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October 31, 2000

Abstract

We present a kinematic analysis program for spatial higher pairs whose parts rotate around or translate along fixed spatial axes. The part geometry is specified in a parametric boundary representation consisting of planar, cylindrical, and spherical patches bounded by line segments and circular arcs. The program consists of two parts: contact curve computation and configuration space computation. We perform kinematic analysis by the configuration space, which needs contact curves as an input. The configuration space computation follows the method that we developed for planar pairs. In this paper, we analyze each type of contact by hand to obtain low-degree equations that are readily solvable in closed form or numerically. The results are stored in a table that is parameterized by the part geometry and motion type. The program obtains contact equations for all pairs of part features from the table, solves them to obtain contact curves, and composes the curves to obtain the configuration space. The computations take a few seconds on a workstation. We demonstrate the program on design scenarios involving spatial systems and planar systems with axis misalignment.
1 Introduction

This paper presents research in kinematic analysis of spatial higher pairs. Higher pairs are common in mechanical design. Gears and cams are used in all types of mechanical systems. Ratchets, indexers, and other specialized pairs are used in low-torque precision mechanisms, such as sewing machines, copiers, cameras, and VCRs. Higher pairs are more versatile than lower pairs because they can realize multiple functions. They are usually cheaper, lighter, more compact, and more robust than actuators. When manufacturing variation and wear are taken into account, lower pairs must be analyzed as higher pairs, as in pin joints with play.

Higher pairs are more complex than lower pairs. A lower pair imposes a simple, permanent contact constraint, whereas a higher pair imposes complex constraints that vary with the part configuration. When two part features (faces, edges, or vertices) touch, the part motions are constrained to prevent the touching features from overlapping. The constraints are algebraic equations in the part degrees of freedom. They also depend on the feature shapes, hence change when the contact point shifts to another feature. Although every pair of features might touch, most potential contacts are blocked by other contacts.

The complexity of higher pairs makes kinematic analysis difficult. The analyst needs to identify the configurations (if any) where each pair of features touch, the contact constraints, and the configurations where contacts change. There are usually far too many contacts for manual analysis, as we will see below. Prior research does not provide an effective computer algorithm for these tasks.

We have developed a fast, robust kinematic analysis program for planar higher pairs based on configuration space computation [15, 14]. Configuration space is a geometric representation of kinematics that is common in robot motion planning [10]. We have shown that configuration space is a useful model of higher pair kinematics. A mechanical system is analyzed by computing a configuration space for each of its kinematic pairs. The configuration spaces describe all possible part contacts. They encode quantitative information, such as part motion paths, and qualitative information, such as failure modes.

In the previous kinematic analysis program [15, 14], we coped with the complexity of planar higher pair kinematics through a combination of scope restriction, manual analysis, and computational geometry. The contact constraint complexity is controlled by restricting the part features to line segments and circular arcs, which suffice for most applications. We solved each type of contact constraint in closed form and stored the solutions in a table parameterized by the feature geometry and motion type. The program formulates constraints for all pairs of features and instantiates the solutions from the table. The solutions yield the configurations in which the features would touch if there were no other features to interfere. The program computes the contact interactions by computational geometry. This is the most expensive computation because in the worst case the number of interactions is quadratic in the number of contacts, which in turn is quadratic in the number of part features. Nevertheless, it is performed in under a second on a workstation in all the examples we have analyzed so far.
In this paper, we extend the program to spatial higher pairs, relying on scope restriction, manual analysis, and computational geometry to control the computational complexity. The parts are specified in a parametric boundary representation with planar, cylindrical, and spherical patches bounded by line segments and circular arcs. Each part rotates around or translates along a fixed spatial axes. Most spatial higher pairs fall into this class, according to our survey of 2,500 mechanisms from an encyclopedia [9] and according to our experience. The main exceptions are precision gears and cams, which require spline surfaces, since non-fixed axis higher pairs are rare. The algorithm also supports spatial tolerancing of planar systems: planar pairs can be analyzed as spatial pairs to study axis misalignment due to manufacturing variation, assembly error, or wear.

We analyzed each type of contact by hand. There are roughly ten times more cases than in planar systems and the equations are much more complex. The main contribution of this paper is a derivation of low-degree algebraic contact equations for every type of contact. Most are solvable in closed form, while the rest are readily solvable by standard numerical methods. Prior derivations, such as Baraff [1], provide high-degree systems of equations that are impractical to solve.

The rest of the paper is organized as follows. Section 2 describes prior research. Section 3 illustrates the configuration space method of kinematic analysis. Section 4 and 5 describe the kinematic analysis algorithm and the contact constraints, respectively. Section 6 shows the examples and Section 7 contains a discussion of our results and plans for future work.

### 2 Prior research

Table 1 summarizes prior kinematic analysis research and provides representative references. Most research focuses on efficient algorithms for multi-body systems with lower pairs, such as linkages and manipulators [4, 7, 19]. Higher pairs are classified as planar or spatial and as fixed-axes (one degree of freedom per part) or general. Fixed-axes planar pairs are by far the most common,

<table>
<thead>
<tr>
<th>Type</th>
<th>Dofs</th>
<th>Part geometry</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>lower pair</td>
<td>1-3</td>
<td>planar</td>
<td>Schiehlen [19], Erdman [4]</td>
</tr>
<tr>
<td>(linkages)</td>
<td></td>
<td>spatial</td>
<td>Haug [7]</td>
</tr>
<tr>
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<td>1</td>
<td>cams, gears</td>
<td>Angeles [6], Litvin [12]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>general</td>
<td>Sacks-Joskowicz [15]</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>lines and arcs</td>
<td>Sacks [14]</td>
</tr>
<tr>
<td>spatial</td>
<td>1</td>
<td>cams, gears</td>
<td>Angeles [6], Litvin [12]</td>
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<tr>
<td></td>
<td></td>
<td>planes, cylinders</td>
<td>This paper</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>polyhedral</td>
<td>Donald [3]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>general</td>
<td>Joskowicz-Taylor [8]</td>
</tr>
</tbody>
</table>

Table 1: Prior research in kinematic analysis.
followed by fixed-axes spatial pairs then by planar pairs with three degrees of freedom [9]. General spatial higher pairs with six degrees of freedom are rare.

Previous research focuses on specialized fixed-axes pairs, mainly gears [12] and cams [6]. It studies contacts between pairs of complex features, such as involutes, helical gears, or spatial cams. The contacts are permanent or change in a known sequence. We developed kinematic analysis algorithms for fixed-axes planar higher pairs [15] and for general planar pairs [14] based on configuration space computation. Robot motion planning research [2, 8, 10] provides algorithms for a polyhedral robot moving amidst fixed polyhedral obstacles. Graphics research [1, 13, 11] provides fast collision detection algorithms for polyhedra in support of simulation. Although a limited form of kinematic analysis, collision detection does not address many aspects of mechanical design.

3 Examples

We illustrate the configuration space method of kinematic analysis with three examples. The first example is a planar dwell gear pair (Figure 1a). The driver is a concentric arc segment mounted on a gear sector of diameter 240mm. The follower is a star-shaped disk mounted on a gear sector of diameter 100mm. The arc segment, disk, and gears are each 20mm thick. Both parts rotate around their centers. As the driver rotates, the gear sector engages and turns the gear wheel. When the sector disengages, the concentric arc segment engages one of the 16 concave surfaces of the star-shaped disk, thereby locking it in a dwell position. A full rotation of the driver advances the follower by 1/16th of a rotation.

We review the planar kinematic analysis method that appears in our previous paper [15]. The configuration space is a two-dimensional manifold whose coordinates are the part degrees of freedom: \( \theta \) is the driver orientation and \( \omega \) is the follower orientation (Figure 1b). The dot marks the configuration displayed in part a. Configuration space partitions into blocked space where the parts overlap (the grey area), free space where they do not touch (the white area), and contact space where they touch without overlap (the black curves in between). Contact space partitions into contact curves that represent contact between pairs of part features.

The free and contact spaces encode the pair kinematics. They consist of 16 channels, each of which has a horizontal segment and a slanted segment connected by necks along the lines \( \theta = 0 \) and \( \theta = 3 \) radians. The horizontal segments contain the configurations where the driver arc locks the follower disk, hence \( \theta \) changes and \( \omega \) is nearly constant. The left/right segment boundaries represent the contacts when the wheel turns clockwise/counterclockwise. The narrow gap between them shows the amount of backlash. The slanted segments contain the configurations where the gears engage, hence \( \omega \) is a nearly linear function of \( \theta \). The gap between the upper and lower boundaries shows backlash.

