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Analytical Depoissonization and its Applications

Philippe Jacquet

Wojciech Szpankowski
Purdue University, spa@cs.purdue.edu

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Wojciech Szpankowski

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Abstract

In combinatorics and analysis of algorithms often a Poisson version of a problem (called further Poisson model or poissonization) is easier to solve than the original one, which we name here as the Bernoulli model. Poissonization is a technique that replaces the original input (e.g., think of balls thrown to urns) by a Poisson process (e.g., think of balls arriving according to a Poisson process to urns). More precisely, analytical Poisson transform maps a sequence (e.g., characterizing the Bernoulli model) into a generating function of a complex variable. However, after poissonization one must depoissonize in order to translate the results of the Poisson model into the original (i.e., Bernoulli) model. We present in this paper several analytical depoissonization results that fall into the following general scheme: if the Poisson transform has an appropriate growth in the complex plane, then an asymptotic expansion of the sequence can be expressed in terms of the Poisson transform and its derivatives evaluated on the real line. Not unexpectedly, actual formulations of depoissonization results depend on the nature of the growth, and thus we have polynomial and exponential depoissonization theorems. Renormalization (e.g., as in the central limit theorem) introduces another twist that led us to formulate the so called diagonal depoissonization theorems. Finally, we illustrate our results on numerous examples from combinatorics and the analysis of algorithms and data structures (e.g., combinatorial assemblies, digital trees, multiaccess protocols, probabilistic counting, selecting a leader, data compression, etc.).

Key Words: Poissonization, depoissonization, Bernoulli and Poisson models, analytical combinatorics, analysis of algorithms, Cauchy integral formula, saddle point method, limiting distributions, Mellin transform of complex variable.
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1 Introduction

In some algorithms (e.g., sorting and hashing are prominent examples) \( n \) elements are
placed randomly into \( m \) "bins". This paradigm is often called the "balls and urns" model.
Questions arise such as how many urns are empty, how many balls are required to fill up all
urns, etc. It is easy to see that the occupancy of urns are not independent (e.g., think all of
balls falling into one urn, thus we know that all the remaining urns are empty). To overcome
this difficulty an interesting probabilistic technique called poissonization was suggested.
Namely, it is assumed that balls are "generated" according to a Poisson process \( N \) with
mean \( z = n \). Due to some unique properties of the Poisson process, the stream of balls
are now placed independently in every urn, thus overcoming the above mention difficulty.
Observe, however, that poissonization has its own problems since one must "extract" the
original results from the Poisson model, that is, depoissonize the Poisson model.

To the best of our knowledge, poissonization was introduced by Marek Kac [30] a half
a century ago when investigated the deviations between theoretical and empirical distribu­
tions. Recently, poissonization was further popularized in the context of analysis of
algorithms and combinatorial structures by Aldous [1], Arratia and Tavaré [2], Gonnet [19],
Gonnet and Munro [20], Holst [22], Jacquet and Régnier [25], Jacquet and Szpankowski
[26, 28], Rais et al. [43], Fill et al. [13], Kirschenhofer et al. [32], and others.

Before we spell out more succinctly our depoissonization results, we first describe another
scenario. In the analysis of algorithms and/or enumeration of combinatorial structures,
often the original problem – which we further call the Bernoulli model – is represented by
a recurrence equation or by a functional/differential equation. For example, this situation
arises in a large class of algorithms involving a splitting process and/or digital trees (cf.
[11, 13, 14, 24, 25, 26, 27, 28, 32, 33, 34, 37, 43, 44, 50, 51]). Imbedding this splitting process
into a Poisson process leads often to a more tractable functional/differential equations.
This was called by Gonnet and Munro [20] the Poisson Transform. We call this technique
analytical poissonization.

More formally, let \( g_n \) be a characteristic of the Bernoulli model of size \( n \). The Poisson
transform is defined as \( \tilde{G}(z) = E g_N = \sum_{n \geq 0} g_n \frac{z^n}{n!} e^{-z} \), that is, the input \( n \) becomes a Poisson
variable \( N \) with mean \( z \) when \( z \geq 0 \). If \( \tilde{G}(z) \) is known, one can extract the coefficient
\( g_n = n! [z^n](\tilde{G}(z)e^z) \) (cf. [20, 21, 40]). However, in most interesting situation \( \tilde{G}(z) \) satisfies
a complicated functional/differential equation that is difficult to solve exactly. Nevertheless,
one can find an asymptotic expansion of \( \tilde{G}(z) \) for \( z \rightarrow \infty \) on real axis and bound \( \tilde{G}(z) \) in
the complex plane. Then, one aims at finding an asymptotic expansion of \( g_n \) from the
asymptotics of $\tilde{G}(z)$. This is called the *analytical depoissonization*. Depoissonization is relatively easy when the Poisson transform $\tilde{G}(z)$ is a meromorphic or algebraic function (cf. [19, 20, 41]) since one can either apply Cauchy residue theorem or singularity analysis of Flajolet and Odlyzko [17]. In this paper we assume that $\tilde{G}(z)$ is an entire function, and our goal is to provide easily applicable tools for depoissonization. In passing, we observe that probabilistic depoissonization faces the same kind of difficulties (cf. [1, 22]).

One may ask *why Poisson process*. Why not to imbed our Bernoulli model into another process? This seems to be a consequence of certain unique properties of the Poisson process that we discuss in some details in the next section. Briefly: Poisson process is the only process that has stationary and independent increments, and which is orderliness (no group arrives) [46]. Actually, in analysis of splitting process two other properties are even more important, namely: superposition of renewal processes is renewal if and only if it is Poisson; and a thinning process in which arrivals are accepted or rejected on an independent basis is Poisson if the original process is Poisson, too. For the ball and urns model another property is of some importance. It says that once we know that there are $n$ arrivals from a Poisson process in a time interval of duration $T$, then these arrivals are distributed uniformly within $(0,T)$.

There is no much literature on the analytical depoissonization, and most published results are tailored to specific problems. Early results of Jacquet and Régnier [24, 25, 44] are similar in spirit to ours, and are special cases of our general results. Jacquet and Szpankowski [26], and Rais et al. [43] used some simple version of depoissonization. The result of this paper have been partially motivated by our depoissonization results of [28] where we dealt with a nonlinear multiplicative differential/functional equation, and a new and sophisticated tools had to be applied to solve the problem. However, to the best of our knowledge none of the result proposed in this paper has been published before in its current form.

Results of this paper can be summarized as follows: We start with a brief account on analytical poissonization (cf. Section 2). Then – after surveying some problems of combinatorics and analysis of algorithms where poissonization/depoissonization plays a significant role (cf. Section 3) – we present an exhaustive list of depoissonization results (cf. Sections 4 and 5). We end up with some applications of our findings (cf. Section 6): In particular, we discuss conflict resolution algorithms, digital trees, randomized algorithm for electing a leader, probabilistic counting, as well as a general depoissonization result for a class of linear functional equations.

Our main results are presented in Sections 4 and 5, and all of them are new (or at
least presented in a stronger version). While our goal is to prepare a readable account on
the depoissonization and its usefulness to (precise) analysis of algorithms and analytical
combinatoric, we aspire to present rigorous derivations of depoissonization results so this
paper could be used as a reliable reference on depoissonization. Actual formulations of
the depoissonization statements depend on the growth of the Poisson transform in a cone
around the real axis. In particular, we have polynomial and exponential depoissonization
theorems (cf. Theorems 1, 2 and 3). Renormalization (e.g., as in the central limit theorem)
introduces another twist that led us to formulate the so called diagonal depoissonization
theorems (cf. Theorems 4, 5 and 8). The most general depoissonization result is presented in
Theorem 5 which is used to prove all our depoissonization findings not dealing with limiting
distributions. A corresponding general depoissonization tool for limiting distributions is
proposed in Theorem 8 which is ultimately aimed at general central limit theorems that are
also discussed in this paper (cf. Theorems 7 and 9).

2 Poisson Transform

We briefly discuss poissonization technique: First, we review probabilistic poissonization
and explain its successes in solving some problems. Then, we briefly discuss analytical
poissonization, and introduce some definitions that are used throughout the paper.

2.1 Probabilistic Poissonization

Consider a combinatorial structures in which \( n \) objects are randomly distributed into some
locations, e.g., one can think of \( n \) balls thrown into urns. The objects are not necessary
uniformly distributed among the locations (cf. digital trees example below). We call such
a setting the Bernoulli model. Let \( X_n \) be a characteristic of the model (e.g., the number of
throws needed to fill up all urns, moment generating function of the number of nonempty
urns, etc.).

Next, we define the Poisson model. Let \( N \) be a random variable distributed according
to Poisson with parameter \( z \geq 0 \), that is, \( \Pr\{N = k\} = e^{-z} \frac{z^k}{k!} \). Let \( X_N \) be the above
characteristic defined in the Poisson model in which the deterministic
input (i.e., \( n \)) is
replaced by the Poisson variable \( N \) with parameter \( z = n \). Then by definition

\[
\bar{X}(z) := \mathbb{E}X_N = \sum_{n\geq 0} \mathbb{E}(X_N \mid N = n) e^{-z} \frac{z^n}{n!}
\]

\[
= \sum_{n\geq 0} \mathbb{E}X_n e^{-z} \frac{z^n}{n!}, \quad (1)
\]
and we use this formula to define $\tilde{X}(z)$ for arbitrary complex $z$. Throughout, we make the following assumption (with an exception of our most general depoissonization finding, namely Theorem 5 where we slightly modify this assumption):

(A) The sum in (1) converges absolutely for every $z$. Hence, the Poisson characteristic $\tilde{X}(z)$ is an entire function of the complex variable $z$.

Before we proceed, we must justify the last equality in (1), that is, we show that the conditional average $E(X_N|N = n)$ can be replaced by the unconditional $EX_n$ which is exactly the Bernoulli model characteristic. This follows from the following three unique properties of the Poisson process that we discuss next.

We first define superposition and thinning or splitting process of a renewal (point) process. Consider two (stationary) renewal processes (cf. [46]), say $N_1$ and $N_2$. Then, the superposition process $N = N_1 + N_2$ consists of all renewal points of both processes. To define the splitting or thinning process, take a single renewal process and for each point decide independently whether to omit it (thinning) or to direct it to one of two (or more) outputs (splitting). For more precise definitions see Ross [46]. The following three properties are well known (cf. [46]):

(P1) A stationary renewal process is the superposition of some independent renewal processes only if the process is Poisson.

(P2) A thinning or splitting process is Poisson with parameter $zp$ where $p$ is probability of thinning if the original process is Poisson with parameter $z$.

(P3) Let $N(t)$ denote Poisson arriving points in the interval $(0, t)$. Then,

$$\Pr\{N(x) = k \mid N(t) = n\} = \binom{n}{k} \left(\frac{x}{t}\right)^k (1 - x/t)^{n-k}$$

where $x \leq t$. In other words, the points of a Poisson process are uniformly and independently distributed in $(0, t)$ conditioned on $n$ arrivals in this interval.

The last property is sometimes called "random occurrence of conditional Poisson process".

2.2 Analytical Poissonization

Equation (1) can be viewed as a definition of the Poisson transform (cf. [20]) when it is analytically continued to complex $z$. Indeed, assuming we know $\tilde{X}(z)$ we can obtain $EX_n$ as the coefficient at $z^n/n!$ of $X(z) = \tilde{X}(z)e^z$. The reader is referred to [19, 21, 40] for some
Table 1: Some Poisson Transforms and Their Properties

<table>
<thead>
<tr>
<th>$g_n$</th>
<th>$\tilde{G}(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const</td>
<td>Const</td>
</tr>
<tr>
<td>$(-1)^n$</td>
<td>$e^{-2z}$</td>
</tr>
<tr>
<td>$\alpha^n$</td>
<td>$e^{(\alpha-1)z}$</td>
</tr>
<tr>
<td>$n(n-1)(n-k+1)$ additive $(p,q)$-splitting on $F_n$ and $G_n$; $p+q=1$</td>
<td>$F(pz)+G(qz)$</td>
</tr>
<tr>
<td>multiplicative $(p,q)$-splitting on $F_n$ and $G_n$</td>
<td>$F(pz)G(qz)$</td>
</tr>
</tbody>
</table>

results on the exact or approximate "reverse" Poisson transform. Our goal is different: We are aiming at presenting easily verifiable conditions that allow to extract asymptotically $\mathbb{E}X_n$ from $\tilde{X}(z)$ provided we only know $\tilde{X}(z)$ asymptotically for $z \to \infty$ on the real axis and can bound the growth of $\tilde{X}(z)$ in the complex plane.

Before we discuss some examples a word about notation. We write $g_n$ as a generic notation for a sequence characterizing the Bernoulli model (e.g., $g_n = \mathbb{E}X_n$ or in general $g_n = \mathbb{E}f(X_n)$ for some function $f(\cdot)$). Then, we denote by $\tilde{G}(z)$ or $\mathcal{P}(g_n; z)$ its Poisson transform, that is,

$$\tilde{G}(z) = \mathcal{P}(g_n; z) := \sum_{n \geq 0} g_n \frac{z^n}{n!} e^{-z}.$$

Section 3 below describes several Poisson transforms arising in combinatorics and the analysis for algorithms and data structures.

However, poissonization/depoissonization is also a useful tool for studying limiting distributions. In this case, we consider a random variable $X_n$, and its generating function $G_n(u) = \mathbb{E}u^{X_n}$ for some complex $u$. Then, the Poisson transform of $G_n(u)$ is $\tilde{G}(z, u) = \sum_{n \geq 0} G_n(u) \frac{z^n}{n!} e^{-z}$. In general, we may define double-indexed sequence $g_{n,k}$ (e.g., think of $g_{n,k} = \mathbb{P}(X_n = k)$), and then the Poisson transform becomes

$$\tilde{G}(z, u) = \mathcal{P}(G_n(u); z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-z} \sum_{k=0}^{\infty} g_{n,k} u^k.$$

We discuss several examples of this kind below in Sections 3.2 – 3.5.

Let us present now some Poisson transforms and their properties. Let $g_n$ be an arbitrary sequence. In the sequel, we say that a sequence $g_n$ is Bernoulli additive $(p,q)$-splitting of
sequences \( f_n \) and \( h_n \) if \( g_n = \sum_{k=0}^{n} (\binom{n}{k}) p^k q^{n-k} (f_k + h_{n-k}) \), where the pair \((p, q)\) is the split probability vector with \( p + q = 1 \). In this case, we have \( \tilde{G}(z) = \tilde{F}(pz) + \tilde{H}(qz) \) with \( \tilde{F}(z) \) and \( \tilde{H}(z) \) being Poisson transforms of sequences \( f_n \) and \( h_n \), respectively. This property extends in a straightforward manner to \((p_1, \ldots, p_\ell)\)-splittings of arbitrary \( \ell \) sequences \( f^n_1 \) with probability vector \( p_1, p_1 + \cdots + p_\ell = 1 \). Indeed, \( \tilde{G}(z) = \tilde{f}_1^1(p_1z) + \cdots + \tilde{f}_\ell(p_\ell z) \) with \( \tilde{f}_i(z) \) being the Poisson transform of the sequence \( f^n_i \). In the same spirit, one can define multiplicative \((p, q)\)-splitting: \( g_n = \sum_{k=0}^{n} (\binom{n}{k}) p^k q^{n-k} f_k h_{n-k} \). Then, \( \tilde{G}(z) = \tilde{F}(pz)\tilde{H}(qz) \).

Table 1 summarizes these findings.

A careful reader should conclude from the above table that \( g_n \) is asymptotically equivalent to \( \tilde{G}(n) \) (i.e., \( g_n \sim \tilde{G}(n) \)) when \( n \to \infty \) in all cases except when \( g_n = \alpha^n \) (think of \( \alpha = -1 \)). This cannot be a coincidence, and in this paper we systematically explore this fact.

3 Poissonization in Combinatorics and Analysis of Algorithms

In this section, we discuss several examples from the fields of combinatorics and analysis of algorithms where poissonization and depoissonization turn out to be useful, if sometimes not the only technique, that may produce a result. In the sequel, we also try to show that the Poisson transform can be derived directly from the probabilistic analysis of the problem at hand with the omission of intermediate steps (and tedious algebraic manipulations) unavoidable when dealing with the Bernoulli model.

3.1 Conflict Resolution Algorithms

Imagine a (infinite) collection of (distributed) users trying to send messages among themselves. There is no coordination among them except that a transmission of a fixed length packet must start at the beginning of a time slot. When only one user sends a message, then a successful transmission takes place. Otherwise, there is a conflict, and all users involved in it must apply an algorithm to solve the conflict (i.e., conflict resolution algorithm). We consider two possible solution, namely: tree-type algorithm (cf. [4, 11, 10, 21, 38, 47, 50] and interval searching algorithm (cf. [5, 26, 52]).

In the tree-type algorithm, all colliding users, say \( n \) of them, flip a biased coin and only those who got head are allowed to transmit in the next slot, while the others must wait until the former group solved their subconflicts. Let \( L_n \) be the length of the conflict resolution session provided \( n \) users are initially involved in the conflict. Let also \( p \) be the probability of flipping the head, thus \( q = 1 - p \) flipping the tail. Clearly, \( L_n \) satisfies the
following recurrence

\[ L_n = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (L_k + L_{n-k}) \quad n \geq 2, \]

with \( L_0 = L_1 = 1. \)

Actually, there are several modifications to the basic tree-type algorithm (cf. [50]) that lead to a more general recurrence. Namely: given \( L_0 \) and \( L_1 \), for \( n \geq 2 \) the following holds

\[ L_n = a_n + \beta \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (L_k + L_{n-k}), \tag{3} \]

where \( a_n \) is a given sequence (also called the additive term), and \( \beta > 0 \) is constant. Let \( L(z) \) and \( A(z) \) be the exponential generating functions of \( L_n \) and \( a_n \), respectively. An easy calculation reveals that

\[ L(z) = \beta L(zp)e^{zp} + \beta L(zq)e^{zp} + A(z) = l_0 - l_1 z, \]

where \( l_0, l_1 \) are constant depending on the initial condition. It is not clear how to solve this functional equation unless we rewrite it in terms of the Poisson generating functions. Then:

\[ \tilde{L}(z) = \beta \tilde{L}(zp) + A(z) - l_0 e^{-z} - l_1 ze^{-z}, \tag{4} \]

which becomes a linear additive functional equation extensively studied in [16, 11, 32, 37, 50, 51].

It is also interesting to rederive probabilistically the above functional equation. Observe first that the conflict resolution algorithm can be represented as a binary digital tree (i.e., trie as discussed in Section 3.3). Then, the Poisson stream of packets with mean \( z \) is split into the left subtree and the right subtree. By property (P2), the left subtree receives a Poisson stream of strings with mean \( pz \), while the right subtree receives an independent Poisson stream with mean \( qz \). The root of the tree contributes \( A(z) \). Taking now into account the initial condition, one immediately derives the functional equation (4) without any algebraic manipulation.

Now, we briefly describe the interval searching algorithm (cf. [5, 26, 52]). This time access to the channel is controlled by a window based mechanism we describe below. This window will be referred to as the enabled interval (EI). Let \( S_i \) denote the starting point for the \( i \)-th EI, and \( t_i \) is the corresponding starting point for the conflict resolution interval (CRI), where CRI represents the number of slots needed to resolve a collision. Roughly speaking, at each step of the algorithm, we compute the endpoints of the EI based on the
outcome of the channel. Details can be found in [5, 26, 52]. The parameters of interest are: the length of the conflict resolution interval \( T_n \), the fraction of resolved interval \( W_n \), and the number of resolved packets \( C_n \). To derive functional equations for these parameters, we immediately consider the Poisson model. For example, \( \tilde{C}(z) \) for \( C_n \) satisfies (cf. [5, 26, 52])

\[
\tilde{C}(z) = \left(1 + (1 + z/2)e^{-z/2}\right) \tilde{C}(z/2) .
\]

Indeed, if there is at most one packet in the first half of EI – which happens with probability \((1 + z/2)e^{-z/2}\) in the Poisson model – then we explore the second half of the interval, and this leads to \( \tilde{C}(z/2) \). Otherwise, we explore the first half of the interval, thus the first term in the above equation.

In summary, fishing in the Poisson stream helps! In general, one can easily derive the following generic functional equation for all interested parameters of this algorithm:

\[
f(z) = \beta \left(1 + (1 + z/2)e^{-z/2}\right) f(z/2) + a(z)
\]

where \( a(z) \) is some function, and \( \beta > 0 \) is a constant. We return to this functional equation in Section 6.

3.2 Combinatorial Assemblies of Labelled Structures

Let us first consider classical (labelled) combinatorial structures such as permutations decomposed into cycles; partitions of a finite set; mappings decomposed into connected components, and graphs decomposed into connected components. We adopt here Wilf’s language [54] and call structures and set of components as “decks” and “hands”, respectively. For example, in the permutation assembly a deck is represented by a cyclic permutation of, say \( n \) elements, while hands are cyclic decomposition of the permutation.

Let now \( h_{k,n} \) denote the number of decompositions (hands) of weight \( n \) that have exactly \( k \) components (e.g., permutations of \( [n] = \{1, \ldots, n\} \) into \( k \) cycles), and \( d_n \) is the number of different structures of weight \( n \) (e.g., cyclic permutations of \( [n] \)). Clearly, \( h_n = \sum_{k \geq 0} h_{k,n} \) is the total number of hands of weight \( n \) (e.g., total number of permutations – not necessary cyclic permutations!). We write \( H_n(z) \) and \( D(z) \) for the exponential generating functions for hands and decks enumerations, respectively. Also,

\[
H(z, u) = \sum_{n,k \geq 0} h_{n,k} \frac{z^n}{n!} u^k .
\]

According to our notation, the Poisson transforms are respectively \( \tilde{H}(z, u) = e^{-z}H(z, u) \) and \( \tilde{H}_n(z) = e^{-z}H_n(z) \).
There is often a complicated relationship between the enumeration of hands \( h_{n,k} \) and decks \( d_n \), but their exponential generating functions (as well as the Poisson counterparts) are simply related as follows (cf. [53, 54])

\[
H(z, u) = e^{uD(z)},
\]

and clearly \( H(z) = e^{D(z)} \). The Poisson distribution is hiding a little bit behind these formulas, but there is one as noted in [2]. In passing, we mention that (7) is usually used to obtain exact enumeration of hands, however, we are more interested in asymptotic enumerations, that is, how to extract asymptotic behavior of \( h_n \) from \( \bar{H}(z) \) (knowing for example \( D(z) \)).

