rho\mu} = -\frac{1}{2}\nabla^{(0)\rho}h_{\rho\mu} + \frac{1}{2}\nabla^{(0)\rho}[g^{(2)}_{\rho\sigma}g^{(0)\sigma\lambda}g^{(2)}_{\lambda\mu}] - \frac{1}{4}\nabla^{(0)\rho}[g^{(2)}_{\rho\mu}(g^{(0)\alpha\beta}g^{(2)}_{\alpha\beta})] + \frac{1}{8}\nabla^{(0)\rho}[g^{(0)}_{\rho\mu}(g^{(0)\alpha\beta}g^{(2)}_{\alpha\beta})^2]$$
$$- \frac{1}{8}\nabla^{(0)\rho}[g^{(0)}_{\rho\mu}(g^{(0)\xi\sigma}g^{(2)}_{\sigma\kappa}g^{(0)\kappa\lambda}g^{(2)}_{\lambda\xi})] - \kappa T^{(2)}_{\mu 4} + \frac{2}{3}\kappa\nabla^{(0)}_{\mu}T^{(0)}_{44}. \tag{51}$$

The near-boundary analysis of the field equations allows the boundary divergences of the action, now regulated at the surface $\rho = \epsilon$, to be determined. The regulated action is given by

$$\Gamma_{\text{Reg.}} = \int d^4x \int_{\rho=\epsilon} d\rho \, \det E\Big[\frac{1}{2\kappa}R + \Lambda - \frac{Z_A}{4}F_{MN}F^{MN} + CE^{MNRST}F_{MN}F_{RS}A_T - DE^{MNRST}R^X{}_{YMN}R^Y{}_{XRS}A_T\Big] + \frac{1}{\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}K(\gamma). \tag{52}$$

Applying the Fefferman-Graham form of the metric and employing the near-boundary solutions to the field equations the divergent terms in the action are isolated as

$$\begin{aligned}\Gamma_{\text{Reg.}} &= \int d^4x \int_{\rho=\epsilon} d\rho \sqrt{-g^{(0)}}\Big\{-\frac{1}{\rho^3}\frac{6R_5^3}{\kappa} - \frac{1}{\rho}\frac{R_5^3}{16\kappa} \times \Big[\Big(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\Big) + \frac{2\kappa Z_A}{R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\Big] + \ldots\Big\} \\ &= \int d^4x\sqrt{-g^{(0)}}\Big\{-\frac{1}{\epsilon^2}\frac{3R_5^3}{\kappa} + \ln(\epsilon/R_5^2)\frac{R_5^3}{16\kappa} \times \Big[\Big(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\Big) + \frac{2\kappa Z_A}{R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\Big]\Big\} \\ &\quad + O(\epsilon^0).\end{aligned} \tag{53}$$

The holographically renormalized action is defined by choosing the near-boundary counterterm action $\Gamma_{\text{Counterterms}}$ to cancel the divergent terms in the regulated action and to impose normalization conditions on the remaining finite terms so that

$$\Gamma_{\text{Sub.}} = \Gamma_{\text{Reg.}} + \Gamma_{\text{Counterterms}}, \tag{54}$$

where, after inverting the near-boundary expansion of the fields to write the boundary quantities in terms of the tensors at the surface $\rho = \epsilon$, the near-boundary counterterm action is found to be

$$\Gamma_{\text{Ct.-terms}} = \frac{3}{\kappa R_5}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma} - \frac{R_5}{4\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}R(\gamma) - \ln(\epsilon/R_5^2)\frac{R_5^3}{16\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\Big[(R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{3}R^2(\gamma)) + \frac{2\kappa Z_A}{R_5^2}F_{\mu\nu}(A)F^{\mu\nu}(A)\Big], \tag{55}$$

where the induced metric $\gamma_{\mu\nu} = (R_5^2/\rho)g_{\mu\nu}|_{\rho=\epsilon}$ on the near-boundary surface is used in the counterterm action. The finite holographic normalization is chosen through the dimensionless ratio in the logarithmic counterterm as (ϵ/R_5^2). A different normalization such as $(\epsilon/\tau R_5^2)$, with $\tau \in \mathbb{R}^+$, will correspond to a finite boundary term in the action,

$$\Gamma_{\text{Improve}} = \ln\tau\frac{R_5^3}{16\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\Big[\Big(R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{3}R^2(\gamma)\Big) + \frac{2\kappa Z_A}{R_5^2}F_{\mu\nu}(A)F^{\mu\nu}(A)\Big], \tag{56}$$

that will lead to finite holographic improvements to the currents that do not alter the form of the scale and chiral R anomalies and are consistent with the current Ward identities as discussed in the next section.