We perform a spatial analysis to assess the sensitivity of the pair to axis misalignment. The axis misalignment transforms the contacts between planar features into spatial ones. For example, arc/arc contact becomes cylinder/cylinder or cylinder/arc contact. Tilting the follower axis by two
Figure 1: (a) Planar dwell gear pair and (b) its nominal configuration space

degrees around local $x$-axis narrows the channels (reduced play) (Figures 1c and 1d), while tilting it by the same degrees around local $y$-axis causes jamming (Figures 1e and 1f). Manual analysis is daunting: for the case of tilting around local $y$-axis, the contact space consists of 2,168 contact curves and the blocked space contains another 13,500 unrealizable contacts.

The second example is a spatial indexing pair (Figure 2). The driver consists of two cylindrical guides of radius 100mm connected by a slanted crossover. It rotates around an axis through its center and perpendicular to the cylindrical plates. The follower is a gear of radius 100mm with 21 rectangular teeth that rotates around an axis through its center and perpendicular to its surface. The two axes are perpendicular. One full rotation of the driver advances the follower by one tooth. The crossover rotates the follower then the guides lock it for the remainder of the driver rotation.

The configuration space coordinates are the gear angle $\theta$ and the driver angle $\omega$. It contains 21 vertical channels where the gear engages the driver cylinders. The top of each channel is connected to the bottom of the channel on the right by a diagonal region where the driver crossover advances the gear. Unlike the first example, the nominal system exhibits spatial contacts. The pair has 380 realizable contacts out of 2877 possible contacts. The design goal is to optimize the part dimensions and axes of rotation to minimize play. To do so, we change the part parameters and observe the results on the configuration space: some parameter changes will reduce the play, others will cause jamming, while others will have no effect.

The third example is a spatial Geneva pair (Figure 3). The cam is a plate with a pin and a half cylinder mounted on it. The follower is a hollow hemisphere of radius 120mm with four evenly
Figure 1: (Contd.) (c) spatial dwell gear pair with a two-degree axis tilt around local $z$-axis, (d) its configuration space, (e) two-degree axis tilt around local $y$-axis, and (f) its configuration space
spaced slots and circular cutouts. The cam rotates around an axis through the cylinder center and the follower rotates around a vertical axis orthogonal to it. As the cam rotates, the pin engages a follower slot and rotates it by a quarter of a turn. When the pin disengages, the cam cylinder engages a follower cutout and prevents it from moving until the pin engages the next slot. The configuration space coordinates are the cam angle $\theta$ and the follower angle $\omega$. The contact space contains four diagonal regions where the pin drives the follower, and four horizontal regions where the cylinder engages the cutouts. The design task is to find a pin angle and a slot clearance that guarantee the correct contact sequence. For example, if the pin reaches a slot too soon, it hits the side and blocks.

4 Kinematic analysis

The kinematic analysis program for spatial fixed-axes pairs consists of two parts: contact curve computation and configuration space computation. The program computes contact curves for all pairs of part features and composes the curves to obtain the free, blocked, and contact spaces. Curve computation is the contribution of this paper and is described next. Composition uses standard numerical and geometric algorithms that carry over from the planar algorithm, hence is not discussed. Configuration space computation follows the method that we developed for planar pairs [15].

Contact curves are computed in four stages: equation formulation, equation solving, inequality formulation, and inequality solving. There are one equation and several inequalities for each
type of pair, for example rotating cylindrical patch/translating spherical patch. The feature types (cylindrical patch/spherical patch) determine the structure of the equations and the part degrees of freedom (rotation/translation) determine the variables.

The contact equation states that the algebraic features touch (cylinder/sphere in our example). The features are the algebraic surfaces that underlie the part faces (planes, cylinders, and spheres), the algebraic curves that underlie the edges (lines and circles), and the vertices. For the case of planes, cylinder, and lines, algebraic features are infinite. Two algebraic features touch if they possess a common point where they are tangent. The definition of tangency depends on the feature type. The equations are solved in closed form, except for two cases that are solved numerically.

The inequalities specify that the contact point lies on the features, which are the finite part of the algebraic features. They also depend on the feature types. For example, a line segment touches a hemisphere when the line is tangent to the sphere and the contact point lies on the segment and on the hemisphere. The inequalities are solved numerically.

4.1 Equations

We have constructed a table of contact equations and solution methods. The table contains two algorithms per pair type. One algorithm computes points on the contact curve: it solves for one variable (part degree of freedom) for a given value of the other. The other algorithm computes the structure of the solution space: the number of contact curves and their endpoints. Figure 4 shows a canonical example in which a single contact generates four curves: two from \( p_0 \) to \( p_1 \) and two
Figure 4: Contact equation with four curves.

The next section contains the derivations for every entry in the contact table. But first we illustrate the process on a representative case: a rotating cylinder $C(\theta)$ whose cylinder axis is parallel to the rotation axis and a translating sphere $S(y)$ (Figure 5a). The cylinder rotates around the global $z$ axis and the sphere translates along a line parallel to the $y$ axis. The cylinder degree of freedom is the angle $\theta$ between its local $x$ axis and the global $x$ axis. The sphere degree of freedom is the global $y$ coordinate of its reference point $q$. Figure 5b shows the projection of the cylinder and the sphere onto the $xy$-plane, consisting of two circles.

(a) (b)

Figure 5: Rotating cylinder/translate sphere contacts.

The contact curve consists of the points in $(\theta, y)$ space where the cylinder and the sphere touch. Figure 6a shows the case when the cylinder angle is $\theta_0$, denoted $C(\theta_0)$, and the contacting
spheres are \( S(y_a) \) and \( S(y_b) \). Figure 6b shows the corresponding points \((\theta_a, y_a)\) and \((\theta_a, y_b)\) on the contact curves. We need to compute points on the contact curve and to find the curve endpoints. In Figure 6b, the endpoints are \( \theta_0 \), \( \theta_1 \), and \( \theta_2 \). At these \( \theta \) values, the cylinder contacts the swept surface of the translating sphere (compare with Figure 5b).

(a)  
(b)

Figure 6: Contact curve consists of four branches

We formulate the contact equation in a manner analogous to C-obstacle (configuration space obstacle) computation in robotics [10]. Consider two moving parts \( \tilde{a} \) and \( \tilde{b} \). Each part reference point traces a parametric circle or line depending on whether the part rotates or translates. Denote these curves with \( \tilde{a}(\theta) \) and \( \tilde{b}(\omega) \) where the parameters \( \theta \) and \( \omega \) denote rotation angles or translation distances. Denote the corresponding transformed parts as \( \tilde{a}(\theta) \) and \( \tilde{b}(\omega) \). The contact curve of \( \tilde{a} \) and \( \tilde{b} \) consists of the \((\theta, \omega)\) configurations where \( \tilde{a}(\theta) \) and \( \tilde{b}(\omega) \) touch.

When we compute the contact angle (or contact distance) of moving object \( \tilde{b} \) with given object \( \tilde{a}(\theta_a) \), we follow the steps:

(a)  
(b)
1. Shrink the object $\tilde{b}$ to its skeleton

2. Compute $C$-obstacle of $\tilde{a}(\theta_a)$ with respect to $\tilde{b}$.

3. Compute the angle (distance) of the moving trajectory $b(\omega)$ which makes the skeleton of $\tilde{b}$ contact the $C$-obstacle of $\tilde{a}(\theta_a)$.

When $\theta = \theta_a$, the parameter value $\omega = \omega_a$ where the skeleton of $\tilde{b}$ at $b(\omega_a)$ contacts the $C$-obstacle of $\tilde{a}(\theta_a)$ consists of the point $(\theta_a, \omega_a)$ on the contact curve. When the object $\tilde{b}$ is a sphere with radius $r$, we shrink it to its center point. Then, the $C$-obstacle of $\tilde{a}(\theta_a)$ with respect to $\tilde{b}$ is an offset of $\tilde{a}(\theta_a)$ with offset radius $r$. We compute the contact angle (distance) of $\tilde{b}$ with $\tilde{a}(\theta_a)$ by intersecting $b(\omega)$ with the $C$-obstacle of $\tilde{a}(\theta_a)$. If the object $\tilde{a}$ is a cylinder with radius $r$, we shrink it to the cylinder axis. The $C$-obstacle of $\tilde{a}(\theta_a)$ with respect to $\tilde{b}$ is an offset of $\tilde{a}(\theta_a)$ with offset distance $r$. We compute the contact angle (distance) of $\tilde{b}$ with $\tilde{a}(\theta_a)$ by computing the contact angle (distance) of the cylinder axis of $\tilde{b}$ with the $C$-obstacle of $\tilde{a}(\theta_a)$.

We compute the endpoints of contact curves as follows:

1. Shrink the object $\tilde{a}$ to its skeleton

2. Compute the swept surface of object $\tilde{b}(\omega)$, denoted $Ub(\omega)$.

3. Compute the $C$-obstacle of $Ub(\omega)$ with respect to $\tilde{a}$.

4. Compute the angle (distance) of the moving trajectory $a(\theta)$ that makes the skeleton of $\tilde{a}$ contact the $C$-obstacle of $Ub(\omega)$.

