Some Thoughts on Ultimate Stability Condition for $G|G|1$ Queue

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SOME THOUGHTS ON ULTIMATE STABILITY
CONDITION FOR GI/GI QUEUE

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Let $N^t$ be queue length at the $t$-th departure epochs. Then

$$N^{t+1} - N^t = A^t - D^t \quad (1)$$

where $A^t$ and $D^t$ represent the number of arrivals and departures in the $k$-th epoches. Then (1) implies

$$EN^{t+1} - EN^t = EA^t - ED^t = EA^t - Pr \{N^t > 0 \}. \quad (2)$$

since $ED^t = Pr \{N^t > 0 \}$.

To study properties of (2) we need the following lemma.

**Lemma.** Let $a_n$ be nonnegative sequence, i.e., $a_n \geq 0$. Then, the following can be proved.

(i) If $\limsup_{n \to \infty} a_n = \infty$, then $\limsup (a_{n+1} - a_n) \geq 0$.

(ii) If $\limsup_{n \to \infty} a_n < \infty$, then $\liminf (a_{n+1} - a_n) \leq 0$.

(iii) For any $a_n \geq 0$ and $b_n \geq 0$

$$\limsup_{n \to \infty} (a_n - b_n) \leq \limsup_{n \to \infty} a_n - \liminf_{n \to \infty} b_n \quad (3a)$$

$$\liminf_{n \to \infty} (a_n - b_n) \geq \liminf_{n \to \infty} a_n - \limsup_{n \to \infty} b_n \quad (3b)$$

**Proof.** We prove first (i). Since $\limsup a_n = \infty$, then there exists an increasing subsequence $a_{n_k}$ tending to infinity. Hence $a_{n_k} - a_{n_{k-1}}$. But then

$$0 \leq a_{n_k} - a_{n_{k-1}} = (a_{n_k} - a_{n_{k-1}}) + (a_{n_{k-1}} - a_{n_{k-2}}) + \cdots + (a_{n_1} + 1 - a_{n_{k-1}}) \quad (4)$$
so (4) implies that at least one of the terms in the parenthesis of the RHS of (4) must be positive. So, we can pick up in each set of integers \((n_k, n_{k-1})\), \(k = 1, 2, \ldots\), such an index \(n_j \in (n_k, n_{k-1})\) that \(a_{n_j} - a_{n_j - 1} \geq 0\). This implies that \(\lim \sup (a_{n+1} - a_n) \geq 0\).

The proof for (ii) goes in the same manner as above. In particular, \(\lim \sup a_k < \infty\), i.e., \(\lim \sup a_n = \inf \sup (a_n, a_{n+1}, a_{n+2}, \ldots)\). So \(\sup (a_n, a_{n+1})\) is nonincreasing, hence \(a_{n_k} - a_{n_{k-1}} \leq 0\). This can be expressed as in (4), so at least one term in (4) is respective, and this implies \(\lim \inf (a_{n+1} - a_n) \leq 0\).

Finally, part (iii) is standard and follows from the following known fact about \(\lim \sup\), namely \(\lim \sup (a_n + b_n) \leq \lim \sup a_n + \lim \sup b_n\); for (3b) we note that \(\lim \sup c \cdot a_n = c \lim \inf a_n\). ■

Now we are ready to formulate our main result.

**Theorem.** For a general G/GI/1 queue the following holds.

(i) If \(\lim \sup EN' < \infty\), then \(\lim \inf EA' \leq \lim \sup Pr\{N' > 0\}\)

(ii) If \(\lim \sup EN' = \infty\), then \(\lim \sup EA' \geq \lim \inf Pr\{N' > 0\}\).

**Proof.** We prove first (i). From (2), Lemma (ii) and (3b) we have,

\[
0 \geq \lim \inf (EN'^{t+1} - EN') = \lim \inf (EA' - Pr\{N' > 0\}) \geq \lim \inf EA' - \lim \sup Pr\{N' > 0\}
\]

which implies (i).

The second thesis comes from (2), Lemma (i) and (3b), i.e.,

\[
0 \leq \lim \sup (EN'^{t+1} - EN') = \lim \sup (EA' - Pr\{N' > 0\}) \geq \lim \sup EA' - \lim \inf Pr\{N' > 0\}
\]

and this immediately proves (ii). ■