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Quadric and Cubic Hypersurface Parameterization

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Abstract

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1 Introduction

Background: Rationality of the algebraic curve or surface is a restriction where advantages are obtained from having both the implicit and rational parametric representations, [6]. For example, an algebraic surface is represented implicitly by the single polynomial equation \( f(x, y, z) = 0 \) and parameterically by the three equations \( x = G_1(s, t), y = G_2(s, t), z = G_3(s, t) \). When the \( G_i, i = 1, \ldots, 3 \) are rational functions, i.e. ratio of polynomials, the parametric representation is known as a rational parametric representation. While the rational parametric representation of a curve or surface allows greater ease for transformation and shape control, the implicit representation is preferred for testing whether a point is on the given curve or surfaces and is further conducive to the direct application of algebraic techniques. Simpler algorithms are possible when both representations are available. For example, a straightforward method exists for computing curve - curve (and surface - surface intersections), when one of the curves (respectively surfaces), is in implicit form and the other is in parametric form. For further details, see for example [5].

Numerous facts on rational algebraic curves and surfaces can be gleaned from books and papers on analytic geometry, algebra and algebraic geometry, see for example [9, 11, 12, 13]. In the case of plane curves, all degree two algebraic curves (conics) are rational. For degree three algebraic curves (cubics): while all singular cubics are rational, the nonsingular cubics only allow a parameterization...
having a square root of rational functions. Small subsets of degree four and higher algebraic curves are rational. For example, degree four curves (quartics) with a triple point or three double points and degree five curves (quintics) with two triple points or six double points, etc., are rational. In general, a necessary and sufficient condition for the rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: $G = 0$, where $G$ is the genus of the curve measuring the deficiency of the given curve’s singularities from its maximum allowable limit. See for example, [12].

For the case of surfaces, all degree two algebraic surfaces (quadrics or conicoids), are rational. All degree three surfaces (cubic surfaces or cubicoids), except the cylinders of nonsingular cubic curves and the cubic cone, have a rational parameterization, with the exceptions again only having a parameterization of the type which allows a single square root of rational functions. Most algebraic surfaces of degree four and higher are also not rational, although parameterizable subclasses can be identified. For example, degree four surfaces with a triple point such as the Steiner surfaces or degree four surfaces with a double curve such as the Plucker surfaces are rational. In general, a necessary and sufficient condition for the rationality of an algebraic surface of arbitrary degree is given by Castelnuovo’s criterion: $P_a = P_2 = 0$, where $P_a$ is the arithmetic genus and $P_2$ is the second plurigenus.

The Parameterization Problem: Given implicit representations of quadric and cubic hypersurfaces, in $n$ dimensional space, $n \geq 3$, construct corresponding parametric representations.

Prior Work: Various algorithms have been given for constructing the rational parametric equations of implicitly defined algebraic curves and surfaces. The idea of parameterizing a conic $C_2$ is to fix a simple point $p$ on $C_2$ and take a one parameter family of lines $L(t)$ through $p$. The lines $L(t)$ then intersect $C_2$ in only one additional point $q(t)$, yielding the rational parameterization. The symbolic intersection of $L(t)$ with $C_2$ can be efficiently achieved by mapping the fixed point $p$ to infinity, along one of the coordinate axis directions, via a linear transformation [1]. The idea for parameterizing singular cubics $SC_3$ is to take lines through the singular point $s_p$ on the cubic. The actual algorithm is again based on mapping, via simple transformations, any simple point $p$ on $SC_3$ to infinity, and furthermore the singular point $s_p$ is never explicitly computed [2].

Computational methods have also been given for constructing parametric equations of the intersection space curves of two degree two surfaces by [7] using the fact that the pencil of quadrics contains a ruled surface and by [8], via the computation of eigenvalues of matrices of quadratic
forms. The parameterization algorithms presented in [3] and [4] are applicable for irreducible rational plane algebraic curves of arbitrary degree, and irreducible rational space curves arising from the intersection of two algebraic surfaces of arbitrary degree. The parameterization techniques, essentially, reduce to solving systems of homogeneous linear equations and the computation of Sylvester resultants, see for e.g. [12].

The geometric technique of constructing parametric representations of quadrics (or conicoids) is identical to that of the conics. The intersection of lines through a fixed, simple point \( p \) on the quadric \( Q_2 \) can again be efficiently achieved by a linear transformation, (mapping the point \( p \) to infinity along one of the coordinate axis directions) [1].