The resulting $\theta$ values are the endpoints of the contact curves. When $b(\omega)$ is a rotating sphere, the swept surface of $\tilde{b}(\omega)$ is a torus. The offset of a torus is another torus, so the $C$-obstacle of $Ub(\omega)$ with any offset radius consists of two tori. When $\tilde{b}$ translates, the $C$-obstacle of $Ub(\omega)$ consists of cylinders. If $\tilde{b}$ is a translating cylinder, the swept surface consists of two planes, and the $C$-obstacle of the swept surface also consists of planes. If $\tilde{b}$ is an axis-aligned rotating cylinder, the $C$-obstacle of the swept surface consists of either cylinders or planes.

The $C$-obstacle approach allows us to compute most spatial contacts via planar computational geometry. We project the relevant geometry onto an appropriate plane or intersect it with a cutting plane. Moreover, most of the planar computations are performed by intersecting conic sections.

For example, our rotating cylinder/translator sphere pair is projected onto the $xy$ plane where it becomes a rotating circle/translator circle pair. The contact curve is then computed by planar circle/line intersection. Given the rotated cylinder $C(\theta_a)$, the contact sphere is computed as follows. We compute the $C$-obstacle of $C(\theta_a)$ with respect to the sphere and project it onto the $xy$-plane to obtain a circle. We project the center trajectory of the sphere onto $xy$-plane to obtain a line. If the line/circle intersection points are $y_a$ and $y_b$, the contact configurations are $(\theta_a, y_a)$ and $(\theta_a, y_b)$ (Figures 6 and 7a). We simplify the endpoint computation analogously. The swept surface of the translating sphere is a cylinder whose $C$-obstacle is another cylinder. The projection
of this C-obstacle onto $xy$-plane is two lines, and the projection of the trajectory of the rotating cylinder is a circle. By intersecting the lines with the circle, we derive the contact curve endpoints (Figure 7b).

Table 2 shows the notations for geometric primitives which will be used in this paper.

### 4.2 Inequalities

We derive contact inequalities for each type of face and edge. The two touching features are treated independently. Features are represented parametrically. Spheres are parameterized with spherical coordinates, cylinders with cylindrical coordinates, circles with polar coordinates, and lines and planes with affine coordinates. The inequalities state that the parameter values of the contact point fall in the parameter range. We invert the parameterization and apply the inverse to the contact point to obtain these values. For example, a circular arc is parameterized as $d = o + a \cos u + b \sin u$ with $d$ the contact point, $o$ the circle center, and $a$ and $b$ perpendicular unit vectors in the base plane. We solve for $u$ to obtain the contact inequalities $u_1 \leq \arccos(a \cdot (d - o)) \leq u_2$ with $[u_1, u_2]$ the parameter interval. The other features also have standard closed-form inverses.

The contact curves are obtained by combining contact equation solving with the inequalities. The contact equations are solved at closely spaced points to approximate the solution curves to a specified accuracy (typically 0.1% relative accuracy). The inequalities are evaluated at each point and the points that do not satisfy them are removed. The curve endpoints are those points where an inequality changes sign. They are computed by bisection search on the interval between the first invalid point and the last valid point.
5 Contact equations

We derive contact equations for the cases shown in Table 3. The features we deal with are general cylinder, axis-aligned cylinder, sphere, plane, and vertical circle. The vertical circle is a circle whose normal is parallel to the rotation axis. The axis-aligned cylinder is either a vertical cylinder or a horizontal cylinder, where vertical cylinder is a cylinder whose axis is parallel with rotation axis, and horizontal cylinder is a cylinder whose axis passes through the rotation axis orthogonally. A general cylinder is a cylinder that is not an axis-aligned cylinder. The contact equation for two general cylinders is a polynomial of degree 6, and is not covered. Line segments and vertices are modeled as cylinders and spheres of radius zero. Plane/plane and plane/cylinder contacts occur at finite sets of configurations, hence do not form contact curves. We do not consider cylinder/plane or plane/plane contacts because they are subsumed by their boundary contacts. We discuss the case where both parts rotate; translating parts yield similar, but simpler equations. It is possible for the contact curve between two features to consist of several vertical lines. This case requires special handling that is not described.

We represent the rotation axes of two features as $u$ and $v$. We represent the reference point trajectory of rotating features as $m + Ra(\theta)$ and $n + Rb(\omega)$ where $a(\theta)$ and $b(\omega)$ are unit spatial circles

$$a(\theta) = M_1(\cos \theta, \sin \theta, 0)^T,$$
$$b(\omega) = M_2(\cos \omega, \sin \omega, 0)^T,$$

with $M_1$ and $M_2$ rotation matrices and with $(a(\theta), u) = 0$ and $(b(\omega), v) = 0$. So, $m + Ra(\theta)$ consists of a circle with center $m$, radius $R$, and normal vector $u$, and $n + Rb(\omega)$ consists of a circle with center $n$, radius $R$, and normal vector $v$. Table 4 lists the geometric operations used to compute the contact curve endpoints.

5.1 Vertical Cylinder and Vertical Cylinder Contact

Given vertical cylinder $C_1(\theta_0) = C_r(p, u)$, we compute the contact angle of rotating vertical cylinder $C_2(\omega) = C_r(n + Ra(\theta), v)$. The main idea has following steps: i) project the cylinder $C_1(\theta_0)$ onto the orthogonal plane to the the axis of $C_2(\omega)$ and ii) compute the rotation angle $\omega_i$ such that $C_2(\omega_i)$ contacts with the projection of $C_1(\theta_0)$. By this way, we reduce the cylinder and cylinder contact problem to the line and circle contact problem. The line and circle contact problem is solved by intersecting a circle with lines.

Figure 8a shows a cylinder $C_1(\theta_0)$ and a rotating vertical cylinder $C_2(\omega)$. Figure 8b shows the projection of $C_1(\theta_0)$ onto the orthogonal plane to the axis of $C_2(\omega)$. The contact angle of $C_2(\omega)$ about $C_1(\theta_0)$ is the same as the contact angle of the axis of $C_2(\omega)$ about the $s$-offset of $C_1(\theta_0)$, where $s$ is the radius of $C_2(\omega)$. The $s$-offset of the cylinder $C_1(\theta_0)$ is the cylinder $C_{r+s}(p, u)$. We project the offset cylinder of $C_1(\theta_0)$ onto the plane $P(p, v)$ which always cuts the axis of $C_2(\omega)$.
<table>
<thead>
<tr>
<th>Features</th>
<th>Contact Angle</th>
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<tbody>
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<td></td>
<td></td>
<td>Parallel Rotation axes</td>
<td>Non-Parallel Rotation axes</td>
<td>Contact Curve Endpoints</td>
</tr>
<tr>
<td>Part 1</td>
<td>Part 2</td>
<td></td>
<td></td>
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<td>g. cylinder</td>
<td>n-</td>
<td>n-</td>
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</tr>
<tr>
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<td>a. cylinder</td>
<td>c+</td>
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<tr>
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<td>v. circle</td>
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<td>v. circle</td>
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</tr>
</tbody>
</table>

a. cylinder: Axis-aligned cylinder (vertical or horizontal cylinder)
g. cylinder: General cylinder which is not axis-aligned
v. circle: Circle whose normal is parallel to the rotation axis
c: Closed form solution exists
n: Numerical solution exists
+: Implemented in the system
-: Not implemented in the system

Table 3: System coverage.
<table>
<thead>
<tr>
<th>Feature 1</th>
<th>Feature 2</th>
<th>Contact configuration</th>
<th>Contact Curve Endpoints</th>
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<td>sphere</td>
<td>h. cylinder</td>
<td>circle/circle intersection</td>
<td>torus/circle intersection</td>
</tr>
<tr>
<td>sphere</td>
<td>v. circle</td>
<td>plane/circle intersection</td>
<td>tangent line to a circle</td>
</tr>
<tr>
<td>plane</td>
<td>v. circle</td>
<td>plane/circle intersection</td>
<td>tangent line to a circle</td>
</tr>
<tr>
<td>v. circle</td>
<td>v. circle</td>
<td>circle/circle intersection</td>
<td>line or circle/circle intersection</td>
</tr>
</tbody>
</table>

Table 4: Operations for computing contact curve endpoints.