3.3 Digital Trees and Algorithms on Words

Digital trees are data structures suitable to store data (keys) represented by a sequence of symbols from a finite alphabet. We assume a binary alphabet, and we think of a key (string) as a (possibly infinite) sequence of zeros and ones (cf. [33, 37]).

There are three types of digital trees, namely: trie, Patricia trie (PAT), and digital search tree (DST). In tries and Patricia tries the keys are stored in external nodes while internal nodes are used only to branch out. More precisely, the branching policy at any level, say \( k \), is based on the \( k \)-th symbol of a string. For example, for a binary alphabet \( \Sigma = \{0, 1\} \), if the \( k \)-th symbol in a string is "0", then we branch-out left in the trie, otherwise we go to the right. This process terminates the first time we encounter a different symbol between a string that is currently being inserted into the trie and all other strings already in the trie. In other words, the access path from the root to an external node (a leaf of a trie) is the minimal prefix of the information contained in this external node; it is minimal in the sense that this prefix is not a prefix of any other strings (cf. [33, 37]). In Patricia trees all unary nodes are collapsed into one, that is, Patricia is a binary tree with all internal nodes having degree two. In a digital search tree keys (strings) are directly stored in nodes, and hence external nodes are eliminated. The branching policy is the same as in tries. Figure 1 illustrates these definitions.

In the Bernoulli model, one assumes that the number of keys is fixed and equal to \( n \). The parameters of interests are:

- typical depth \( D_n \), i.e., the length of a path from the root to a randomly selected (external) node,
- height \( H_n \), i.e., maximum path from the root to a terminal node,
Figure 1: A trie, Patricia trie and a digital search tree (DST) built from the following four strings $S_1 = 11100\ldots$, $S_2 = 10111\ldots$, $S_3 = 00110\ldots$, and $S_4 = 00001\ldots$.

- total path length $L_n$, i.e., sum of all paths from the root to (external) nodes,
- size of the tree $S_n$, i.e., number of nodes.

We derive now some functional equations for the above parameters in the Poisson model. We start with tries. Let $D_n(u) = \mathbb{E}u^D_n$, $S_n(u) = \mathbb{E}u^S_n$, and $L_n(u) = \mathbb{E}u^L_n$ be the probability generating functions for the depth $D_n$, size $S_n$ and the total path $L_n$ in the Bernoulli model. We also define $H_n^k = \Pr\{H_n \leq k\}$. The appropriate Poisson transforms satisfy the following equations:

\[
\begin{align*}
\bar{D}(z, u) &= u(p\bar{D}(zp, u) + q\bar{D}(zq, u)) + (1 - u)e^{-z}, \\
\bar{H}^k(z) &= \bar{H}^{k-1}(zp)\bar{H}^{k-1}(zq), \quad \bar{H}^0(z) = (1 + z)e^{-z}, \\
\bar{S}(z, u) &= u\bar{S}(zp, u)\bar{S}(zq, u) + (1 - u)e^{-z}, \\
\bar{L}(z, u) &= \bar{L}(zp, u)\bar{L}(zq, u) + z(1 - u)e^{-z}.
\end{align*}
\]

The above equations can be derived from their corresponding recurrences by (sometimes tedious) algebraic manipulations. But, a significant simplification is possible for the Poisson model. Indeed, let us consider for example the functional equation (8) for the Poisson generating function of the depth $D_n$. Since digital trees are recursive structures, and due to properties (P1)-(P3) of the Poisson process, we immediately observe that the Poisson transform of the depth of the left tree is $D(zp, u)$ while for the right subtree is $D(zq, u)$. Furthermore, since the subtrees are one level lower than the root, we have the factor $u$. Finally, we either compute the typical depth of the left subtree (and this happens with probability $p$) or the right subtree (with probability $q$). The initial conditions add $(1-u)e^{-z}$.
Thus, the functional equation (8) follows. The other equations can be derived by the same (almost) semi-automatic method.

The functional equations for Patricia tries are harder to solve since often they contain additional unknown terms. In particular, the corresponding functional equation for the depth, height and total path length are (observe that the size is constant in the Bernoulli model, thus uninteresting):

\[
\begin{align*}
\tilde{D}(z, u) &= u(p\tilde{D}(zp, u) + q\tilde{D}(zq, u)) + (1 - u)pD(zp, u)e^{-qz} + qD(zq, u)e^{-pz}, \\
\tilde{H}^k(z) &= \tilde{H}^{k-1}(zp)\tilde{H}^{k-1}(qz) + \tilde{H}(zp)e^{-qz} + \tilde{H}(zq)e^{-pz}, \\
\tilde{L}(z, u) &= \tilde{L}(zup, u)\tilde{L}(zuq, u) + \tilde{L}(zq, u) + \tilde{L}(zq, u).
\end{align*}
\]

where \(\tilde{H}^k(z) = (H^k(z) - 1)e^{-z}\) and \(\tilde{L}(z, u) = (L(z, u) - 1)e^{-z}\).

Finally, we discuss digital search trees. This time the functional equations become differential functional equations, thus making the problems much more challenging. We present them for the exponential generating functions instead of Poisson transforms since they have slightly simpler forms. With the definitions as above, we obtain the following differential-equations

\[
\begin{align*}
\frac{\partial D(z, u)}{\partial z} &= u(pD(zp, u) + qD(zq, u)) + 1, \\
\frac{dH^k(z)}{dz} &= H^{k-1}(zp)H^{k-1}(qz), \\
\frac{\partial L(z, u)}{\partial z} &= L(zup, u)L(zuq, u).
\end{align*}
\]

with \(H^0(z) = (1 + z)\).

### 3.4 Leader Election Algorithm

The following elimination process (cf. [13, 42]) has several applications, such as the election of a leader in a distributed or parallel system (a practise exercised when a token is lost or when synchronization is lost in a token-passing ring-connected computer network). A group of \(n\) people (users) wishes to identify a leader by tossing fair coins. All \(n\) people who throw heads are losers and do not participate any more in the election, those who throw tails are candidate leaders and should flip their coins again. The process is repeated among candidate leaders until one leader is identified. If at any stage all remaining candidate leaders throw heads, the tosses are considered inconclusive and they all participate again as candidate leaders in the next round of coin tossing.
It is easy to notice that this election process can be represented by a special trie which was named by Prodinger [42] the *incomplete trie* since only one side of this tree is developed. Let $H_n$ be the number of tosses until a leader is elected, that is, it is the path length in the incomplete trie from the root to the furthest terminal node on the left (assuming we move to the left when we toss a tail). If $G_n(u) = E u^{H_n}$ denotes its probability generating function and $\tilde{G}(z,u)$ its Poisson generating function, then the following functional equation can be derived (cf. [13, 42])

$$G_n(u) = \frac{u}{2^n} \sum_{k=0}^{n} \binom{n}{k} G_k(u) - \frac{u}{2^n} + \frac{u G_n(u)}{2^n}, \quad (18)$$

$$\tilde{G}(z,u) = u(1 + e^{-z/2}) \tilde{G} \left( \frac{z}{2}, u \right) + e^{-z} [(1 + z)(1 - u) - u e^{z/2}] . \quad (19)$$

Furthermore, the Poisson mean $\tilde{X}(z) = E H_N$ and the Poisson second factorial moment $\tilde{W}(z) = E H_N (H_N - 1)$ satisfy the following functional equations:

$$\tilde{X}(z) = \tilde{X}(z/2) (1 + e^{-z/2}) + 1 - e^{-z} - z e^{-z} \quad (20)$$

$$\tilde{W}(z) = \tilde{W}(z/2) (1 + e^{-z/2}) + 2 \tilde{X}(z/2) (1 + e^{-z/2}) \quad (21)$$

and $\text{Var} \ H_N = \tilde{W}(z) + \tilde{X}(z) - [\tilde{X}(z)]^2$.

### 3.5 Generalized Probabilistic Counting

In some applications (cf. [14]) one needs to estimate quickly the cardinality of a large set. A probabilistic counting is a possible solution. To estimate the cardinality $n$ of a set (with replications) every element of the set is hashed into a binary string of size $m$ (the choice of $m$ is easy, and $m = 5 + \log n$ suffices). The bitwise OR-composition of modified hashed strings is used to build the so called bitmap and to obtain the estimate $R_n$ of the cardinality $n$ of a set. More precisely, the position of the leftmost zero in the bitmap string approximates $\log_2 n$ (for details see below and [14]).

We consider a generalized probabilistic counting as in [32] in which the bitmap is $(d+1)$-ary string instead of a binary one, where $d$ is an integer parameter of the scheme. We can describe this scheme as follows: Let us consider an empty bitmap string, that is, with all positions filled by zeros. Assume that $n$ objects (e.g., data, persons, etc.) can randomly insert (hit) a 1 at any position of the bitmap, however, the probability of hitting the $j \geq 0$ position is equal to $2^{-j-1}$. In terms of probabilistic counting, this means that the probability of the occurrence of the pattern like $0^j 1 \cdots$ is equal to $2^{-j-1}$ since 0 and 1 are equally likely ($0^j$ denotes $j$ consecutive zeros). Every object can hit only one time. In addition, we count
the number of hits in any position of the bitmap, but we count the number of hits only up to some value \( d+1 \), where \( d \) is a given parameter. In other words, the bitmap is a \( d+1 \)-ary string. The parameter of interest is the length \( R_{n,d} \) of the longest run of \( d+1 \) symbols in front of the bitmap. More precisely:

\[
R_{n,d} = \min\{k : \text{bitmap}(k) < d+1 \text{ and for all } 0 \leq i < k \text{ bitmap}(i) = d+1\}.
\]

Let \( G_n(u) = \mathbb{E}u^{R_{n,d}} \), and \( \bar{G}(z,u) \) be its Poisson transform. As in [32] we observe that

\[
G_n(u) = \sum_{k=0}^{\infty} \binom{n}{k} 2^{-n} + u \sum_{k=0}^{n-d-1} \binom{n}{k} 2^{-n} G_k(u), \tag{22}
\]

\[
\bar{G}(z,u) = u f_d(z/2) \bar{G}(z/2,u) + (u-1)(f_d(z/2) - 1), \tag{23}
\]

where \( f_d(z) = 1 - e_d(z)e^{-z} \) and

\[
e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!}, \tag{24}
\]

is the truncated exponential function.

### 4 General Depoissonization Theorems

In this section, we present our main results. In particular, we derive conditions under which the Bernoulli model characteristics can be inferred from its corresponding Poisson model. The plan for this section is as follows: We start with some definitions and notations followed by a presentation of our main depoissonization results (with an exception of distributional results that we move to the next section). Then, we discuss several technical lemmas needed to prove different version of our findings. The last subsection contains proofs. We first give a detailed proof our our basic depoissonization results (cf. Theorem 1), and then present a general depoissonization tool (cf. Theorem 5) from which all other results will follow.

#### 4.1 Definitions and Notations

As discussed in Section 3, our goal is to study asymptotic expansion of a sequence \( g_n \) through its Poisson transform \( \bar{G}(z) = \mathcal{P}(g_n;z) \). Extension to distributions requires to investigate double-index sequence \( g_{n,k} \) (e.g. \( g_{n,k} = \mathbb{P}(X_n = k) \) or \( g_{n,k} = \mathbb{E}e^{tX_n}/\sqrt{V_k} \) for some sequence \( V_k \)), and its Poisson transform is denoted as \( \bar{G}(z,u) = \mathcal{P}(g_{n,k};z,u) \). In the latter case, it is often more convenient to investigate the Poisson transform \( \bar{G}_k(z) = \mathcal{P}(g_{n,k};z) \) defined as

\[
\bar{G}_k(z) = \sum_{n=1}^{\infty} g_{n,k} \frac{z^n}{n!} e^{-z}. \tag{25}
\]
We also write $G(z) = \tilde{G}(z)e^z$ for the standard exponential generating function of $g_n$.

As discussed above, our goal is to infer asymptotics of the Bernoulli model from the asymptotic behavior of the Poisson generating function as $z \to \infty$. However, in most cases the behavior of $\tilde{G}(z)$ as $z \to \infty$ depends on the region in the complex plane in which $z \to \infty$. Usually, we must restrict the asymptotics of $\tilde{G}(z)$ to a cone around the positive axis of $\Re(z)$.

Two types of cones play a special role in our analysis, namely the so called linear cones and polynomial cones. They are defined below.

**Definition 1**

(i) A linear $S_\theta$ cone is defined as

$$ S_\theta = \{z : |\arg z| \leq \theta, \quad |\theta| < \pi/2 \} . $$

(ii) A polynomial cone $C(D, \delta)$ is defined as

$$ C(D, \delta) = \{z = x + iy : |y| \leq Dx^\delta, \quad 0 < \delta \leq 1, \quad D > 0 \} . $$

Observe that that is, the polynomial cone becomes a linear cone.
Remark 1. There is another asymptotically equivalent definition of polynomial cones that is useful in some of our derivations. Indeed, let $|z| > 1$, and observe that for $z \in C(D, \delta)$ there exists a constant $D'$ such that

$$D' \theta \leq \tan \theta \leq D|z|^{\delta - 1} \leq D|z|^{\delta - 1}$$

where $\theta = \arg(z)$. Thus, we could define $C(D, \delta)$ as

$$C(D', \delta) = \{ z : |\arg(z)| \leq D'|z|^{\delta - 1}, \quad 0 < \delta \leq 1, \quad D' > 0 \} , \quad (28)$$

and in some of our proofs we use this definition of the polynomial cone.

In Figure 2 we show an example of a linear cone $S_{\theta_n}$ (shadow area) and a polynomial cone $C(d, \delta)$. Observe that the circle of radius $n$ and center $O$ intersects the polynomial cone at the angle $\theta_n$ such that $\theta_n \sim Dn^{\delta - 1}$ when $n \to \infty$.

4.2 Main Results

We consider now a sequence $g_n$ and its Poisson transform $\tilde{G}(z) = P(g_n, z)$. Throughout, we assume that $\tilde{G}(z)$ is an entire function. Our goal is to extract asymptotically $g_n$ from $\tilde{G}(z)$. By Cauchy's formula we have

$$g_n = \frac{n!}{2\pi i} \oint \frac{\tilde{G}(z) e^z}{z^{n+1}} dz = \frac{n!}{n^n 2\pi} \int_{-\pi}^\pi \tilde{G}(ne^{it}) \exp\left(ne^{it}\right) e^{-n^it} dt . \quad (29)$$

All of our depoissonization results will follow from the above by a careful estimation of the integral using a saddle point method.

We now begin presenting our findings. We start with a basic depoissonization result that holds in a linear cone with a polynomial bound on $\tilde{G}(z)$ (cf. Theorem 1). Then, we extend it to polynomial cones, more general bounds, and full asymptotic expansion of $g_n$ versus $\tilde{G}(n)$ (cf. Theorems 2, 3, 4) will be presented. The proof of Theorem 1 can be found in Subsection 4.4.1 where we strive to use only elementary tools. All remaining main results are proved in Subsection 4.4.2 by a simple application of a general depoissonization tool, namely: Theorem 5.

**Theorem 1 Basic Depeissonization Lemma.**

Let $\tilde{G}(z)$ be the Poisson transform of a sequence $g_n$ that is assumed to be an entire function of $z$. We postulate that in a linear cone $S_{\theta}$ ($0 < \pi/2$) the following two conditions simultaneously hold for some real numbers $A, B, R > 0$, $\beta$, and $\alpha < 1$:

\[ (1) \quad \text{For } z \in S_{\theta} \]

\[ |z| > R \quad \Rightarrow \quad |\tilde{G}(z)| \leq B|z|^\beta , \quad (30) \]
For \( z \notin S_\delta \)

\[
|z| > R \implies |\tilde{G}(z)e^z| \leq A\exp(\alpha |z|) .
\]  

Then,

\[
g_n = \tilde{G}(n) + O(n^{\beta-1})
\]

for large \( n \).

This basic depoissionization result can be extended in three directions. Namely:

(i) we replace the linear cone by a polynomial cone;

(ii) we provide full asymptotic expansion for \( g_n \) in terms of \( \tilde{G}^{(k)}(n) \) where \( \tilde{G}^{(k)}(n) \) denotes the \( k \)th derivative of \( \tilde{G}(z) \) at \( z = n \);

(iii) we generalize the polynomial bound on \( \tilde{G}(z) \) to certain exponential bounds (see also Section 5 for more about this kind of an extension).

Such extensions are needed in some applications, e.g., for the analysis of the number of phrases in the Lempel-Ziv parsing algorithm [28, 55, 57, 56] and/or redundancy of the Lempel-Ziv code as discussed in [35, 57].

The next theorem refers to the first and second extensions discussed above.

**Theorem 2 General Depoissonization Lemma.**

Consider a polynomial cone \( C(D, \delta) \) with \( 1/2 < \delta \leq 1 \). Let the following two conditions hold for some numbers \( A, B, R > 0 \) and \( \alpha > 0, \beta, \gamma \):

(1) For \( z \in C(D, \delta) \)

\[
|z| > R \implies |\tilde{G}(z)| \leq B|x|^\beta \Psi(|x|) ,
\]

where \( \Psi(x) \) is a slowly varying function, that is, such that for fixed \( t \) \( \lim_{x \to \infty} \frac{\Psi(tx)}{\Psi(x)} = 1 \) (e.g., \( \Psi(x) = \log^d x \) for some \( d > 0 \));

(O) For all \( z = \rho e^{i\theta} \) in the complex plane with \( \theta \leq |\arg(z)| \leq \pi \)

\[
\rho = |z| > R \implies |\tilde{G}(z)e^z| \leq A\rho^n \exp((1 - \alpha \theta^2)\rho) ,
\]

Then, for every nonnegative integer \( m \)

\[
g_n = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \tilde{G}^{(j)}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n))
\]

\[
= \tilde{G}(n) + \sum_{k=1}^{m} \sum_{i=1}^{k} b_{i,k} n^i \tilde{G}^{(k+i)}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n))
\]
where \( b_{ij} \) are as the coefficients of \( \exp(x \log(1 + y) - xy) \) at \( x^iy^j \), that is:

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}x^iy^j = \exp(x \log(1 + y) - xy) .
\]  

(36)

such that \( b_{ij} = 0 \) for \( j < 2i \).

**Remark 2.** (i) To visualize the expansion (35) we present it below for \( m = 6 \) and \( \delta = 1 \) (i.e., in a linear cone)

\[
\begin{align*}
g_n &= \tilde{G}(n) - \frac{1}{2} n \tilde{G}(2)(n) + \frac{1}{3} n \tilde{G}(3)(n) + \frac{1}{8} n^2 \tilde{G}(4)(n) - \frac{1}{4} n \tilde{G}(4)(n) - \frac{1}{6} n^2 \tilde{G}(5)(n) - \frac{1}{48} n^3 \tilde{G}(6)(n) + \frac{13}{72} n^2 \tilde{G}(6)(n) + \frac{1}{24} n^3 \tilde{G}(7)(n) + \frac{1}{384} n^4 \tilde{G}(8)(n) - \\
&\quad - \frac{1}{6} n \tilde{G}(6)(n) - \frac{11}{60} n^2 \tilde{G}(7)(n) - \frac{17}{288} n^3 \tilde{G}(8)(n) - \frac{1}{144} n^4 \tilde{G}(9)(n) - \frac{1}{384} n^5 \tilde{G}(10)(n) + O(n^{-6}) .
\end{align*}
\]

(37)

(ii) It is not difficult to notice that indeed \( b_{ij} = 0 \) for \( j < 2i \). Let \( f(x,y) = \exp(x \log(1 + y) - xy) \). Observe that \( f(xy^{-2}, y) \) is analytical at \( x = y = 0 \), hence its Laurent expansion possesses only \( x^iy^j-2i \) with nonnegative powers leading to \( j \geq 2i \) for non zero coefficients \( b_{ij} \), as desired. \( \square \)

In some applications (cf. next section) polynomial growth of \( |\tilde{G}(z)| \) is a too severe restriction. This is remedied in the next theorem.

**Theorem 3 EXPONENTIAL DEPOISSONIZATION LEMMA.**

Let the conditions of Theorem 2 be satisfied with condition (1) replaced by

\[
|\tilde{G}(z)| \leq A \exp(B|z|^\beta)
\]

for some \( 1 - \delta \leq \beta \leq \frac{1}{2} \) and constants \( A > 0 \) and \( B \). Then, for every integer \( m \geq 0 \)

\[
g_n = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \tilde{G}^{(j)}(n) + O(n^{-(m+1)(1-2\beta)} \exp(Bn^\beta))
\]

(38)

for large \( n \).

Finally, when studying limiting distributions (in particular, the central limit theorem) one must study a double indexed sequence \( g_{n,k} \) in order to obtain asymptotics of the “diagonal” sequence \( g_{n,n} \). The next result deals with such a case. (For example, we can set \( g_{n,k} = \text{E} e^{X_{n}/\sqrt{V_k}} \) where \( X_n \) is a sequence of random variable and \( V_k \) is a sequence representing the variance of \( X_k \).) We recall that \( \tilde{G}_k(z) = \sum_{n=0}^{\infty} g_{n,k} \frac{z^n}{n!} e^{-z} \).
Theorem 4 Diagonal DePoissonization Lemma.