Parameterizing cubicoids, essentially relies on being able to generate rational curves (straight lines, conics or singular cubics) on the cubicoid surface. The parameterization method of [10] for cubic surfaces (or cubicoids), generates lines \( L \) on the cubic surface \( CS_3 \). Lines \( L \) are computed by intersecting \( L \) in parametric form with \( SC_3 \), which requires the simultaneous solution a nonlinear system of four equations in four unknowns. In [2] lines and singular cubics are generated on \( CS_3 \) by an efficient method of intersecting \( SC_3 \) with tangent planes.

Results: In this paper we present a combination of both algebraic and numerical techniques to construct parametric representations of all quadric and cubic hypersurfaces (except hypersurfaces which are cylinders or cones), in \( n \)-dimensional space, for \( n \geq 3 \).

2 Preliminaries

A point in complex projective space \( CP^n \) is given by a nonzero homogeneous coordinate vector \((X_0, X_1, \ldots, X_n)\) of \( n+1 \) complex numbers. A point in complex affine space \( CA^n \) is given by the non-homogeneous coordinate vector \((x_1, x_2, \ldots, x_n) = (\frac{X_1}{X_0}, \frac{X_2}{X_0}, \ldots, \frac{X_n}{X_0})\) of \( n \) complex numbers. The set of points \( Z^n_\mathbb{C}(f) \) of \( CA^n \) whose coordinates satisfy a single non-homogeneous polynomial equation \( f(x_1, x_2, \ldots, x_n) = 0 \) of degree \( d \), is called an \( n-1 \) dimension, affine hypersurface of degree \( d \). The hypersurface \( Z^n_\mathbb{C}(f) \) is also known as a flat or a hyperplane, a \( Z^n_\mathbb{H}(f) \) is known as a quadric hypersurface, and a \( Z^n_\mathbb{L}(f) \) is known as a cubic hypersurface. The hypersurface \( Z^n_d \) is a curve of degree \( d \), a \( Z^n_2 \) is known as a surface of degree \( d \), and \( Z^n_4 \) is known as a threefold of degree \( d \). A hypersurface \( Z^n_d \) is reducible or irreducible based upon whether \( f(x_1, x_2, \ldots, x_n) = 0 \) factors or not, over the field of complex numbers. A rational hypersurface \( Z^n_\mathbb{C}(f) \), can additionally
be defined by rational parametric equations which are given as \( x_1 = G_1(u_1, u_2, \ldots, u_{n-1}), x_2 = G_2(u_1, u_2, \ldots, u_{n-1}), \ldots, x_n = G_n(u_1, u_2, \ldots, u_{n-1}) \), where \( G_1, G_2, \ldots, G_n \) are rational functions of degree \( d \) in \( u = (u_1, u_2, \ldots, u_{n-1}) \), i.e., each is a quotient of polynomials in \( u \) of maximum degree \( d \).

3 Quadric Hypersurfaces

Method 1: Consider the implicit representation of a quadric hypersurface, (which is neither a cylinder or a cone)

\[
Z^n(f): \sum_{i_1+i_2+\ldots+i_n \leq 2} a_{i_1,i_2,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} = 0 \tag{1}
\]

We assume that all quadratic terms of \( Z^n(f) \) are present, for otherwise there exists a trivial parametric representation.

1. Choose a simple point \((\alpha_1, \ldots, \alpha_n)\) on \( Z^n(f) \) and apply a linear coordinate transformation

\[
y_j = x_j - \alpha_j, \quad j = 1 \ldots n\tag{2}
\]

to make the hypersurface pass through the origin. Applying the linear transformation (2) to equation (1) yields

\[
Z^n(f_1): \sum_{1 \leq i_1+i_2+\ldots+i_n \leq 2} b_{i_1,i_2,\ldots,i_n} y_1^{i_1} \ldots y_n^{i_n} = 0 \tag{3}
\]

with the constant term \( b_{000\ldots0} = 0 \). That is

\[
Z^n(f_1): b_{100\ldots0}y_1 + b_{010\ldots0}y_2 + \ldots + b_{000\ldots1}y_n
+ \sum_{i_1+i_2+\ldots+i_n = 2} b_{i_1,i_2,y_1^{i_1} \ldots y_n^{i_n}} = 0 \tag{4}
\]

