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Abstract

To solve the real nonsingular linear system \( Ax = b \) (1) on parallel and vector machines, we consider multisplitting methods, \( m \)-step preconditioners and \( m \)-step additive preconditioners, generalizing some of the results and methods developed in previous related works. In particular we generalize the method and the corresponding convergence results in [14], and determine suitable relaxed \( m \)-step preconditioners ([1], [6]) treating also the problem of minimizing the related condition number, with respect to the relaxation (extrapolation) parameter involved, in various cases. We also generalize the theory for determining suitable \( m \)-step additive preconditioners [2] and finally we solve completely the problem of determining the optimum SOR-additive iterative method [2] for 2-cyclic positive definite matrices.

Key words and phrases: multisplitting methods, \( m \)-step preconditioners, extrapolation method, successive overrelaxation (SOR) method.

1 Introduction

For solving the large nonsingular linear system of equations

\[ Ax = b, \tag{1.1} \]

where \( A \in \mathbb{R}^{n,n} \), \( b \in \mathbb{R}^n \), parallel iterative methods, called multisplitting methods, were introduced in [12]. According to [12], given a multisplitting of \( A \)

\[ A = M_k - N_k, \quad \det(M_k) \neq 0, \quad k = 1(1)p, \tag{1.2} \]

the corresponding multisplitting method is defined by

\[ x^{(m+1)} = \sum_{k=1}^{p} D_k M_k^{-1} N_k x^{(m)} + \sum_{k=1}^{p} D_k M_k^{-1} b, \quad m = 0, 1, 2, \ldots, \tag{1.3} \]

where \( D_k \) is a diagonal matrix, with \( D_k \geq 0 \), \( k = 1(1)p \), and \( \sum_{k=1}^{p} D_k = I \). Setting

\[ H = \sum_{k=1}^{p} D_k M_k^{-1} N_k \quad \text{and} \quad G = \sum_{k=1}^{p} D_k M_k^{-1}, \tag{1.4} \]

(1.3) takes the form

\[ x^{(m+1)} = H x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \tag{1.5} \]

where \( c = Gb \). Moreover we have

\[ H = I - GA. \tag{1.6} \]

According to [18], Thm. 2.6, p. 68, (1.5) is consistent with (1.1). Furthermore (1.5) is completely consistent with (1.1) iff \( G \) is nonsingular. From now on we assume that (1.5) is completely consistent with (1.1); hence it is obvious that (1.5) can be obtained using the splitting

\[ A = G^{-1} - G^{-1} H. \tag{1.7} \]

It is well known that (1.5) converges to \( A^{-1} b \) for any starting vector \( x^{(0)} \) iff \( \rho(H) < 1 \), where \( \rho(\cdot) \) denotes spectral radius. Convergence results of (1.5), under various assumptions, can be found in the literature (see, e.g., [4], [5], [7], [8], [11], [12], [14], [16], [17]).

In [1], [6] for the linear system (1.1), where \( A \) is positive definite (cf. [18], p. 21) a splitting \( A = M - N, \det(M) \neq 0 \), is considered, where \( M \) is positive definite and \( \rho(M^{-1}N) < 1 \), and the associated preconditioning matrix or \( m \)-step preconditioner is defined by

\[ M_m = M(I + G + G^2 + \ldots + G^{m-1})^{-1}, \quad m > 1, \tag{1.8} \]

where \( G = M^{-1}N \). If \( A \approx M \), then \( M_m \) is an improved approximation to \( A \) and is used instead of \( M \) for accelerating the rate of convergence of Chebyshev and Conjugate Gradient methods. Also in
for the same purpose \( m \)-step additive preconditioners are defined, which are connected with the multisplitting method (1.5) for \( p = 2 \) and \( D_1 = D_2 = \frac{1}{2} I \). In particular, in [2] the SOR-additive preconditioner is defined and an optimal value \( \omega_{opt} \) for the parameter \( \omega \) of the 2-cyclic SOR-additive iterative method is also determined.

In the present paper we give in Section 2 two theorems concerning the convergence of the method (1.5), when: (i) \( A \) in (1.1) satisfies \( A^{-1} \geq 0 \) and (1.2) are weak regular splittings (cf. [3]) and (ii) \( A \) is positive definite and (1.2) are \( P \)-regular splittings (see [13]). Also in Section 2 we generalize the two-splitting method (method of the arithmetic mean) treated in [14] and prove some theorems which generalize Thms 1, 2, 3 in [14]. In Section 3 we give a method for finding a suitable \( m \)-step preconditioner \( M_m, m \geq 1 \), for system (1.1). The given preconditioner contains a parameter \( \omega \) and we determine in more than half of the cases the optimal value of \( \omega \) so that the condition number of \( M_m^{-1} A \) is minimized. We also generalize the procedure given in [2] for defining \( m \)-step additive preconditioners and prove a theorem giving sufficient conditions for determining suitable additive preconditioners. Finally, in Section 4 we completely solve the problem of determining the optimal \( \omega \) of the SOR-additive iterative method studied in [2]. As we show the theoretical analysis in [2] concerning this problem was not complete.

2 Convergence Results

We consider the linear system (1.1) and the multisplitting method (1.5). Then we obtain the following results which are useful in the sequel (see also Thm 1 (a), (b) in [12] and Thm 1 and Cor 1 in [17]).

Theorem 2.1

If in (1.1) \( A^{-1} \geq 0 \) and (1.2) are weak regular splittings of \( A \), then (1.7) is also a weak regular splitting of \( A \); hence (1.5) converges \((\rho(H) < 1)\).

Proof

It follows from Thm 1 and Cor 1 in [17]. \( \Box \)

Theorem 2.2

If \( A \) in (1.1) is positive definite, (1.2) are \( P \)-regular splittings of \( A \) and \( D_k = a_k I \) \((a_k \geq 0, \sum_{k=1}^{p} a_k = 1)\), then (1.7) is also a \( P \)-regular splitting of \( A \); hence (1.5) converges.