III. HOLOGRAPHIC CURRENTS

According to gravity/gauge duality, the expectation values of the boundary currents, the energy-momentum tensor $\theta_{\mu\nu}(x)$ and the R-symmetry current $R_\mu(x)$, in the presence of their respective external sources $g^{(0)}_{\mu\nu}(x)$ and $A^{(0)}_\mu(x)$ are found by varying the boundary sources in the on-shell renormalized action as

$$\delta\Gamma[g^{(0)}_{\mu\nu}, A^{(0)}_\mu] = \int d^4x\sqrt{-g^{(0)}}\Big[\frac{1}{2}\theta_{\mu\nu}(x)\delta g^{(0)\mu\nu}(x) + R_\mu(x)\delta A^{(0)\mu}(x)\Big]. \tag{57}$$

This can be accomplished by varying the sources in the regulated near-boundary action and counterterms, then taking the $\epsilon \to 0$ limit of the subtracted action. The variation of the surface $\rho = \epsilon$ sources γ and A for the on-shell, near-boundary regulated Einstein-Maxwell-Chern-Simons action (i.e. the second order in derivative part of the action, the mixed gauge-gravity Chern-Simons term has higher order derivatives and will be treated separately) yields

$$\begin{aligned}\delta\Gamma^{EMCS}_{\text{Reg.}} &= \int_{\rho=\epsilon} d^5x \det E\Big\{\frac{1}{2\kappa}\nabla_R(\nabla^R g_{MN}\delta g^{MN} - \nabla_S\delta g^{RS}) + \nabla_M[\delta A_L(-Z_A F^{ML} + 8CE^{MLRST}\partial_R A_S A_T)]\Big\} \\ &= -\frac{2}{\kappa}\int_{\rho=\epsilon} d^4x\frac{\rho}{R_5}\frac{\partial}{\partial\rho}[\delta\sqrt{-\gamma}] + \frac{1}{2\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}[K_{\mu\nu}(\gamma) - \gamma_{\mu\nu}K(\gamma)]\delta\gamma^{\mu\nu} \\ &\quad - \int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\delta A_\mu\Big[2Z_A\frac{\rho}{R_5}\gamma^{\mu\nu}\frac{\partial}{\partial\rho}A_\nu + 8CE^{\mu\nu\rho\sigma}(\gamma)A_\nu\partial_\rho A_\sigma\Big].\end{aligned} \tag{58}$$

In order for there to be a well-defined variational problem, the extrinsic curvature term must be added to cancel the undetermined source variation $\partial/\partial\rho(\delta\gamma^{\mu\nu})$ terms. The variation of the near-boundary extrinsic curvature term, $\Gamma_{\rm K} = 1/\kappa \int_\epsilon d^4x\sqrt{-\gamma}K(\gamma)$, is found to be

$$\delta\Gamma_{\rm K} = \frac{2}{\kappa}\int_\epsilon d^4x \frac{\rho}{R_5}\frac{\partial}{\partial\rho}(\delta\sqrt{\gamma}). \tag{59}$$

In addition, the variation of the counterterms,

$$\begin{aligned}\delta\Gamma_{\rm Ct.-Terms} = &-\frac{3}{2\kappa R_5}\int_\epsilon d^4x\sqrt{-\gamma}\gamma_{\mu\nu}\delta\gamma^{\mu\nu} - \frac{R_5}{2\kappa}\int_\epsilon d^4x\sqrt{-\gamma}\frac{1}{2}[R_{\mu\nu}(\gamma) - \frac{1}{2}\gamma_{\mu\nu}R(\gamma)]\delta\gamma^{\mu\nu}\\ &- R_5 Z_A \ln\frac{\epsilon}{R_5^2}\int_\epsilon d^4x\sqrt{-\gamma}\delta\gamma^{\mu\nu}[F_{\mu\rho}F_{\nu\sigma}\gamma^{\rho\sigma} - \frac{1}{4}\gamma_{\mu\nu}F^2] + R_5 Z_A \ln\frac{\epsilon}{R_5^2}\int_\epsilon d^4x\sqrt{-\gamma}\delta A_\sigma\nabla_\rho F^{\rho\sigma},\end{aligned} \tag{60}$$

are needed to provide a finite $\epsilon \to 0$ limit for the Einstein and Maxwell terms contributions to the currents as well as the action (both Chern-Simons terms are finite as are their contributions to the currents).

Combining these terms, $\Gamma^{EMCS}_{\rm Sub.} = \Gamma^{EMCS}_{\rm Reg.} + \Gamma_{\rm K} + \Gamma_{\rm Counterterms}$, it is found that the subtracted action is given by

$$\begin{aligned}\delta\Gamma_{\rm Sub.} = &\frac{\epsilon}{R_5^2}\int_\epsilon d^4x\sqrt{-\gamma}\delta\gamma^{\mu\nu}\frac{1}{2}\Big[\frac{R_5^2}{\epsilon\kappa}(K_{\mu\nu}(\gamma) - K(\gamma)\gamma_{\mu\nu}) - \frac{3R_5}{\epsilon\kappa}\gamma_{\mu\nu} - \frac{R_5^3}{2\epsilon\kappa}\Big(R_{\mu\nu}(\gamma) - \frac{1}{2}R(\gamma)\gamma_{\mu\nu}\Big)\\ &- 2R_5 Z_A \ln\frac{\epsilon}{R_5^2}\frac{R_5^2}{\epsilon}\Big(F_{\mu\rho}F_{\nu\sigma}\gamma^{\rho\sigma} - \frac{1}{4}\gamma_{\mu\nu}F^2\Big)\Big]\\ &- \int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\delta A_\mu\Big[2Z_A\frac{\rho}{R_5}\gamma^{\mu\nu}\frac{\partial}{\partial\rho}A_\nu + 8CE^{\mu\nu\rho\sigma}(\gamma)A_\nu\partial_\rho A_\sigma - R_5 Z_A\ln\Big(\frac{\epsilon}{R_5^2}\Big)\nabla_\rho F^{\rho\mu}\Big].\end{aligned} \tag{61}$$