2. Now, there must be at least one nonzero coefficient amongst the linear terms in equation (4). Otherwise the origin is a singular point for the surface and this contradicts the earlier assumption. Without loss of generality, let \( b_{100\ldots0} \neq 0 \). Then apply the linear transformation

\[
x_1 = b_{100\ldots0}y_1 + b_{010\ldots0}y_2 + \ldots + b_{000\ldots1}y_n
x_j = y_j, \quad j = 2 \ldots n\tag{5}
\]
which makes the $z_1 = 0$, the tangent hyperplane of $Z'_2(f_1)$ at the origin. This yields

$$Z'_2(f_2) : z_1 + \frac{b_{020}...0}{b^2_{100...0}} z_1^2 + \left[ \frac{b_{002}...0 + \frac{b^2_{010}...0}{b^2_{100...0}}} {b^2_{100...0}} \right] z_2^2$$

$$+ \ldots + \left[ \frac{b_{000}...2 + \frac{b_{000}...1}{b^2_{100...0}}} {b^2_{100...0}} \right] z_n^2$$

$$+ \sum_{i_1 + \ldots + i_n = 2, i_k = i_{k'}, 1 \leq i \leq k \neq k'} c_{i_1 i_2 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n} = 0 \quad (6)$$

3. To equation (6), apply the linear coordinate transformation which maps the origin to infinity, along the $z_1$ axis. Namely,

$$z_1 = \frac{1}{w_1}$$

$$z_j = \frac{w_j}{w_1}, \quad j = 2, \ldots, n \quad (7)$$

This yields

$$Z'_2(f_3) : \frac{1}{w_1} + \frac{b_{020}...0}{b^2_{100...0}} \frac{1}{w_1^2} + \left[ \frac{b_{002}...0 + \frac{b^2_{010}...0}{b^2_{100...0}}} {b^2_{100...0}} \right] \frac{w_2^2}{w_1^2}$$

$$+ \ldots + \left[ \frac{b_{000}...2 + \frac{b_{000}...1}{b^2_{100...0}}} {b^2_{100...0}} \right] \frac{w_n^2}{w_1^2}$$

$$+ \frac{1}{w_1^2} \sum_{i_1 + \ldots + i_n = 2, i_k = i_{k'}, 1 \leq i \neq k \neq k'} c_{i_1 i_2 \ldots i_n} w_2^{i_2} w_3^{i_3} \ldots w_n^{i_n} = 0 \quad (8)$$

4. Clearing the denominator of equation (8 and simplifying the expression for $Z'_2(f_3)$ yields

$$w_1 = \frac{b_{020}...0}{b^2_{100...0}} - \sum d_{i_2 \ldots i_n} w_2^{i_2} w_3^{i_3} \ldots w_n^{i_n}$$

$$= g_2(w_2 \ldots w_n) \quad (9)$$

Hence from transformation (7) above, we obtain

$$z_1 = \frac{1}{g_2(w_2 \ldots w_n)}$$

$$z_j = \frac{w_j}{g_2(w_2 \ldots w_n)}, \quad j = 2, \ldots, n \quad (10)$$

From transformation (5) we obtain,

$$y_1 = \frac{1 - b_{010}...0 w_2 - b_{001}...0 w_3 - \ldots - b_{000}...1 w_n}{b_{100...0} g_2(w_2 \ldots w_n)}$$

$$y_j = z_j, \quad j = 2, \ldots, n. \quad (11)$$

Finally from transformation (2) we obtain,

$$z_j = y_j + \alpha j, \quad j = 1, \ldots, n. \quad (12)$$
as rational functions of the parameters $w_2, \ldots, w_n$, a rational parameteric representation of the quadric hypersurface.

**Method 2:** Consider again the implicit representation (1) of a quadric hypersurface.

1. Choose a simple point $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ on $Z^n(f)$ and apply a linear coordinate transformation

   $y_j = x_j - \alpha_j, \quad j = 1, \ldots, n$  \hspace{1cm} (13)

   to make the resulting hypersurface pass through the origin. This yields

   \[ Z^n(f_1) : \sum_{i_1 + \cdots + i_n = 1} b_{i_1, \ldots, i_n} y_1^{i_1} \cdots y_n^{i_n} + \sum_{i_1 + \cdots + i_n = 2} c_{i_1, \ldots, i_n} y_1^{i_1} y_2^{i_2} \cdots y_n^{i_n} = 0 \]  \hspace{1cm} (14)

2. Apply the homogenizing transformation

   \[ y_j = \frac{Y_j}{Y_0} \quad j = 1, \ldots, n \]  \hspace{1cm} (15)

   to $Z^n(f_1)$ and clear the denominator $Y_0^2$ to yield

   \[ Z^n(F_1) : Y_0 \sum_{i_1 + \cdots + i_n = 1} b_{i_1, \ldots, i_n} Y_1^{i_1} \cdots Y_n^{i_n} + \sum_{i_1 + \cdots + i_n = 2} c_{i_1, \ldots, i_n} Y_1^{i_1} Y_2^{i_2} \cdots Y_n^{i_n} = 0 \]  \hspace{1cm} (16)