Proof

From the hypothesis \( M_k \) is nonsingular and \( M_k + N_k \) is positive real (see [18], Thm 2.9, p. 24), i.e., \( M_k + N_k + (M_k + N_k)^T \) is positive definite or equivalently \( M_k + M_k^T - A, k = 1(1)p, \) is positive
definite ($C^T$ denotes the transpose of $C$). Since $A$ is positive definite, according to [18], Thm 5.3, p. 79, it suffices to show that

$$M + M^T - A = \frac{1}{2}[M + N + (M + N)^T]$$

(2.1)

is positive definite, where $M = G^{-1}$, $N = G^{-1}H$ ($A = M - N$), or equivalently that

$$M^{-1}(M + M^T - A)M^{-T} = M^{-T} + M^{-1} - M^{-1}AM^{-T} =: Q$$

(2.2)

is positive definite. Thus we have

$$Q = \sum_{k=1}^{p} a_k(M_k^{-T} + M_k^{-1} - M_k^{-1}AM_k^{-T}) + \sum_{k=1}^{p} a_kM_k^{-1}AM_k^{-T}$$

The matrix $S_1 \equiv \sum_{k=1}^{p} a_k(M_k^{-T} + M_k^{-1} - M_k^{-1}AM_k^{-T}) = \sum_{k=1}^{p} a_kM_k^{-1}(M_k + M_k^T - A)M_k^{-T}$ is positive definite, since $a_k \geq 0$ and $M_k^{-1}(M_k + M_k^T - A)M_k^{-T}$, $k = 1(1)p$, is positive definite. Moreover, for the symmetric matrix $S_2 \equiv Q - S_1$ we have

$$S_2 = \left(\sum_{k=1}^{p} a_kM_k^{-T} \right) - \sum_{k=1}^{p} a_kM_k^{-1}AM_k^{-T}$$

$$= \sum_{k,j=1}^{p} a_k a_j M_k^{-1}AM_k^{-T} - \sum_{k,j=1}^{p} a_k a_j M_k^{-1}AM_j^{-T}$$

$$= \sum_{k,j=1}^{p} a_k a_j[M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T}].$$

Hence

$$2S_2 = S_2 + S_2^T$$

$$= \sum_{k,j=1}^{p} a_k a_j(M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T}) + \sum_{k,j=1}^{p} a_k a_j(M_k^{-1}AM_k^{-T} - M_j^{-1}AM_k^{-T})$$

$$= \sum_{k,j=1}^{p} a_k a_j[M_k^{-1}AM_k^{-T} - M_k^{-1}AM_j^{-T} + M_j^{-1}AM_j^{-T} - M_j^{-1}AM_k^{-T}]$$

$$= \sum_{k,j=1}^{p} a_k a_j \left[(M_k^{-1} - M_j^{-1})A(M_k^{-1} - M_j^{-1})^T \right].$$

$S_2$, as a sum of nonnegative definite matrices, is nonnegative definite. This implies that $Q$ is positive definite and that $A = G^{-1} - G^{-1}H$ is a $P$-regular splitting of $A$; hence $\rho(H) < 1$. □

Remarks

i) As one can see the proof in Theorem 2.2 parallels that of Thm 1(b) in [12]. However, it is based on a simpler (equivalent) theorem than that in [12]. This makes the corresponding expressions for
\( S_1 \) and \( S_2 \) be simpler and easier to handle. ii) Note that \( S_2 \) may be nonnegative definite iff all \( M_j, j = 1(1)p, \) share a common eigenvalue-eigenvector pair.

In the following a generalization, in various directions, of the method of the arithmetic mean of [14] is suggested. Consider the splittings of \( A \)

\[
A = M_k - N_k, \quad \text{det}(M_k) \neq 0, \quad k = 1(1)2q,
\]

where

\[
M_k = \frac{1}{\omega} D + W_k - L, \quad N_k = (\frac{1}{\omega} - 1) D + W_k + U, \quad k = 1(1)q,
\]

and

\[
M_k = \frac{1}{\omega} D + W_k - U, \quad N_k = (\frac{1}{\omega} - 1) D + W_k + L, \quad k = q + 1(1)2q.
\]

In (2.4), (2.5) \( W_k \) is a diagonal matrix, \( W_k > 0, k = 1(1)2q, \) and \( \omega \) a real positive parameter. For the corresponding multisplitting method (1.5), where \( p = 2q \) and \( M_k \) is given by (2.4), (2.5), \( k = 1(1)2q, \) we prove the theorems below, which generalize Thms 1, 2, 3 in [14]. We simply mention that in [14], \( p = 2, \omega = 1, W_k = \rho W (\rho > 0, W > 0), \) and \( D_1 = D_2 = \frac{1}{2} I. \)

**Theorem 2.3**

If \( A \) in (1.1) is an irreducibly diagonally dominant \( L \)-matrix ([15], p. 23 and [18], p. 42), then the multisplitting method (1.5), where \( p = 2q, M_k \) is given by (2.4), (2.5), \( k = 1(1)2q, \) and \( 0 < \omega < 1, \) converges.

**Proof**

The matrix \( M_k \) is nonsingular, since \( D > 0, W_k > 0 \) and \( \omega > 0, k = 1(1)2q. \) According to the hypothesis (see [15], Cor 1, p. 85) \( A \) is a nonsingular \( M \)-matrix with \( A^{-1} > 0. \) Obviously \( M_k \) is a strictly diagonally dominant \( L \)-matrix, \( k = 1(1)2q; \) hence \( M_k \) is an \( M \)-matrix and therefore \( M_k^{-1} \geq 0, k = 1(1)2q. \) We also have \( N_k \geq 0, k = 1(1)2q. \) Consequently, (2.3) are regular splittings of \( A \) and hence weak regular splittings of \( A. \) Now, by Thm 2.1 we have \( \rho(H) < 1. \)

**Theorem 2.4**

Let \( A \) in (1.1) be a positive real matrix. Then the multisplitting method (1.5), where \( p = 2q, M_k \) is given by (2.4), (2.5) with \( \omega = 1 \) and \( W_k = \rho_k I, k = 1(1)2q, D_k = a_k I \) and

\[
\rho_k > \begin{cases} 
\max\{0, -\frac{\mu_m}{\lambda_m}\} & \text{for } k = 1(1)q \\
\max\{0, -\frac{\nu_m}{\lambda_m}\} & \text{for } k = q + 1(1)2q,
\end{cases}
\]

where \( \lambda_m \) is the smallest eigenvalue of \( A + A^T \) and \( \mu_m, \nu_m \) are the smallest eigenvalues of the matrices \((D - L)(D - L)^T - UU^T\) and \((D - U)(D - U)^T - LL^T, \) respectively, converges.
Proof