<table>
<thead>
<tr>
<th>Feature 1</th>
<th>Feature 2</th>
<th>Contact Curve</th>
<th>Contact Curve Endpoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>v. cylinder</td>
<td>v. cylinder</td>
<td>6 2</td>
<td>-</td>
</tr>
<tr>
<td>v. circle</td>
<td>v. cylinder</td>
<td>8 numerical</td>
<td>-</td>
</tr>
<tr>
<td>h. cylinder</td>
<td>v. cylinder</td>
<td>6 2</td>
<td>-</td>
</tr>
<tr>
<td>v. cylinder</td>
<td>v. circle</td>
<td>8 numerical</td>
<td>-</td>
</tr>
<tr>
<td>sphere</td>
<td>v. cylinder</td>
<td>4 2</td>
<td>-</td>
</tr>
<tr>
<td>sphere</td>
<td>h. cylinder</td>
<td>4 1</td>
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</tr>
<tr>
<td>sphere</td>
<td>v. circle</td>
<td>4 2</td>
<td>-</td>
</tr>
<tr>
<td>plane</td>
<td>v. circle</td>
<td>2 1</td>
<td>-</td>
</tr>
<tr>
<td>v. circle</td>
<td>v. circle</td>
<td>2 2</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5: Comparison of polynomial degrees for two methods.
orthogonally. Then, the contour of the projection consists of two lines $L = L(p \pm (r+s)d/\|d\|, u')$, where $d = u \times v$ and $u' = -\langle v, u \rangle v + u$ (Figure 8c).

The intersection of the swept rotation axes of $C_2(\omega)$ with the plane $P(p, v)$ consists of a circle $K$ with radius $R_2$ and center $n'$, where $n' = n + (p - n, v) v$. We compute the intersection point $q$ between the circle $K$ and the set of lines $L$. Then, the contact angle of $C_2(\omega)$ with $C_1(\theta_0)$ is the angle of $q - n'$ with basis vectors for rotating $C_2(\omega)$.

Figure 8: Contact Angle of Rotating Vertical Cylinder for a Given Cylinder.

When two cylinders $C_1 = C_r(p, u)$ and $C_2 = C_s(q, v)$ contact each other, we determine the contact point as follows. The pair of the closest points between two axes of cylinders, $L_1 = p + su$ and $L_2 = q + tv$, are $c_1 = p + s_c u$ and $c_2 = q + t_c v$, where

$$
t_c = \frac{\langle u, v \rangle \langle p - q, u \rangle + \langle p - q, v \rangle}{\langle u, v \rangle^2 - 1},$$
$$s_c = \langle u, v \rangle t_c - \langle p - q, u \rangle.
$$

The contact point between $C_1$ and $C_2$ is the intersection point of the line between $c_1$ to $c_2$ with the projection of $C_r(\theta_0)$.
cylinder $C_1$:

$$c_1 + r \frac{c_2 - c_1}{||c_2 - c_1||}.$$

We compute the endpoints of contact curve by line and circle intersection, too. The swept surface of $C_2(\omega)$ consists of two cylinders. We offset the outer cylinder with the offset distance $r$, and the inner cylinder with the offset distance $-r$, where $r$ is the radius of the $C_1(\theta)$. When $P$ is a plane which orthogonally cuts the rotation axis of $C_1$, the projection of offset cylinders onto $P$ consists of four lines or two circles. The plane $P$ cuts the swept axes of $C_1$ in a circle, and the intersection points between this circle and projection of offset swept surface of $C_2(\omega)$ determine the endpoints of contact curves (Figure 9).

Figure 9: Contact Curve Endpoints of two Vertical Cylinders.
### 5.2 Horizontal Cylinder and Vertical Cylinder Contact

Given cylinder $C_1(\theta_0) = C_r(p, u)$, the computation of the contact angle of rotating vertical cylinder $C_2(\omega) = C_z(n + Rb(\omega), v)$ is already explained in Section 5.1. The contact point between two cylinders are also computed in the same way explained in Section 5.1.

The endpoints of contact curve of horizontal cylinder $C_1(\theta) = C_r(m, a(\theta))$ with the rotating vertical cylinder $C_2(\omega) = C_z(n + Rb(\omega), v)$ is computed by following steps: i) offset $C_2(\omega)$ with offset distance $r$, ii) compute the swept surface of $r$-offset $C_2(\omega)$, denoted $C'_2(\omega)$, and iii) compute the rotation angle of the axis of $C_1(\theta)$, i.e., $L(m, a(\theta))$, which tangentially contact with $C'_2(\omega)$.

We do not consider the case when the rotation axes of two features are orthogonal; that is, we assume that $(u, v) \neq 0$. When two rotation axes $u$ and $v$ are not orthogonal, the intersection between $C'_2(\omega)$ and $P(m, u)$ is an ellipse with minor axis $u \times v$, major axis $(u, v) v - u$, minor radius $R\pm(r+s)$, major radius $(R\pm(r+s))/ (u,v)$, and center $p_0$, where $p_0 = n + (m-n, u)\nu$ (Figure 10).

We may assume that the intersection ellipse $E = C_2(\omega) \cap P(m, u)$ is given in the standard position as

$$E : ax^2 + by^2 = 1, \quad (1)$$

and the rotating line $L(m, a(\theta))$ is contained in $xy$-plane with rotation center $(x_0, y_0)$. The contacting line from the point $(x_0, y_0)$ to ellipse $E$ is computed as follows. The gradient of the ellipse $E$ is $(ax, by)$; thus, from the equation:

$$(x - x_0, y - y_0) \cdot (ax, by) = 0,$$

we derive the following equation:

$$1 - ax_0 x - by_0 y = 0. \quad (2)$$

Using Equations 1 and 2, we compute the angles of rotating axis $L(m, a(\theta))$ which tangentially touches $C'_2(\omega)$, and these angles consist of the endpoints of contact curve between $C_1(\theta)$ and rotating cylinder $C_2(\omega)$. So, we need only the square root operation to compute the endpoints of contact curve.

### 5.3 Horizontal Cylinder and Horizontal Cylinder Contact

Given cylinder $C_1(\theta_0) = C_r(p, u)$, let’s compute the contact angle of rotating horizontal cylinder $C_2(\omega) = C_z(n, b(\omega))$, whose rotation axis $L(n, b(\omega))$ and rotates around $v$. The contact angle of $C_2(\omega)$ about $C_1(\theta_0)$ is the same as the contact angle of $L(n, b(\omega))$ with $s$-offset of the cylinder $C_1(\theta_0)$, and this is the contact problem between a rotating line and an fixed ellipse (Figure 11). Let’s denote the $s$-offset of $C_1(\theta_0)$ as $C'_1(\theta_0)$, and the swept plane of rotating $L(n, b(\omega))$ as a plane $P(n, v)$. 

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Figure 10: Contact Curve Endpoints of Rotating Horizontal Cylinder for Rotating Vertical Cylinder.

We do not consider the case when the rotation axes of two features are orthogonal; that is, we assume that \( \langle u, v \rangle \neq 0 \). When two rotation axes \( u \) and \( v \) are not orthogonal, the intersection between \( C'(\theta_0) \) and \( P(n, v) \) is an ellipse with minor axis \( u \times v \), major axis \( \langle u, v \rangle v - u \), minor radius \( r + s \), major radius \( (r + s)/\langle u, v \rangle \), and center \( p_0 \), where \( p_0 = p + \langle n-p,v \rangle u \). Section 5.2 explains the way to compute the contact angle of rotating line with an ellipse.

Figure 11: Contact Angle of Horizontal Cylinder for a Given Cylinder.

The contact point between the cylinder \( C_1(\theta_0) \) and \( C_2(\omega) \) is computed using the same way as of contacting two vertical cylinders.

To compute the endpoints of contact curve between \( C_1(\theta) \) and \( C_2(\omega) \), we first compute the swept volume of s-offset \( C_2(\omega) \): the volume bounded by two planes \( P(n \pm (r + s)v, v) \). If the rotation center of \( C_1(\theta) \), \( m \) is on the bounding plane or outside this volume, and the swept plane of rotating axis of \( C_1(\theta) \) intersects with this volume, the curve endpoints consist of \(-\pi\) to
\[ \pi. \text{ Otherwise, the endpoints of contact curve consist of the contact angle of } C_1(\theta)'s \text{ axis with the sphere centered at } n \text{ with radius } r + s. \text{ If } m \text{ is inside the sphere } S_{r+s}(n), \text{ contact curve between } C_1(\theta) \text{ and } C_2(\omega) \text{ does not exist.} \]

### 5.4 Vertical Cylinder and Circle Contact

Given cylinder \( C(\theta_0) = C_r(p, u) \) and rotating circle \( K(\omega) = K_s(n + Rb(\omega), v) \), we compute the contact angle of \( K(\omega) \) with \( C(\theta_0) \). We cut the given cylinder with the sweeping plane of the circle, \( P(n, v) \), where \( n \) is the rotating center of the circle. Then, the intersection consists of an ellipse or two lines (Figure 12). The intersection ellipse is with center \( p_0 = p + \frac{(n - p, v)}{(u, v)}u \), minor axis \( u \times v \), major axis \( (u, v) u - v \), minor radius \( r \), and major radius \( r / (u, v) \). When the intersection consists of two lines, the lines are \( L(p_0 \pm l d, u) \), where \( p_0 = p + (n - p, v) v \), \( l = \sqrt{r^2 - \|p - p_0\|^2} \), and \( d = u \times v / \|u \times v\| \).