Thus the subtracted boundary energy-momentum tensor and R-symmetry current take the form

$$\begin{aligned}\theta^{\rm EMCS}_{{\rm Sub.}\mu\nu}(\epsilon) &\equiv \frac{R_5^2}{\epsilon}\frac{2}{\sqrt{-\gamma}}\frac{\delta\Gamma_{\rm Sub.}}{\delta\gamma^{\mu\nu}} = \frac{R_5^2}{\epsilon\kappa}(K_{\mu\nu}(\gamma) - K(\gamma)\gamma_{\mu\nu}) - \frac{3R_5}{\epsilon\kappa}\gamma_{\mu\nu} - \frac{R_5^3}{2\epsilon\kappa}\Big(R_{\mu\nu}(\gamma) - \frac{1}{2}R(\gamma)\gamma_{\mu\nu}\Big)\\ &\quad - 2R_5 Z_A\ln\frac{\epsilon}{R_5^2}\frac{R_5^2}{\epsilon}\Big(F_{\mu\rho}F_{\nu\sigma}\gamma^{\rho\sigma} - \frac{1}{4}\gamma_{\mu\nu}F^2\Big)\\ R^{{\rm EMCS}\mu}_{\rm Sub.}(\epsilon) &\equiv \frac{1}{\sqrt{-\gamma}}\frac{\delta\Gamma_{\rm Sub.}}{\delta A_\mu} = -2Z_A\gamma^{\mu\nu}\frac{\epsilon}{R_5}\frac{\partial}{\partial\rho}A_\nu + 8CE^{\mu\nu\rho\sigma}(\gamma)A_\nu\partial_\rho A_\sigma + R_5 Z_A\ln\Big(\frac{\epsilon}{R_5^2}\Big)\nabla_\rho F^{\rho\sigma}.\end{aligned} \tag{62}$$

Expanding these expressions as the $\rho = 0$ boundary is approached gives the Einstein-Maxwell-pure $U(1)_R$ gauge field Chern-Simons contribution to the renormalized boundary currents as

$$\begin{aligned}\theta^{\rm EMCS}_{\mu\nu} = \theta^{\rm EMCS}_{{\rm Sub.}\mu\nu}(\epsilon = 0) &= \frac{R_5^3}{\kappa}[2g^{(4)}_{\mu\nu} + h_{\mu\nu} - g^{(2)}_{\mu\nu}g^{(0)\rho\sigma}g^{(2)}_{\rho\sigma} - g^{(0)}_{\mu\nu}(2g^{(0)\rho\sigma}g^{(4)}_{\rho\sigma} + g^{(2)\rho\sigma}g^{(2)}_{\rho\sigma})]\\ &\quad - \frac{R_5^3}{4\kappa}\Big[R^{(0)}_{\mu\lambda}R^{(0)\lambda}{}_{\nu} + R^{(0)}_{\lambda\mu\rho\nu}R^{(0)\lambda\rho} - \frac{1}{2}\nabla^{(0)2}R^{(0)}_{\mu\nu} + \frac{1}{6}\nabla^{(0)}_\mu\nabla^{(0)}_\nu R^{(0)} + \frac{1}{12}g^{(0)}_{\mu\nu}\nabla^{(0)2}R^{(0)2} + \frac{1}{2}R^{(0)}_{\rho\sigma}R^{(0)\rho\sigma}g^{(0)}_{\mu\nu}\\ &\quad - \frac{1}{12}g^{(0)}_{\mu\nu}R^{(0)2} - \frac{1}{2}R^{(0)}\Big(R^{(0)}_{\mu\nu} - \frac{1}{6}g^{(0)}_{\mu\nu}R^{(0)}\Big)\Big]\\ R^{{\rm EMCS}\mu} = R^{{\rm EMCS}\mu}_{\rm Sub.}(\epsilon = 0) &= -2R_5 Z_A(A^{(2)\mu} + B^\mu) - 4CE^{\mu\nu\rho\sigma}(g^{(0)})A^{(0)}_\nu F^{(0)}_{\rho\sigma}.\end{aligned} \tag{63}$$

These are precisely the same expressions obtained by expanding the variation of the action terms $\delta\Gamma^{\rm EMCS} = \lim_{\epsilon\to 0}\delta\Gamma_{\rm Sub.}$ at the boundary directly with

$$\begin{aligned}\theta^{\rm EMCS}_{\mu\nu} &\equiv \frac{2}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma^{\rm EMCS}}{\delta g^{(0)\mu\nu}}\\ R^{{\rm EMCS}\mu} &\equiv \frac{1}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma^{\rm EMCS}}{\delta A^{(0)}_\mu}.\end{aligned} \tag{64}$$

Finally, the contribution to the currents due to the higher derivative mixed gauge-gravity Chern-Simons term can be determined directly from the variation of its action at the boundary as it is finite

$$\begin{aligned}\delta\Gamma_{\rm RRA} &= -4D\int d^5x\partial_P[{\rm det}E E^{PQRSL}g_{QM}A_L R^X{}_{NRS}\nabla_X\delta g^{MN}]\\ &\quad + 4D\int d^5x\partial_X[{\rm det}E E^{PQRSL}g_{QM}\delta g^{MN}\nabla_P(A_L R^X{}_{NRS})].\end{aligned} \tag{65}$$