Let \( \tilde{G}_n(z) \) be a sequence of Poisson transforms of \( g_{n,k} \) which are assumed to be a sequence of entire functions of \( z \). Consider a polynomial cone \( C(D, \delta) \) with \( 1/2 < \delta \leq 1 \). Let the following two conditions hold for some \( A > 0 \), \( B, R > 0 \) and \( \alpha > 0 \), \( \beta \), and \( \gamma \):

(1) For \( z \in C(D, \delta) \) and

\[
|z| \in (n - Dn^\delta, n + Dn^\delta) \implies |\tilde{G}_n(z)| \leq Bn^\beta |\Psi(n)| ,
\]

where \( \Psi(x) \) is a slowly varying function.

(2) For \( z \) outside the polynomial cone:

\[
|z| = n \implies |\tilde{G}_n(z)e^z| \leq n^\gamma \exp(n - An^\alpha) ,
\]

Then, for large \( n \)

\[
g_{n,n} = \tilde{G}_n(n) + O(n^{\beta-(2\delta-1)\Psi(n)}) .
\]

More generally, for every nonnegative integer \( m \)

\[
g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{m} b_{ij} n^m \tilde{G}_n^{(j)}(n) + O(n^{\beta-(m+1)(2\delta-1)\Psi(n)})
\]

where \( \tilde{G}_n^{(j)}(n) \) denotes the \( j \)th derivative of \( \tilde{G}_n(z) \) at \( z = n \).

Finally, we address the issue of the necessity of conditions (1) and (2). Although we further generalize them below (cf. Section 5), we claim that some restrictions on the growth of \( \tilde{G}(z) \) are necessary in order to allow depoissonization as the following two examples show.

Example 1. Condition (2) does not hold

Let \( g_n = (-1)^n \), thus \( \tilde{G}(z) = e^{-2\pi} \) and condition (1) is true for any \( \beta \) and for any \( \theta < \pi/2 \). But, in this case the condition (2) outside the cone \( \mathcal{S}_\theta \) does not hold because \( \tilde{G}(z)e^z = e^{\pi i} \) for \( \arg(z) = \pi \). Clearly, \( g_n \) is not \( O(n^\beta) \) for any \( \beta < 0 \).

Example 2. Condition (1) is violated

If \( g_n = (1 + t)^n \) for \( t > 0 \), then \( \tilde{G}(z) = e^{tz} \). Condition (2) holds for some \( \theta \) such that \( (1 + t) \cos \theta < 1 \). But condition (1) inside the cone \( \mathcal{S}_\theta \) does not hold because \( \tilde{G}(z) \) does not have a polynomial growth. And, as a matter of fact, \( g_n \) is not equivalent to \( \tilde{G}(n) \). □
4.3 Technical Lemmas

In the preparation for the proofs of the above results, we present in this section a series of technical lemmas that are of their own interest.

The lemma below is a well-known result concerning the Gaussian integral and can be found in textbooks (e.g., N. G. Bruijn [3]):

**Lemma 1** The following identities are true:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-tx^2} dx = \begin{cases} 
0 & k = 1, 3, 5, \ldots \\
\frac{\Gamma\left(\frac{1}{2} - k/2\right)}{(k/2)!} & k = 2, 4, 6, \ldots
\end{cases}
\]  

(43)

and

\[
\int_{\theta}^{\infty} x^k e^{-tx^2} dx = O \left(e^{-\frac{1}{2}tx^2}\right)
\]

(44)

where \(\theta\) is a positive number.

The next lemma is a simple extension of the Cauchy estimate on derivatives of an analytical function.

**Lemma 2** Suppose that \(f\) is an analytical function in a domain containing the disc \(B(z_0, r) = \{z : |z - z_0| \leq r\}\), for some \(z_0 \in \mathbb{C}\) and \(r > 0\). Suppose further that \(|f(z)| \leq M\) when \(z \in B(z_0, r)\). Then,

\[|f^{(k)}(z_0)| \leq k!Mr^{-k}, \quad k \geq 0.
\]

\(f^{(k)}(z)\) being the \(k\)th derivative of \(f(z)\).

**Proof.** It follows directly from the Cauchy estimate:

\[
f^{(k)}(z_0) = \frac{k!}{2i\pi} \oint \frac{f(z)dz}{(z - z_0)^{k+1}},
\]

where the integration being done along the circle of radius \(r\) and center \(z_0\).

The following lemma is crucial for our analysis. It bounds derivatives of \(\tilde{G}(z)\) inside a cone. We formulate it for linear cones as well as for polynomial cones.

**Lemma 3** (i) **LINEAR CONE.** Let \(\theta_0 < \pi/2\) and \(\xi > 0\) be such that for all \(z \in S_{\theta_0}:

\[
|z| > \xi \quad \Rightarrow \quad |G(z)| \leq B|z|^\beta
\]

(45)

for some real \(\beta\). Then, for all \(\theta < \theta_0\) there exists \(B'\) and \(\xi' > \xi\) such that for all positive integers \(k\) the following holds

\[
|z| > \xi' \quad \Rightarrow \quad |G^{(k)}(z)| \leq k!(B')^k|z|^\beta-k.
\]

(46)
(ii) Polynomial Cone. Let for all $z \in C(D, \delta)$ there exist $\xi, B > 0$ such that $|G(z)| \leq B|z|^\beta$ for $|z| > \xi$. Then, there exists a smaller cone $C(D', \delta)$ with $D' < D$ such that for some $\xi' > \xi$ and a constant $B'$ one obtains

$$|z| > \xi' \implies |G^{(k)}(z)| \leq k!(B')^k|z|^{\beta-k\delta}.$$  \hfill (47)

**Proof.** It directly follows from the previous lemma. We prove here only part (i) since part (ii) can be derived in a similar manner, and as a matter of fact we have already proved it in [28] as Fact 5. We place a circle of radius $\Omega|z|$ for some $\Omega > 0$ at the closest corner of the cone $S_0$ just after $\Re(z) = \xi$ as shown in Figure 3. We finally obtain $\Omega = \sin(\theta_0 - \theta)$. Then, by previous lemma with $M = \Omega$ we immediately prove (46).

For part (ii) we set $\Omega = D'|z|^{\delta}$ for $z \in C(D', \delta)$ such that the circle of radius $\Omega$ and center $z$ is included in $C(D, \delta)$. \hfill \blacksquare

In the proof of our main results, we often will deal with derivatives of $G(ne^{i\theta}/\sqrt{n})$ for which we need a uniform bound. To handle it efficiently, we introduce below a special class of functions whose derivatives behave nicely on some intervals. We shall use the following throughout the paper:

---

**Figure 3:** Illustration to Lemma 3.
**Definition 2** Let \( f^{(k)}(x) \) denote the \( k \)th derivative of a function \( f(x) \). We say that a sequence of functions \( f_n : I_n \to \mathbb{C} \) defined on subintervals \( I_n \) belongs to the class \( \mathcal{D}_k(\omega) \) for \( \omega \) real if there exist \( D > 0 \) such that for all integers \( j \leq k \) and \( x \in I_n \) we have \(|f_n^{(j)}(x)| \leq Dn^{-j\omega}\).

We illustrate the above definition in the example below:

**Example 3. Functions belonging to \( \mathcal{D}(\omega) \).**

(i) Let \( f \) be infinitely differentiable on \([-1,1]\), and define \( f_n(x) = f(xn^{-\omega}) \) on \( I_n = [-n^\omega, n^\omega] \). Clearly, \( f_n \in \mathcal{D}_k(\omega) \) for every \( k \geq 0 \).

(ii) Let \( F_n \) be an analytic function defined in \( \{ z : |z| \leq n^\omega \}, \ n \geq 1 \), such that the sequence \( F_n \) is uniformly bounded. Then the restriction \( f_n \) of \( F_n \) to \( I_n = [-\frac{1}{2}n^\omega, \frac{1}{2}n^\omega] \) belongs to \( \mathcal{D}_k(\omega) \) for every \( k \geq 0 \). \( \square \)

The next lemma presents some simple properties of the class \( \mathcal{D}_k(\omega) \). Its proof is trivial, so we omit it.

**Lemma 4** (i) If \( f_n \) belongs to \( \mathcal{D}_k(\omega) \), then \( f_n \) belongs to \( \mathcal{D}_k(\omega') \) for all \( \omega' \leq \omega \).

(ii) If \( f_n \) and \( g_n \) belong to class \( \mathcal{D}_k(\omega) \), and if \( H(x,y) \) is a function which is \( k \) times continuously differentiable, then \( H(f_n,g_n) \) belongs to \( \mathcal{D}_k(\omega) \).

Consequently if \( f_n \) and \( g_n \in \mathcal{D}_k(\omega) \), then \( f_n + g_n \) and \( f_n \times g_n \in \mathcal{D}_k(\omega) \).

To prove our general asymptotic expansion like in Theorem 2, we must study bounds and Taylor’s expansions of the following function

\[
h_n(t) = \exp(n(e^{it\sqrt{n}} - 1 - it/\sqrt{n})) .
\]  

One can interpret \( h_n(t) \) as the kernel of the Cauchy integral (29) (cf. also (53)), thus not surprisingly this function often appears in our depoissonization theorems. Its property are described in the next lemma.

**Lemma 5** The following statements hold:

(i) For \( t \in [-\pi/\sqrt{n}, \pi/\sqrt{n}] \) there exists \( \mu > 0 \) such that \(|h_n(t)| \leq e^{-\mu t^2} \), where \( \mu \) is a constant.

(ii) For complex \( t \) such that \(|t| \leq Bn^{\frac{1}{2}} \) for some \( B > 0 \), the sequence of functions \( F_n(t) = h_n(t)e^{-t^2/2} \) are bounded and belong to \( \mathcal{D}_k(\frac{1}{B}) \) for any integer \( k \).

(iii) Let \( \xi_{ij} \) be defined as

\[
\sum_{ij} \xi_{ij}x^iy^j = \exp(y(e^{ix} - 1 - ix + \frac{1}{2}x^2)) .
\]
Then, for all nonegative integers $k$, there exists $\nu_k$ such that for all $t \in [-\log n, \log n]$,

$$h_n(t) = e^{-t^2/2} \left( 1 + \sum_{i=3}^{2k} \sum_{j=1}^k \xi_{ij} t^i n^{j-i/2} + O(\nu_k \log^{3(k+1)} n^{-k+1/2}) \right).$$

**Proof:** Part (i) is easy, and in fact, $\mu = 1/\pi^2$ suffices. Point (ii) is a little more intricate. According to Lemma 4 it suffices to prove that the sequence of function $h_n(t) \in \mathbb{R}$, and then refer to the fact that the sequence of exponential of $h_n(t)$ is still in $\mathbb{R}$.

Denoting $e^{ix} - 1 - ix + x^2/2 = \ell(x)$, we observe that the sequence of functions $\ell(t/\sqrt{n})$ belongs to $\mathbb{K} \left( \frac{1}{2} \right)$ for $x = O(\sqrt{n})$ and any integer $k \geq 1$. Therefore, the $i$th derivative for $3 \leq i \leq k$ of $n\ell(t/\sqrt{n})$ is $O(n^{1-i/2})$, which is $O(n^{-i/6})$ for all $i \geq 3$. In particular, the third derivative is $O(n^{-1/2})$. But, we observe also that by successive integrations the first derivative of $n\ell(t/\sqrt{n})$ is $O(n^{-1/6})$, the second derivative is $O(n^{-1/3})$, and $n\ell(t/\sqrt{n})$ is $O(1)$, because the first two derivatives of $\ell(t)$ and $\ell(0)$ are zero at $t = 0$ by the construction. Hence, part (ii) follows.

Finally, to prove part (iii), we proceed as follows: Let, as before, $e^{ix} - 1 - ix + x^2/2 = \ell(x)$. There exists $\mu_3$ such that $|\ell(x)| \leq \mu_3|x|^3$ holds for all real $x$, and therefore the absolute value of $n\ell(t/\sqrt{n})$ is smaller than $\mu_3(\log n)^3 n^{-1/2}$ for $t = O(\log n)$. Clearly, we can write $n\ell(t/\sqrt{n}) = \log h_n(t) + t^2/2$, and thus due to the fact that $n\ell(t/\sqrt{n})$ remains uniformly bounded for $t = O(\log n)$, the following Taylor expansion of $h_n(t) \exp(t^2/2)$ with respect to $n\ell(t/\sqrt{n})$ is valid:

$$h_n(t) \exp(t^2/2) = 1 - n\ell(t/\sqrt{n}) + \frac{n^2}{2} (\ell(t/\sqrt{n}))^2 + \frac{n^{k-1}}{(k-1)!} (\ell(t/\sqrt{n}))^{k-1} + O \left( (\ell(t/\sqrt{n}))^k n^k \right)$$

(50)

We now expand with respect to $t$ each function $\ell^i(x)$ up to degree $k + 2i$. Replacing each $\ell^i(t/\sqrt{n})$ in (50) by the corresponding Taylor expansion with respect to $t$ leads to the reminder $O(t^{2i+k}n^{-i-k/2})$. Note that $n^{-i}$ cancels the factor $n^i$ in front of $\ell^i(t/\sqrt{n})$ in (50). Regrouping the rests yields the desired expansion for some $\xi_{ij}$. To identify the expression (49), we just observe that we have $\sum_{ij} \xi_{ij} x^i y^j = \exp(yx(x))$ by formal identification $y = n$ and $x = t/\sqrt{n}$ for any $t$. $\blacksquare$

Furthermore, when proving general depoissonization Theorems 2, we need an extension of Lemma 1 which is presented next.

**Lemma 6** For nonnegative $H$, and $\beta$, let a sequence of complex functions $F_n(x)$ defined on $x \in [-H \log n, H \log n]$ belong to the class $\mathbb{K}_k(\beta)$ for any $k \geq 1$. Then,

$$\frac{1}{\sqrt{2\pi}} \int_{-H \log n}^{H \log n} F_n(x) e^{-x^2/2} dx = \sum_{i=0}^{k-1} \frac{1}{2^4i!} F_n^{(2i)}(0) + O(n^{-2k}\beta).$$

(50)
Proof: By Taylor's expansion
\[ F_n(x) = \sum_{i=0}^{k-1} \frac{x^i}{i!} F^{(i)}(0) + \Delta_n(x)x^k \]
due to \( F_n \in \mathcal{B}_k(\beta) \) for any \( k \). Thus,
\[
\frac{1}{\sqrt{2\pi}} \int_{-H \log n}^{H \log n} F_n(x)e^{-x^2/2} dx = \sum_{i=0}^{k-1} \frac{F^{(i)}(0)}{i! \sqrt{2\pi}} \int_{-H \log n}^{H \log n} x^i e^{-x^2/2} dx + O(n^{-k\beta})
\]
Observe that \( F_n^{(i)}(0) = O(n^{-i\beta}) \) since \( F_n \in \mathcal{B}_k(\beta) \) and \( i \leq k \). Furthermore, by Lemma 1 changing the limit of integration in the above to \( \pm \infty \), one introduces an error of order \( O(e^{-(H \log n)^2/4}) \) which decreases faster than any polynomial. This completes the proof after some algebra.

Finally, for our exponential depoissonization results (cf. Theorem 3, and Theorems 8 and 9 of Section 5), we replace the above lemma by the following one which is much harder to prove:

**Lemma 7** Let \( \beta_1 \geq 0 \) \( \beta_2 \) such that \( 2\beta_1 + \beta_2 < 0 \). Let \( D > 0 \) and \( \gamma > \beta_1 \). Let \( F_n(z) \) be a sequence of complex analytical functions defined for \( z \in J_n \) where \( J_n \) is the set of complex \( z \) such that both \( \Re(z) \) and \( \Im(z) \) belong to \([-Dn^\gamma, Dn^\gamma]\). We assume that \( F_n(0) = 0 \) and that there exists a real number \( B > 0 \) such that \( |F_n(0)| \leq Bn^{\beta_1} \) and the second derivatives \( |F_n^{(2)}(z)| \leq Bn^{\beta_2} \) uniformly in \( z \in J_n \). Then:
\[
\frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(z)]e^{-\frac{1}{2}z^2} dz = \exp \left( \frac{1}{2}(F_n^{(1)}(0))^2 \right) \left( 1 + O(n^{2\beta_1+\beta_2}) \right)
\]

Proof: Using the first order Taylor's expansion of \( F_n(z) \) around \( z = 0 \), for all \( z \in J_n \), we have the following estimate:
\[
F_n(z) = F_n'(0)z + \Delta_n(z)z^2.
\]
where \( \Delta_n(z) \) is the reminder that can be computed as:
\[
\Delta_n(z) = \int_0^1 F_n^{(2)}(zt)(1-t) dt.
\]
Multiplying the function \( \exp[F_n(z)]e^{-\frac{1}{2}z^2} \) by \( \exp[-\frac{1}{2}(F_n'(0))^2] \) yields \( \exp[-\frac{1}{2}(x - F_n'(0))^2 + \Delta_n(x)x^2] \). Thus:
\[
\frac{\exp[-\frac{1}{2}(F_n^{(1)}(0))^2]}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(z)]e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp \left( -(x - F_n'(0))^2 + \Delta_n(x)x^2 \right) dx
\]
Moving the contour of integration to a horizontal line containing \( F_n'(0) \) we obtain the following expression:
\[
\int_{-Dn^\gamma}^{Dn^\gamma} \exp[-\frac{1}{2}(x - F_n'(0))^2 + \Delta_n(x)x^2] dx
\]
By our hypotheses \( |\Delta_n(z)| \leq Bn^{\beta_2} \) for \( z \in J_n \) and \( |F_n'(0)| < Bn^{\beta_1} \). Therefore, the two last integrals on the right hand side of (51) are of order 
\[
Bn^{\beta_1} \exp \left( -\frac{1}{2} (Dn^\gamma)^2 + (Bn^{\beta_1})^2 + BD^2n^{\beta_2+2\gamma} + B^3n^{\beta_2+2\beta_1} \right)
\]
which decreases to zero faster than any polynomial. But, after substitution \( z - F_n'(0) = x \), we can now rewrite the first integral of the right hand side of (51) as

\[
\int_{-Dn^\gamma}^{Dn^\gamma} \exp \left( -\frac{1}{2} x^2 + R_n(x)(x + F_n'(0))^2 \right) dx
\]

with \( R_n(x) = \Delta_n(z + F_n'(0)) \). Let us take \( \gamma' \in ]\beta_1, -\beta_2/2[ \) such that \( \gamma' < \gamma \). Notice that \( \gamma' > \beta_1 \) and \( 2\gamma' + \beta_2 < 0 \). We can again split the integral (4.3) into three parts as follows:

\[
\int_{-Dn^\gamma}^{Dn^\gamma} \exp \left( -\frac{1}{2} x^2 + R_n(x)(x + F_n'(0))^2 \right) dx = 
\int_{-Dn^\gamma}^{Dn^\gamma} \exp \left( -\frac{1}{2} x^2 + R_n(x)(x + F_n'(0))^2 \right) dx
\]

Observe that \( |R_n(x)| \leq Bn^{\beta_2} \) since \( x + F_n'(0) \in J_n \), thus in the two last integrals of the above quantity \( R_n(x)(x + F_n'(0))^2 \) can be estimated as follows (for \( x > Dn^\gamma \) when \( n \) is large enough)

\[
|R_n(x)(x + F_n'(0))^2| \leq Bn^{\beta_2} |x| + Bn^{\beta_1} x^2 \leq 4Bn^{\beta_2} x^2 \leq \frac{1}{4} x^2,
\]

which by Lemma 1 leads to an estimate \( O(e^{-D^2n^{2\gamma'}/8}) \) of the integrals.

Coming back to the first integral of the right-hand side of (52), namely:

\[
\int_{-Dn^\gamma}^{Dn^\gamma} \exp \left( -\frac{1}{2} x^2 + R_n(x)(x + F_n'(0))^2 \right) dx
\]

The quantity \( |R_n(x)(x + F_n'(0))^2| \leq B(B + D)^2n^{\beta_2+2\gamma'} \) tends to zero as \( n \to \infty \) due to \( \beta_2 + 2\gamma' < 0 \), thus \( \exp[R_n(x)(x + F_n'(0))^2] = 1 + O(R_n(x)(x + F_n'(0))^2) \). The above expression becomes:

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\[
\int_{-D_n^v_1}^{D_n^v_1} \exp \left( -\frac{1}{2} x^2 \right) + R_n(x)(x + F_n'(0))^2 \, dx
\]
\[
= \int_{-D_n^v_1}^{D_n^v_1} \exp \left( -\frac{1}{2} x^2 \right) \left( 1 + O(R_n(x)(x + F_n'(0))^2) \right) \, dx
\]
\[
= \int_{-D_n^v_1}^{D_n^v_1} \exp \left( -\frac{1}{2} x^2 \right) \, dx
\]
\[
+ \int_{-D_n^v_1}^{D_n^v_1} \exp \left( -\frac{1}{2} x^2 \right) O(R_n(x)(x + F_n'(0))^2) \, dx
\]

If we expand the limit of the integral \( \int_{-D_n^v_1}^{D_n^v_1} \exp \left( -\frac{1}{2} x^2 \right) \, dx \) to \( \pm \infty \), then by Lemma 1 we only introduce an error of order \( \exp \left( -(D_n^v)^2 / 4 \right) \) that decays to zero faster than any polynomial. Then, such an integral has classical value \( \sqrt{2\pi} \).