Since \( A \) is positive real, we have that \( A \) is nonsingular, \( B \equiv A + A^T \) is positive definite and \( D > 0 \). Consequently \( M_k \) is nonsingular, \( k = (1) 2q \), since \( \rho_k > 0 \). Moreover we have \( \lambda_m > 0 \). The matrices \( C_1 \equiv (D - L)(D - L)^T - UU^T \) and \( C_2 \equiv (D - U)(D - U)^T - LL^T \) are symmetric and for any \( z \in \mathbb{R}^n, z \neq 0 \), we have

\[
\frac{z^T (\rho_k B + C_1) z}{z^T z} \geq \rho_k \lambda_m + \mu_m, \quad \frac{z^T (\rho_k B + C_2) z}{z^T z} \geq \rho_k \lambda_m + \nu_m. \tag{2.7}
\]

Because of (2.6), (2.7) implies that the matrices \( \rho_k B + C_1, k = (1) q \), and \( \rho_k B + C_2, k = q + (1) 2q \), are positive definite. Setting \( G_k = M_k^{-1} N_k, k = (1) 2q \), it can be shown that

\[
\rho_k B + C_1 = M_k (I - G_k G_k^T) M_k^T, \quad k = (1) q \tag{2.8}
\]

and

\[
\rho_k B + C_2 = M_k (I - G_k G_k^T) M_k^T, \quad k = q + (1) 2q. \tag{2.9}
\]

From (2.8), (2.9) we have that \( I - G_k G_k^T, k = (1) 2q \), are positive definite; hence the eigenvalues of \( G_k G_k^T \) belong to \([0,1), k = (1) 2q \). Thus we obtain \( ||G_k||_2 = |\rho(G_k G_k^T)|^{1/2} < 1, k = (1) 2q \), and

\[
||H||_2 = || \sum_{k=1}^{2q} a_k G_k ||_2 \leq \sum_{k=1}^{2q} a_k ||G_k||_2 < \sum_{k=1}^{2q} a_k = 1,
\]

implying that the method converges. \( \square \)

Theorem 2.5

If \( A \) in (1.1) is a positive definite matrix, then the multisplitting method (1.5), where \( p = 2q, M_k \) is given by (2.4), (2.5), \( D_k = a_k I \) and \( 0 < \omega < 2 \), converges.

Proof

In this case we have \( U = LT \) and \( A = D - L - L^T, D > 0 \). The splittings (2.4), (2.5) are \( P \)-regular splittings, since \( M_k \) is nonsingular and \( M_k + N_k + (M_k + N_k)^T = 2(M_k + M_k^T - A) = 2((\frac{2-\omega}{\omega}) D + 2W_k), k = (1) 2q \). Thus by Thm 2.2 we obtain the desired result. \( \square \)

3 m-Step Preconditioners

We consider the linear system (1.1), where \( A \) is positive definite. If

\[
A = M - N, \quad \det(M) \neq 0, \tag{3.1}
\]

then using the iterative method


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\[ Mx^{(m+1)} = Nx^{(m)} + b, \quad m = 0, 1, 2, \ldots, \]

we solve in every iteration a linear system of the form

\[ My = c. \quad (3.2) \]

It is known that \( M \) is chosen so that it approximates \( A \) as well as possible (\( A \approx M \)) and \( \rho(G) < 1 \), where \( G = M^{-1}N \). Choosing a positive definite \( M (A \approx M) \) with \( \rho(G) < 1 \), we can find improved approximations to \( A \) using the Neumann expansion (see e.g., [1], [2], [6])

\[ A^{-1} = (I - G)^{-1}M^{-1} = (I + G + G^2 + \ldots)M^{-1}. \quad (3.3) \]

Thus we have

\[ A \approx M_m = M(I + G + G^2 + \ldots + G^{m-1})^{-1}, \quad m \geq 1. \quad (3.4) \]

It can be shown (see Thm 3.1 of [6]), that under the above assumptions \( M_m \) is also positive definite and therefore \( M_m^{-1} \) is usually used to accelerate convergence of the Conjugate Gradient method. The matrix \( M_m \) is the preconditioning matrix or \( m \)-step preconditioner. One comment here: In Thm 1 of [1], it was proved that for \( m \) odd the hypothesis "\( A \) and \( M \) are positive definite" is sufficient for \( M_m \) to be positive definite. However, this hypothesis does not guarantee that \( M_m \) will be a better than \( M \) approximation to \( A \), since then

\[ M_m^{-1}N_m = M_m^{-1}(M_m - A) = I - (I + G + \ldots + G^{m-1})(I - G) = G^m. \]

Therefore the condition \( P(G) < 1 \) should be included in our assumptions for all \( m \) (odd or even).

Taking into consideration the theory mentioned previously (see also [10]), in order to find suitable \( m \)-step preconditioners for (1.1), we can work as follows: We choose some positive definite matrix \( M \) and write \( A = M - N \). Then \( G = M^{-1}N \) has real eigenvalues \( \lambda_i, i = 1(1)n \), such that \( \lambda_i < 1, i = 1(1)n \). Suppose that \( \lambda_i \) are ordered as \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). We consider now the splitting

\[ A = \tilde{M} - \tilde{N}, \quad (3.5) \]

where \( \tilde{M} = \frac{1}{\omega}M \). As is known the splitting (3.5) defines the extrapolated method based on the original splitting. Obviously \( \tilde{M} \) is positive definite for \( \omega > 0 \) and it is \( \rho(\tilde{M}^{-1}\tilde{N}) < 1 \) iff \( 0 < \omega < \frac{2}{1 - \lambda_1} \). Hence an \( m \)-step preconditioner, which is positive definite and approximates \( A \) well, is given by

\[ \tilde{M}_m = \tilde{M}(I + \tilde{G} + \tilde{G}^2 + \ldots + \tilde{G}^{m-1})^{-1}, \quad m \geq 1, \quad (3.6) \]