The second term when expanded in terms of the Fefferman-Graham metric can be seen to vanish at the $\rho = 0$ boundary:

$$\delta\Gamma_{\rm RRA}^{\rm second\ term} = -2D\int d^4x\delta g^{\mu\nu}\sqrt{-g}[E^{\xi\sigma\pi\lambda}(g)F_{\pi\lambda}\{\rho g_{\xi\mu}(-2g''_{\sigma\nu} + g'_{\nu\rho}g^{\rho\zeta}g'_{\zeta\sigma}) + \rho g_{\xi\nu}(-2g''_{\sigma\mu} + g'_{\mu\rho}g^{\rho\zeta}g'_{\zeta\sigma})\} + \rho E^{\rho\sigma\xi\lambda}(g)A'_\lambda\{g_{\xi\mu}(\nabla_\sigma g'_{\nu\rho} - \nabla_\rho g'_{\nu\sigma}) + g_{\xi\nu}(\nabla_\sigma g'_{\mu\rho} - \nabla_\rho g'_{\mu\sigma})\}] \to 0, \quad \text{as} \quad \rho \to 0. \tag{66}$$

The first expression for the variation of the mixed Chern-Simons action for finite $\rho = \epsilon$ has an undetermined $\partial/\partial\rho\delta\gamma^{\mu\nu}$ term in it. However, this contribution vanishes on the $\rho = 0$ boundary contrary to the extrinsic curvature case discussed earlier

$$\delta\Gamma_{\rm RRA}^{\rm first\ term} = -4D\int d^4x\sqrt{-g^{(0)}}\frac{1}{2}\delta g^{(0)\mu\nu}E^{\xi\rho\sigma\lambda}(g^{(0)}) \times \{\nabla^{(0)}_\alpha[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}R^{(0)\alpha}{}_{\nu\rho\sigma} + g^{(0)}_{\xi\nu}R^{(0)\alpha}{}_{\mu\rho\sigma})] + 2\nabla^{(0)}_\rho[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}g^{(2)}_{\mu\sigma})] - 2A^{(0)}_\lambda(g^{(0)}_{\xi\mu}\nabla^{(0)}_\rho g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}\nabla^{(0)}_\rho g^{(2)}_{\mu\sigma})\}. \tag{67}$$

This leads to the final mixed gravitational and $U(1)_R$ gauge field Chern-Simons contribution to the energy-momentum tensor of the form

$$\theta^{\rm CS}_{\mu\nu} = \frac{2}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma_{\rm RRA}}{\delta g^{(0)\mu\nu}} = -4DE^{\xi\rho\sigma\lambda}(g^{(0)})\{\nabla^{(0)}_\alpha[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}R^{(0)\alpha}{}_{\nu\rho\sigma} + g^{(0)}_{\xi\nu}R^{(0)\alpha}{}_{\mu\rho\sigma})] + 2\nabla^{(0)}_\rho[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}g^{(2)}_{\mu\sigma})] - 2A^{(0)}_\lambda(g^{(0)}_{\xi\mu}\nabla^{(0)}_\rho g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}\nabla^{(0)}_\rho g^{(2)}_{\mu\sigma})\}. \tag{68}$$

Thus the complete renormalized boundary energy-momentum tensor and R-symmetry current are determined as

$$\begin{aligned}\theta_{\mu\nu} = \theta^{\rm EMCS}_{\mu\nu} + \theta^{\rm CS}_{\mu\nu} &= \frac{R_5^3}{\kappa}[2g^{(4)}_{\mu\nu} + h_{\mu\nu} - g^{(2)}_{\mu\nu}g^{(0)\rho\sigma}g^{(2)}_{\rho\sigma} - g^{(0)}_{\mu\nu}(2g^{(0)\rho\sigma}g^{(4)}_{\rho\sigma} + g^{(2)\rho\sigma}g^{(2)}_{\rho\sigma})] \\ &- \frac{R_5^3}{4\kappa}[R^{(0)}_{\mu\lambda}R^{(0)\lambda}{}_\nu + R^{(0)}_{\lambda\mu\rho\nu}R^{(0)\lambda\rho} - \frac{1}{2}\nabla^{(0)2}R^{(0)}_{\mu\nu} + \frac{1}{6}\nabla^{(0)}_\mu\nabla^{(0)}_\nu R^{(0)} + \frac{1}{12}g^{(0)}_{\mu\nu}\nabla^{(0)2}R^{(0)2} \\ &+ \frac{1}{2}R^{(0)}_{\rho\sigma}R^{(0)\rho\sigma}g^{(0)}_{\mu\nu} - \frac{1}{2}R^{(0)}R^{(0)}_{\mu\nu}] - 4DE^{\xi\rho\sigma\lambda}(g^{(0)})\{\nabla^{(0)}_\alpha[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}R^{(0)\alpha}{}_{\nu\rho\sigma} + g^{(0)}_{\xi\nu}R^{(0)\alpha}{}_{\mu\rho\sigma})] \\ &+ 2\nabla^{(0)}_\rho[A^{(0)}_\lambda(g^{(0)}_{\xi\mu}g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}g^{(2)}_{\mu\sigma})] - 2A^{(0)}_\lambda(g^{(0)}_{\xi\mu}\nabla^{(0)}_\rho g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}\nabla^{(0)}_\rho g^{(2)}_{\mu\sigma})\} \\ R^\mu = R^{{\rm EMCS}\mu} &= -2R_5Z_A(A^{(2)\mu} + B^\mu) - 4CE^{\mu\nu\rho\sigma}(g^{(0)})A^{(0)}_\nu F^{(0)}_{\rho\sigma}.\end{aligned} \tag{69}$$