To complete the proof, it suffices to estimate

\[
\int_{-D_n^v_1}^{D_n^v_1} O(R_n(x)(x + F_n'(0))^2) \exp \left( -\frac{1}{2} x^2 \right) .
\]

But this can be bounded from the above by

\[
\int_{-\infty}^{\infty} O(B_n^{\beta_2})(x^2 + 2|F_n'(0)||x| + |F_n'(0)|^2 \exp \left( -\frac{1}{2} x^2 \right) .
\]

which is bounded by \( O(n^{\beta_2}) \left( 4|F_n'(0)| + (1 + |F_n'(0)|^2) \sqrt{2\pi} \right) \) and this yields \( O(n^{\beta_2 + 2\beta_1}) \).

The lemma is proved. \( \blacksquare \)

Finally, when dealing with limiting distributions (cf. Theorem 9 of Section 5), we need a slight extension of Levy's continuity theorem. The lemma below seems to be well known but since we could not find a reference, we present it with a short proof.

**Lemma 8 (Analytical Levy Theorem)** Let \( Y_n \) and \( Y \) be respectively a sequence of real random variables and a real random variable such that \( g_n(t) = \mathbb{E}[e^{tY_n}] \) and \( g(t) = \mathbb{E}[e^{tY}] \) are their moment generating functions defined in a real neighbourhood of 0. Suppose that \( \lim_{n \to \infty} g_n(t) = g(t) \) for \( t \) belonging to such a real neighbourhood of 0. Then, \( Y_n \) converges to \( Y \) both in distribution and in moments.

**Proof:** In order to show the convergence in both distribution and moments, it suffices to prove that \( g_n(t) \) and \( g(t) \) are defined in a complex neighbourhood of 0, and \( g_n(t) \) converge to \( g(t) \) in such a complex neighbourhood. Indeed in this case \( g_n(t) \) and \( g(t) \) are analytical,
therefore due to Levy's Continuity Theorem, the convergence in distribution holds since we have convergence for imaginary \( t \), and \( g(t) \) is continuous at 0. The convergence in moments is easier since the \( k \)th moment of \( Y_n \) is exactly \( g^{(k)}(0) \) and the convergence of analytical functions implies the convergence of their derivatives.

Let us first extend the definition of \( g_n(t) \) to complex neighbourhood of \( t = 0 \) defined by \( t \) such \( \Re(t) \) belongs to the real neighbourhood. Then,

\[
|e^{tY_n}| < e^{-\Re(t)Y_n} + e^{\Re(t)Y_n}
\]

Therefore, \( g_n(t) \) exists and is bounded in a complex neighbourhood of 0. Clearly, \( g_n(t) \) and \( g(t) \) are analytical functions. Let us take a compact complex neighbourhood of 0. We know that \( |g_n(t)| \) is uniformly bounded in this neighbourhood by, say, a number \( A > 0 \). We redefine this neighbourhood by removing all point with distance smaller than \( c > 0 \) of the boundary of the former neighbourhood, where \( \varepsilon \) is arbitrarily small. Due to Cauchy estimate the derivatives \( |g'_n(t)| \) are also uniformly bounded by \( Ae^{-1} \), and therefore \( g_n(t) \) are bounded and uniformly continuous. By Ascoli's Theorem, from every sequence of \( g_n(t) \) we can extract a convergent subsequence. This limit function can only be the unique analytical continuation of \( g(t) \), thus \( g_n(t) \) converges to \( g(t) \) in this complex neighbourhood of 0.

4.4 Proofs of Main Results

In this subsection we present proofs of the main results. For the reader's convenience we first present a detailed proof of Theorem 1 that is elementary and self contained (i.e., we try to avoid even technical results of the previous subsection). Then, in the next subsection we present a general depoissonization tool theorem (cf. Theorem 5) that is used to prove all of the remaining theorems in a uniform manner.

4.4.1 Proof of Basic Depoissonization Theorem

We now prove Theorem 1. The proof relies on the Cauchy formula (29). By Stirling's approximation \( n! = n^ne^{-\frac{n}{2}}n(1 + O(1/n)) \), thus (29) becomes

\[
g_n = (1 + O(1/n))\sqrt{\frac{n}{2\pi}} \int_{-\pi}^{\pi} \tilde{G}(ne^it) \exp \left(n \left(e^{it} - 1 - it\right)\right) dt
\]

\[
= (1 + O(1/n))(I_n + \varepsilon_n)
\]

where

\[
I_n = \sqrt{\frac{n}{2\pi}} \int_{-\theta}^{\theta} \tilde{G}(ne^it) \exp \left(n \left(e^{it} - 1 - it\right)\right) dt
\]
\[ E_n = \sqrt{\frac{n}{2\pi}} \int_{[e^{it}, e^{it}]} \tilde{G}(ne^{it}) \exp \left( \frac{n}{2} \left( e^{it} - 1 - it \right) \right) dt \]

\[ = \frac{n^n e^{-n/2\pi n}}{n!2\pi i} \int_{[e^{it}, e^{it}]} \frac{\tilde{G}(z) e^{z}}{z^{n+1}} dz . \]

We estimate the above two integrals. We begin with the latter. Observe that by condition (O) (cf. (31)), we obtain

\[ |E_n| \leq A' \sqrt{2\pi n} e^{-\alpha n} \]

where \( A' \) depends only on \( A \) and \( \xi \), thus \( E_n \) exponentially decays to zero.

Having \( E_n \) "unter hat", we can turn our attention to integral \( I_n \) which is more intricate to handle. First of all, we substitute \( t = t/\sqrt{n} \) to get

\[ I_n = \frac{1}{\sqrt{2\pi}} \int_{-\theta/\sqrt{n}}^{\theta/\sqrt{n}} \tilde{G}(ne^{it/\sqrt{n}}) \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) dt . \]  

Let \( h_n(t) = \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) \). We need to estimate \( h_n(t) \) in the interval \( t \in [-\theta/\sqrt{n}, \theta/\sqrt{n}] \), and find the Taylor expansion of it in a smaller interval, say for \( t \in [-\log n, \log n] \). The latter restriction is necessary since \( \frac{1}{\sqrt{n}} = O(1) \) for \( t \in [-\theta/\sqrt{n}, \theta/\sqrt{n}] \). Thus, we can split the integral \( I_n \) into two terms \( I_n' \) and \( I_n'' \), that is, \( I_n = I_n' + I_n'' \) where

\[ I_n' = \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} \tilde{G}(ne^{it/\sqrt{n}}) \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) dt , \]

\[ I_n'' = \frac{1}{\sqrt{2\pi}} \int_{[-\theta/\sqrt{n}, -\log n]} \tilde{G}(ne^{it/\sqrt{n}}) \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) dt + \]

\[ + \frac{1}{\sqrt{2\pi}} \int_{[\log n, \theta/\sqrt{n}]} \tilde{G}(ne^{it/\sqrt{n}}) \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) dt . \]

To estimate the second integral \( I_n'' \) we observe that \( |h_n(t)| \leq e^{-\mu t^2} \) for \( t \in [-\theta/\sqrt{n}, \theta/\sqrt{n}] \), where \( \mu \) is a constant (in fact, \( \mu = 1/n^2 \) suffices). Thus, by Lemma 1 and condition (I) we immediately show that \( I_n'' = O(n^\beta e^{-\mu \log^2 n/2}) \) which decays faster than any polynomial.

Now, we estimate \( I_n' \). Observe first that for \( t \in [-\log n, \log n] \) we can write

\[ h_n(t) = \exp \left( \frac{n}{2} \left( e^{it/\sqrt{n}} - 1 - it/\sqrt{n} \right) \right) \]

\[ = e^{-t^2/2} \left( 1 - \frac{it^3}{6\sqrt{n}} + \frac{t^4}{24n} - \frac{it^6}{72n} + O \left( \frac{\log^3 n}{n^{3/2}} \right) \right) . \]

Furthermore, using (I) and Lemma 3 for \(|z| > (1 + \Omega)\xi \) and \( z \in S_{\theta'} \) for \( \theta' < \theta \) we have \(|\tilde{G}'(z)| \leq B_1 |z|^\beta - 1 \) and \(|\tilde{G}''(z)| \leq B_2 |z|^{\beta - 2} \), for some constants \( B_1 \) and \( B_2 \). Thus, we can expand \( \tilde{G}(ne^{it/\sqrt{n}}) \) around \( t = 0 \) as

\[ \tilde{G}(ne^{it/\sqrt{n}}) = \tilde{G}(n) + it\sqrt{n}\tilde{G}'(n) + \Delta_n(t)t^2 \]
where $|\Delta_n(t)| \leq (B_1 + B_2)n^{\beta-1}$. In summary, the integral $I_n'$ becomes

$$
I_n' = \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} e^{-t^2/2} (\tilde{G}(n) + \tilde{G}'(n)it\sqrt{n}) \left(1 - \frac{it^3}{6\sqrt{n}} + \frac{t^4}{24n} - \frac{it^6}{72n}\right) dt +
$$

$$
+ \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} e^{-t^2/2} \Delta_n(t) t^2 h_n(t) dt +
$$

$$
+ \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} e^{-t^2/2} \left(\tilde{G}(n) + \tilde{G}'(n)it\sqrt{n}\right) O\left(\frac{\log^2 n}{n\sqrt{n}}\right) dt.
$$

To complete the proof we must estimate the above three integrals. From Lemmas 1 and 3 we assess that the first integral is equal to $G(n) + O(n^{\beta-1})$. Thus it suffices to show that the last two integrals are $O(n^{\beta-1})$. Indeed, using an estimate on $\Delta_n(t)$ we observe that the absolute value of second integral is smaller than $(B_1 + B_2)n^{\beta-1}$. Finally, the last integral is $O(n^{\beta-3/2} \log^{9/2} n)$. All together the error becomes $O(n^{\beta-1})$. Theorem 1 is proved.

### 4.4.2 Basic Depoissonization Tool Theorem

In this subsection we prove all remaining theorems by an application of a general result that we formulate next:

**Theorem 5 Basic Diagonal Depoissonization Tool**

Let $\tilde{G}_k(z)$ be the Poisson transform of a sequence $g_{n,k}$ that is assumed to be a sequence of analytical functions for $|z| \leq n$. We postulate that the following three conditions simultaneously hold for some $H > 0, \gamma, \text{and integer \(m\):}

1. For $z \in C(D, \delta)$, such that $|z| = n$: $|\tilde{G}_n(z)| \leq F(n)$,

2. The sequence of function $f_n(t) = \tilde{G}_n(ne^{it\sqrt{n}}) / F(n)$ defined for $t \in [-H \log n, H \log n]$ belongs to the class $\mathcal{D}_m(\gamma)$ for some integer $m \geq 0$,

3. For $z \notin C(D, \delta)$ and $|z| = n$: $|\tilde{G}_n(z)e^z| \leq p(n)e^n F(n)$ where $p(n)$ decays faster than $n^{-\frac{3}{2} - k \min\{\gamma, \frac{1}{2}\}}$, i.e., $p(n) = o(n^{-\frac{3}{2} - k \min\{\gamma, \frac{1}{2}\}})$. Then, for large $n$

$$
g_{n,n} = \tilde{G}_n(n) + O(n^{-\min\{\gamma, \frac{1}{2}\}})F(n), \quad (54)
$$

and more generally for any integer $m \geq 0$

$$
g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{m} b_{ij} n^i \tilde{G}_n^{(j)}(n) + O(n^{-(m+1)\min\{\gamma, \frac{1}{2}\}})F(n), \quad (55)
$$

where $b_{ij}$ are defined in (36).
Proof. The proof again relies on the Cauchy integral that we split into $I_n$ and $E_n$ as before, where $I_n$ is the integral over $t$ such that $ne^{it}$ is inside $C(D, \delta)$, and $E_n$ is the integral for the $ne^{it}$ outside the cone, that is:

$$I_n = \frac{\omega_n}{2\pi} \int_{-Dn^{\delta-1}/2 \pm \frac{\pi}{2}}^{Dn^{\delta-1}/2 \pm \frac{\pi}{2}} \tilde{G}_n(ne^{it}) \exp \left(n \left(e^{it} - 1 - it/\sqrt{n}\right)\right) dt$$

$$E_n = \frac{n!}{2\pi i} \int_{|z|=\frac{Dn^{\delta-1}/2}{2\pi}} \tilde{G}_n(z) \frac{e^z}{2^{n+1}} dz$$

$$= \frac{\omega_n}{2\pi} \int_{|t|=\frac{Dn^{\delta-1}/2}{2\pi}} \tilde{G}_n(ne^{it}) \exp \left(n \left(e^{it} - 1 - it/\sqrt{n}\right)\right) dt,$$

where $\omega_n = n!n^{-n}e^{-n(2\pi)^{-1}} = 1 + O(1/n)$ by the Stirling formula. We estimate the above two integrals. We begin with the latter. Observe that by condition (O) we have $|E_n|/F(n) \leq \sqrt{2\pi n}p(n) = o(n^{-k \min\{\gamma, 1/6\}})$ which is negligible compared to the error term.

The evaluation of $I_n$ is more intricate. First of all, we substitute $t = t'/\sqrt{n}$ to get

$$I_n = \frac{\omega_n}{\sqrt{2\pi}} \int_{-Dn^{\delta-1}/2}^{Dn^{\delta-1}/2} \tilde{G}_n(ne^{it/\sqrt{n}}) \exp \left(n \left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt$$

$$= \frac{\omega_n}{\sqrt{2\pi}} \int_{-Dn^{\delta-1}/2}^{Dn^{\delta-1}/2} \tilde{G}_n(ne^{it/\sqrt{n}}) h_n(t) dt$$

where $h_n(t)$ is defined in (48) and analyzed in Lemma 5. We further split the integral $I_n$ into $I'_n$ and $I''_n$ as follows

$$I'_n = \frac{\omega_n}{\sqrt{2\pi}} \int_{-H \log n}^{H \log n} \tilde{G}_n(ne^{it/\sqrt{n}}) \exp \left(n \left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt,$$

$$I''_n = \frac{\omega_n}{\sqrt{2\pi}} \int_{|t|\leq \frac{Dn^{\delta-1}/2}{2\pi}, |t| \leq |\log n|} \tilde{G}_n(ne^{it/\sqrt{n}}) \exp \left(n \left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt$$

$$+ \frac{\omega_n}{\sqrt{2\pi}} \int_{|t|\leq |\log n|, |t| \leq \frac{Dn^{\delta-1}/2}{2\pi}} \tilde{G}_n(ne^{it/\sqrt{n}}) \exp \left(n \left(e^{it/\sqrt{n}} - 1 - it/\sqrt{n}\right)\right) dt.$$

To estimate the second integral $I''_n$ we use the fact that $|h_n(t)| \leq e^{-\mu^2}$ with $\mu > 0$ for $t \in [-Dn^{\delta-1}/2, Dn^{\delta-1}/2]$ as shown in Lemma 5. Thus, by Lemma 1 and condition (I) we immediately obtain that $I''_n/F(n) = O(e^{-\mu^2 n/2})$ which decays faster than any polynomial.

Now, we estimate $I'_n$. Observe first that since $h_n(t)e^{t/2} \in \mathbb{D}_m(\frac{\mu}{2})$ for any $m$ and $\tilde{G}(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma)$, therefore their product $R_n(t) = h_n(t)e^{t/2} \tilde{G}_n(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma_2)$ where $\gamma_2 = \min\{\gamma, 1/6\}$. In view of this, by Lemma 6 we arrive at

$$I'_n/F(n) = \frac{\omega_n}{\sqrt{2\pi}} \int_{H \log n}^{H \log n} R_n(t)e^{-t^2/2} dt = \omega_n(F_n(0) + O(n^{-\gamma})),$$
and consequently \( g_n = \tilde{G}_n(n) + O(n^{-7})F(n) \) (observing that \( \omega_n = 1 + O(n^{-1}) \)) which proves (54).

To prove our general result (55), we apply Lemma 6 for any \( k \) to obtain

\[
g_{n,n}/F(n) = \omega_n \sum_{i=0}^{k-1} \frac{1}{2^{i+1}} R_n^{(2i)}(0) + O(n^{-2k-7})
\]

replacing the derivatives of \( R_n(0) \) by their actual values (that involve the derivatives of \( \tilde{G}(z) \) at \( z = n \)), and noting that all odd powers of \( n^{1/2} \) disappear (since we are considering only derivatives of \( F_n(t) \) of even order), multiplying by \( \omega_n F(n) \), replacing \( \omega_n \) by the Stirling expansion, we finally obtain

\[
g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{i+m} \sum_{k=0}^{m} \sum b_{ij} n^{i+m} \tilde{G}^{(j)}(n) + O(n^{-(m+1)\min\{1, \frac{1}{2}\}})F(n)
\]

where we formally changes \( m + 1 = 2k \). Notice that the terms in the expansion do not involve \( m \) since the Stirling expansion and the expansion of Lemma 6 do not contain \( m \).

Remark 3. (i) We have already explained in the proof of Lemma 5 a way to identify the coefficients \( b_{ij} \) in our general depoissonization statement, but here we provide another more detailed derivation. We can write \( g_{n,n} = \frac{n!}{2\pi i} \tilde{G}_n(z)e^{\frac{z-n}{n^2}}. \) Finding Taylor’s expansion of \( \tilde{G}_n(z) = \sum_k \tilde{G}_n^{(k)}(z)(z^n)k/k! \) and formally replacing each \( \tilde{G}_n^{(k)}(z) \) by \( y^k \) we obtain the identity:

\[
\tilde{G}_n(z) = \sum_{k \geq 0} (1/k!)(z-n)^k y^k = \exp(y(z-n)).
\]

Therefore

\[
g_{n,n} = n!\{z^n\}(e^z \tilde{G}_n(z)) = n!\{z^n\}(e^{-yn}e^{1+y}) = (1+y)^n e^{-ny}.
\]

On the other hand, \( g_{n,n} = \sum b_{ij} n^{i+m} \) and this provides the desired identification of the coefficients \( b_{ij} \). There is a more classical proof of the above based on Stirling numbers which is presented in Appendix A.

(ii) A careful reader will notice that there is a minor problem in the above proof which is more pedagogical than mathematical. When we apply Lemma 6 for some given \( k \) and expand the derivatives, we obtain an expansion of the indicated type, but not all of the coefficients \( b_{ij} \) are the right ones. For example, with \( k = 2 \) (expanding to \( F_n'' \)), there appears a term \(-\frac{1}{2} \tilde{G}_n''(n) \) (which is absent from the final result since \( b_{01} = 0 \)); this term is cancelled when expanding one step further, but then other coefficients are wrong. The explanation is, of course, that you get the right coefficients for all terms that matter; the discrepancies
only appear in terms which anyhow can be absorbed in the error term, and there is nothing wrong with the final result. □

Having the depoisonization tool of Theorem 5, we can now finally prove our remaining theorems by identifying the function $F(n)$ (see condition (I)) and finding the right $\gamma$ in condition (L) of Theorem 5.

We start with the proof of Theorem 4 which immediately implies Theorem 2 after setting $g_{n,n'} = g_n$. First of all, observe that conditions (I) and (O) of Theorem 4 imply conditions (I) and (O) of Theorem 5 when we set $F(n) = Bn^\beta$. Thus, to complete the proof we must verify condition (L) of Theorem 5 and find such that $f_n(t) = \mathcal{G}_n(n \exp(it/\sqrt{n}) n^{-\beta}) \in \mathcal{B}_m(\gamma)$ for any integer $m \geq 1$. But, by Lemma 3(ii) for all integer $m$ and for all $z$ belonging to a smaller polynomial cone $C(D', \delta)$ with $D' < D$ there exists $B_m$ such that $|\mathcal{G}_n^{(m)}(z)| \leq B_m |z|^{\beta-\delta(1)}$. Thus, after setting $z = n \exp(it/\sqrt{n})$ we immediately prove that the sequence of functions $f_n(t)$ belongs to $\mathcal{B}_m(\delta - \frac{1}{2})$ for all integer $m$. Set now $\gamma = \min\{\delta - \frac{1}{2}, \frac{1}{2}\}$. Then, by Theorem 5 there is some $m' > m$ such that

$$g_{n,n} = \sum_{i=0}^{m'} \sum_{j=0}^{m'} b_{ij} n^i \mathcal{G}_n^{(j)}(n) + O(n^{\beta-m'\gamma}) .$$

The last delicate point is to obtain the correct error term of Theorem 4. But this can be achieved by setting $m' = \lceil (m + 1)(\delta - 1)/\gamma \rceil$ for given value $m$. This will lead to the correct error term equal to $O(n^{\beta-(m+1)(2\delta-1)})$. Indeed, the additional terms obtained are those with $i > m$ and $j > i + m$. The corresponding coefficients are $b_{ij} n^i \mathcal{G}_n^{(j)}(n)$ with $b_{ij} \neq 0$ for $j \geq 2i$. But, they orders are:

$$n^{\beta+i-j\delta} \leq n^{\beta+i-2i\delta} = n^{\beta-i(2\delta-1)} \leq n^{\beta-(m+1)(2\delta-1)} ,$$

and this proves Theorems 4 and 2.