3. Now in (16) there exists some nonzero coefficient of the quadratic terms $Y_1^2, Y_2^2, \ldots, Y_n^2$. Without loss of generality, let that be $b_{2000\ldots0} \neq 0$. Then set $Y_1 = 1$, a dehomogenizing transformation to yield

   \[ Z^n(F_2) : Y_0 \sum_{i_1 + \cdots + i_n = 1} Y_2^{i_2} \cdots Y_n^{i_n} + \sum_{i_1 + \cdots + i_n = 2} Y_2^{i_2} \cdots Y_n^{i_n} = 0 \]  \hspace{1cm} (17)

   from where we obtain

   \[ Y_0 = \frac{-\sum_{i_1 + \cdots + i_n = 2} c_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}}{\sum_{i_1 + \cdots + i_n = 1} b_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}} \]  \hspace{1cm} (18)

4. Using (18) and (15) with $Y_1 = 1$ we obtain

   \[ y_1 = \frac{1}{Y_0} = \frac{-\sum_{i_1 + \cdots + i_n = 1} b_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}}{\sum_{i_1 + \cdots + i_n = 2} c_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}} \]  \hspace{1cm} (19)

   \[ y_j = \frac{Y_j}{Y_0} = \frac{-Y_j \sum_{i_1 + \cdots + i_n = 2} b_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}}{\sum_{i_1 + \cdots + i_n = 2} c_{i_1, \ldots, i_n} Y_2^{i_2} \cdots Y_n^{i_n}}, \quad j = 2, \ldots, n \]

   and finally using (13) we obtain

   \[ x_j = y_j + \alpha_j \quad j = 1, \ldots, n. \]  \hspace{1cm} (20)

   explicit parametric equations with parameters $Y_2, \ldots, Y_n$. 
4 Cubic Hypersurfaces

Consider the general implicit equation of a cubic hypersurface

\[ Z_3^n(f) : \sum_{i_1+i_2+...+i_n \leq 3} a_{i_1i_2...i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n} = 0 \]  (21)

1. Choose a simple point \((\alpha_1, \alpha_2, ..., \alpha_n)\) on \(Z_3^n(f)\) and apply the linear coordinate transformation

\[ y_j = x_j - \alpha_j, \quad j = 1, ..., n \]  (22)

which translates the hypersurface \(Z_3^n(f)\) to pass through the origin. This yields

\[ Z_3^n(f_1) : \sum_{i_1+i_2+...+i_n=1} b_{i_1i_2...i_n} y_1^{i_1} y_2^{i_2} ... y_n^{i_n} + \sum_{i_1+i_2+...+i_n=2} b_{i_1i_2...i_n} y_1^{i_1} y_2^{i_2} ... y_n^{i_n} + \sum_{i_1+i_2+...+i_n=3} b_{i_1i_2...i_n} y_1^{i_1} y_2^{i_2} ... y_n^{i_n} = 0 \]  (23)

2. Apply the linear transformation

\[ z_1 = b_{000...0} y_1 + b_{001...0} y_2 + ... + b_{000...1} y_n \]
\[ z_j = y_j, \quad j = 1, ..., n \]  (24)

which makes \(z_1 = 0\) to be the new tangent hyperplane to the hypersurface at the origin. The hypersurface \(Z_3^n(f_1)\) of equation (23) then becomes

\[ Z_3^n(f_2) : z_1 + z_1 \sum_{0<i_2+...+i_n \leq 2} c_{i_2...i_n} z_2^{i_2} ... z_n^{i_n} + z_1^2 \sum_{i_1+...+i_n=1} d_{i_1...i_n} z_1^{i_1} ... z_n^{i_n} + \sum_{i_2+...+i_n=2} s_{i_2...i_n} z_2^{i_2} ... z_n^{i_n} + \sum_{i_3+...+i_n=3} t_{i_3...i_n} z_3^{i_3} ... z_n^{i_n} \]  (25)

3. Intersecting the hypersurface \(Z_3^n(f_2)\) with the tangent hyperplane \(z_1 = 0\) yields

\[ Z_3^{n-1}(f_3) : \sum_{i_2+...+i_n=2} s_{i_2...i_n} z_2^{i_2} ... z_n^{i_n} + \sum_{i_3+...+i_n=3} t_{i_3...i_n} z_3^{i_3} ... z_n^{i_n} = 0 \]  (26)