Exploiting the near-boundary solutions to the field equations found in Sec. II the anomalous divergence of the R-symmetry current is found to be

$$\nabla_\mu R^\mu = CE^{(0)\mu\nu\rho\sigma}F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} + DE^{(0)\mu\nu\rho\sigma}R^{(0)\xi\zeta}_{\mu\nu}R^{(0)}_{\xi\zeta\rho\sigma}. \tag{70}$$

The energy-momentum tensor has contributions from gravity and matter flowing into the boundary of the form

$$\nabla^\mu\theta^{\rm EMCS}_{\mu\nu} = -2R_5^3T^{(2)}_{\nu4} = -2R_5^3(T^{(2)\,\rm Maxwell}_{\nu4} + T^{(2)\rm D}_{\nu4}),$$
$$\nabla^\mu\theta^{\rm CS}_{\mu\nu} = 2R_5^3T^{(2)\rm D}_{\nu4} + A^{(0)}_\nu\mathcal{A}^{\rm D}_R, \tag{71}$$

with the mixed Chern-Simons contribution to the R-symmetry anomaly given by

$$\mathcal{A}^{\rm D}_R = DE^{\mu\nu\rho\sigma}(g^{(0)})R^{(0)\xi\zeta}{}_{\mu\nu}R^{(0)}_{\rho\sigma\xi\zeta}. \tag{72}$$

From these follows the Ward identity relating the diverence of the energy-momentum tensor with that of the R-symmetry current as

$$\nabla^\mu\theta_{\mu\nu} = F^{(0)}_{\mu\nu}R^\mu + A^{(0)}_\nu\nabla_\mu R^\mu. \tag{73}$$

The trace [taken with $g^{(0)\mu\nu}$] for the various contributions to the energy-momentum tensor is found from the field equations to be

$$\theta^{{\rm EMCS}\mu}{}_\mu = \frac{R_5^3}{\kappa}\left[\frac{1}{8}\left(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right) + \frac{\kappa Z_A}{4R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\right] \equiv \mathcal{A}, \tag{74}$$

while

$$\theta^{{\rm CS}\mu}{}_\mu = 0. \tag{75}$$

Thus the renomalized boundary energy-momentum tensor has the anomalous trace

$$\theta_\mu{}^\mu = \mathcal{A}. \tag{76}$$

These results agree with the general diffeomorphism and R-symmetry transformations of the action. From the definition of the boundary currents, Eq. (57), it is found that the action is invariant under diffeomorphism transformations

$$\delta g^{(0)\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu,$$
$$\delta A^{(0)\mu} = -\nabla^\mu \xi_\nu A^{(0)\nu} - \xi_\nu \nabla^\nu A^{(0)\mu}, \tag{77}$$

so that

$$\delta\Gamma = \int d^4x\sqrt{-g^{(0)}}\xi^\nu[-\nabla^\mu\theta_{\mu\nu} + F^{(0)}_{\mu\nu}R^\mu + A^{(0)}_\nu\nabla_\mu R^\mu] = 0, \tag{78}$$

as follows from Eq. (73). The anomalous R-symmetry variation of the action follows directly from the R-current divergence equation. For the R-symmetry transformations

$$\delta g^{(0)\mu\nu} = 0, \qquad \delta A^{(0)\mu} = \partial^\mu\omega, \tag{79}$$

Equation (57) implies

$$\delta\Gamma = -\int d^4x\sqrt{-g^{(0)}}\omega\nabla^{(0)\mu}R_\mu = -\int d^4x\sqrt{-g^{(0)}}\omega[CE^{(0)\mu\nu\rho\sigma}F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} + DE^{(0)\mu\nu\rho\sigma}R^{(0)\xi\zeta}_{\mu\nu}R^{(0)}_{\xi\zeta\rho\sigma}]. \tag{80}$$

The energy-momentum trace anomaly Eq. (76) implies the Weyl scale transformation,

$$\delta g^{(0)\mu\nu} = 2\sigma g^{(0)\mu\nu}, \qquad \delta A^{(0)\mu} = 0, \tag{81}$$

anomaly for the renormalized action

$$\delta\Gamma = \int d^4x\sqrt{-g^{(0)}}\sigma(x)\theta^\mu{}_\mu(x) = \int d^4x\sqrt{-g^{(0)}}\sigma(x)\mathcal{A}(x) = \int d^4x\sqrt{-g^{(0)}}\sigma(x)\frac{R_5^3}{\kappa}\left[\frac{1}{8}\left(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right) + \frac{\kappa Z_A}{4R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\right]. \tag{82}$$

It is possible to improve the definition of the currents by adding a finite boundary counterterm to the action given by Eq. (56) which at the $\rho = 0$ boundary becomes

$$\Gamma_{\text{Improve}} = \ln\tau\frac{R_5^3}{2\kappa}\int d^4x\sqrt{-g^{(0)}}\left[\frac{1}{8}\left(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right) + \frac{\kappa Z_A}{4R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\right]. \tag{83}$$

Its variation at the boundary is given by

$$\delta\Gamma_{\text{Improve}} = \ln\tau\frac{R_5^3}{2\kappa}\int d^4x\sqrt{-g^{(0)}}\frac{1}{2}\delta g^{(0)\mu\nu}4h_{\mu\nu} - \ln\tau\frac{R_5^3}{2\kappa}\int d^4x\sqrt{-g^{(0)}}\delta A^{(0)}_\nu\frac{\kappa Z_A}{R_5^2}\nabla^{(0)}_\mu F^{(0)\mu\nu}. \tag{84}$$