When the circle radius is \( s \), we use the \( \pm s \)-offset of the ellipse (or lines) and the circle center trajectory intersection to compute the contact angle of circle for given cylinder. But, this intersection needs 8th order polynomial solving, so we use a numerical method to solve this case.

We introduce a numerical method to compute the intersection between an offset ellipse and a circle as follows. Without loss of generality, we may assume that given ellipse is centered at the origin of \( xy \)-plane, and has \( x \)-axis and \( y \)-axis as a minor axis and a major axis, respectively. The \( s \)-offset of an ellipse \( E = (a \cos \alpha, b \sin \alpha) \) is represented as follows:

\[
E_s = (x(\alpha), y(\alpha)) = (a \cos \alpha, b \sin \alpha) + \frac{s(b \cos \alpha, a \sin \alpha)}{\sqrt{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha}}.
\]
To intersect this with a circle, we compute the bounding box of $E_s$. To compute the bounding box of the curve segment of $E_s$, we have to compute the equations: $dx(\alpha)/d\alpha = 0$ and $dy(\alpha)/d\alpha = 0$, but these equations do not have a closed-form solution. We use another simpler method to compute bounding box of the offset ellipse.

First, we split $E_s$ for two end points of curve segment be two vertices of the bounding box. When the ellipse is offset with positive offset distance, we split $E_s$ at the points which $\alpha = 0, \frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$. After $E_s$ is split into four segments, when we subdivide each segment, the bounding box of it is determined by its two end points. When $s < 0$, we need more splitting. We split the $E_s$ at the point which the offset distance and the curvature radius the ellipse $E$ are the same. When ellipse $E$ is given as $(a \cos \alpha, b \sin \alpha)$, the curvature of $E$ is as follows:

$$\kappa(\alpha) = \frac{ab}{(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)^{3/2}}.$$  

The curvature radius of $E$ is $s$, when

$$\alpha = \arcsin \left( \pm \sqrt{\frac{(ab)^{2/3} - b^2}{a^2 - b^2}} \right).$$

After we split the curve at these points, we finally get the curve segments of offset ellipse whose bounding boxes are determined by two end points of each curve segment (see Figure 13).

Let's assume that given circle is centered at $(x_0, y_0)$ with radius $R$. When the cylinder axis and circle rotation axis are almost orthogonal, the major radius of ellipse will be close to infinite. To deal with this case, we subdivide the offset ellipse again at the point which meets $y = y_0 \pm R$. The intersection point of a line $y = k$ and the $s$-offset of the ellipse $E$ is computed as $(a \cos \theta, b \sin \theta)$, where $\theta = \arcsin(x) \pm \arcsin(\pm \sqrt{(ab)^{2/3} - b^2} / a^2 - b^2)$ for the solution of following equation:

$$b^2(a^2 - b^2)x^4 - 2bk(a^2 - b^2)x^3 + ((k^2 - s^2)a^2 + b^2(b^2 - k^2))x^2 - 2b^2kx + k^2b^2 = 0.$$  

The intersection point of a line $x = k$ and the $s$-offset of the ellipse $E$ is computed as $(a \cos \theta, b \sin \theta)$, where $\theta = \arccos(x)$ for the solution of following equation:

$$a^2(b^2 - a^2)x^4 - 2ak(b^2 - a^2)x^3 + ((k^2 - s^2)b^2 + a^2(a^2 - k^2))x^2 - 2a^2kx + k^2a^2 = 0.$$  

With these two equations, we can intersect the bounding box of circle and offset ellipse intersection points (Figure 14). We split the offset ellipse at these points again.

After we split the offset ellipse at minimum and maximum $x, y$ values and intersection points with bounding box of given circle, for each split curve segment, we decide whether the bounding box of it intersects with given circle or not. If they intersect each other, we subdivide the curve segment until it has a designated small size. When the bounding box size is small enough, we approximate the curve segment to polyline and intersects the polyline with given circle. Algorithm 6 shows this subdivision procedure.
Figure 13: Splitting of Offset Ellipse.

Figure 14: Bounding Boxes of Split Curve Segments on Offset Ellipse.
Algorithm: Intersect (Off-Ellipse $E(\alpha_{\text{min}}, \alpha_{\text{max}}, a, b)$, Circle $K$)

/* Return the intersection points between circle $K$ and curve segment of offset-ellipse from angle $\alpha_{\text{min}}$ to $\alpha_{\text{max}}$ with minor radius $a$ and major radius $b$ */

begin
    if the size of BoundingBox($E$) is smaller than $c_0$ then
        Output Linear Approximation of $E$ and $K$ intersection;
    else if BoundingBox($E$) is outside $K$ or inside $K$ then
        Output end points of $E$ and $K$ intersection;
    else
        begin
            Intersect ($E(\alpha_{\text{min}}, (\alpha_{\text{min}} + \alpha_{\text{max}})/2, a, b)$, $K$);
            Intersect ($E((\alpha_{\text{min}} + \alpha_{\text{max}})/2, \alpha_{\text{max}}, a, b)$, $K$);
    end
end

Table 6: Subdivision Procedure for Offsetted Ellipse and Circle Intersection.

When the offsetted ellipse and the circle intersect at a point $p_0$, the contact point between the corresponding cylinder and circle is the point on the ellipse whose offset is $p_0$.

The contact curve endpoints of the circle with rotating cylinder are compute by offsetted ellipse and circle intersection also. The boundary of swept circle consists of two circles $K_{R+}(n, v)$. We project the boundary circle to the plane $P$ which contains the rotation center $m$ of $C(\theta)$ and with normal vector parallel to the axis of $C(\theta)$. The projection of a spatial circle $K_{R}(n, v)$ onto a plane $P(m, u)$ is an ellipse (Figure 15). The ellipse is with minor axis $- (u, v) v + u$, major axis $u \times v$, minor radius $R(u, v)$, major radius $R$, and center $n'$, where $n' = p + (n - p, v) v$. For the rotating cylinder $C(\omega)$ whose radius is $s$, we compute $s$-offset of outer ellipse and $-s$-offset of inner ellipse, and we intersect the offsetted ellipses with the cylinder center trajectory. If the intersection point is $q$, the contact curve end point is an angle of $q - m$ with the basis vectors of rotating cylinder $C(\omega)$.

5.5 Horizontal Cylinder and Circle Contact

Contact angle and contact point are computed in the same way as the case when vertical cylinder and circle contact.

The contact curve endpoints are computed by shrinking horizontal cylinders as lines and offset the boundary circles of swept circle as tori. If we compute the angle of the line which contacts with the tori, the angles consist of the contact curve endpoints.
5.6 Sphere and Horizontal Cylinder Contact

Given sphere $S(\theta_0) = S_r(p)$, we compute the contact angle of rotating cylinder $C(\omega) = C_s(n, b(\omega))$ as follows. The contact angle of $C(\omega)$ with the sphere $S_r(p)$ is the same as that of cylinder axis $L(n, b(\omega))$ with the offsetted sphere $S'(\theta_0) = S_{r+}(p)$ (Figure 16(a)).

For the cylinder $C(\omega)$, $n$ is the rotation center and $v$ is the rotation axis. When we cut the sphere $S'(\theta_0)$ with the plane $P(n, v)$, the intersection curve consists of two circles with a common center point $p_0 = p + (v, n - p)$ and normal vector $v$, and radii $\sqrt{(r \pm s)^2 - \|p - p_0\|^2}$, respectively. The contact angle of the line from $n$ to these circles are the contact angle of horizontal cylinder for given sphere.

When a circle is given as $(x - x_0)^2 + (y - y_0)^2 = \delta^2$, the tangent line from origin $(0, 0)$ to the circle has the angle from $x$-axis $\tan^{-1}(y_0, x_0) \pm \alpha$, where $\alpha = \cos^{-1}((x_0^2 + y_0^2 - \delta^2)/(x_0^2 + y_0^2))$.

The contact point between a sphere $S_r(p)$ and a cylinder $C_s(n, v)$ is computed as follows. The closest point of the cylinder axis, $l(n, v)$, from the sphere center point $p$ is $q$:

Then the contact point is one of the points $p'$ whose distance from $q$ is $s$.