where \( \tilde{G} = \tilde{M}^{-1}\tilde{N} \) and \( \omega \in (0, \frac{2}{1 - \lambda_1}) \). Certainly \( \tilde{M}_m \) depends on \( \omega \) and the problem as how to choose \( \omega \) for a fixed \( m \), so that the condition number \( k(\tilde{M}_m^{-1}A) \) of \( \tilde{M}_m^{-1}A \) is as small as possible, arises. It is easy to show that
\[ I - \hat{M}^{-1}_m A = \hat{G}^m = [(1 - \omega)I + \omega G]^m \]  

hence 

\[ k(\hat{M}^{-1}_m A) = \frac{\max_i \mu_i^{(m)}}{\min_i \mu_i^{(m)}}, \]  

where \( \mu_i^{(m)}, i = 1(1)n, \) are the eigenvalues of \( \hat{M}^{-1}_m A. \) We note that the eigenvalues of \( \hat{G} \) are ordered as 

\[ -1 < 1 - \omega + \omega \lambda_1 \leq 1 - \omega + \omega \lambda_2 \leq \ldots \leq 1 - \omega + \omega \lambda_n < 1. \]  

Because of (3.7) we have 

\[ k(\hat{M}^{-1}_m A) = \max_i \{1 - [1 - \omega + \omega \lambda_i]^m\} \frac{m}{\min_i \{1 - [1 - \omega + \omega \lambda_i]^m\}}, \quad m \geq 1. \]  

It can be shown, as in [1], that 

\[ k(\hat{M}^{-1}_m A) = \begin{cases} \frac{1-(1-\omega+\omega\lambda_1)^m}{1-(1-\omega+\omega\lambda_n)^m}, & \text{if } m \text{ is odd}, \\ \frac{1-\min_i [1-\omega+\omega\lambda_i]^m}{1-\max_i [1-\omega+\omega\lambda_i]^m}, & \text{if } m \text{ is even}, \end{cases} \]  

where \( \omega \in (0, \frac{2}{1 - \lambda_1}). \)

The problem of finding \( \min_{\omega} k(\hat{M}^{-1}_m A) \) seems to be not an easy one in the general \( m \) odd case. In the sequel we solve first this problem for \( m = 1 \) (trivial case), \( m = 3 \) and for any even \( m \geq 2. \) The results are given in Thms 3.1 and 3.3. In these theorems it is assumed that \( \lambda_1 < \lambda_n, \) for if \( \lambda_1 = \lambda_n, \) then \( k(\hat{M}^{-1}_m A) = 1 \) for all \( m \) and all permissible values of \( \omega. \)

**Theorem 3.1**

The condition number \( k_m = k_m(\omega) \) of \( \hat{M}^{-1}_m A, \) given by (3.11), for \( m = 1 \) is independent of \( \omega \) and is given by \( k_1 = \frac{\nu_1}{\nu_n}, \) while for \( m = 3, \) is minimized with respect to \( \omega \) for 

\[ \omega = \omega_{\text{opt}} = \frac{\nu_1 + \nu_n - \sqrt{\nu_1^2 + \nu_n^2 - \nu_1 \nu_n}}{\nu_1 \nu_n}, \]  

where \( \nu_1 = 1 - \lambda_1, \nu_n = 1 - \lambda_n. \)

**Proof**

For \( m = 1 \) the result is trivially obtained. For \( m = 3 \) it can be shown after some manipulation that 

\[ \text{sign} \left( \frac{\partial k_3(\omega)}{\partial \omega} \right) = \text{sign} \left( \phi(\omega) \right), \]  

where
\[ \phi_3(\omega) = -\nu_1 \nu_n \omega^2 + 2(\nu_1 + \nu_n)\omega - 3. \] (3.13)

The two roots of \( \phi_3(\omega) \) are real and are given by

\[ \rho_1 = \frac{\nu_1 + \nu_n + \sqrt{\nu_1^2 + \nu_n^2 - \nu_1 \nu_n}}{\nu_1 \nu_n}, \quad \rho_2 = \frac{\nu_1 + \nu_n - \sqrt{\nu_1^2 + \nu_n^2 - \nu_1 \nu_n}}{\nu_1 \nu_n}. \] (3.14)

It can be proved that \( 0 < \rho_2 < \frac{2}{\nu_n} < \rho_1 \). Moreover \( \frac{\partial k_3}{\partial \omega} < 0 \) if \( 0 < \omega < \rho_2 \) while \( \frac{\partial k_3}{\partial \omega} > 0 \) if \( \rho_2 < \omega < \frac{2}{\nu_n} \). Hence \( \min k_3(\omega) = k_3(\rho_2) \) and our assertion follows. \( \Box \)

Remarks:

(i) For \( m = 1 \) the extrapolation parameter (damping factor) \( \omega \) was used in conjunction with the Jacobi iteration matrix in [10]. Thm 3.1 effectively shows that if \( \omega \) is kept fixed during the iterations no improvement over the original preconditioner should be expected! (ii) For odd \( m \geq 5 \) the function \( \phi_m(\omega) \) is a polynomial of degree \( m - 1 \) whose sign determination as \( \omega \) varies in \( (0, \frac{2}{\nu_n}) \) seems not an easy problem to study. This is what makes the whole problem difficult to solve.

To derive the optimal results for even \( m \geq 2 \) first we introduce the notation "\( a \sim b \)" to denote that the expressions \( a \) and \( b \) are of the same sign and then state and prove the lemma below, a basic key to the proof of one of our main results.

Lemma 3.1:

For any even \( m \geq 2 \) the function

\[ \phi_m \equiv \phi_m(x) := \frac{x^{m-1} - x^m}{1 - x^m}, \quad x \in (-1, 1) \] (3.15)

is a strictly increasing function of \( x \) in \( (-1, 1) \).