Finally, we prove Theorem 3. As before, the proof relies on Theorem 5 with $F(n) = \exp(Bn^\beta)$. Conditions (I) and (O) are again easy to verify, so we only need to check condition (L) of Theorem 5, that is, we must decide about the growth of $f_n(t) = \mathcal{G}(n \exp(it/\sqrt{n}) \exp(-Bn^\beta))$. Actually, we prove that $f_n(t) \exp(Bn^\beta) \in \mathcal{B}(\frac{1}{2} - \beta)$. For this we extend Lemma 3, and will show that $|\mathcal{G}^{(m)}(z)| \leq B_m |z|^m(\beta-1) \exp(B|z|^\beta)$. We use again the Cauchy formula to obtain for some constant $\Omega$

$$|\mathcal{G}^{(k)}(z)| \leq \max_{|\omega|=|z|} |\mathcal{G}(z + \omega)| \Omega |z|^m(1-\beta) .$$

Notice that $z + \omega$ still belongs to the polynomial cone since $1 - \beta \leq \delta$, and therefore exponential estimate $|\mathcal{G}(z + \omega)| < A \exp(B|z + \omega|^\beta)$ is still valid. Due to the fact that the
function \( z^\beta \) has derivative \( O(z^{\beta-1}) \), we can derive as follows:

\[
\max_{|\omega|=\Omega|z|^{1-\beta}} |\tilde{G}(z+\omega)| \leq A \exp(|B||z|^\beta + O(z^{\beta-1}||z|^{1-\beta}\Omega)) \\
\leq A' \exp(|B||z|^\beta)
\]

for some constant \( A' \). Thus, \( |\tilde{G}^{(k)}(z)| \leq A' \exp(B|z|^\beta) \). To complete the proof, we must establish the error term. Let \( \gamma = \min\{\frac{1}{2} - \beta, \frac{1}{6}\} \). An application of Theorem 5 leads to the error term \( O\left(n^{-m'\gamma} \exp(Bn^\beta)\right) \) for some integer \( m' \geq 0 \). To establish the right error term \( O(n^{-(m+1)(1-2\beta)} \exp(Bn^\beta)) \) we follow the same approach as before: That is, we set \( m' = [(m+1)(1-2\beta)/\gamma] \) which introduces additional terms of the value \( b_{ij} n^i \tilde{G}^{(j)}(n) \) that contribute \( n^{i-j(1-\beta)} \exp(Bn^\beta) \). But, since \( b_{ij} \neq 0 \) for \( j \geq 2i \),

\[
n^{i-j(1-\beta)} \exp(Bn^\beta) \leq n^{i-2i(1-\beta)} \exp(Bn^\beta) \leq n^{-i(1-2\beta)} \exp(Bn^\beta) \leq n^{-m(1-2\beta)} \exp(Bn^\beta),
\]

and this completes the proof.

Remark 4. One should notice that in condition (I) in Theorem 5 we bound \( \tilde{G}_n(z) \) only for \( |z| = n \), not for \( |z - n| \leq Dn^\delta \). Because of this we need additional assumption (L). This is the strongest depoissionization theorem that we could prove. But, as pointed to us by S. Janson, if we replace our condition (I) by a weaker one, namely:

(I') For \( z \in C(D, \delta) \), such that \( |z - n| \leq Dn^\delta \), \(|\tilde{G}_n(z)| \leq A|F(n)|;

then, condition (L) follows directly from the above as easy to notice. The following example shows that there are sequences for which our Theorem 5 works while the above weaker does not: Let \( g_{n,k} = 1 + n!/(k + \log k)^n \), so that \( \tilde{G}_k(z) = 1 + e^{-z}(1 - \frac{z}{k + \log k})^{-1} \). Clearly, for \( |z| = n \) we have \( \tilde{G}(z) = O(1) \), and condition (L) holds since for every \( m \geq 0 \)

\[
|\tilde{G}^{(m)}(z)| = O(n^{m/2}e^{-n\log n}),
\]

hence \( \tilde{G}_n(ne^{id}/\sqrt{n}) \in \mathbb{B}_n(\gamma) \) for any \( \gamma \) real. Thus, we can apply Theorem 5, and indeed \( g_{n,n} = \tilde{G}(n) + O(1/n) = 1 + O(1/n) \). On the other hand, observe that the above condition (I') does not hold for our choice of \( \tilde{G}_k(z) \).

5 Limiting Distribution

Hereafter, we use depoissionization idea, as in Theorems 1 and 2, to establish limiting distributions for some dependent random variables. Let \( X_n \) be a sequence of integer random variables and \( X_N \) its corresponding Poisson driven sequence, where \( N \) is a Poisson random
variable with mean $z$. Let $\tilde{G}(z, u) = \mathbb{E}uX_N = \sum_{n=0}^{\infty} \mathbb{E}uX_n \frac{z^n}{n!} e^{-z}$ be its Poisson transform. We introduce also the Poisson mean $\tilde{X}(z)$ and the Poisson variance $\tilde{V}(z)$ as

$$\tilde{X}(z) = \tilde{G}'(z, 1),$$
$$\tilde{V}(z) = \tilde{G}''(z, 1) + \tilde{X}(z) - \left( \tilde{X}(z) \right)^2,$$

where $\tilde{G}'(z, 1)$ and $\tilde{G}''(z, 1)$ denote respectively the first and the second derivative of $\tilde{G}(z, u)$ with respect to $u$ at $u = 1$.

**Remark 5.** For historical reasons in the analysis of algorithms and combinatorics we restrict $X_N$ to (non-negative) integer values. But, of course, $X_N$ can be extended to non-integer real values, and then $\tilde{G}(z, t) = \mathbb{E}e^{tX_N}$ and one would redefine $\tilde{X}(z) = \tilde{G}'(z, 0)$ and $\tilde{V}(z) = \tilde{G}''(z, 0) - (\tilde{X}(z))^2$. In the sequel, we assume that $X_N$ is integer valued for simplicity of the presentation. Furthermore, multidimensional random variables can be handled in a similar fashion, but due to the length of this paper we shall not pursue this topic in the current paper.

### 5.1 Relationships Between Poisson and Bernoulli Mean and Variance

Before we proceed, we must understand a relationship between the Poisson mean $\tilde{X}(z)$ and variance $\tilde{V}(z)$ of $X_N$, and the Bernoulli mean $\mathbb{E}X_n$ and variance $\text{Var} X_n$. The next result describes such a relationship. It is actually a corollary to our main depoissonization findings, namely Theorem 2. We restrict our considerations only to linear cones, but extension to polynomial cones is easy and left to the interested reader.

**Theorem 6** Let $\tilde{X}(z)$ and $\tilde{V}(z)$ satisfy conditions (I) and (O) of Theorem 2 with $\beta \leq 1$, that is, $\tilde{X}(z) = O(z^\beta \Psi(z))$ and $\tilde{V}(z) = O(z^\beta \Psi(z))$ in a linear cone $S_\beta$, where $\Psi(z)$ is a slowly varying function. Then, the following holds

$$\mathbb{E}X_n = \tilde{X}(n) - \frac{1}{2} n \tilde{X}''(n) + O(n^{\beta-2} \Psi(n)) \quad (56)$$
$$\text{Var} X_n = \tilde{V}(n) - n[\tilde{X}'(n)]^2 + O(n^{\beta-1} \Psi(n)) \quad (57)$$

for large $n$.

**Proof.** The asymptotic expansion (56) follows directly from Theorem 2 for $n = 1$. To derive (57) observe that the Poisson transform of $\mathbb{E}X_n^2$ is $\tilde{V}(z) + (\tilde{X}(z))^2$, thus again by Theorem 2

$$\mathbb{E}X_n^2 = \tilde{V}(n) + (\tilde{X}(n))^2 - \frac{1}{2} n \tilde{X}''(n) - n[\tilde{X}'(n)]^2 - n \tilde{X}''(n) \tilde{X}(n) + O(n^{\beta-2} \Psi(n))$$
$$= \tilde{V}(n) + \left((\tilde{X}(n))^2 - n \tilde{X}''(n)\tilde{X}(n) + O(n^{2\beta-2} \Psi(n))\right) - n[\tilde{X}'(n)]^2 + O(n^{\beta-1} \Psi(n))$$

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where the last error term is a consequence of \( n \tilde{V}^{(2)}(n) = O(n^{\beta-1} \Psi(n)) \). Since \( \text{Var} X_n = E X_n^2 - E^2 X_n \), the result follows. \( \square \)

We illustrate the above theorem on a simple example.

Example 4. I.I.D. Random Variables

Let \( Z_1, \ldots, Z_n \) be a sequence of independently identically distributed random variables with generating function \( P(u) = E u^{Z_1} \) and mean \( \mu \) and variance \( \sigma^2 \). The generating function of \( X_n = Z_1 + \cdots + Z_n \) is \( \tilde{G}_n(u) = (P(u))^n \). Observe that \( E X_n = n\mu \) and \( \text{Var} X_n = n\sigma^2 \). If we consider the Poisson transform of \( X_n \) we obtain \( \tilde{G}(z,u) = \exp(z(P(u) - 1)) \), and \( \tilde{X}(z) = z\mu, \tilde{V}(z) = (\mu^2 + \sigma^2)z \). Thus, \( \text{Var} X_n = \tilde{V}(n) - n[\tilde{X}'(n)]^2 \), as predicted by Theorem 6. \( \square \)

5.2 Limiting Distribution Results with Polynomial Bounds

When dealing with distributions, one must estimate \( G_n(u) \) from the Poisson transform \( \tilde{G}(z,u) \). If \( u \) belongs to a compact set, then our previous depoissonization results can be directly applied. For example, we have the following two corollaries that are quite useful in some analysis (cf. [13, 29, 43]). We again only consider linear cones.

Corollary 1 Suppose \( \tilde{G}_k(z) = \sum_{n=0}^{\infty} g_{n,k} z^n e^{-z} \), for \( k \) belonging to some set \( K \), are entire functions of \( z \). Let for some constants \( A, B, R > 0, \beta \) and \( \alpha < 1 \) the following two conditions holds uniformly in \( k \in K \):

(1) For \( z \in S_\theta \)

\[ |z| > R \Rightarrow |\tilde{G}_k(z)| \leq B|z|^\beta, \]  

(2) For \( z \notin S_\theta \)

\[ |z| > R \Rightarrow |\tilde{G}_k(z)e^z| \leq A \exp(\alpha |z|) . \]

Then, uniformly in \( k \in K \)

\[ g_{n,k} = \tilde{G}_k(n) + O(n^{\beta-1}) . \]

for large \( n \), and the error estimate does not depend on \( K \).

Proof. In fact, the above corollary is a direct consequence of our previous proofs. Never­
theless, we show below how it can be concluded from Theorem 5. Indeed, let us assume contrary that the thesis of the corollary, that is (60), does not hold. In other words, there is a subsequence \( (n_i, k_i) \) such that

\[ \lim_{t \to \infty} \left| \frac{g_{n_i, k_i} - \tilde{G}_{k_i}(n_i)}{n_i^{\beta-1}} \right| = \infty . \]
Observe that \( n_i \) cannot be bounded. Indeed, if the subsequence \( n_i \) would be bounded, then in this case the uniform boundness of \( G_k(z) \) (by our assumption (I)) in any compact set yields that the \( g_{n,k} \) are uniformly bounded if \( n_i \) are uniformly bounded (it suffice to bound the integrand in \( g_{n,k} = \frac{n_i}{2\pi r} \int G_k(z)e^{z-n-1}dz \)), which contradicts (61). So assume now that \( n_i \) is unbounded and strictly increasing, and define for a nonegative integer \( m \)

\[
\tilde{H}_m(z) = \begin{cases} 
0 & m \neq n_i \\
\tilde{G}_{k_i}(z) & m = n_i 
\end{cases}
\]

Then, \( h_{n_i,n_i} = g_{n_i,k_i} \) for all \( i \). Clearly, \( \tilde{H}_m(z) \) satisfies assumptions of Theorem 5 (since it satisfies condition (I) and (O) of the corollary which — as we already pointed out in our Remark 4 — imply conditions (I), (O) and (L) of Theorem 5). Thus,

\[
h_{n_i,n_i} = g_{n_i,k_i} = \tilde{G}_{k_i}(n_i) + O(n_i^{\beta-1})
\]

which is the desired contradiction. ■

The next corollary is a direct consequence of Corollary 1 and it already found several applications in the analysis of algorithms.

**Corollary 2** Let \( \tilde{G}(z, u) \) satisfy hypothesis of Theorem 1, i.e. for some numbers \( \theta < \pi/2, A, B, \xi > 0, \beta, \) and \( \alpha < 1 \) (I) and (O) hold for all \( u \) in a set \( \mathcal{U} \). In particular, for \( u \in \mathcal{U} \) condition (I) and (O) hold. Then,

\[
G_n(u) = \tilde{G}(n, u) + O(n^{\beta-1})
\]

uniformly in \( u \in \mathcal{U} \).

**Proof.** It directly from Corollary 1 since \( \tilde{G}_u(z) = \tilde{G}(z, u) \) and \( u \in K \) where \( K \) is an arbitrary set defined in Corollary 1. ■

While the above corollaries are quite useful in deriving limiting distributions (cf. some examples discussed in the next section), there are cases when condition (I) is not applicable. In fact, for many problems encountered in the analysis of algorithms \( \log G(z, u) \) increases in a polynomial manner (cf. [23, 24, 25, 28]), thus \( \tilde{G}(z, u) \) grows exponentially fast, and none of the theorems of previous sections can handle such a case. We need, therefore, another depoissionization result to handle these cases. This is discussed below.
5.3 Central Limit Theorem for the Poisson Model

First of all, we present below a central limit theorem for the Poisson model. It is later extended to the Bernoulli model in Theorem 9 which is our strongest result.

**Theorem 7** Let $X_N$ be a characteristic of the Poisson model with $\tilde{G}(z, u) = EuX_N$, mean $\tilde{X}(z)$ and variance $\tilde{V}(z)$. Let for $z \to \infty$ in a polynomial cone $C(D, \delta)$ with $\delta > 1/2$: and for $u$ belonging to a neighbourhood $U$ of $u = 1$ of the complex plane the following holds

$$\log \tilde{G}(z, u) = O(A(z)) ,$$

where $A(z)$ is such that

$$\lim_{z \to \infty} \frac{A(z)}{\sqrt{V_3/2}(z)} = 0$$

provided $\lim_{z \to \infty} V(z) = \infty$ and $\liminf_{z \to \infty} |V(z)| > 0$. Then for all complex $\tau$ in the vicinity of $0$:

$$\tilde{G} \left( z, e^{\tau} / \sqrt{\tilde{V}(z)} \right) e^{-\tau \tilde{X}(z) / \sqrt{\tilde{V}(z)}} = e^{\frac{z^2}{2}} \left( 1 + O \left( \frac{A(z)}{\sqrt{V_3/2}(z)} \right) \right) ,$$

that is, $X_N = (X_N - \tilde{X}(z)) / \sqrt{\tilde{V}(z)}$ converges in distribution and in moments to the standard normal distribution.

**Proof.** Developing $\log \tilde{G}(z, u)$ into Taylor's expansion around $u = 1$, we obtain

$$\log \tilde{G}(z, u) = (u - 1) \frac{\partial}{\partial u} \log \tilde{G}(z, 1) + \frac{(u - 1)^2}{2} \frac{\partial^2}{\partial u^2} \log \tilde{G}(z, 1) + R(z, u)$$

where $R(z, u)$ is the reminder in the Taylor's expansion. Observe that

$$\frac{\partial}{\partial u} \log \tilde{G}(z, 1) = \tilde{X}(z) ,$$

$$\frac{\partial^2}{\partial u^2} \log \tilde{G}(z, 1) = \tilde{V}(z) - \tilde{X}(z) .$$

The reminder can be expressed into two different forms corresponding to the two cases above (cf. [45])

$$R(z, u) = \frac{1}{2} \int_{1}^{u} (u - v)^2 \frac{\partial^3}{\partial u^3} \log \tilde{G}(z, u) dv ,$$

$$R(z, u) = \frac{(u - 1)^3}{2\pi i} \oint (v - u)(v - 1)^3 dv .$$

Now, we apply (62) to the above to find

$$R(z, u) = O((u - 1)^3 A(z))$$
which, after substitution \( u = e^{\tau/\sigma(z)} \), finally leads to

\[
\tilde{G}(x, e^{\tau/\sigma(z)}) e^{-\tilde{X}(x)/\sigma(z)} = e^{\tau^2/2} \left( 1 + O \left( \tau^3 A(z)/\sigma^3(z) \right) \right)
\]

where \( \sigma(z) = \sqrt{V(z)} \). This completes the proof. •

5.4 Limiting Distribution Results for the Bernoulli Model

Depoissonization of the case discussed in Theorem 7 does not follow yet from any of the results presented in Section 4. This is due to the fact that in Theorem 7 we normalized \( u = e^{t/\sqrt{V(z)}} \). We need another tool that resembles our idea of diagonalization of Theorem 5. Thus, we present one more depoissonization result that is used for limiting distribution when some kind of normalization is used.

**Theorem 8 Diagonal exponential depoissonization tool.**

Let \( \tilde{G}_k(z) \) be the Poisson transforms of a sequence \( g_{n,k} \), and they are assumed to be a sequence of entire functions of \( z \). Let \( \log \tilde{G}_k(z) \) exist in a a polynomial cone \( C(D, \delta) \) with \( 1/2 < \delta \leq 1 \). We suppose that there exists \( \beta \in [\delta - 1/2, 4/3(\delta - 1/2)] \) such that the condition (I) of Theorem 2 is replaced by:

(I) for all \( z \in C(D, \delta) \) such that for \( |z| \in [n - Dn^\delta, n + Dn^\delta] \)

\[
|\log \tilde{G}_n(z)| \leq Bn^\beta
\]

for some constant \( B > 0 \). In addition, condition (O) becomes

(0) for all \( z \notin C(D, \delta) \) such that \( |z| = n \):

\[
|\tilde{G}_n(z)e^z| \leq \exp(n - An^\alpha)
\]

for some \( \alpha > \beta \). Then, for all \( \varepsilon > 0 \):

\[
g_{n,n} = \tilde{G}_n(n) \exp \left[ -\frac{n}{2} (L'_n(n))^2 \right] \left( 1 + O(n^{3\beta - 4(\delta - 1/2) + \varepsilon}) \right)
\]

where \( L_n(z) = \log \tilde{G}_n(z) \) and \( L'_n(z) = \tilde{G}'_n(z)/\tilde{G}_n(z) \) is the first derivative of \( L_n(z) \).

**Proof:** We use Lemma 7 with \( F_n(t) = \log(h_n(t)) + t^2/2 + L_n(ne^{it\sqrt{n}} - L_n(n)) \). Under condition (I), the sequence of functions \( L_n(ne^{it\sqrt{n}}n^{-\beta}) \) defined on \( \{ t : \Re(t) & \Im(t) \in [-D'n^\delta-1/2, D'n^\delta-1/2]\} \) belong to \( D_m(\delta - 1/2) \) (more precisely: it belongs to the natural extension of \( D_m(\delta - 1/2) \) to functions of complex variable) for any integer \( m \geq 0 \). Let
\( \gamma < \delta - \frac{1}{2} \). As before, using Cauchy's formula, we arrive at

\[
g_{n,n} = \frac{\tilde{G}_n(n)\omega_n}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(x)]e^{-x^2/2}dx + \]

\[
+ \frac{\omega_n}{\sqrt{2\pi}} \int_{[Dn^{\gamma}, Dn^{\delta-\frac{1}{2}}]} \tilde{G}_n(ne^{ix/\sqrt{n}})h_n(x)dx + O(n^{1/2}\exp(-An^{\alpha})) ,
\]

with \( \omega_n = n!n^{-n}e^{n(2\pi n)^{-1}} = 1 + O(1/n) \). Our aim is to apply Lemma 7 to the first integral above. Denoting \( F'_n(x) \) and \( F''_n(x) \) as the first and second derivative of \( F_n(x) \), we must estimate \( F_n(0) \), \( F'_n(0) \) and \( F''_n(0) \) for \( \{\Re(x) & \Im(x) \in [-Dn^{\delta-1/2}, Dn^{\delta-\frac{1}{2}}]\} \) in order to identify the constant \( \beta_1 \) and \( \beta_2 \). Observe that by definition of \( F_n(t) \), we have \( F_n(0) = 0 \) and \( F'_n(0) = i\sqrt{n}L'_n(n) \). But, by the same argument as in the proof of Lemma 3, we deduce that \( L'(n) = O(n^{\beta-\delta}) \), and hence \( F'_n(0) = O(n^{\beta_1}) \) with \( \beta_1 = \beta - \delta + 1/2 \).

In order to estimate the second derivative \( F''_n(x) \) of \( F_n(x) \), we observe that it is the sum of the second derivative of \( \log(h_n(x)e^{x^2/2}) \) and second derivative of \( L_n(ne^{ix/\sqrt{n}}) \). The second derivative of \( \log(h_n(x)e^{x^2/2}) \) is exactly \( 1 - e^{ix/\sqrt{n}} \) which is \( O(n^{\gamma-1/2}) \) for \( x = O(n^{\gamma}) \). Since the second derivative of \( L_n(ne^{ix/\sqrt{n}}) \) is \( O(n^{\beta-2\gamma+1}) \), we identify \( \beta_2 = \max\{\gamma-1/2, \beta-2\gamma+1\} \).

Let us now fix \( \gamma = \beta_1 + \epsilon \) with \( 0 < \epsilon < 4(\delta - 1/2) - 3\beta \). We still have \( \gamma \leq \delta - 1/2 \) since \( \gamma < \beta_1 + 4(\delta - 1/2) - 3\beta = 3(\delta - 1/2) - 2\beta \leq \delta - 1/2 \).