To compute the contact curve endpoints, we shrink the rotating sphere as a rotating vertex, and offset the rotating cylinder as much as the radius of sphere. Given horizontal cylinder $C(\omega)$ with radius $r$, when the rotation center is $n$ and the rotation axis is $v$, the swept surface consists of two planes $P(n \pm (r + s)v, v)$. The contact curve endpoints are computed by intersecting $P(n \pm (r + s)v, v)$ with $K_{r}(m, u)$ which is the center trajectory of rotating sphere (Figure 16(b)).
Figure 16: Sphere and Horizontal Cylinder Contact.
5.7 Sphere and Vertical Cylinder Contact

Given sphere $S_r(p)$, the contact angle of rotating cylinder $C(\omega) = C_s(n + Rb(\omega), v)$ is the same as the contact angle of cylinder axis $L(n + Rb(\omega), v)$ with the $\pm s$-offset of the sphere $S_r(p)$. When we cut the swept surface of rotating cylinder's axis with the plane $P(p, v)$, there is a circle $K_R(n', v)$, where $n' = n + (p - n, v) v$. This plane also cuts the $s$-offset of sphere into two circles $K_{r\pm s}(p, v)$. The intersection points of these circles $q \in K_R(n', v) \cap K_{r\pm s}(p, v)$ are used to compute contact angles. The angle of $q - n'$ with the basis vectors is the contact angle (Figure 17).

The contact point of a sphere and a vertical cylinder is computed in the same way as the case of sphere and horizontal cylinder contact.

The contact curve endpoints are computed by intersecting the rotating sphere’s center trajectory with the offset swept surface of rotating vertical cylinders. We intersect the sphere’s center trajectory with the swept surface of offset cylinder (Figure 18). Let’s assume that the rotating sphere has a center trajectory $K_{R_1}(m, u)$. The swept surface of $r$-offset of the rotating cylinder consists of two cylinders: $C_{\delta}(n, v)$, where $\delta = R_2 \pm (r + s)$.

The plane which contains the sphere center trajectory is $P(n, v)$. When we cut the cylinder $C_{\delta}(n, v)$ with the plane $P(n, v)$, the intersection consists of ellipses or lines. If $(u, v) = 0$, the intersection consists of lines (Figure 18(a)). In this case the intersection points between the circle
$K_{R_1}(m, u)$ and $C\delta(n, v)$ are as follows:

$$Q = m + x\frac{n' - m}{\|n' - m\|} + yv,$$

where $x = \|n' - m\|\pm\sqrt{(R_2 \pm (r+s))^2 - \|n' - n''\|^2}$, $y = \pm\sqrt{R_1^2 - x^2}$, $n'' = n + (m - n, v)v$, and $n' = n'' + (m - n', u)u$.

When $(u, v) \neq 0$, the intersection consists of two ellipses (Figures 18(b)-(c)). We can represent the plane $P(m, u)$ as $P: u_x x + u_y y + u_z z = (m, u) = 0$. Then two ellipses have the common center $n' = P \cap (n + \beta v) = n + \frac{(u, m - n)}{u, v}v$. They have the common minor and major axis, where the minor axis is $b_{x2} = u \times v / \|u \times v\|$ and the major axis is $b_{y2} = (\langle u, v \rangle u - v) / (\| \langle u, v \rangle u - v\|)$. The minor and major radii of inner ellipse are $R_2 - (r + s)$ and $|R_2 - (r + s)| / \langle u, v \rangle$, respectively. The minor and major radii of outer ellipse are $R_2 + (r + s)$ and $|R_2 + (r + s)| / \langle u, v \rangle$, respectively.

When $A$ and $B$ are minor, major radii of the ellipse, respectively, the intersection between the ellipse $E: B^2 x^2 + A^2 y^2 = A^2 B^2$ and the circle $K: (c_x + R_1 \cos \theta, c_y + R_1 \sin \theta)$, where $c_x = (m - n', b_{x2})$ and $c_y = (m - n', b_{y2})$, consists of the points:

$$Q = m + R_1(\cos \theta_1 b_{x2} + \sin \theta_1 b_{y2}),$$

where $\theta_1$ is the solution of the following equation:

$$B^2 R_1^2 \cos^2 \theta + A^2 R_1^2 \sin^2 \theta + 2B^2 c_x R_1 \cos \theta + 2A^2 c_y R_1 \sin \theta + B^2 c_x^2 + A^2 c_y^2 - A^2 B^2 = 0.$$
5.8 Sphere and Sphere Contact

Given sphere \( S_r(p) \), we compute the contact angle of \( S(\omega) = S_e(n + r_2 b(\omega)) \). To compute contact angle, we shrink the rotating sphere \( S(\omega) \) to a rotating point, and we offset the given sphere \( S_r(p) \) as much as the radius of sphere \( S(\omega) \). Then, by intersecting the center trajectory of \( S(\omega), n + R_2 b(\omega) \), with the swept surface of \( \pm r \)-offset \( S_e(p) \), i.e. \( S_{r \pm e}(p) \), consists of the contact angle (Figure 19).

There is a plane \( P(n, v) \) which contains the center trajectory of \( S(\omega) \). When we cut two spheres \( S_{r \pm e}(p) \) with \( P(n, v) \), the intersection circles have a common center point \( p' = p - (p - n, v) v \). They also have a common normal vector \( v \) and radii \( \sqrt{(r \pm s)^2 - (n - p, v)^2} \), respectively.

We compute the intersection between two circles \( (x - A)^2 + y^2 = l^2 \) and \( x^2 + y^2 = R_2^2 \), where \( A = \|p' - n\| \), the contact angle is the angle of \( q = x b_x + y b_y \), where \( b_x = \frac{p'r - n}{\|p' - n\|} \) and \( b_y = \frac{v \times b_x}{\|v \times b_x\|} \), in the plane with basis vectors \( b_x \) and \( b_y \).

\[ \theta = \text{atan2}(\langle q, b_y \rangle, \langle q, b_x \rangle) \]

![Diagram of sphere and sphere contact](image)

**Figure 19:** Sphere and Sphere Contact.

The contact point between two spheres \( S_r(p) \) and \( S_e(q) \) is one of the points \( p \pm t \frac{q - p}{\|q - p\|} \), whose distance from \( q \) is \( s \).

We compute the contact curve endpoints between two rotating spheres as the intersection between the center trajectory of \( S_1(\theta) \), and \( \pm r \)-offset of swept surface of \( S_2(\omega) \). Let’s denote the center trajectory of \( S_1(\theta) \) as a circle \( K(\theta) = m + R_1 (\cos \theta b_{z_1} + \sin \theta b_{y_1}) \). The offset of the swept surface of \( S_2(\omega) \) consists of two tori \( T_{R_2, r \pm e}(n, v) \) (Figure 20).

To simplify the computation of torus and circle intersection, we translate and rotate the tori to be positioned at standard position (Figure 21). In this case, \( K(\theta) \) is transposed to a circle with center point:

\[ m' = ((m - n, b_{z_2}), (m - n, b_{y_2}), (m - n, v)) \]

with basis vectors \( b_{z_1}', b_{y_1}', \) and \( u' \). The basis vectors are as follows:

\[ b_{z_1}' = ((b_{z_1}, b_{z_2}), (b_{z_1}, b_{y_2}), (b_{z_1}, v)) \]
\[ b'_{y_1} = \{(b_{y_1}, b_{z_2}), (b_{y_1}, b_{y_2}), (b_{y_1}, v)\} \]
\[ u' = \{(u, b_{x_2}), (u, b_{y_2}), (u, v)\}. \]

Then, we can represent the torus as follows:

\[
(x'^2 + y'^2 + z'^2 + R_i^2 - (r \pm s)^2)^2 - 4R_i^2(x'^2 + y'^2) = 0,
\]

and circle as:

\[
m' + R_i (\cos \theta b'_{x_1} + \sin \theta b'_{y_1}).
\]

We intersect the torus and circle by solving quartic equation:

\[
(m'^2 + R_i^2 + 2R_i (\cos \theta \langle m', b'_{x_1} \rangle + \sin \theta \langle m', b'_{y_1} \rangle) + R_i^2 - (r \pm s)^2)^2 - 4R_i^2((m'_{x_2} + R_i (\cos \theta b'_{x_1} + \sin \theta b'_{y_1}))^2 + (m'_{y_2} + R_i (\cos \theta b'_{x_1} + \sin \theta b'_{y_1}))^2) = 0.
\]
5.9 Sphere and Circle Contact

Given sphere $S_r(p)$, we compute the contact angle of rotating circle $K(\omega) = K_s(n + Rb(\omega), v)$. When the plane which contains the circle $K(\omega)$ is $P(n, v)$, the intersection of this plane with given sphere consists of a circle with a center point $p' = p + (n - p, v) v$, normal vector $v$, and a radius $l = \sqrt{r^2 - (n - p, v)^2}$ (Figure 22(a)). We compute the contact angle by offsetting this circle with offset distance $\pm s$, where $s$ is the radius of rotating circle $n + Rb(\omega)$ and then intersect offset circles with the center trajectory of rotating circle $K_R(n, v)$ (Figure 22(b)).