Proof

Differentiating (3.15) with respect to \( x \) we obtain

\[ \frac{\partial \phi_m}{\partial x} \sim (m - 1) - mx + x^m = (m - 1)(1 - x) - x(1 - x^{m-1}). \] (3.16)

If \( x \in (-1, 0] \), the rightmost expression in (3.16) is positive since \( 1 - x > 0 \), \( -x \geq 0 \) and \( 1 - x^{m-1} > 0 \), implying that \( \phi_m \) strictly increases in \( (-1, 0] \). For \( x \in [0, 1) \) let

\[ z \equiv z(x) := (m - 1) - mx + x^m, \quad x \in [0, 1). \] (3.17)

Then on differentiation we take
\[
\frac{\partial z}{\partial x} = -m(1 - x^{m-1}) < 0
\]
and therefore \(z(x)\) strictly decreases in \([0,1)\) with \(\lim_{x\to 1^-} z(x) = 0\), and \(z(0) = m - 1 > 0\). Hence \(z(x)\) takes on positive values only and by virtue of (3.17) and (3.16) so does \(\frac{\partial \phi_m}{\partial x}\). Consequently \(\phi_m\) strictly increases in \([0,1)\).

In the sequel we state and prove two theorems that solve the problem of determining the optimal extrapolation parameter for all even \(m \geq 2\).

**Theorem 3.2:**

Let the eigenvalues \(\lambda_i, i = 1(1)n\), of \(\hat{G}\) in (3.7) satisfy

\[\begin{align*}
-1 < -\lambda_n = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1, \\
(\lambda_1 \leq 0 \leq \lambda_n).
\end{align*}\]  

(3.18)

Then the condition number \(k_m = k_m(\omega)\) of \(\tilde{M}_m^{-1} A\), given by (3.11) for even \(m \geq 2\), is minimized with respect to \(\omega \in (0, \frac{2}{1-\lambda_1})\) for

\[\omega_{opt} = 1.\]  

(3.19)

**Proof**

Let \(\lambda_i\) and \(\lambda_{i+1}\), \(i \in \{1,2,\ldots,n-1\}\) be the absolutely smallest nonpositive and nonnegative eigenvalues of \(G\), respectively. Two cases are distinguished depending on the sign of \(\lambda_i + \lambda_{i+1}\).

Case I: Let \(\lambda_{i+1} + \lambda_i < 0\). (The subcase \(\lambda_{i+1} + \lambda_i = 0\) can be trivially examined after the analysis is complete.) We subdivide the interval for \(\omega, (0, \frac{2}{1-\lambda_1})\), into a number of (at most \(2n+1\)) subintervals. For continuity arguments to apply all of them are taken to be closed, except the first and the last ones. The subdivision points are

\[
\frac{1}{1-\lambda_1}, \frac{2}{1-\lambda_1}, \frac{1}{1-\lambda_2}, \frac{2}{1-\lambda_2}, \frac{1}{1-\lambda_3}, \ldots, \frac{1}{1-\lambda_i}, \frac{2}{1-\lambda_i}, \frac{2}{1-\lambda_{i+1}}, \frac{1}{1-\lambda_{i+1}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots
\]

The last point is either \(\frac{1}{1-\lambda_j}\) for some \(j \in \{i+1, i+2, \ldots, n\}\) iff \(\frac{2}{1-\lambda_j} < \frac{2}{1-\lambda_1} \leq \frac{2}{1-\lambda_{i+1}}\) or \(\frac{2}{2-\lambda_j-\lambda_i}\) for some \(j \in \{i+2, i+3, \ldots, n\}\) iff \(\frac{2}{2-\lambda_j-\lambda_i} < \frac{2}{1-\lambda_1} \leq \frac{1}{1-\lambda_i}\). Let \(I_1, I_2, I_3, \ldots, I_{2i}, I_{2i+1}, I_{2i+2}, \ldots\) be the successive subintervals of \((0, \frac{2}{1-\lambda_1})\) defined by these points. Let also

\[\lambda_k(\omega) := 1 - \omega + \omega \lambda_k, \quad k = 1(1)n.\]  

(3.20)

As can be readily checked, the ordering of the eigenvalues \(\lambda_k(\omega)\) of \(\hat{G} \equiv G\omega\) is the same as that of the \(\lambda_k\)'s in (3.18). We then claim that: "\(k_m = k_m(\omega)\) is a strictly decreasing function of \(\omega\) in each subinterval \(I_{\ell}\), \(\ell = 1(1)2i + 1\), and a strictly increasing one in each \(I_{\ell}\), \(\ell \geq 2i + 2\)." The proof of our claim will prove (3.19). For this we shall distinguish four cases: (a) \(\omega \in I_{\ell}\), \(\ell = 2(2)i\), (b) \(\omega \in I_{\ell}\),
\[ \ell = 1(2)2i + 1, \text{ (c) } \omega \in I_{\ell}, \ell = 2i + 2, 2i + 4, \ldots, \text{ and (d) } \omega \in I_{\ell}, \ell = 2i + 3, 2i + 5, \ldots. \]

In case (a), \( \omega \in \left[ \frac{1}{1-\lambda_k}, \frac{2}{2-\lambda_k-\lambda_{k+1}} \right], k = \ell/2. \) It can be readily checked that \( \lambda_k(\omega) \) and \( \lambda_{k+1}(\omega) \) are, respectively, the absolutely smallest nonpositive and nonnegative eigenvalues of \( G_\omega \) with \( 0 \leq -\lambda_k(\omega) \leq \lambda_{k+1}(\omega). \)

On the other hand \( 0 \leq -\lambda_1(\omega) \leq \lambda_n(\omega). \) So, \( k_m(\omega) \) will be given by the expression

\[ k_m(\omega) = \frac{1 - \lambda_k^m(\omega)}{1 - \lambda_n^m(\omega)}. \quad (3.21) \]

Since \( m \) is even, and both \( \lambda_k(\omega) \) and \( \lambda_n(\omega) \) strictly decrease with \( \omega \) increasing it is concluded that the numerator and the denominator of the expression in (3.21) decreases and increases, respectively, making \( k_m(\omega) \) be a strictly decreasing function of \( \omega \in I_{\ell}. \) In case (b), \( \omega \in \left[ \frac{2}{2-\lambda_{k+1}-\lambda_k}, \frac{1}{1-\lambda_k} \right], k = \frac{\ell + 1}{2}. \) (\( I_1 \) is open on the left with bound 0 and \( I_{2i+1} \) is closed on the right with bound 1.)