To apply Lemma 7, we must check that the conditions on \( \gamma, \beta_1 \) and \( \beta_2 \) required in Lemma 7 are actually satisfied. In particular, \( 2\beta_1 + \beta_2 < 3\beta - 4(\delta - 1/2) + \epsilon \) (we already know that \( 3\beta - 4(\delta - 1/2) + \epsilon < 0 \), and then

\begin{itemize}
  \item \( \beta_1 \geq 0 \): by hypothesis \( \beta \geq \delta - 1/2 \);
  \item \( \gamma > \beta_1 \): by hypothesis about \( \epsilon \);
  \item Moreover we claim that \( 2\beta_1 + \beta_2 \leq 3\beta - 4(\delta - 1/2) + \epsilon \) (we already know that \( 3\beta - 4(\delta - 1/2) + \epsilon < 0 \)): we must check both case \( \beta_2 = \beta_1 + \epsilon - 1/2 \) and \( \beta_2 = \beta - 2(\delta - 1/2) \):
    \begin{enumerate}
      \item case \( \beta_2 = \beta_1 + \epsilon - 1/2 \): \( 2\beta_1 + \beta_2 \) becomes \( 3\beta - 3(\delta - 1/2) - 1/2 + \epsilon \) which is equal to \( 3\beta - 4(\delta - 1/2) + \epsilon + \delta - 1 \), and since \( \delta \leq 1 \) we obtained the desired result;
      \item case \( \beta_2 = \beta - 2(\delta - 1/2) \): \( 2\beta_1 + \beta_2 \) becomes \( 3\beta - 4(\delta - 1/2) \), as needed.
    \end{enumerate}
\end{itemize}

In summary: all conditions of Lemma 7 hold and we can apply Lemma 7 to yield:

\[
\frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(x)]e^{-x^2/2}dx = \exp\left[\frac{1}{2}(F'_n(0))^2\right](1 + O(n^{2\beta_1+\beta_2})).
\]

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By identifying $\frac{1}{2}(F_n'(0))^2$ with $-\frac{n}{2}(L_n'(n))^2$, we prove that
\[ g_{n,n} = \omega_n \bar{G}_n(n) \exp[-\frac{n}{2}(L_n'(n))^2](1 + O(n^{2\beta_1 + \beta_2}) + \frac{\omega_n}{\sqrt{2\pi}} \int_{|x| \leq D_n\gamma_n, D_n\gamma_n^{1-\frac{1}{2}}} \exp[L_n(ne^{ix}/\sqrt{n}) - L_n(n)] h_n(x) dx + O(\exp(-An^\alpha)) \]

The logarithm of $\bar{G}_n(n) \exp[-\frac{n}{2}(L_n'(n))^2]$ is of order $n^\beta$ (the order of $-\frac{n}{2}(L_n'(n))^2$ is $n^{2\beta - 2\delta + 1}$ which is smaller). The order of the last integral above is negligible since $|L_n(ne^{ix}/\sqrt{n}) - L_n(n)| = O(n^{\beta_1 - 3})$ and this leads to an estimate
\[ \exp \left( L_n(ne^{ix}/\sqrt{n}) - L_n(n) \right) h_n(x) = \exp(O(n^{\beta_1})e - \mu x^2) , \]
and therefore the integrals are of order $n^{\delta - \frac{1}{2}} \exp[Dn^{\gamma}(O(n^{\beta_1}) - \mu Dn^\gamma)]$ which decay faster than any polynomial since $\beta_1 < \gamma$. Finally, $O(\exp(-An^\alpha))$ decreases faster than any polynomial in $n$ since $\alpha > \beta$, and can also be neglected.

The error term $O(1/n)$ introduced by the Stirling approximation (i.e., $\omega_n = 1 + O(1/n)$) is swallowed by the order $O(n^{3\beta - 4(\beta - 1/2) + \varepsilon})$ which is larger than $O(n^{-1+\varepsilon})$. This completes the proof. \[ \Box \]

With the help of the depoissonization lemma we can now invert Theorem 7 to derive the normal limiting distribution for the Bernoulli model, that is, for $X_n$ and its generating function $G_n(u) = E u^{X_n}$. We recall that the mean $EX_n$ and the variance $\text{Var} X_n$ of the Bernoulli model can be obtained from the mean $\bar{X}(z)$ and the variance $\bar{V}(z)$ of the Poisson model as discussed in Theorem 6.

Let us consider the Poisson generating function $\bar{G}(z, u)$ such that inside a polynomial cone $C(D, \delta)$ the logarithm of $\bar{G}(z, u)$ exists, and we denote $L(z, t) = \log \bar{G}(z, e^t)$. We have the standard expansion:
\[ L(z, t) = \bar{X}(z)t + \bar{V}(z)\frac{t^2}{2} + R(z, t)t^3 \quad (68) \]
where $R(z, t)$ the remainder in Taylor's expansion of $L(z, t)$. We denote $V_n = \bar{V}(n) - n \left( \bar{X}'(n) \right)^2$ which by Theorem 6 is asymptotically equal to the variance in the Bernoulli model.

**Theorem 9** Let $\chi$ and $\nu$ such that $3\nu/2 > \chi > \nu > 0$ and $\beta = \chi - \nu/2 \in [\delta - 1/2, 4/3(\delta - 1/2)]$ with $\frac{1}{3} < \delta \leq 1$. Suppose that there exist $D > 0$, $B > 0$ and $F > 0$ such that for all real $|t| < F$ the following three conditions hold:

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(I) \( z \in C(D, \delta) \):

\[
L(z, t) | \leq B |z|^\delta, \quad (69)
\]

\[
R(z, t) | \leq B |z|^\delta \quad (70)
\]

for some constant \( B > 0 \), where \( R(z, t) \) is the remainder term in (68).

(0) For all \( z \notin C(D, \delta) \) such that \( |z| = n \):

\[
|\tilde{G}_n(z)e^z| \leq \exp(n - An^\alpha) \quad (71)
\]

for some \( \alpha > \chi - \nu/2 \).

(V) For some \( B > 0 \)

\[
V_n > Bn^\nu.
\]

Then, for all real \( t \) such that \( |t| < F \), and any \( \varepsilon > 0 \):

\[
G_n(e^{t/\sqrt{V_n}}) \exp \left(-\tilde{X}(n) \frac{t}{\sqrt{V_n}}\right) = e^{-t^2/2} \left(1 + O(tn^{3\beta-4(\delta-1/2)+\varepsilon})\right)
\]

In words: \( Y_n = (X_n - \tilde{X}(n))/\sqrt{V_n} \) converges in distribution and in moments to the standard normal distribution.

**Proof.** Let first assume \( t \) real such that \( t \in (-F, F) \). Since \( L(z, t) \) and \( R(z, t) \) are both \( O(z^\delta) \), it is clear that for \( z \in C(D, \delta) \) the mean \( \tilde{X}(z) \) and the variance \( \tilde{V}(z) \) are also \( O(z^\delta) \). Consequently, the derivative with respect to \( z \) of these quantities are all \( O(z^{\delta-\delta}) \). Let us now fix \( t \). We write \( \tilde{G}_k(z) = \tilde{G}(z, e^{t/\sqrt{V_k}}) \) and \( L_k(z) = L(z, t/\sqrt{V_k}) \). Notice that \( t/\sqrt{V_k} \) will fall and stay in \((-F, F)\) when \( k \to \infty \). In the sequel we suppose that \( k \) is large enough to have this property. When \( z = O(n) \) in the cone we have \( L_n(z) = O(tn^{\beta-\nu/2}) \). Therefore, conditions of Theorem 8 are fulfilled: with \( \beta = \chi - \nu/2 \). We have \( 3\beta - 4(\delta - 1/2) < 0 \) and for all \( \varepsilon > 0 \):

\[
G_n(e^{t/\sqrt{V_n}}) = \tilde{G}_n(n) \exp \left(-n/2(L_n'(n))^2\right) \left(1 + O(tn^{3\beta-4(\delta-1/2)+\varepsilon})\right)
\]

In order to complete the proof, we use the expansion

\[
\log \tilde{G}_n(n) = L(n, t/\sqrt{V_n})
\]

\[
= \tilde{X}(n) \frac{t}{\sqrt{V_n}} + \tilde{V}(n) \frac{t^2}{2V_n} + t^2 \frac{R(n, t)}{V_n^{3/2}}
\]

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According to our estimate on $R(n,t)$ and $V_n$ the last term is $n^{x-3/2\nu}$ which tends to zero. We also have the estimate

$$L'_n(n) = \tilde{X}'(n) \frac{t}{\sqrt{V_n}} + t^2 O(n^{x-\delta-\nu})$$

which leads to

$$n(L'_n(n))^2 = n(\tilde{X}'(n))^2 \frac{t^2}{V_n} + O(n^{2x-2\delta-3/2\nu}) .$$

Since $2x-2\delta-3/2\nu < 0$ the error term converges to zero. In other words, we obtain

$$G_n(e^{t/\sqrt{V_n}}) = \exp \left[ \tilde{X}(n) \frac{t}{\sqrt{V_n}} + \frac{\tilde{V}(n) - n(\tilde{X}'(n))^2}{2} + O(n^{3\beta-4(\delta-1/2) + \epsilon}) \right]$$

with $3\beta - 4(\delta - 1/2) < 0$.

Thus, we proved that for a real $t \in (-F, F)$ the function $g_n(t) = Ec^{Y_n}$ where $Y_n = (X_n - \tilde{X}(n))/\sqrt{V_n}$ converges to $e^{\frac{1}{2}t^2}$. Convergence of $Y_n$ in moments follows directly from Lemma 8. This completes the proof. \(\blacksquare\)

6 Applications of Depoissonization Results

In this section, we apply our depoissonization theorems of previous sections to various problems arising in combinatorics and analysis of algorithms and data structures. We start with a general result (cf. Theorem 10) that provides a simple tool to depoissonize a class of linear functional equations often appearing in the analysis of digital data structures and algorithms (cf. Section 6.1). Then, we deal with problems where depoissonization of moments is required (cf. Sections 6.2–6.4), and finally we present some depoissonization results for limiting distributions (cf. Sections 6.5–6.9).

6.1 Depoissonization of a Linear Functional Equation

Several problems discussed in Section 3 can be reduced to the following general linear functional equation

$$\tilde{G}(x,u) = \beta a(x/2,u)\tilde{G}(x/2,u) + b(x,u) , \quad (72)$$

where $|\beta| \leq 1$, and $u$ is either fixed (cf. (5), (6), (20), (21), etc.) or $u$ belongs to a compact neighborhood $U(u_0)$ of $u_0$ (cf. (8), (12), (19), (23), etc.). Iterating (72), we obtain a general solution of the above equation (cf. [26, 32, 50])

$$\tilde{G}(x,u) = \sum_{n=0}^{\infty} \beta^n b(x^{2^{-n}}, u) \prod_{k=1}^{n} a(x^{2^{-k}}, u) \quad (73)$$

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provided $\tilde{G}(0,u) = 0$, and all the series above converge. Define

$$\varphi(z,u) = \prod_{j=0}^{\infty} a(z2^j, u), \quad (74)$$

if the infinite product in (74) converges. Then, the general solution (73) can be rewritten as

$$\tilde{G}(z,u)\varphi(z,u) = \sum_{n=0}^{\infty} \beta^n b(z2^{-n}, u)\varphi(z2^{-n}, u). \quad (75)$$

Solution (75) is often used to derive asymptotics of the functional equation (73), and in particular we used it to solve some problems discussed in Section 3 (e.g., (5), (6), and (20)-(21)) as already presented in [18, 26, 32]. This is possible since the sum in (75) falls under the so called harmonic sum that can be handled by the Mellin transform technique (cf. [15]). In general, this allows to derive an asymptotic solution to the above equation for $z \to \infty$, and then "depoissonize" to recover the original sequence.

To apply our depoissonization findings of the previous sections, one must establish conditions (I) and (O), that is, some upper bounds for the Poisson transform inside and outside a cone, respectively. A technique we shall use to demonstrate such bounds for equation (73) was proposed for the first time in [23, 25], and in a more formal way in [28, 43]. It is based on the mathematical induction over the so called increasing domains that we define next. In fact, we can treat a more general functional equation, namely (for simplicity we drop the variable $u$)

$$\tilde{G}(z) = \gamma_1(z)\tilde{G}(zp) + \gamma_2(z)\tilde{G}(zq) + t(z), \quad (76)$$

where $\gamma_1(z)$, $\gamma_2(z)$ and $t(z)$ are functions of $z$ such that the above equation has a solution. The reader is referred to [11, 47] for conditions on these functions under which a solution of (76) exists. We further assume that $p + q = 1$ (cf. (4)), but in fact $zp$ and $zq$ could be replaced by more general functions that form a semigroup of substitutions under the operation of composition of functions (cf. [11, 47]). In passing, we observe that the above more general equation arises in non-blocking conflict resolution algorithms (cf. [11, 47]) and asymmetric leader election algorithm (cf. [29]).

Let us define for integers $m = 0, 1, \ldots$ and a constant $\lambda$ such that $0 < \max\{p, q\} \leq \lambda^{-1} < 1$, a sequence of increasing domains (cf. Figure 4) $D_m$ as

$$D_m = \{z : \xi \leq |z| \leq \xi \lambda^{m+1}\}$$

for some constant $\xi > 0$. Observe that

$$z \in D_{m+1} \implies pz, qz \in D_m. \quad (77)$$
The last property is crucial to apply mathematical induction over $m$ in order to establish appropriate bounds on $\bar{G}(z)$ over a whole complex plane, so we can apply either Theorem 1 or Theorem 2.

The next result presents a "depoissonization lemma" for the functional equation (76) in the case $p + q = 1$.

**Theorem 10** Consider the functional equation (76) with $p + q = 1$, that is,

$$\bar{G}(z) = \gamma_1(z)\bar{G}(zp) + \gamma_2(z)\bar{G}(zq) + t(z)$$

which is postulated to have an entire solution. Let also for some positive $A, \beta, \theta, \xi$ and $0 < \eta < 1$ the following conditions hold for $|z| > \xi$:

1. For $z \in S_\theta$ ($0 < |\theta| < \pi/2$)

   $$|\gamma_1(z)|p^\beta + |\gamma_2(z)|q^\beta \leq 1 - \eta,$$

   $$|t(z)| \leq A\eta|z|^{\beta};$$

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(O) For \( z \notin S_0 \) we request that for some \( \alpha < 1 \) the following three inequalities are true

\[
|\gamma_1(z)|e^{p\Re(z)} \leq \frac{1}{3} e^{\alpha|z|}, \tag{80}
\]
\[
|\gamma_2(z)|e^{p\Re(z)} \leq \frac{1}{3} e^{\alpha|z|}, \tag{81}
\]
\[
|\ell(z)|e^{p\Re(z)} \leq \frac{1}{3} e^{\alpha|z|}. \tag{82}
\]

Then,

\[
g_n = \tilde{G}(n) + O(n^{p-1}) \tag{83}
\]

or more generally, for all nonnegative integer \( m \)

\[
g_n = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} \tilde{G}^{(j)}(n) + O(n^{p-m-1}) \tag{84}
\]

where \( g_n = [z^n] \left( n! e^{z\tilde{G}(z)} \right) \).

**Proof.** We apply Theorem 1 or Theorem 2 (with \( \delta = 1 \)). For this, we need to establish the following bounds

\[
|\tilde{G}(z)| \leq B|z|^p \tag{85}
\]

for \( z \in S_0 \), and

\[
|\tilde{G}(z)e^z| \leq e^{\alpha|z|} \tag{86}
\]

for \( z \notin S_0 \) with \( \alpha < 1 \). It suffices to prove that under (78)-(79) and (80)-(82) the above two conditions take place.

The proof is by induction over the increasing domains \( D_m \). Let us first consider \( z \in S_0 \). Define \( \hat{D}_m = D_m \cap S_0 \). We take \( B \) large enough and greater than \( A \) such that for \( m = 0 \) the above inequalities hold for \( |z| \leq \xi \). We also take \( \alpha \geq \cos \theta \). Let us assume now that inequalities (78) and (79) are satisfied in \( \hat{D}_m \), and we prove that they also hold in a larger region, namely \( \hat{D}_{m+1} \), thus proving (85) in \( S_0 \). Assume that \( z \in \hat{D}_{m+1} \sim \hat{D}_m \). But by property (77) we have \( z p, z q \in \hat{D}_m \), and we can invoke our induction hypothesis. Hence, taking into account (78)-(79) we conclude from equation (76) that

\[
|\tilde{G}(z)| \leq |\gamma_1(z)||\tilde{G}(zp)| + |\gamma_2(z)||\tilde{G}(zq)| + |\ell(z)|,
\]

\[
\leq B|\gamma_1(z)||z|^p + B|\gamma_2(z)||z|^q + |\ell(z)|,
\]

\[
\leq B(1 - \eta)|z|^p + B\eta|z|^q = B|z|^p
\]

which is the desired result.
Assume now that \( z \notin S_\theta \) and we aim at proving (86). We first observe that \( |e^z| = e^{\Re(z)} \leq e^{\alpha|z|} \) where \( \alpha \geq \cos \theta \geq \cos(\arg(z)) \). Now, the induction over the increasing domains can be applied as before, however, this time we consider \( D_m = D_m \cap S_\theta \) where \( S_\theta \) is the complementary set to \( S_\theta \). Observe that

\[
|\tilde{G}(z)^2| \leq |\gamma_1(z)||\tilde{G}(zp)e^{2\Re(z)}|e^{2\Re(z)}| + |\gamma_2(z)||\tilde{G}(zq)e^{2\Re(z)}|e^{2\Re(z)}| + |t(z)e^z|,
\]

\[
\leq |\gamma_1(z)||\tilde{G}(zp)|e^{\alpha|z|}e^{\Re(z)} + |\gamma_2(z)||\tilde{G}(zq)|e^{\alpha|z|}e^{\Re(z)} + |t(z)|e^{\Re(z)},
\]

where the last inequality follows from (80)-(82) and \( \Re(z) = |z| \cos(\arg(z)) \leq |z| \cos \theta \leq \alpha|z| \).

This completes the proof.

Below, we discuss some examples that illustrate Theorem 10. The first set of examples deals with moments (cf. Sections 6.2–6.4). Next, we turn our attention to limiting distributions (cf. Sections 6.5–6.9).

### 6.2 Conflict Resolution algorithm

In Section 3.1 we described several conflict resolution algorithms. While certain functional equations arising in these applications can be handled without Poisson transforms, equation (5), or more generally (6), requires poissonization. Let us, as an example, consider (5) which is repeated below

\[
\tilde{C}(z) = \left( (1 + z/2)e^{-z/2} \right) \tilde{C}(z/2).
\]

This equation is of the form of (72), thus has (73) as a solution. Jacquet and Szpankowski [26] proved that for large \( z \in S_\theta \) the above equation attains asymptotically the following solution \( \tilde{C}(z) = D + P(\log z) + O(1/z) \) where \( D \) is a constant computed in [26], and \( P(\log z) \) is a fluctuating function with a small amplitude.

Can we depoissonize this solution? One must check conditions (I) and (O) of Theorem 1. But, \( \tilde{C}(z) = O(1) \) in a cone \( S_\theta \) by the results of [26] just cited, thus we need only to verify (O) outside the cone \( S_\theta \). By Theorem 10 (e.g., see (80)) we need to show the existence of \( \xi \) such that for \( |z| > \xi \) the following holds

\[
(1 + |z|/2)e^{\Re(z)/2} \leq e^{\alpha|z|/2}
\]

which is clearly true for large enough \( \xi \) since \( \alpha \geq \cos \theta \). Thus, by Theorem 10 (in fact, by Theorem 1) we conclude that \( G_n = D + P(\log n) + O(n^{-1}) \) (which is stronger than in [26] where only \( O(1/\sqrt{n}) \) error term was established).
6.3 Leader Election Algorithm or Incomplete Tries

In Section 3.4 we discussed a leader election algorithm due to Prodinger [42] (cf. also [13]). In [13] it was shown that the first moment $X(z)$ and the second factorial moment $W(z)$ of the number of steps required to elect a leader in the Poisson model satisfy the following functional equations (cf. (18)-(20))

$$
X(z) = X(z/2)(1 + e^{-z/2}) + 1 - e^{-z} - ze^{-z},
$$

$$
W(z) = W(z/2)(1 + e^{-z/2}) + 2X(z/2)(1 + e^{-z/2}).
$$

These equations fall under (72) and therefore depoissonization can be obtained via Theorem 10. Indeed, in a cone $S_\theta$ the authors of [13] proved that for large $z$ and any $R > 0$

$$
X(z) = \log_2 z + \frac{1}{2} - \delta_1(\log z) + O(|z|^{-R}),
$$

$$
W(z) = \log_2^2 z + \frac{\pi^2}{6\log^2 2} - \frac{1}{6} - \frac{2\gamma_1}{\log^2 2} - \frac{\gamma^2}{\log^2 2} + \delta_2(\log z) + O(|z|^{-R}),
$$

with $\delta_1(\cdot)$ and $\delta_2(\cdot)$ being fluctuating functions with small amplitudes. Here the constants $(-1)^k \gamma_k/k!$, $k \geq 0$, are the so-called Stieltjes constants, with

$$
\gamma_k := \lim_{m \to \infty} \left( \sum_{t=1}^{m} \frac{\ln^k t}{t} - \frac{\ln^{k+1} m}{k+1} \right);
$$

in particular, $\gamma_0 = \gamma = 0.577215 \ldots$ is Euler's constant and $\gamma_1 = -0.072815 \ldots$. Since both Poisson transforms grow in a poly-log fashion, condition (1) of Theorem 1 is automatically satisfied. As a matter of fact, the above asymptotics were derived through Mellin transform, thus the above formally holds only for $z$ real (since Mellin results were originally defined only for $z$ real). In Appendix B we introduce a complex Mellin transform, and show that all results valid for real $z$ can be easily translated into complex $z$, as needed for most of our depoissonization results. In passing, we should also mention that alternatively we can use analytical continuation to extend the above asymptotics to complex $z$. However, using complex Mellin approach is more powerful, as we shall see in Section 6.6.