![Diagram of Sphere and Circle Contact](image)

The contact point between a sphere $S_r(p)$ and a circle $K_s(q, v)$ is one of the points $q' = \frac{q - q'}{||q - q'||}$, where $q' = p + (q - p, v) v$ and $r' = \sqrt{r^2 - (q - p, v)^2}$.

To compute the contact curve endpoints, we follow three steps: i) compute the swept surface of rotating circle, ii) offset the swept surface with offset distance $r$, where $r$ is the radius of rotating sphere, and iii) intersect the offset surface with the center trajectory of rotating sphere. When the rotating circle $K(\omega)$ has a center trajectory $n + Rb(\omega)$, normal vector $v$, and a radius $s$, the swept surface consists of two boundary circles $K_{\pm r}(n, v)$ and the planar surface between these two circles. The $r$-offset of this swept surface consists of two tori and two planes (Figure 23).

5.10 Plane and Sphere Contact

Given plane $P(\theta_0) = P(p, u)$, we compute the contact angle of $S(\omega) = S_s(n + Rb(\omega))$. We offset the plane with offset distance $\pm s$, where $s$ is the radius of given sphere. Then, the contact angle of $S(\omega)$ with $P(\theta_0)$ is the same as the contact angle of the center of $S(\omega)$ with the offset plane (Figure 24). The $\pm s$-offset of the plane $P(p, u)$ consists of two planes $P(p \pm su, u)$, and the center trajectory of the sphere $S(\omega)$ is a circle $n + Rb(\omega)$.

The intersection between the plane set $P(p \pm su, u)$ and the center trajectory of sphere $n + Rb(\omega)$ is the value of $\omega$ which is the solution of following equation: $(p \pm su - (n + Rb(\omega)), u) = \ldots$
Figure 23: Offset of Swept Circle.

Figure 24: Contact Angle of Rotating Sphere for a Plane.

The contact point between a plane $P(p, u)$ and a sphere $S_x(q)$ is $q + (p - q, u) u$.

To compute contact curve endpoints, we offset given rotating planes with offset distance $\pm s$ and shrink the rotating sphere as a point, then we find the plane which tangentially touches the center trajectory of the rotating sphere. Let's assume that the center trajectory of the sphere is given as a circle $K$ with radius $R_1$, center point $O$, and is contained on $xy$-plane. When the rotation axis for plane is $u$, let's denote the rotation basis vectors as $b_x$ and $b_y$, where $(b_x, b_y) = (b_x, u) = 0$.

If the plane is parallel to the rotation axis $u$, we can represent the rotating plane as $P(m + R_1 a(\theta), a(\theta))$, where $a(\theta) = M_1 (\cos \theta, \sin \theta, 0)^T$. Then, the intersection line of rotating plane with $xy$-plane is

$$(\cos \theta b_{xz} + \sin \theta b_{yz}) x + (\cos \theta b_{xy} + \sin \theta b_{yy}) y = \langle m, a(\theta) \rangle + R_1.$$
and the contact angle of the intersection line with the circle \( K \) is \( \theta \) which satisfies following equation:

\[
R_2^2((\cos \theta b_{xx} + \sin \theta b_{yx})^2 + (\cos \theta b_{xy} + \sin \theta b_{yy})^2) = ((m, a(\theta)) + R_1)^2.
\]

Otherwise, we can represent the rotating plane as \( P(m, N(\theta)) \), where \( N(\theta) = (\cos \theta N_x - \sin \theta N_y) b_x + (\sin \theta N_x + \cos \theta N_y) b_y + N_z u \). The intersection line of rotating plane with \( xy \)-plane is

\[
A(\theta)x + B(\theta)y = (m, N(\theta)),
\]

where \( A(\theta) = (\cos \theta N_x - \sin \theta N_y) b_x + (\sin \theta N_x + \cos \theta N_y) b_y + N_z b_z \) and \( B(\theta) = (\cos \theta N_x - \sin \theta N_y) b_x + (\sin \theta N_x + \cos \theta N_y) b_y + N_z b_y \), and the contact angle of the intersection line with the circle \( K \) is \( \theta \) which satisfies following equation:

\[
R_2^2(A(\theta)^2 + B(\theta)^2) = (m, N(\theta))^2.
\]

### 5.11 Plane and Circle Contact

We compute the contact angle of the rotating circle \( K_r(\mathbf{n} + R_2b(\omega), \mathbf{v}) \) for given plane \( P(\theta_0) = P(p, u) \). To compute contact angle, we offset given plane with offset distance \( \pm l \), where \( l = \frac{s \sin(\cos^{-1}(\langle u, v \rangle))}{(u, v)} \) (Figure 25). The \( \pm l \)-offset plane consists of two planes: \( P(p \pm lu, u) \). The intersection of the circle center trajectory \( \mathbf{n} + R_2b(\omega) \) and \( \pm l \)-offset plane is computed by solving the following equation:

\[
\langle b(\omega), u \rangle = \langle p - \mathbf{n}, u \rangle \pm l,
\]

and the contact angle is \( \omega \) value which satisfies this equation.

![Figure 25: Contact Angle of Circle for Given Plane.](image)

The contact point between a plane \( P(p, u) \) and a circle \( K_r(\mathbf{q}, \mathbf{v}) \) is one of the points \( \mathbf{q} \pm s\frac{(v, u) v - u}{\|v, u\|} \), which is on the plane \( P(p, u) \).

The contact curve endpoints are computed by finding tangential plane to the boundary of swept curve of rotating circle. The boundary of swept rotating circle consists of two circles \( K_{R \pm \epsilon}(\mathbf{n}, \mathbf{v}) \).
The intersection between the rotating plane and the main plane of these two circles consists of a line, and we compute the tangentially touching line to the boundary circle in the same way as explained in the plane and sphere contact case.

5.12 Circle and Circle Contact

Given circle \( K_1(\theta_0) = K_r(p, u) \), we compute the angle of rotating circle \( K_2(\omega) = K_s(n + R_2 b(\omega), v) \) which contacts with \( K_1(\theta_0) \). Let \( Q \) denote a set of intersection points between \( K_1(\theta_0) \) and the plane \( P(n, v) \). The contact angle is the set of \( \omega \) values which corresponds to the intersection points \((n + R_2 b(\omega)) \cap K_s(q, v)\), where \( q \in Q \) (Figure 26).

![Figure 26: Contact Angle of a Circle for Given Circle.](image)

The contact point between two circles \( K_r(p, u) \) and \( K_s(q, v) \) is one of the points \( q \pm s \frac{(v, u) v - u}{\| (v, u) v - u \|} \), whose distance from \( p \) is \( r \).

Let's denote the sweeping surfaces of \( K_1(\theta) \) and \( K_2(\omega) \) as \( K_1 \) and \( K_2 \), respectively:

\[
K_1 = \bigcup_{\theta} K_r(m + R_1 a(\theta), u) = \bigcup_{\theta} K_1(\theta)
\]

\[
K_2 = \bigcup_{\omega} K_s(n + R_2 b(\omega), v) = \bigcup_{\omega} K_2(\omega)
\]

Then, we compute the contact curve endpoints of \( \theta \), where \( K_1(\theta) \cap K_2 \neq \emptyset \), as follows: i) Compute the set of line segments \( L = K_1 \cap K_2 \) and ii) Compute \( \theta \) values which corresponds to the intersection points \( K_r(m + R_1 a(\theta), u) \cap (r\text{-offset of } L \text{ on } P(m, u)) \), (Figure 27).

6 More Examples

In this section we show seven examples of gear or follower/cam pairs and the configuration spaces of them. Each configuration space in this section consists of blocked space, free space, and con-
Figure 27: Contact Curve Endpoints of Two Circles.

tact space represented by grey area, white area, and black curve, respectively. The dot in each configuration space marks the configuration of the related parts.

Figure 28a shows a pair of orthogonal spatial gears, where the driver rotates by the angle $\theta$ and the follower rotates by the angle $\omega$. Each gear consists of a cylindrical plate with five evenly spaced teeth. The height of each tooth is 6.87mm, and each tooth consists of a cylindrical patch topped by truncated sphere. The gears rotate around their rotation axes, which are orthogonal to each other. Rotating one gear causes the other rotates the same angle. We show the configuration space of orthogonal spatial gears in Figure 28b. The contact space consists of the contacts between tooth sides, caps, and boundary of tops which are computed by cylinder/cylinder, cylinder/sphere, cylinder/circle, sphere/circle, and circle/circle contact. It shows a nearly linear gear ratio. The free space with initial angle (2.13, -1.60) consists of one narrow channel that quantifies gear play.