Now \( -\lambda_{k-1}(\omega) \geq \lambda_k(\omega) \geq 0, \) so that \( k_m(\omega) \) will be given again by (3.21). However, this time both terms of the fraction strictly increase with \( \omega. \) Thus, differentiating with respect to \( \omega \) one obtains

\[ \frac{\partial k_m}{\partial \omega} \sim \frac{(1 - \lambda_k^m(\omega))(1 - \lambda_k)\lambda_k^{m-1}(\omega)}{1 - \lambda_k^m(\omega)} = \frac{(1 - \lambda_k^m(\omega))(1 - \lambda_n)\lambda_n^{m-1}(\omega)}{1 - \lambda_n^m(\omega)} = \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_n(\omega)), \quad (3.22) \]

because of \( \omega(1 - \lambda_j) = 1 - \lambda_j(\omega), j = k, n, \) and in view of (3.15). Since \( \omega \) varies in \( I_{2k-1} \subset (0,1] \) and \( \lambda_k(\omega) \leq \lambda_n(\omega) \) Lemma 3.1 applies, implying that \( \frac{\partial k_m}{\partial \omega} \leq 0, \) with equality concerning limiting cases only. Therefore \( k_m(\omega) \) strictly decreases in \( I_{2k-1}. \) In case (c), where \( I_\ell, \ell = 2i + 2, 2i + 4, \ldots, \) is of the general type \( \left[ \frac{1}{1-\lambda_k}, \frac{2}{2-\lambda_k-\lambda_{k+1}} \right], k = \ell/2, \) except the first and maybe the last interval, we have a similar situation to that of case (a). This time \( k_m(\omega) \) is given by the expression

\[ k_m(\omega) = \frac{1 - \lambda_k^m(\omega)}{1 - \lambda_n^m(\omega)}. \quad (3.23) \]

Since \( \lambda_k(\omega) \geq 0 \geq \lambda_1(\omega) \) and both \( \lambda_k(\omega) \) and \( \lambda_1(\omega) \) decrease with \( \omega \) increasing, \( k_m(\omega) \) strictly increases with \( \omega. \) In case (d) we have a similar situation to that in case (b). The interval \( I_\ell, \ell = 2i + 3, 2i + 5, \ldots, \) is of the general type \( \left[ \frac{2}{2-\lambda_{k-1}-\lambda_k}, \frac{1}{1-\lambda_k} \right], k = (\ell - 1)/2, \) except maybe the last one, and \( k_m \) is given by (3.23), where this time \( 0 \geq \lambda_k(\omega) \geq \lambda_1(\omega), \) so both terms of the fraction in (3.23) decrease with \( \omega \) increasing. On differentiation we have a series of relationships similar to those in (3.22) but this time

\[ \frac{\partial k_m}{\partial \omega} \sim \phi_m(\lambda_k(\omega)) - \phi_m(\lambda_1(\omega)). \quad (3.24) \]

Based now on Lemma 3.1 we have again the desired result, namely \( k_m(\omega) \) strictly increases on \( I_\ell. \)

Summarizing the conclusions of cases (a)-(d) leads to (3.19).

**Case II:** In case \( \lambda_{i+1} + \lambda_i > 0 \) we work in a similar way as in Case I. This time \( 1 \in \left[ \frac{1}{1-\lambda_i}, \frac{2}{2-\lambda_i-\lambda_{i+1}} \right] \) and we have \( 2i \) subintervals to the left and at most \( 2(n - i) + 1 \) ones to the right of 1. The function \( k_m(\omega) \) behaves in exactly the same way as before in the subintervals which are to the left and to
$k_m(\omega)$ behaves in exactly the same way as before in the subintervals which are to the left and to the right of 1 as is readily checked and consequently we arrive at exactly the same conclusion. This completes the proof of our theorem. \(\square\)

Suppose now that the eigenvalues of $G$ in (3.7) satisfy

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1,$$

that is without the further assumption $\lambda_n = -\lambda_1$ of Thm 3.2. Suppose also that we extrapolate $G$ using any parameter $\omega \in (0, \frac{2}{1-\lambda_1})$. The answer now to the question “What is the value of $\omega_{opt}$ in this case?” can be given almost immediately. Having in mind the fact that “The extrapolation with a parameter $\omega_2$ of an extrapolation with parameter $\omega_1$ is also an extrapolation with parameter $\omega = \omega_2 \omega_1$”, which can be easily checked (see also [9]), leads us to writing $\omega$ as $\omega = \omega_2 \omega_1$, where $\omega_1 = \frac{2}{\lambda_1 - \lambda_n}$. The eigenvalues $\lambda'_i = 1 - \omega_1 + \omega_1 \lambda_i$, $i = 1(1)n$, of $G_{\omega_1}$ satisfy

$$-1 < -\lambda'_i = \lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_n < 1,$$ 
that is all the assumptions of Thm 3.2. So, extrapolation of $G_{\omega_1}$ becomes optimal iff $\omega_2 = 1$. Thus we have just proved:

**Theorem 3.3:**

Let the eigenvalues of $G$ in (3.7) satisfy (3.25). Then the condition number $k_m = k_m(\omega)$ of $M_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized with respect to $\omega \in (0, \frac{2}{1-\lambda_1})$ for

$$\omega_{opt} = \frac{2}{2 - \lambda_1 - \lambda_n}.$$  \hspace{1cm} (3.27)

As an immediate corollary we have

**Corollary 3.1:**

If $A$ is real symmetric positive definite and point (or block) 2-cyclic consistently ordered and $M$, in the splitting $A = M - N$, is the diagonal (or the block diagonal part corresponding to the block partitioning) of $A$, then the condition number $k_m = k_m(\omega)$ of $M_m^{-1}A$, given by (3.11) for even $m \geq 2$, is minimized for $\omega_{opt} = 1$.

Note: If the only information available on the spectrum of $G$ is its spectral radius $\rho(G) = \lambda_n < 1$, then $\omega_{opt}$ should be taken to be 1.