4. Consider \( u = (u_1, ..., u_k), k \leq n - 2, \) parameter family of lines, passing through the origin and lying in the hyperplane \(z_1 = 0\). These lines are given by

\[ z_{i+2} = u_i z_2, \quad 1 \leq i \leq k \]
\[ z_j = z_2, \quad k < j \leq n - 2 \]  (27)
5. Intersect these lines given by equation (27) with \( Z_{3}^{-1}(f_{3}) \) of equation (25) to yield

\[
    z_{2} = -\frac{\sum_{i_{2} \ldots i_{n}=2} s_{2} \ldots i_{n} u_{1}^{i_{2}} \ldots u_{k}^{i_{n}+2}}{\sum_{i_{1} \ldots i_{n}=3} s_{1} \ldots i_{1}^{i_{2}} \ldots u_{k}^{i_{n}+2}} \tag{28}
\]

which together with (27) above yields a parametric representation of \( Z_{3}^{-1}(f_{3}) \) in terms of parameters \( u = (u_{1}, ..., u_{k}) \).

6. Using the linear transformation (22), (24), the parametric representation of \( Z_{3}^{-1}(f_{3}) \) and \( Z_{1} = 0 \) we can straightforwardly construct a \( u \) parameterization of \( Z_{3}^{-1}(f_{3}) \) in the original space \( (x_{1}, ..., x_{n}) \). Namely

\[
    x_{i} = M_{i}(u) \quad i \leq i \leq n \tag{29}
\]

7. Next choose another simple point \( (\beta_{1}, \beta_{2}, ..., \beta_{n}) \) on \( Z_{3}^{3}(f) \) and repeat steps 1., 2., 3. replacing \( (a_{1}, a_{2}, ..., a_{n}) \) with \( (\beta_{1}, \beta_{2}, ..., \beta_{n}) \). This would yield another \( Z_{3}^{-1}(f_{3}) \) of similar structure as equation (25), viz., the intersection of a corresponding hypersurface \( Z_{3}^{3}(f_{2}) \) with an appropriate tangent hyperplane \( \hat{z}_{1} = 0 \).

8. Analogous to Step 4. above, consider then a \( v = (v_{1}, ..., v_{l}) \), \( l = n - k - 1 \), parameter family of lines, passing through the origin and lying in the hyperplane \( \hat{z}_{1} = 0 \). These lines are again given by

\[
    \hat{z}_{j+2} = v_{j}\hat{z}_{2}, \quad 1 \leq j \leq l \quad \hat{z}_{j} = \hat{z}_{2}, \quad l < j \leq n - 2 \tag{30}
\]

9. Similar to Steps 5. and 6. above, intersect these lines of equation (30) with \( Z_{3}^{-1}(f_{3}) \) to derive a \( v \) parametric representation of \( Z_{3}^{-1}(f_{3}) \) in the original space \( (x_{1}, ..., x_{n}) \). Namely,

\[
    x_{i} = N_{i}(v) \quad 1 \leq i \leq n \tag{31}
\]

10. Finally consider the \( (u, v) \) parameter family of lines in \( (x_{1}, ..., x_{n}) \) space joining points \( (M_{1}(u), M_{2}(u), ..., \) and \( (N_{1}(v), N_{2}(v), ..., N_{n}(v)) \). Namely,

\[
    x_{i} = N_{i}(v) + \frac{(N_{i}(v) - N_{i}(v))}{M_{i}(u) - M_{i}(u)}(x_{1} - M_{i}(u)) \quad 1 \leq i \leq n \tag{32}
\]

11. Intersect these lines of equation (32) with the hypersurface \( Z_{3}^{3}(f) \) to yield

\[
    f(x_{1}, u, v) = 0 \tag{33}
\]

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with degree of $x_1$ to be at most three, i.e., the lines intersect the hypersurface in at most three distinct intersection points.

12. Two of the intersection points lying on the hypersurface $Z^3(f)$ have $x_1$ values $M_1(u)$, and $N_1(v)$, Hence $\frac{f(x_1, u, v)}{(n-M_1)(n-N_1)}$ yields an expression which is linear in $x_1$. Thus $x_1 = R(u, v)$ where $R$ is a rational function in the $l+k = (n-1)$ parameters $u = (u_1, ..., u_k)$, $v = (v_1, ..., v_l)$. Using this together with equation (32) yields a parametric representation of the hypersurface $Z^3(f)$ in terms of the $n-1$ parameters $u, v$.

5 Conclusions and Future Research

Various techniques for parametrizing quadric and cubic hypersurfaces for any dimension $\geq 3$, have been presented. There exist other alternatives and variations to these basic methods. Algorithms for parametrizing general quartics and higher degree hypersurfaces are as yet unknown. Deriving such parametrizations is a keen area of future research with a number of open algorithmic problems.

References