To complete depoissonization of $X(z)$ and $W(z)$ we must verify condition (O) outside the cone $S_\theta$. This can be done along the steps shown in the example above, but for completeness we deal here with $W(z)$ provided we already proved that outside a cone $S_\theta$ we have $|X(z)e^\alpha| \leq \alpha |z|$ for some $\alpha > \cos \theta$. By Theorem 10 it suffices to show that for sufficiently large $\xi$ the following holds

$$
2(1 + e^{-\Re(z)/2})e^{\Re(z)/2} \leq \frac{1}{2} e^{\alpha |z|/2}.
$$
But, the above trivially holds since $\Re(z) = |z| \cos(\arg(z)) < \alpha |z|$ for some $\alpha > 0$. This completes the depoissonization and leads to:

**Proposition 1 (Prodinger [42], Fill et al. [13])** Define $L := \ln 2$ and $\chi_k := 2\pi i k / L$. Then:

\[
\begin{align*}
E[H_n] &= \log_2 n + \frac{1}{2} - \delta_1(\log n) + O\left(\frac{1}{n}\right), \\
\Var[H_n] &= \frac{\pi^2}{6L^2} + \frac{1}{12} - \frac{2\gamma_1}{L^2} - \frac{\gamma^2}{L^2} + \delta_2(\log n) + O\left(\frac{\ln n}{n}\right) \\
&= 3.116695 \ldots + \delta_2(\log n) + O\left(\frac{\ln n}{n}\right).
\end{align*}
\]

where $\delta_1(\cdot)$ is a periodic function of magnitude $\leq 2 \times 10^{-5}$ given by

\[
\delta_1(x) := \frac{1}{L} \sum_{k \neq 0} \zeta(1 - \chi_k) \Gamma(1 - \chi_k) e^{2\pi i k x},
\]

and $\zeta(\cdot)$ and $\Gamma(\cdot)$ denote Riemann's zeta function and Euler's gamma function, respectively. Furthermore, the constants $(-1)^k \gamma_k / k!$, $k \geq 0$, are the so-called Stieltjes constants, with

\[
\gamma_k := \lim_{m \to \infty} \left( \sum_{i=1}^{m} \frac{\ln^k i}{i} - \frac{\ln^{k+1} m}{k+1} \right);
\]

in particular, $\gamma_0 = \gamma = 0.577215 \ldots$ is Euler's constant and $\gamma_1 = -0.072815 \ldots$. The periodic function $\delta_2(\cdot)$ has magnitude $\leq 2 \times 10^{-4}$.

### 6.4 Path Length in Digital Search Trees

This example is to illustrate an application of Theorem 6 in which the leading term of the Bernoulli variance is effected by the correction term of the depoissonization (cf. (57)). Let us consider the path length $L_n$ in a digital search tree (cf. Section 3.3). The Poisson transform of $E u^L_n$ satisfies the differential-functional equation (17). One shows that the Poisson mean $\bar{X}(z) = \bar{G}'(z, 1)$ and the Poisson second factorial moment $\bar{W}(z) = \bar{G}''(z, 1)$ become (cf. [28])

\[
\begin{align*}
\bar{X}(z) + \bar{X}'(z) &= \bar{X}(zp) + \bar{X}(zq) + z, \quad (87) \\
\bar{W}(z) + \bar{W}'(z) &= \bar{W}(zp) + \bar{W}(zq) + 2zp \bar{X}'(zp) + 2zq \bar{X}'(zq) + (\bar{X}'(z))^2. \quad (88)
\end{align*}
\]

Using Mellin transform approach Jacquet and Szpankowski [28] proved that in a polynomial cone $C(D, \delta)$ ($\delta > \frac{1}{2}$)

\[
\bar{X}(z) = \frac{z}{h} \log z + \frac{z}{h} \left( \gamma - 1 + \frac{h_2}{2h} - \alpha - \delta_1(\log z) \right) + O(1), 
\]

(89)
\[ \tilde{V}(z) = \frac{x \log^2 z}{h^2} + \frac{2x \log z}{h^3} \left( \gamma h + h_2 - \frac{h^2}{2} - \alpha h - h \delta_1(\log z) \right) + O(z) , \quad (90) \]

where \( \tilde{V}(z) = \tilde{W}(z) + \tilde{X}(z) \), and \( h = -p \log p - q \log q, \ h_2 = p \log^2 p + q \log^2 q, \ \gamma = 0.577215 \ldots \) is Euler's constant, and finally

\[ \alpha = -\sum_{k=1}^{\infty} \frac{p^{k+1} \log p + q^{k+1} \log q}{1 - p^{k+1} - q^{k+1}} . \]

In the above \( \delta_1(z) \) is a fluctuating function with a small amplitude.

In order to depoissonize \( \tilde{X}(z) \) and \( \tilde{W}(z) \) we first observe that (87) and (88) are differential-functional equations which do not fall into our general functional equation treated in Theorem 10. But, fortunately, the method of increasing domains (presented in the proof of Theorem 10) still works. Since we deal with Poisson mean and Poisson variance we are in the framework of Theorem 6. Nevertheless, one must verify conditions (I) and (O) for \( \tilde{X}(z) \) and \( \tilde{W}(z) \). The condition (I) inside the cone follows directly from our asymptotics (89)-(90) and either analytical continuation or an application of the complex Mellin transform (cf. Appendix B). Thus, we must prove a bound for \( \tilde{X}(z)e^z \) and \( \tilde{W}(z)e^z \) outside a polynomial cone \( C(D, \delta) \), that is, to verify condition (O) of Theorem 2.

Let us analyze only the Poisson mean \( \tilde{X}(z) \), however, we consider a more general differential-functional equation (arising in the analysis of the depth in a generalization of digital search trees known as b-digital search trees [18, 36]), namely:

\[ X^{(b)}(z) = X(z)p e^{\omega q} + X(z)q e^{z p} + z e^{z} \]

where \( X^{(b)}(z) \) denotes the \( b \)th derivative of \( X(z) = \tilde{X}(z)e^z \). Observe that the above equation can be alternatively represented as

\[ X(z) = \int_0^z \int_0^{w_2} \cdots \int_0^{w_b} \left( X(w_1 p) e^{w_1 q} + X(w_1 q) e^{z w_1 p} + z w_1 \right) dw_1 dw_2 \cdots dw_b \]

provided \( X^{(k)}(z) = 0 \) for all \( k < b \) which we assume to hold. We must prove \( |X(z)| \leq e^{a|z|} \) for \( z \notin S_\theta \) for \( \alpha < 1 \). As in the proof of Theorem 10 we apply mathematical induction over increasing domains \( D_m \). To carry out the induction, we first define \( \overline{D}_m = D_m \cap \overline{S}_\theta \) where \( \overline{S}_\theta \) denotes points in the complex plane outside \( S_\theta \). Since \( X(z) \) is bounded for \( z \in \overline{D}_0 \), thus the initial step of induction holds. Let us now assume that for \( z \in \overline{D}_m \) we have \( |X(z)| \leq e^{a|z|} \) for some \( m \) and \( \alpha < 1 \). We intend to prove that \( |X(z)| \leq e^{a|z|} \) for \( z \in \overline{D}_{m+1} \). Indeed, let \( z \in \overline{D}_{m+1} \). If also \( z \in D_m \), then the proof is completed. So let us now assume that \( z \in \overline{D}_{m+1} - \overline{D}_m \). Since then \( zp, zq \in \overline{D}_m \), we can use our induction hypothesis together
with the above integral equation to obtain the following estimate for $|z| > \xi$ where $\xi$ is large enough

$$|X(z)| \leq |z|^b \left( e^{\xi z}((\rho a + \xi \cos \theta) + e^{\xi z}((\rho a + \rho \cos \theta) + e^{\xi z} \cos \theta) \right).$$

Let us now define $\alpha > \cos \theta$ such that the following three inequalities are simultaneously fulfilled

$$|z|^b e^{\xi |z| (\rho a + \xi \cos \theta)} \leq \frac{1}{3} e^{\alpha |z|},$$

$$|z|^b e^{\xi |z| (\rho a + \rho \cos \theta)} \leq \frac{1}{3} e^{\alpha |z|},$$

$$|z|^b e^{\xi |z| \cos \theta} \leq \frac{1}{3} e^{\alpha |z|}.$$

Then, for $z \in \mathbb{D}_{m+1}$ we have $|X(z)| \leq e^{\alpha |z|}$, as needed to verify condition (O) of Theorem 1. In a similar manner we can handle $\overline{W}(z)$.

Now, we are in a position to apply Theorem 6. Interestingly enough, the leading term in the variance of the Poisson model is cancelled resulting in $\text{Var } L_n = O(n \log n)$ (instead of $O(z \log^2 z)$ as in the Poisson model). More precisely:

**Proposition 2** (Jacquet and Szpankowski [28]) *With the notation as above, we obtain*

$$\mathbb{E} L_n = \frac{n}{h} \left( \log n + \frac{h_2}{2h} + \gamma - 1 - \alpha + \delta_0 (\log m) \right)$$

$$+ \frac{1}{h} \left( \log n + \frac{h_2}{2h} - \gamma - \log p - \log q + \alpha \right) + O(1)$$

$$\text{Var } L_n = \frac{h^2 - h_2}{h^3} n \log n + O(n)$$

*for $p \neq q$, where $h = -p \log p - q \log q$ and $h_2 = p \log^2 p + q \log^2 q$.*

In passing, we should point out that that in the case $p = q = \frac{1}{2}$ the variance is $O(n)$ as proved in [31], and in fact the term in front of $n$ contains a fluctuating function. We should add that the techniques of [31] and [28] are different even if both use Mellin transforms.

### 6.5 Leader Election Algorithm Revisited

We refer to Section 3.4 or Section 6.3 above for a description of the problem. Hereafter, we are interested in the limiting distribution of the height $H_n$ which represents the number of tosses before a leader is elected (cf. [13, 42]). The generating function of $H_n$ is given by (18), and its Poisson transform by (19) which we repeat below for the reader's convenience

$$\tilde{G}(z, u) = u(1 + e^{-z/2})\tilde{G} \left( \frac{z}{2}, u \right) + e^{-z}(1 + z)(1 - u) - ue^{z/2}.$$

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Actually, using the Mellin transform approach, we can solve asymptotically the above for large $z$ to obtain (cf. [13])

$$
\tilde{G}(z, u) = -\frac{z^{\log u}}{\ln 2} ((1 - u)\Gamma(1 - \log u)\zeta(1 - \log u) + \delta(z, u)) (1 - e^{-z}) + O(z^{-M}) \quad (94)
$$

where $|u| < 1/2$ and $M$ is a large positive number, and $\delta(z, u)$ is a fluctuating function with a small amplitude. The above can be used to obtain exact and asymptotic formulæ for the distribution of $H_n$. We concentrate here on the limiting distribution, if it exists. First, however, we must verify depoissonization conditions (I) and (O) to assure that we can appeal to Corollary 2. But, the functional equation falls under the general functional equation treated in Theorem 10, thus an easy verification lead us to a conclusion that all conditions for depoissonization are in place. In particular, $\tilde{G}(z, u) = O(1)$ since $|u| < 1$, and therefore by Theorem 10 we conclude that $G(u) = \tilde{G}(u, u) + O(u^{-1})$ (for details see [13]).

Knowing that depoissonization of $\tilde{G}(z, u)$ works, we may ask about the limiting distribution of $H_n$. From (94) and the Cauchy formula we have for an integer $k$

$$
\Pr\{H_n \leq \log n + k\} = \Pr\{H_n \leq [\log n + k]\}
= -\frac{1}{2\pi i \ln 2} \int \frac{1}{z^{[\log_2 n] - k - 1}} \Gamma(1 - \log z)\zeta(1 - \log z)dz
+ \int \frac{O(n^{-1})}{(1 - z)z^{[\log_2 n] + k}}dz
$$

where $[\log_2 n] = \log_2 n - [\log_2 n]$. Observe that the last integral is $O(n^{-1})$. Indeed, it suffices to enlarge the circle of integrating to $z = 1$ (more precisely: we can take $z = 2^j$ where $j = 1, 2, \ldots$), which are contributed by the $\Gamma(\cdot)$ function, and one additional pole at $z_0 = 1$ coming from Riemann’s zeta function. The residue at $z_0$ is $-\ln 2$. The residue at $z_j$, for $j = 1, 2, \ldots$ is

$$
\frac{(-1)^j \ln 2}{(j-1)!} 2^j [\log_2 n - k] \zeta(1 - j)
$$

where $[\log_2 n] = \log_2 n - [\log_2 n]$. The special values of Riemann’s zeta function appearing in the last expression are related to the Bernoulli numbers $B_r$, $r = 0, 1, 2, \ldots$, as follows: $\zeta(0) = -1/2$, $\zeta(1 - 2k) = -B_{2k}/(2k)$, for $k = 1, 2, 3, \ldots$, and $B_{-2k} = 0$, for $k = 1, 2, \ldots$. Subsequently we arrive at

$$
\Pr\{H_n \leq \log n + k\} = 1 - 2^{[\log_2 n] - k - 1} + \sum_{j=1}^{\infty} 2^j [\log_2 n - k] \frac{B_{2j}}{(2j)!} + O\left(\frac{1}{n}\right).
$$
The remaining sum is expressible in terms of the classical generating function of the Bernoulli numbers
\[
\frac{y}{e^y - 1} = \sum_{j=0}^{\infty} \frac{B_j y^j}{j!},
\]
and this yields

**Proposition 3 (Fill et al. [13])** Uniformly over all integers \(k\),

\[
\Pr\{H_n \leq [\log_2 n] + k\} = \frac{2^{[\log_2 n]-k}}{\exp(2([\log_2 n]-k))} + O\left(\frac{1}{n}\right)
\]

as \(n \to \infty\), where \([\log_2 n]\) = \log_2 n - [\log_2 n].

Finally, we observe that the function \([\log_2 n]\) is dense on the interval \([0,1]\) but not uniformly dense, hence a limiting distribution of \(H_n\) does not exist (but formula above presents the asymptotic distribution). The reader is referred to [13] for more details where the above result was derived for the first time (but, with a weaker error term, namely \(O(1/\sqrt{n})\)).

An interesting extension was recently considered by Janson and Szpankowski [29] who analyzed the asymmetric leader election process in which \(p\) is the probability of survival at each round \((p \neq \frac{1}{2})\). In this case, the Poisson transform \(\tilde{G}(z,u)\) satisfies the following functional equation

\[
\tilde{G}(z,u) = u\tilde{G}(pz,u) + u\tilde{G}(qz,u)e^{-pz} + (1-u)ze^{-z}.
\]

Asymptotic solution of this equation is much more intricate. We observe that the Poisson transform \(\tilde{G}_k(z) = \sum_{n \geq 0} \Pr\{H_n = k\} \frac{z^n}{n!} e^{-z}\) fulfills

\[
\tilde{G}_0(z) = ze^{-z},
\]
\[
\tilde{G}_{k+1}(z) = \tilde{G}_k(pz) + e^{-pz}\tilde{G}_k(qz), \quad k \geq 1.
\]

This yields the following solution as proved in [29]

\[
\tilde{G}_k(z) = p^k z \int_0^{p^{-k}} e^{-p z t} d\mu(t),
\]

where \(\mu(t)\) is defined as follows: Partition the positive real axis into an infinite sequence of consecutive intervals \(I_0, I_1, \ldots\) such that \(I_k\) has length \((q/p)^{s(k)}\), where \(s(k)\) is the number of 1’s in the binary expansion of \(k\). Thus \(I_0 = [0,1], I_1 = [1,1+q/p], \ldots\). Note that the total length of the first \(2^m\) intervals \((I_0, \ldots, I_{2^m-1})\) is \(p^{-m}\), and that these \(2^m\) intervals are obtained by repeated subdivisions of \([0,p^{-m}]\), each time dividing each interval in the...
proportions $p : q$. Given these intervals, define $\mu$ by putting a point mass $|I_k|$ at the right endpoint of $I_k$, for each $k = 0, 1, \ldots$

To obtain an asymptotic distribution for $H_n$, one must depoissonize $\tilde{G}_k(z)$ and then apply Corollary 1. Observe that only a slight generalization of Theorem 10 is required to handle the functional equation (96). Conditions of Theorem 10 are easy to check. However, since we have an explicit formula for $\tilde{G}_k(z)$ (cf. (97)), we may check condition (I) and (O) directly. For the reader's convenience, we prove now that inside a cone $S_\theta$ we have $\tilde{G}_k(z) = O(1)$ uniformly over all natural numbers $k$. Indeed, we first evaluate the function $F(x)$ defined for positive $x$ as

$$F(x) = x \int_0^\infty e^{-xt}d\mu(t).$$

Let, without losing generality, $q \geq p$, and set $\alpha = q/p \geq 1$. Noting that the interval $I_j$ satisfy: $\alpha \leq |I_j| \leq \alpha^j$, we can estimate

$$F(x) := x \int_0^\infty e^{xt}d\mu(t) = x \sum_{j=1}^\infty e^{-x|I_1|+\ldots+|I_j||I_j|}$$

$$\leq x \sum_{j=1}^\infty e^{-x\alpha^j} \alpha^j = \frac{x}{e^{ax}\log \alpha - 1}.$$

Clearly, the above is bounded when we replace $x$ either by $|x|^p$ or $\Re(x)$ in a cone $S_\theta$. This establishes condition (I). Conditions (O) can be proved in a similar manner (cf. [29] for details).

Now, we are ready to apply Corollary 1 to arrive at, as already proved in [29]:

Proposition 4 (Janson and Szpankowski [29]) The following holds, uniformly for all integers $k$,

$$\Pr\{H_n \leq k\} = F(p^kn) + O(n^{-1}),$$

where $F(x)$ is defined in (98). In particular, when $k = \lfloor \log_p n \rfloor + \kappa$ where $\kappa$ is an integer, then for large $n$ the following asymptotic formula is true uniformly over $\kappa$

$$\Pr\{H_n \leq \lfloor \log_p n \rfloor + \kappa\} = p^{\kappa - \lfloor \log_p n \rfloor} \int_0^\infty e^{-ip^{\kappa - \lfloor \log_p n \rfloor}}d\mu(t) + O \left( \frac{1}{n} \right),$$

where $\lfloor \log_p n \rfloor = \log_p n - \lfloor \log_p n \rfloor$.

Observe that for $p = q = \frac{1}{2}$ we obtain

$$F(x) = x \sum_{j=1}^\infty e^{-jx} = \frac{x}{e^x - 1},$$

and our results coincide with those of [13] and our (95).
6.6 Probabilistic Counting

In Section 3.5 we described a generalized probabilistic counting algorithm. We argued that to estimate a cardinality of set, one can measure a quantity \( R_{n,d} \) whose generating function and Poisson generating function satisfy functional equations (22) and (23), respectively. In this example, we show how to depoissonize \( \tilde{G}(z, u) \) to obtain an asymptotic distribution (as above, we shall see that the limiting distribution of \( R_{n,d} \) does not exist).

Let \( G_n(u) = \mathbb{E}u^{R_{n,d}} \) and \( \tilde{G}(z, u) = \mathbb{E}u^{R_{n,d}} \) be the probability generating function and its Poisson transform, respectively. As in [32] we observe that

\[
\tilde{G}(z, u) = uf_d(z/2)\tilde{G}(z/2, u) + (u - 1)(f_d(z/2) - 1),
\]

where \( f_d(z) = 1 - e_d(z)e^{-z} \) and \( e_d(z) = 1 + \frac{z}{1!} + \cdots + \frac{z^d}{d!} \) is the truncated exponential function.

Our goal is to estimate the limiting distribution through depoissonization. First of all, we consider \( \tilde{H}(z, u) = \tilde{G}(z, u) - 1 \) instead of \( \tilde{G}(z, u) \) which satisfies the following functional equation:

\[
\tilde{H}(z, u) = uf_d(z/2)\tilde{H}(z/2, u) + (u - 1)f_d(z/2).
\]

This is a linear functional equation of type (72), hence (73) is its solution which becomes in this case:

\[
\phi(x)\tilde{H}(z, x) = (z - 1)\sum_{n=0}^{\infty} x^n \phi(x2^{-n-1}),
\]

where

\[
\phi(z) = \prod_{j=0}^{\infty} f_d(z 2^j) = \prod_{j=0}^{\infty} \left(1 - e_d(z 2^j)e^{-z 2^j}\right).
\]

This implies

\[
\frac{1 - \tilde{G}(z, u)}{1 - u} = \sum_{k=0}^{\infty} u^k \frac{\phi(z2^{-k-1})}{\phi(z)},
\]

and finally

\[
\tilde{k}_k(z) = [u^k] \left( \frac{\tilde{H}(z, u)}{1 - u} \right) = e^{-\mathbb{E}} \sum_{N=0}^{\infty} \frac{\Pr\{R_{n,d} > k\} z^N}{N!} = \frac{\phi(z2^{-k-1})}{\phi(z)}.
\]

To evaluate the limiting distribution we apply Corollary 1 to \( \tilde{k}_k(z) \). We must verify conditions (I) and (O). The condition (O) outside the cone \( S_0 \) where \( \theta < \pi/2 \) is easy to check, and it follows directly from the functional equation, as we already seen in previous sections. The difficulty is to prove a bound on \( \tilde{k}_k(z) \) inside the cone \( S_0 \). We use the complex Mellin transform as discussed in Appendix B to establish the following bound:
Lemma 9 For all $z \in S_\theta$ with $z \neq 0$ and $|\theta| < \pi/2$ we have uniformly

$$|\varphi(z)| < A$$

(99)

where $A$ is a constant. Also, for large $z$ we have $\varphi(z) = 1 + O(z^{-M})$ for any $M > 0$.

Proof. First of all, we observe that for large $z$ the series in $\varphi(z)$ converges exponentially fast, and one immediately proves that $\varphi(z) = 1 + O(z^{-M})$ for any $M > 0$. Thus, the only difficult part is to prove (99) for $z$ close to zero, which we consider next. For simplicity of presentation we only assume $d = 0$. We use complex Mellin transform. Let $\ell(z) = \log \varphi(z)$.