We close this section by noting that the idea in [2] for defining $m$-step additive preconditioners of (1.1), where $A$ is positive definite, can be generalized. For this we consider the multisplitting

$$A = P_k - Q_k, \quad \det(P_k) \neq 0, \quad k = 1(1)p,$$  \hspace{1cm} (3.28)

and the iteration matrix $H$ of the corresponding multisplitting method (1.5) with $D_k = a_k I$, $k = 1(1)p$. Setting

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\[ G_k = P_k^{-1}Q_k, \quad M^{-1} = \sum_{i=1}^{p} a_i P_i, \quad (3.29) \]

then

\[ H = \sum_{i=1}^{p} a_i G_i \quad (3.30) \]

and the \(m\)-step additive preconditioner is defined by

\[ M_m = M(I + H + H^2 + \ldots + H^{m-1})^{-1}, \quad m \geq 1, \quad (3.31) \]

provided that \(M_m\) is positive definite (and \(A \approx M_m\)). We note that the \(m\)-step additive preconditioner is an \(m\)-step preconditioner (see (3.4)) related to the splitting defining a multisplitting method. Certainly, if \(M\) is positive definite and \(\rho(H) < 1\), then \(M_m\) is also positive definite and \(A \approx M_m\). In the following Theorem we give sufficient conditions for \(M_m\) to be positive definite.

**Theorem 3.4**

Let \(A\) in (1.1) be positive definite and

\[ A = P_k - Q_k, \quad k = 1(1)2q, \quad (3.32) \]

where

\[ P_{q+i} = P_i^T, \quad i = 1(1)q. \quad (3.33) \]

If the splittings (3.32) for \(k = 1(1)q\) are \(P\)-regular splittings of \(A\), then the \(m\)-step additive preconditioner (3.31), where

\[ M = \left( \sum_{i=1}^{2q} a_i P_i^{-1} \right)^{-1}, \quad a_i = \frac{1}{2q}, \quad i = 1(1)2q, \quad H = \sum_{i=1}^{2q} a_i G_i, \quad G_i = P_i^{-1}Q_i, \quad (3.34) \]

is positive definite.

**Proof**

Since (3.32) for \(k = 1(1)q\) are \(P\)-regular splittings and (3.33) holds, it follows that (3.32) for \(k = q+1(1)2q\) are also \(P\)-regular splittings of \(A\). Thus we have that \(P_k + Q_k + (P_k+Q_k)^T = 2(P_k + P_k^T - A)\) is positive definite, \(k = 1(1)2q\). Consequently \(P_k + P_k^T\) is positive definite, \(k = 1(1)2q\). Moreover, using (3.33), we find

\[ M^{-1} = \sum_{i=1}^{2q} a_i P_i^{-1} = \frac{1}{2q} \sum_{i=1}^{q} (P_i^{-1} + P_{q+i}^{-1}) = \frac{1}{2q} \sum_{i=1}^{q} [(P_i^{-1})^T (P_i^T + P_i) P_i^{-1}] \quad (3.35) \]
Since \( P_i + P_i^T \) is positive definite, \( i = 1(1)q \), and \( M^{-1} \) is a sum of positive definite matrices, \( M^{-1} \) and hence \( M \) is positive definite. Moreover it is \( \rho(H) < 1 \) by Thm 2.2. Now, using Thm 3.1 of [6] we obtain the desired result. □

4 Optimum SOR-Additive Iterative Method

We again consider system (1.1), where

\[ A = D - L - L^T \]  

(4.1)

and \( A \) is positive definite. Given the splittings \( A = P_k - Q_k, k = 1, 2 \), with

\[ P_1 = \frac{1}{\omega}(D - \omega L), \quad P_2 = P_1^T = \frac{1}{\omega}(D - \omega L^T) \]  

(4.2)

and \( \omega \neq 0 \) a real parameter, it can be shown that \( A = P_1 - Q_1 \) is a \( P \)-regular splitting of \( A \), if \( 0 < \omega < 2 \). Hence Thm 3.4 for \( q = 1 \) (see also Thm 2.2) implies that the SOR two-splitting method or SOR-additive method [2]

\[ x^{(m+1)} = H x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \]  

(4.3)

where

\[ H = H(\omega) = \frac{1}{2}(G_1 + G_2), \quad c = \frac{1}{2}(P_1^{-1} + P_2^{-1})b, \quad G_i = P_i^{-1} Q_i, \quad i = 1, 2, \]  

(4.4)

converges. Under the further assumption that \( A \) has the 2-cyclic form

\[ A = \begin{bmatrix} D_1 & -X \\ -X^T & D_2 \end{bmatrix} \]  

(4.5)

\( (D_1, D_2 \) are diagonal matrices), it was proved in [2] that if \( \lambda \) is an eigenvalue of \( H \), then

\[ \lambda = \frac{1}{2}[(\omega^2 \mu^2 + \omega(2-\omega)\mu + 2(1-\omega))], \]  

(4.6)

where \( \mu \) is an eigenvalue of the Jacobi iteration matrix \( J = I - D^{-1}A \) for \( A \). It is noted that \( J \) has real eigenvalues, which occur in \( \pm \) pairs and \( \rho(J) < 1 \). Moreover it was shown in [2] that

\[ \min_{0<\omega<2} \rho(H(\omega)) = \rho(H(\omega_{opt})), \]  

where

\[ \omega_{opt} = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}, \quad \mu_m = \rho(J). \]  

(4.7)

We observe first that \( \lim_{\mu_m \to 0^+} \mu = 0 \) for all the eigenvalues \( \mu \) of \( J \) and from (4.7) we obtain

\[ \lim_{\mu_m \to 0^+} \lambda = 1 - \omega, \]  

which means that the optimum \( \omega \) satisfied \( \lim_{\mu_m \to 0^+} \omega_{opt} = 1 \). On the other hand, (4.7) for \( \mu_m = 0 \) gives
This observation suggests that the theoretical determination of the optimum value of \( \omega \) must be reconsidered. In what follows we give the complete solution to this problem and the results are contained in the following theorem.

**Theorem 4.1**

If \( A \) in (1.1) is positive definite, \( A = D - L - L^T \) and has the form (4.5), then the optimum value \( \omega_{\text{opt}} \) for \( 0 < \omega < 2 \) of the SOR-additive method defined by (4.3) is given by

\[
\omega_{\text{opt}} = \begin{cases} 
\frac{1 - \sqrt{1 - 2\mu_m^2}}{\mu_m^2}, & \text{if } 0 < \mu_m \leq \frac{1}{\sqrt{6}} \\
\frac{\mu_m - \frac{3}{4} + \sqrt{3 - 2\mu_m^2}}{\frac{3}{4} + \mu_m + \mu_m^2}, & \text{if } \frac{1}{\sqrt{6}} \leq \mu_m < 1
\end{cases}
\]  

(4.9)

where \( \mu_m = \rho(J) \) and \( J = I - D^{-1}A \).