When the driver is tilted, its configuration space changes. Figure 28c shows the orthogonal spatial gears when the driver is tilted by 10 degrees around local y-axis. The configuration space of this pair is in Figure 28d. The free space with initial angle (2.30, -1.69) consists of a narrow strip whose shape is slightly different and shifted to the positive $\theta$ direction from that of Figure 28b.

When the tooth height gets taller, the change in configuration space gets more significant. In Figures 29a and 29b, we show the orthogonal gear pairs for the case when each tooth height is 7.48mm and its configuration space, respectively. The free space with initial angle (1.49, -2.01) is narrower and less straight than that of tooth height 6.87mm. Figure 29c shows the gear pairs when the tooth height is 8.16mm, and Figure 29d shows the configuration space which has free space with initial angle (1.16, -1.67). From the initial angle, if we increase the angle $\omega$ and decrease the angle $\theta$, the free space is blocked. For the corresponding gear pair (See Figure 29c), if we rotate the driver to the negative direction and the follower to the positive direction a little, this pair will be almost jammed.

Figure 30a shows a pair of involute gears with parallel rotation axes, where one gear is a driver with rotation angle $\theta$ and the other is a follower with rotation angle $\omega$. The shapes of two gears are the same, and each gear consists of evenly spaced 16 teeth. The sides of each tooth consists of 6 cylindrical patches connected by 7 line segments and the tops consists of 12 circular arcs and
Figure 28: Contact of Orthogonal Gears
Figure 29: Contact of Orthogonal Gears
7 vertices. A full rotation of one gear advances another gear by full rotation. Figure 30b shows the configuration space of the pair of involute gears and the free space with initial angle (-1.62, 1.61). The contact space consists of almost linear lines, and it shows the linear gear ratio. The contact space consists of the arc/arc and arc/vertex contact of two gears because the rotation axes are parallel.

Figure 30c shows the case when the driver is tilted by 2 degrees around local x-axis. The configuration space has a narrower free space than the nominal case (Figure 30d), where the contact space consists of the arc/cylinder and arc/line contacts, rather than arc/arc and arc/vertex contacts. Figure 30e shows the case when the gear with rotation angle $\theta$ is tilted by 2 degrees around local y-axis. The free space in the configuration space is much narrower now (Figure 30f).

We show the geneva gear pair in Figure 31a. The cam consists of a pin and two concentric cylindrical patches which are connected by two planes. The follower consists of four concave cylindrical patches with four evenly spaced slots. The cam rotates around the center axis of two cylinders and the follower rotates around an axis parallel to the rotation axis of the cam. As the cam rotates, the pin engages a follower slot and rotates it by a quarter of a turn. When the pin disengages, the cam cylinder engages the concave cylindrical part of the follower and prevents it from moving until the pin engages the next slot. The configuration space coordinates are the cam angle $\theta$ and the follower angle $\omega$. Figure 31a shows the case when the rotation angles of this pair is $(-0.02, -3.07)$. The contact space contains four diagonal regions where the pin drives the follower, and four horizontal regions where the cylinder engages the cutouts (Figure 31b). The design task is to find a pin angle and a slot clearance that guarantee the correct contact sequence. For example, if the pin reaches a slot too soon, it hits the side and blocks.

In Figure 31c, the follower is tilted by 3 degrees around local x axis. Figure 31d shows its configuration space, which contains the free space with initial angle $(0.52, 2.40)$. The horizontal region still remains in the free space of the configuration space, but the diagonal region is blocked. This means while the cylinder of the cam is engaging the cutouts, the pair rotates without jam, but when the pin starts to engage the slot, free space between the pair gets narrower and eventually the pin hits the side of the slot.

Figure 31e shows geneva gear pair for the case when the follower is tilted by 3 degrees around local y axis. Figure 31f shows its configuration space with free space generated from initial angle $(-0.31, -2.56)$. This configuration space is not much different from the case when the rotation axes are parallel each other.

Figure 32a is the example of a gear and a slotted wheel pair. The gear consists of a cylindrical plate with evenly spaced twelve teeth. Two planes and two cylindrical patches compose the sides of the tooth, and the tops of the tooth are bounded by circular arcs and line segments. The wheel has one slot: wheel consists of one big cylindrical patch with bounding arcs, and the slot consists of two planes and one cylindrical patch with bounding line segments and arcs. Figure 32b shows the configuration space. One narrow horizontal channel of the free space is the space of angles when the slot of the wheel engages the gear. When the slot engages the gear, the gear may rotate freely. The vertical channels are composed for the case when the wheel engages between two teeth of the
Figure 30: Contact of Parallel Gears
Figure 31: Contact of 2 dimensional Geneva Gear
The contact space of the bevel gear pair consists of the vertex/plane and line/line contact curves. A full rotation of one gear advances another gear by full rotation.

Figures 33c and 33e show the bevel gear pairs for cases when the driver (gear with rotation angle $\theta$) is tilted by 2 degrees around local $x$ axis and local $y$ axis, respectively. Their configuration spaces are shown in Figure 33d and 33f, respectively. Figures 33d and 33f show the free space with initial angle $(1.54, -1.37)$ and $(1.38, -1.17)$ for each tilting case.

Figure 34a shows the example of half bevel gears pair. The bevel gear with rotation angle $\theta$ has only 8 teeth, and this gear drives the other gears. Each follower has evenly spaced 15 teeth, and the followers share the same rotation axis with rotation angle $\omega$ and $\psi$, respectively. Figure 34b and 34c show the configuration spaces of this pair. When the teeth of the driver engage the teeth of the gear with rotation angle $\omega$, the gear with rotation angle $\psi$ may rotate freely. If the teeth of the driver engage the teeth of the gear with rotation angle $\psi$, the gear with rotation angle $\omega$ has no restriction to be rotated.

The last example is a cam/follower pair (See Figure 35a). The cam has a constant-breadth profile consisting of three arc segments, cylindrical patches, lines, and six vertices. The follower
has a rectangular profile and consists of four planes which are bounded by line segments. Nominal, the cam rotates around global z-axis and the follower translates along global y-axis. Each cam rotation makes the follower go up and down three times. The narrow gap between the parts prevents jamming. Figure 35b shows the configuration space of cam/follower pair. Contact space contact between the cam and the lower and upper follower planes. Each consists of three pairs of alternating vertex/line and arc/line contact curves because the rotation axis of cam and follower are parallel in this case.

Figures 35c and 35d show the cam/follower pair and its configuration space when the cam is tilted by 2 degrees around local x-axis. In this case, the free space is narrower than the nominal case. The contact space consists of vertex/plane, arc/plane, and cylinder/line contacts.

7 Conclusion

We have presented a kinematic analysis program for spatial fixed-axes higher pairs. This program frees designers from manual derivation of contact curves and contact changes, helps them understand complex part interactions, and facilitates function validation. The program handles parts in a parametric boundary representation with planar, cylindrical, and spherical patches bounded by line segments, circular arcs, and vertices. We derived low-degree algebraic contact equations that
Figure 33: (Continued) Contact of Bevel Gears
Figure 34: Contact of Half Bevel Gears
Figure 35: Contact of Follower and Cam
are readily solvable in closed-form or numerically. We implemented every case, except for pairs of general cylinders.

We combined configuration space obstacle computation with planar projection to derive low-degree contact equations, which are key to a robust, efficient algorithm. The configuration space formulation allows us to solve the contact equations in closed form, except for general cylinder/general cylinder and cylinder/circle pairs. This formulation also helps us compute the contact curve endpoints, which define the contact curve structure. The endpoints are computed numerically for cylinder and circle contacts and in closed form otherwise.

We illustrate the program on ten examples: dwell gear pairs with parallel and skewed axes, a spatial indexer, a spatial Geneva, an orthogonal gear pair, an involute gear pair, a planar Geneva, a gear/slotted wheel pair, a bevel gear pair, a bevel mechanism, and a constant-breath cam/follower pair. The computations take a few seconds on a workstation. All the figures in the paper were produced by the program.

There are several directions for future development and research. One extension is to handle other useful part features, such as helicoids and tori for helical gears and screws. We can approximate these features with planes, cylinders, and spheres or can develop new contact equations for which numerical solutions would probably be required. Another extension is to general spatial pairs. Computing full configuration spaces for six degrees of freedom is impractical and unnecessary. Instead, we plan to develop an incremental algorithm that will construct the portion of the configuration space that is relevant to the design problem. We plan to extend our dynamical simulation [16] and tolerance analysis algorithms [17, 18] from planar to fixed-axes spatial systems. Both algorithms extend to spatial pairs without major changes. We need to validate the impact model for the dynamical simulator and to formulate sensitivity equations for the new contact equations.

References