These variations lead to the energy-momentum tensor, $\tau_{\mu\nu}$, and R-symmetry current, r^μ, improvement terms

$$\tau_{\mu\nu} = \frac{R_5^3}{2\kappa}\ln\tau 4h_{\mu\nu},$$
$$r^\mu = -\frac{R_5^3}{2\kappa}\ln\tau\frac{\kappa Z_A}{R_5^2}\nabla^{(0)}_\nu F^{(0)\nu\mu} = -\frac{R_5^3}{2\kappa}\ln\tau 4\frac{\kappa Z_A}{R_5^2}B^\mu. \tag{85}$$

This is an improvement that is consistent with the diffeomorphism, R-symmetry and scale Ward identities since these improvement terms obey

$$\nabla^{(0)}_\mu r^\mu = 0, \qquad g^{(0)\mu\nu}\tau_{\mu\nu} = 0, \qquad \nabla^{(0)\mu}\tau_{\mu\nu} = F^{(0)}_{\mu\nu}r^\nu. \tag{86}$$

In addition, the completely traceless and divergenceless improvement for the energy-momentum tensor can be obtained from the addition of the finite boundary action term

$$\Gamma_\alpha = \frac{\alpha}{32}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\left[R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{3}R^2(\gamma)\right] = \frac{\alpha}{32}\int_{\rho=0} d^4x\sqrt{-g^{(0)}}\left[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right], \tag{87}$$

with α an arbitrary constant. The new improvement term for the energy-momentum tensor becomes

$$\tau^{(\alpha)}_{\mu\nu} = \frac{2}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma_\alpha}{\delta g^{(0)\mu\nu}} = \alpha\left[h_{\mu\nu} + \frac{1}{4}\kappa T^{(2)}_{\mu\nu}\right]. \tag{88}$$

It also is consistent with the current Ward identities as $r^{(\alpha)}_\mu = 0$ and

$$\nabla^{(0)\mu}\tau^{(\alpha)}_{\mu\nu} = 0, \qquad g^{(0)\mu\nu}\tau^{(\alpha)}_{\mu\nu} = 0. \tag{89}$$

Hence the final expressions for the improved energy-momentum tensor and R-symmetry current is obtained from Eqs. (69), (85), and (88),

$$r^{\mu}_{\text{Improved}} = R^{\mu} + r^{\mu} = -2R_5 Z_A(A^{(2)\mu} + (1+\ln\tau)B^{\mu}) - 4CE^{\mu\nu\rho\sigma}(g^{(0)})A^{(0)}_{\nu}F^{(0)}_{\rho\sigma},$$

$$\begin{aligned}\tau^{\text{Improved}}_{\mu\nu} = \theta_{\mu\nu} + \tau_{\mu\nu} + \tau^{(\alpha)}_{\mu\nu} &= \frac{R_5^3}{\kappa}[2g^{(4)}_{\mu\nu} + (1+2\ln\tau+\alpha)h_{\mu\nu} - g^{(2)}_{\mu\nu}g^{(0)\rho\sigma}g^{(2)}_{\rho\sigma} - g^{(0)}_{\mu\nu}(2g^{(0)\rho\sigma}g^{(4)}_{\rho\sigma} + g^{(2)\rho\sigma}g^{(2)}_{\rho\sigma})] \\ &- \frac{R_5^3}{4\kappa}\Big[R^{(0)}_{\mu\lambda}R^{(0)\lambda}{}_{\nu} + R^{(0)}_{\lambda\mu\rho\nu}R^{(0)\lambda\rho} - \frac{1}{2}\nabla^{(0)2}R^{(0)}_{\mu\nu} + \frac{1}{6}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu}R^{(0)} + \frac{1}{12}g^{(0)}_{\mu\nu}\nabla^{(0)2}R^{(0)2} + \frac{1}{2}R^{(0)}_{\rho\sigma}R^{(0)\rho\sigma}g^{(0)}_{\mu\nu} \\ &- \frac{1}{2}R^{(0)}R^{(0)}_{\mu\nu}\Big] - 4DE^{\xi\rho\sigma\lambda}(g^{(0)})\{\nabla^{(0)}_{\alpha}[A^{(0)}_{\lambda}(g^{(0)}_{\xi\mu}R^{(0)\alpha}{}_{\nu\rho\sigma} + g^{(0)}_{\xi\nu}R^{(0)\alpha}{}_{\mu\rho\sigma})] + 2\nabla^{(0)}_{\rho}[A^{(0)}_{\lambda}(g^{(0)}_{\xi\mu}g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}g^{(2)}_{\mu\sigma})] \\ &- 2A^{(0)}_{\lambda}(g^{(0)}_{\xi\mu}\nabla^{(0)}_{\rho}g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}\nabla^{(0)}_{\rho}g^{(2)}_{\mu\sigma})\} + \frac{\alpha}{4}\kappa\frac{1}{R_5^2}\Big[Z_A F^{(0)}_{\mu\rho}F^{(0)\rho}{}_{\nu} + \frac{Z_A}{4}g^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma}F^{(0)\rho\sigma}\Big].\end{aligned} \tag{90}$$

IV. CONCLUSIONS

The holographic R current and R charges require additional normalizations in order for the R current to belong to the same $\mathcal{N}=1$ SUSY multiplet as the holographic energy-momentum tensor. In addition, a finite wave function renormalization of the $U(1)_R$ gauge field [c.f. Eq. (19)] was needed in order that the holographic contribution of the field strength to the trace anomaly Eq. (2) is consistent with the boundary SCFT trace anomaly. The general structure of the trace anomaly including the subleading corrections was reviewed in the Appendix for the generic non-SUSY case. The $U(1)$ field strength subleading contribution was obtained for generic 4 derivative terms in the action which was used in the introduction to fix the gauge field normalization in the supersymmetric case.