**Proof**

The problem we solve is: Find

\[
\min_{\omega} \max_{\mu} |\lambda|,
\]

where \( \lambda \) is given by (4.6), \( 0 < \omega < 2, \mu \in [-\mu_m, \mu_m] \) and \( \mu_m < 1 \). For this we have that \( \frac{\partial \lambda}{\partial \mu} = 0 \) iff \( \mu = \frac{\omega - 2}{2\omega} \equiv \mu^* \). Moreover,

\[
\mu^* \in [-\mu_m, \mu_m] \quad \text{iff} \quad \omega^* \equiv \frac{2}{1 + 2\mu_m} \leq \omega < 2.
\]

(4.11)

With \( \lambda = \lambda(\mu) \) we find

\[
A = A(\omega) \equiv |\lambda(\mu_m)| = \frac{1}{2} |\omega^2 \mu_m^2 + \omega(2 - \omega)\mu_m + 2(1 - \omega)|,
\]

(4.12)

\[
B = B(\omega) \equiv |\lambda(-\mu_m)| = \frac{1}{2} |\omega^2 \mu_m^2 - \omega(2 - \omega)\mu_m + 2(1 - \omega)|,
\]

(4.13)

\[
C = C(\omega) \equiv |\lambda(\mu^*)| = \begin{cases} 
\frac{1}{8} (\omega^2 + 4\omega - 4), & \text{if } 2(\sqrt{2} - 1) \leq \omega < 2 \\
\frac{1}{8} (4 - 4\omega - \omega^2), & \text{if } 0 < \omega \leq 2(\sqrt{2} - 1)
\end{cases}
\]

(4.14)

Hence

\[
\max_{\mu} |\lambda| = \max\{A, B, C\}.
\]

(4.15)
It can be proved that

(i) If \( 0 < \mu_m < \frac{\sqrt{3}}{2} \) and \( 0 < \omega \leq \omega_1 \equiv \frac{1 - \sqrt{1 - 2\mu_m^2}}{\mu_m} \) or \( \frac{\sqrt{3}}{2} \leq \mu_m < 1 \) and \( 0 < \omega < 2 \), then

\[
B \leq A = \frac{1}{2} [\omega^2 \mu_m^2 + \omega(2 - \omega)\mu_m + 2(1 - \omega)].
\]

(ii) If \( 0 < \mu_m < \frac{\sqrt{2}}{2} \) and \( \omega_1 \leq \omega < 2 \), then

\[
A \leq B = \frac{1}{2} [\omega(2 - \omega)\mu_m - \omega^2 \mu_m^2 - 2(1 - \omega)].
\]

Thus, we distinguish the following cases:

Case I: \( \frac{\sqrt{3}}{2} \leq \mu_m < 1 \). Then it can be shown that \( \omega^* \leq 2(\sqrt{2} - 1) \) and

\[
\max\{A, B, C\} = \begin{cases} 
A & \text{if } 0 < \omega \leq \rho_2 \\
C & \text{if } \rho_2 \leq \omega < 2,
\end{cases}
\]

where

\[
\rho_2 = \frac{\mu_m - \frac{3}{2} + \sqrt{3 - 2\mu_m^2}}{\frac{1}{4} + \mu_m - \mu_m^2}.
\]

Now, we find that \( \frac{\partial A}{\partial \omega} < 0 \) and \( \frac{\partial C}{\partial \omega} > 0 \), implying \( \min_\omega A = A(\rho_2) \) and \( \min_\omega C = C(\rho_2) = A(\rho_2) \).

Hence we obtain \( \omega_{\text{opt}} = \rho_2 \) and \( \min_\omega \max_\mu |\lambda| = A(\rho_2) = C(\rho_2) = \frac{1}{8}(\rho_2^3 + 4\rho_2^2 - 4) \).

Case II: \( 0 < \mu_m < \frac{\sqrt{2}}{2} \). Then it can be shown that:

(i) If \( 0 < \mu_m \leq \frac{1}{\sqrt{6}} \), then \( 2(\sqrt{2} - 1) < \omega_1 \leq \omega^* \).

(ii) If \( \frac{1}{\sqrt{6}} \leq \mu_m < \frac{\sqrt{2}}{2} \), then \( 2(\sqrt{2} - 1) < \omega^* \leq \omega_1 \).

Therefore we must distinguish the following subcases:

Case IIa: \( 0 < \mu_m \leq \frac{1}{\sqrt{6}} \). Then we find

\[
\max\{A, B, C\} = \begin{cases} 
A & \text{if } 0 < \omega \leq \omega_1 \\
B & \text{if } \omega_1 \leq \omega \leq \omega^* \\
C & \text{if } \omega^* \leq \omega < 2,
\end{cases}
\]

and
\[
\min_w A(\omega) = A(\omega_1), \quad \min_w B(\omega) = B(\omega_1), \quad \min_w C(\omega) = C(\omega^*) = B(\omega^*) \geq B(\omega_1).
\]
Hence we have \( \omega_{opt} = \omega_1 \) and \( \min_{\omega} \max_\mu |\lambda| = A(\omega_1) = B(\omega_1) \).

Case IIb: \( \frac{1}{\sqrt{6}} < \mu_m < \frac{\sqrt{2}}{2} \). Then it can be proved that

\[0 < 2(\sqrt{2} - 1) < \omega^* < \rho_2 < \omega_1 < 2\] (4.19)

and

\[
\max\{A, B, C\} = \begin{cases} 
A & \text{if } 0 < \omega \leq \rho_2 \\
C & \text{if } \rho_2 \leq \omega < 2.
\end{cases}
\] (4.20)

As in Case I we find that \( \omega_{opt} = \rho_2 \) and \( \min_{\omega} \max_\mu |\lambda| = A(\rho_2) = C(\rho_2) = \frac{1}{8}(\rho_2^2 + 4\rho_2 - 4) \).

Combining the above results of Cases I, IIa, IIb we obtain (4.8).

References