Observe that (for $d = 0$)

$$\ell(z) = \log(1 - e^{-z}) + \ell(2z).$$

Hence, the Mellin $\ell^*(s)$ of $\ell(z)$ exists in $\Re(s) \in (0, \infty)$ and it becomes (cf. [15])

$$\ell^*(s) = -\frac{\Gamma(s)\zeta(s+1)}{(1 - 2^{-s})}.$$

There is a triple pole at $s = 0$ which gives us for $z \to 0$ (ignoring the fluctuating term)

$$\ell(z) = -\frac{1}{2}(\log^2 z + \log z) + O(1).$$

Thus $\varphi(z) = \exp(\ell(z)) = \exp(-\frac{1}{2}\log^2 z - \frac{1}{2}\log z + O(1))$. To show uniform bound around $z = 0$, let $z = \rho e^{i\theta}$. Then,

$$|\varphi(z)| = \exp(\Re(\ell(z))) = \exp(-0.5(\log^2 \rho - \theta^2 + \log \rho)).$$

Observe that for $\rho < \rho_c$ where

$$\log^2 \rho_c - \pi^2/4 + \log \rho_c > 0$$

the function $\varphi(z)$ is bounded for $\rho$ small, and this completes the proof. \[\square\]

Having the above result, we easily show that

$$\frac{|\varphi(z^{2^{k-1}})|}{|\varphi(z)|} \leq A.$$

which completes the proof of the de poissonization inequality (1). Thus, by Corollary 1 we prove that $\Pr\{R_{n,d} > k\} = \frac{\varphi(n2^{k-1})}{\varphi(n)} + O(1/n)$, or more precisely we obtain the following result of [32] with a better error term:

**Proposition 5 (Kirschenhofer et al. [32])** For any integer $m$

$$\Pr\{R_{n,d} \leq \log_2 n + m - 1\} = 1 - \varphi\left(2^{-m - \{\log_2 n\}}\right) + O(n^{-1})$$

where $\{\log n\} = \log n - \lfloor \log n \rfloor$.  

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6.7 Depth in a PATRICIA Trie

We illustrate in this subsection a usage of the "diagonal depoissonization" with polynomial bound inside a cone, that is, Theorem 4, while in the next two subsections we apply "diagonal exponential depoissonization", that is, Theorems 8 and 9.

Let us consider the depth in a PATRICIA trie (cf. Section 3.3), that is, we study $\tilde{D}(z, u)$ which satisfies the functional equation (12) that we repeat here

$$
\tilde{D}(z, u) = u(p\tilde{D}(zp, u) + q\tilde{D}(zq, u)) + (1 - u)(p\tilde{D}(zp, u)e^{-qz} + q\tilde{D}(zq, u)e^{-pz}).
$$

Observe that the unknown function $\tilde{D}(z, u)$ is involved in a product with $e^{-qz}$ and $e^{-pz}$ which makes the equation complicated, and in fact it does not have a simple explicit solution. Nevertheless, it falls under Theorem 10 with $\gamma_1(z) = up + (1 - u)pe^{-qz}$, $\gamma_2(z) = uq + (1 - u)qe^{-pz}$ and $t(z) = 0$. Furthermore, for $p \neq q$ the asymptotic behavior of $\tilde{D}(z, u)$ inside a cone $S_\theta$ ($z \to \infty$) is determined by the first two terms of the above equation. Using Mellin transform Rais et al. [43] proved that for large $z$ the Poisson depth is normal, that is, for $z$ large in a cone $S_\theta$, and $u = e^t$ where $t$ is complex we have:

$$
e^{-t\bar{X}(z)/\sigma(z)}\tilde{D}(z,e^{t\sigma(z)}) = e^{t^2/2}(1 + O(1/\sigma(z))),$$

where $\bar{X}(z) = O(\log z)$ is the Poisson mean, and $\sigma^2(z) = \bar{V}(z) = O(\log z)$ is the Poisson variance. This result extends to PATRICIA trie similar results of Jacquet and Regnier [24] established for trees.

One now must depoissonize (100) in order to establish the central limit law for the Bernoulli model. First all, an easy application of Theorem 6 leads us to a conclusion that $ED_n \sim \bar{X}(n)$ and $\text{Var } D_n \sim \bar{V}(n)$. In fact, from Szpankowski [51] we know that $ED_n = \frac{1}{h} \log n + O(1)$ and $\text{Var } D_n = h^2/h^2 \log n + O(1)$ where $h = -p \log p - q \log q$ is the entropy, and $h_2 = p \log^2 p + q \log^2 q$. Thus, we only need to depoissonize (100) in order to obtain the limiting distribution. We first observe that the above functional equation falls under Theorem 10 as mentioned above (with $\gamma_1(z) = up + (1 - u)pe^{-qz}$, $\gamma_2(z) = uq + (1 - u)qe^{-pz}$ and $t(z) = 0$). Thus, the mathematical induction over increasing domains works in this case, too, and we can conclude – as the authors of [43] – that $\tilde{D}(z, u) = O(x^{\epsilon})$ for any $\epsilon > 0$ inside a cone $S_\theta$ for $z$ large. This actually completes the depoissonization, and by Corollary 2 we arrive at (cf. [43]):

**Proposition 6 (Rais et al. [43])** For a complex $t$

$$e^{-tED_n/\sqrt{\text{Var } D_n}}e^{tD_n/\sqrt{\text{Var } D_n}} = e^{t^2/2}(1 + O(1/\sqrt{\log n})).$$

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that is, \((D_n - ED_n)/\sqrt{\text{Var } D_n}\) converges in distribution and in moments to the standard normal distribution.

6.8 Size of a Tries

Let us now consider the size \(S_n\) of a trie (i.e., number of internal nodes in a trie built from \(n\) strings). In section 3.3 we show that its Poisson transform \(\tilde{S}(z, u)\) satisfies multiplicative functional equation (10), which we repeat below:

\[
\tilde{S}(z, u) = u\tilde{S}(zp, u)\tilde{S}(zq, u) + (1-u)e^{-z}.
\]

Due to the multiplicative nature of the above functional equation, one should expect larger rate of growth inside a cone, hence our depoissonization results with polynomial growth could not work anymore, and we have to appeal to our exponential depoissonization (cf. Theorems 3 and 8). Indeed, Jacquet and Régnier [25] (cf. also [44]) proved that \(\log \tilde{S}(z, u)\) exists and \(\log \tilde{S}(z, u) = O(z)\) inside a linear cone \(S_{\theta}\) for some \(0 < |\theta| < \pi/2\) and \(u\) complex in a neighbourhood of 1 (equivalently, by setting \(u = e^t\) we consider \(t\) in a complex neighbourhood of 0). Furthermore, Régnier and Jacquet [44] proved that Poisson mean \(X(z) = O(z)\), Poisson variance \(\nu(z) = O(z \log z)\), and the reminder \(R(z, t) = O(z)\) in the expression \(\log \tilde{S}(z, e^t) = tX(z) + t^2\nu(z)/2 + t^3R(z, t)\). A simple application of Theorem 6 shows that the Bernoulli mean \(\text{Var } S_n = O(n)\). In summary, by Theorem 7 the size in the Poisson model (i.e., \(S_N\) where \(N\) is a Poisson distributed random variable) is asymptotically normally distributed. More formally:

\[
e^{-tX(z)/\sqrt{\nu(z)}}\tilde{S}\left(z, e^{t/\sqrt{\nu(z)}}\right) = e^{t^2/2}\left(1 + O\left(\frac{1}{\sqrt{\nu(z)}}\right)\right).
\]

To translate the above into the Bernoulli model, we must apply Theorem 9. We first observe that all assumptions of this theorem hold with \(\chi = 1, \nu = 1\) and \(\delta = 1\): \(\beta = 1/2 \in [1/2, 2/3]\). Thus:

Proposition 7 (Jacquet and Regnier [25]) The normalized size of a trie that is,

\[
\frac{(S_n - ES_n)}{\sqrt{\text{Var } S_n}} \overset{d}{\rightarrow} N(0, 1)
\]

where \(N(0, 1)\) is the standard normal distribution. By Theorem 9 we know that the convergence in moments also holds, and the convergence rate is \(O(n^{\epsilon-1/2})\) for \(\epsilon > 0\) arbitrarily small.
6.9 Path Length in a Digital Search Tree

Finally, we study one of the most challenging problems of this paper, namely, the limiting distribution of the path length $L_n$ in digital search tree (cf. Section 3.3). Its exponential generating function $L(z, u) = \bar{L}(z, u)e^z$ satisfies differential-functional multiplicative equation (17) that we repeat below

$$\frac{\partial L(z, u)}{\partial z} = L(puz, u)L(quz, u).$$

This equation was studied in Jacquet and Szpankowski [28] in order to determine the limiting distribution of the number of phrases in the Lempel-Ziv parsing scheme [57]. Interestingly enough, the growth of $L(z, u)$ is quite complicated. It was proved in [28] that $\log L(z, u)$ exists and $\log L(z, u) = O(z^{\kappa(u)} + \varepsilon)$ for $u$ in a real neighbourhood of 1 and $z$ in a cone $C(D, \delta)$ with $\delta < 1$ arbitrary close to 1. Actually, $\kappa(u)$ satisfies the relation $(pu)\kappa(u) + (qu)\kappa(u) = 1$, and one can show that $\kappa(u) = 1 + h(u - 1) + O((u - 1)^2)$ where $h$ is the entropy. We should also add that this problem motivated us to introduce polynomial cones. In such a cone $C(D, \delta)$ with $\delta < 1$ (but arbitrary close to $\delta = 1$ when $u$ approaching 1), the authors of [28] also showed that any derivative of $\log L(z, u)$ grows like $O(z^{\kappa(u) + \varepsilon})$ for any $\varepsilon > 0$.

Furthermore, in Section 6.4 we studied the Poisson mean $\bar{X}(z)$ and the Poisson variance $\bar{V}(z)$ and proved that $\bar{X}(z) = O(z \log z)$ and $\bar{V}(z) = O(z \log^2 z)$. Thus, by Theorem 7 we immediately see that $L_N$ in the Poisson model is normally distributed with the rate of convergence $O(z^{-1/2 + \varepsilon})$.

Extension of the above result to the Bernoulli model requires substantial work as demonstrated in [28]. Fortunately, we can significantly simplify the proof by applying our powerful Theorem 9. We first observe that (cf. Section 6.4) the Bernoulli variance $V_n = \text{Var} L_n = O(n \log n)$, hence all conditions of Theorem 9 hold with $\chi = 1 + \varepsilon$ and $\nu = 1$, and $\delta = 1 - \varepsilon$ and $\varepsilon > 0$ arbitrarily small. Thus $\beta = 1/2 + \varepsilon$ which belongs to $[1/2 - \varepsilon, 2/3 - 4/3\varepsilon]$ (it suffices to choose $\varepsilon$ small enough). Consequently, the external length in digital search trees is asymptotically normal in distribution and in moments. More formally:

**Proposition 8 (Jacquet and Szpankowski [28])** For arbitrary $\varepsilon > 0$

$$e^{-t\bar{E}L_n/\sqrt{V_n}} \mathbb{E} \left( e^{t\bar{L}_n/\sqrt{V_n}} \right) = e^{\frac{1}{2}t^2} \left( 1 + O \left( \frac{1}{n^{\frac{3}{2}} - \varepsilon} \right) \right)$$

where $\bar{E}L_n$ is given by (91), and $V_n = \text{Var} L_n$ is computed according to (93).

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A  Formal Proof of (36)

We now formally prove the general expression (35) by showing that the coefficients $b_{ij}$ satisfy (36). To identify these coefficients, we are allowed to choose any function $g_n$ that satisfies our hypothesis. For simplicity, we now assume that $g_n = (n)_t$, where $(x)_t = x(x-1) \cdots (x-t+1)$.

In this case $\tilde{G}(z) = z^t$, so we are in the framework of Theorem 2. We use the identity

$$(n)_t = \sum_{i=0}^{i=t} n^{t-i} s(\ell, \ell - i),$$

where the $s(n, k)$'s are the Stirling numbers of first kind (cf. [6]). It is known (cf. [6], Ex. 16, pp. 227) that

$$s(\ell, \ell - i) = \sum_{j=i+1}^{2i} \binom{\ell}{j} s_2(j, j - i),$$

where $s_2(n, k)$ are defined as (cf. [6] Ex. 20 pp. 295)

$$\sum_{n,k} s_2(n, k) t^n u^k / n! = e^{-tu} (1 + t)^u. \quad (101)$$

Noticing that $\frac{G^{(j)}(n)}{j!} = \binom{n}{j} n^{t-j}$, we finally obtain

$$(n)_t = \sum_{i=0}^{i=t} \sum_{j=i+1}^{2i} s_2(j, j - i) n^{t-i} \binom{\ell}{j} = \sum_{i=0}^{i=t} \sum_{j=i+1}^{2i} \frac{G^{(j)}(n)}{j!} s_2(j, j - i) n^{t-i} = \sum_{j=0}^{2t} \sum_{i=0}^{i=t} \frac{G^{(j)}(n)}{j!} s_2(j, i) n^{i}.\quad (101)$$

Thus, $c_{ij} = s_2(j, i) / j! = b_{ij}$, and our result follows from (101).

B  Mellin Transform of a Complex Variable

In various applications of poissonization/depoissonization (and also singularity analysis), one needs to deal with the Mellin transform of a complex analytical function. Since we did not find a proper reference to such an extension (except for existence in [7, 9]; see also [15, 24]), we add this appendix for the completeness of the presentation.

In the theorem below, we remind basic facts about Mellin transform of a function $f(x)$ of a real variable $x$. Its proof is a classical one, and can be found in many places. e.g., [9, 15].
Theorem 11  Let \( f(x) \) be a function defined for a non-negative real \( x \). We assume that \( f(x) \) is twice differentiable, and \( f''(x) = O(x^{-c-2}) \) for \( x \to 0 \) while \( f''(x) = O(x^{-d-2}) \) for \( x \to \infty \) (or more generally: there is an interval \((c, d)\) such that for all \( b \in (c, d)\): \( f''(x) = O(x^{-b-2}) \) as \( x \to 0 \) and \( x \to \infty \)). Then:

(i) The Mellin transform \( M(f, s) = f^*(s) = \int_0^{\infty} f(x)x^{s-1}dx \) exists in the strip \( \Re(s) \in (c, d) \).

(ii) The Mellin transform \( f^*(b + it) \) is absolutely integrable with respect to variable \( t \) from \(-\infty \) to \(+\infty \) as long as \( b \in (c, d) \), that is,

\[
\int_{-\infty}^{\infty} |f^*(b + it)|dt < \infty
\]

(iii) The inverse Mellin transform exists and \( f(x) = \frac{1}{2\pi i} \int_{\Re(s) = b} x^{-s}f^*(s)ds \) for \( b \in (c, d) \).

We now extend the Mellin transform to a complex function \( F(z) \) such that \( f(x) = F(z) \mid_{z: z \geq 0} \) for \( z \) complex.

Theorem 12  Let \((\theta_1, \theta_2)\) be an interval containing 0. Let \( F(z) \) be an analytical function defined for \( z \) in a cone \( S_{\theta_1, \theta_2} = \{ z : \theta_1 < \arg(z) < \theta_2 \} \) such that \( F(z) = O(z^{c}) \) for \( z \to 0 \), and \( F(z) = O(z^{d}) \) for \( z \to \infty \) with \( z \) confined to the cone (or more generally: there is an interval \((c, d)\) such that for all \( b \in (c, d)\): \( F(z) = O(z^{-b}) \) as \( x \to 0 \) and \( x \to \infty \) in the cone).

Then:

(i) For \( \Re(s) \in (c, d) \) the integral

\[
\int_{C_{0}} F(z)z^{s-1}dz
\]

exists and does not depend on the curve \( C_{0} \) which starts at \( z = 0 \) and goes to \( \infty \) inside the cone. We call it the complex Mellin transform, and denote as \( M(F, s) = F^*(s) = \int_{C_{0}} F(z)z^{s-1}dz \).

(ii) The complex Mellin transform \( F^*(s) \) is identical to the real Mellin transform \( f^*(s) \), that is,

\[
\int_0^{\infty} F(x)x^{s-1}dx = f^*(s) = F^*(s) = \int_{C_{0}} F(z)z^{s-1}dz .
\]

Furthermore, \( M(F(za), s) = a^{-s}f^*(s) \) defined on \( S_{\theta_1-\arg(a), \theta_2-\arg(a)} \) with \( a \in S_{\theta_1, \theta_2} \).

(iii) For \( b \in (c, d) \) and \( \theta_1 < t < \theta_2 \), \( f^*(b + it)e^{\theta t} \) are absolutely integrable with respect to variable \( t \) from \(-\infty \) to \(+\infty \). The inverse Mellin exists and is equal to

\[
F(z) = \frac{1}{2\pi i} \int_{\Re(s) = b} z^{-s}f^*(s)ds
\]

for \( b \in (c, d) \).
Proof: Let $f(x)$ be the function of real nonnegative variable $x$ equal to $F(x)$ on the real line. For part (i), observe that the function $f(x)x^{s-1}$ is analytical, thus by Cauchy formula

$$
\oint F(x)x^{s-1}dz = 0
$$

where the integration is over any arbitrary closed curve inside the cone $S_{\theta_1,\theta_2}$. Let $z_1$ and $z_2$ be two points on $L_0$, let $r = |z_1|$ and $R = |z_2|$ and let $L_0(z_1,z_2)$ be the part of $L_0$ between $z_1$ and $z_2$. Let further $A_r$ and $A_R$ be the circular arcs connecting $z_1$ with $r$ and $z_2$ with $R$. We integrate along the closed curve formed by $L_0(z_1,z_2)$, the subinterval $(r,R)$ of the real line and the two arcs $A_r$ and $A_R$. Let us first consider the contribution of $A_R$. Then:

$$
\int_{L_0} F(x)x^{s-1}dz = \int_{I_R} F(x)x^{s-1}dz + \int_{A_R} F(x)x^{s-1}dz.
$$

(102)

Passing to the limit in the above, and noticing that $\int_{A_R} F(x)x^{s-1}dz = O(x^{s-b})$ for any $b \in (R(s),d)$, we conclude that the contribution of the arc vanishes when $R \to \infty$. In a similar manner, we consider the contribution of $A_r z \to 0$. This proves part (i) and the first part of (ii).

For part (ii), we infer immediately from (102), that is,

$$
F^*(s) = \lim_{R \to \infty} \int_{L_0} F(x)x^{s-1}dz = \lim_{R \to \infty} \int_{I_R} F(x)x^{s-1}dz = f^*(s).
$$

To prove the identity $M(F(za),s) = a^{-s}f^*(s)$ one only needs to make the change of variable $za = z'$ in the integration. Notice that function $F(za)$ is now defined in cone $S_{\theta_1-\text{arg}(a),\theta_2-\text{arg}(a)}$.

Point (iii) is a natural consequence of point (ii) by selecting $a = e^{i\theta}$. In this case we immediately obtain that $(e^{i\theta})^* f^*(s)$ is the (real) Mellin transform of function $F(xe^{-i\theta})$ which is absolutely integrable by part (ii) of Theorem 11. Therefore, for all $z \in S_{\theta_1,\theta_2}$ the function $z^{-\theta}f^*(s)$ is absolutely integrable with respect to $\mathcal{H}(s)$. The Mellin inverse identity holds in the complex case, too. □

The above extension enjoys all "nice" properties of Mellin transform of real $x$. In particular, it can be used to obtain asymptotic expansion of $F(z)$ for $z \to 0$ and $z \to \infty$, as we know from the Mellin transform of a function of real variable. Indeed, this is proved formally below:

**Theorem 13** Let $f(z)$ be a function of real variable. Let $f^*(s)$ be its Mellin transform defined in the strip $\Re(s) \in (c,d)$. Let $(\theta_1,\theta_2)$ be an interval containing 0. If for all $b \in (c,d)$ function $f^*(b+it)e^{\theta_1 t}$ and $f^*(b+it)e^{\theta_2 t}$ are absolutely integrable, then
(i) The analytic continuation \( F(z) \) of \( f(x) \) exists in cone \( S_{\theta_1, \theta_2} \) and \( F(z) = O(z^{-b}) \) for \( |z| \to 0 \) and \( |z| \to \infty \) for all \( b \in (c, d) \).

(ii) If the Mellin transform \( f^*(s) \) has all its singularities regular in the strip \( \Re(s) \in (c', d') \) where \( c' < c \) and \( d' > d \), then the function \( F(z) \) has the same asymptotic expansions as \( f(x) \) up to \( z^{-c'} \) at \( 0 \) and up to order \( z^{-d'} \) at \( \infty \).

Proof: Defining \( F(z) = \frac{1}{2\pi i} \int_{\Re(s)=\delta} z^{-s} f^*(s) ds \) one introduced an analytical function which matches \( f(x) \) on the real line. By an elementary estimate of the integrand we have \( F(z) = O(z^{-b}) \) due to assumption regarding the smallness of \( f^*(b + it) \) as \( t \to \pm \infty \). Part (ii) is a simple consequence of (i) and the fact that \( f^*(s) \) is the complex Mellin transform of \( F(z) \), as noticed above.

To illustrate a simple usage of such an extension, let us consider the functional equation studied in Section 6.1, that is:

\[
f(z) = a(z) + f(pz) + f(qz)
\]

where \( a(z) \) is given. Denoting \( f^*(s) \) the Mellin of \( f(x) \) we arrive at

\[
f^*(s) = a^*(s) + p^{-s} f^*(s) + q^{-s} f^*(s)
\]

which leads to \( f^*(s) = a^*(s)(1 - p^{-s} - q^{-s})^{-1} \) and a simply application of the residue technique allows to extract asymptotics of \( f(z) \) for large \( z \) in a cone \( S_{\theta} \). In particular if \( a^*(z) = \alpha(s) \Gamma(s) \) with \( \alpha(s) \) of at most polynomial on \( \pm \infty \), then the asymptotic expansion remain valid in all cone strictly included in \( S_{-\pi/2, \pi/2} \) because \( |\Gamma(it)|^2 = \pi |t \sinh(-\pi t)|^{-1} = O(e^{-\pi|t|}) \) when \( t \) real tends to \( \pm \infty \).

C References

References