The subleading mixed gravitational field-$U(1)_R$ gauge field Chern-Simons term was added to the action as it gave rise to subleading gravitational contributions to the R anomaly. The modifications to the near-boundary solutions to the field equations were then obtained along with the boundary counterterms and normalization required by holographic renormalization. Once the on-shell action was obtained, the Brown-York energy-momentum tensor and R-symmetry current were constructed. The near-boundary solutions were used to secure the Ward identity obeyed by the currents as

$$\nabla^{\mu}\theta_{\mu\nu} = F^{(0)}_{\mu\nu}R^{\mu} + A^{(0)}_{\nu}\nabla_{\mu}R^{\mu}, \tag{91}$$

along with the trace and R anomalies

$$\begin{aligned}\theta^{\mu}{}_{\mu} &= \frac{R_5^3}{\kappa}\Big[\frac{1}{8}\Big(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\Big) + \frac{\kappa Z_A}{4R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\Big] \\ \nabla_{\mu}R^{\mu} &= CE^{(0)\mu\nu\rho\sigma}F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} + DE^{(0)\mu\nu\rho\sigma}R^{(0)\xi\zeta}_{\mu\nu}R^{(0)}_{\xi\zeta\rho\sigma}.\end{aligned} \tag{92}$$

The Ward identities for diffeomorphism invariance of the action then followed as given in Eqs. (77) and (78). Likewise, the R-symmetry transformation of the action was obtained in Eqs. (79) and (80) while the Weyl scaling of the action followed in Eqs. (81) and (82). Since the Ward identities provide the interpretation of the holographic currents as the energy-momentum tensor and R-symmetry current, improvements to the currents were constructed which left the Ward identities unchanged. As explicitly demonstrated, the improvements were expressed as additional finite boundary terms in the action.

Lastly, the fermionic gravitino sector of the conformal supergravity action can be included [22], although left here for future work. The boundary $\mathcal{N}=1$ SCFT includes the supersymmetry current in the supercurrent multiplet and the superconformal anomaly, given by the γ trace of the supersymmetry current, as part of the anomaly multiplet. The central charges a and c also describe the superconformal anomaly. The holographic supersymmetry currents can be constructed and their divergence and trace determined. The Ward identities will then include these fermionic currents as well, while anomaly matching will provide additional consistency checks for the AdS/gauge duality.

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APPENDIX: SUBLEADING HOLOGRAPHIC ANOMALIES

The holographic calculation of the $U(1)$ and trace anomalies for a generic $U(1)$ gauge field coupled to a gravitational theory with a cosmological constant in five dimensions, including four derivative terms, is considered. The bosonic part of the gauged supergravity action relevant for the holographic trace anomaly calculation is obtained by imposing the appropriate SUSY relationships among the generic parameters as presented below. The leading, two derivative part of the action contributing to the trace anomaly takes the form

$$\Gamma^{(1)}_{\text{Leading}} = \int d^5x \det E\Big(\frac{1}{2\kappa}R + \Lambda - \frac{Z_A}{4}F_{MN}F^{MN}\Big). \tag{A1}$$

The relation between the gravitational constant, the cosmological constant, and the radius of AdS space follows from the Einstein equation as

$$\kappa\Lambda R_5^2 = -6. \tag{A2}$$

Three types of four derivative terms are relevant for the holographic calculation of the trace anomaly. The first type includes the curvature-squared terms [5–10]

$$\Gamma^{(1)}_{\text{Subleading}} = \int d^5x \det E[\alpha R^2 + \beta R_{MN}R^{MN} + \gamma R^{MNKL}R_{MNKL}]. \tag{A3}$$

Including these terms, the modified Einstein equation still allows anti-de Sitter space as a solution, but results in an altered relationship between the gravitational constant, the cosmological constant, and the radius of AdS space, now involving also the parameters α, β, and γ, as

$$6\frac{R_5^3}{2\kappa} + \frac{1}{2}\Lambda R_5^5 + 40\alpha R_5 + 8\beta R_5 + 4\gamma R_5 = 0. \tag{A4}$$

Note that for the special values of the parameters $\alpha = 1/6$, $\beta = -4/3$, and $\gamma = 1$, the three curvature-squared terms combine to yield the square of the Weyl tensor, and the relationship between the gravitational constant, the cosmological constant, and the radius of AdS space is seen to reduce to the one that is obtained in the absence of the curvature-squared terms. A second type of four derivative terms couples the square of the gauge field strength tensor to the curvature tensors,

$$\Gamma^{(2)}_{\text{Subleading}} = \int d^5x \det E[\delta R F_{MN}F^{MN} + \xi R^{MN}F_{MK}F_{NL}g^{KL} + \zeta W^{MNKL}F_{MN}F_{KL}], \tag{A5}$$

and a third type includes terms that involve the square of the gauge field strength tensor and two additional derivatives,

$$\Gamma^{(3)}_{\text{Subleading}} = \int d^5x \det E[\mu F_{MN}\nabla^N\nabla_R F^{MR} + \nu\nabla^M F^{NR}\nabla_M F_{NR} + \sigma\nabla^M F^{NR}\nabla_N F_{RM}]. \tag{A6}$$

The holographic trace anomaly is determined by the logarithmically divergent part of the on-shell action as

$$\Gamma_{\text{on-shell}} = \cdots + \frac{1}{2}\ln\left(\frac{\epsilon}{R_5^2}\right)\int dx^4\sqrt{-\det g^{(0)}}\,\mathcal{A} + \cdots, \tag{A7}$$

with the regulating near-boundary surface located at $\rho = \epsilon$ and with minimal subtraction at $\rho = R_5^2$. For generic values of the parameters, the holographic trace anomaly thus obtained reads (see [5–10] for the gravitational contribution)

$$\begin{aligned}\Theta^\mu{}_\mu = &\left(\frac{1}{8}\frac{R_5^3}{2\kappa} + 5\alpha R_5 + \beta R_5 - \frac{1}{2}\gamma R_5\right)W^2 \\ &+ \left(-\frac{1}{8}\frac{R_5^3}{2\kappa} - 5\alpha R_5 - \beta R_5 - \frac{1}{2}\gamma R_5\right)R^2_{\text{GB}} \\ &+ \left(\frac{1}{4}R_5 Z_A - 20\frac{\delta}{R_5} - 4\frac{\xi}{R_5} + 6\frac{\nu}{R_5} - 3\frac{\sigma}{R_5}\right)F^{(0)}_{\mu\nu}F^{(0)\mu\nu},\end{aligned} \tag{A8}$$

where the square of the Weyl tensor and the Einstein-Gauss-Bonnet invariant are defined as

$$\begin{aligned}W^2 &= W^{(0)}_{\mu\nu\kappa\lambda}W^{(0)\mu\nu\kappa\lambda} = R^{(0)}_{\mu\nu\kappa\lambda}R^{(0)\mu\nu\kappa\lambda} - 2R^{(0)}_{\mu\nu}R^{(0)\mu\nu} + \frac{1}{3}R^{(0)2}, \\ R_{\text{GB}}{}^2 &= R^{(0)}_{\mu\nu\kappa\lambda}R^{(0)\mu\nu\kappa\lambda} - 4R^{(0)}_{\mu\nu}R^{(0)\mu\nu} + R^{(0)2}.\end{aligned} \tag{A9}$$

The anomaly in the divergence of the $U(1)$ current J^μ is holographically obtained from the variation of the action under the $U(1)$ gauge transformation $\delta A_T = \partial_T\Omega(x)$. The relevant leading, two derivative pure Chern-Simons term in the action is

$$\Gamma^{(2)}_{\text{Leading}} = C\int d^5x \det E E^{MNRST}F_{MN}F_{RS}A_T, \tag{A10}$$

while the subleading, four derivative mixed Chern-Simons term takes the form

$$\Gamma^{(4)}_{\text{Subleading}} = -D\int d^5x \det E E^{MNRST}R^X{}_{YMN}R^Y{}_{XRS}A_T. \tag{A11}$$

The $U(1)$ anomaly is obtained from

$$\delta\Gamma = -\int d^4x\Omega(x)\mathcal{A}_R, \tag{A12}$$

resulting in

$$\sqrt{-g}\nabla_\mu J^\mu = C\epsilon^{\mu\nu\rho\sigma}F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} + D\epsilon^{\mu\nu\rho\sigma}R^{(0)\xi\zeta}{}_{\mu\nu}R^{(0)}_{\xi\zeta\rho\sigma}. \tag{A13}$$

All of the above terms appear in the bosonic sector of the five-dimensional $\mathcal{N} = 2$ conformal supergravity action [23] further extended with four derivative terms [9,10], $\Gamma = \Gamma^{(1)}_{\text{Leading}} + \Gamma^{(2)}_{\text{Leading}} + \Gamma^{(1)}_{\text{Subleading}} + \cdots + \Gamma^{(4)}_{\text{Subleading}}$, with now the gauge field A_μ corresponding to the gravi-photon and the $U(1)$ current J^μ equal to the R current. In terms of the gravitational constant 2κ, the cosmological constant Λ, the wave function renormalization factor Z_A, and the parameter γ that sets the scale for the subleading terms, supersymmetry forces the remaining parameters to take the values

$$
\begin{aligned}
C &= \frac{1}{12\sqrt{3}} Z_A^{3/2}\sqrt{2\kappa}\left(1 - \frac{8}{3}\gamma\Lambda(2\kappa)^2\right),\\
D &= -\frac{1}{2\sqrt{3}}\gamma Z_A^{1/2}\sqrt{2\kappa},\\
\alpha &= \frac{1}{6}\gamma,\\
\beta &= -\frac{4}{3}\gamma,\\
\delta &= \frac{1}{6}\gamma Z_A(2\kappa),\\
\xi &= -\frac{8}{3}\gamma Z_A(2\kappa),\\
\mu &= -4\gamma Z_A(2\kappa),\\
\nu &= -2\gamma Z_A(2\kappa),\\
\sigma &= -2\gamma Z_A(2\kappa).
\end{aligned}
\tag{A14}
$$

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