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Calculations of Pressure Pulsations in Pipelines In Case of Non-Sinusoidal Input Flows

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To calculate the pressure pulsations in pipelines in frequency domain, it is necessary to linearize the nonlinear damping function $|q|q$, under the condition that the flow $q$ is periodic, but not sinusoidal. The authors have proposed an idea that two equivalent linearization coefficients should be determined in such cases. To verify the idea, the responses of pressure pulsation in a simple pipeline are calculated for several input flows, using three linearization schemes based on this idea. The results are all in good agreement with those by Runge-Kutta-Gill integration method.

**NOMENCLATURE**

- $q_{in}$: dimensionless input flow
- $q$: dimensionless flow in pipe
- $\overline{q}$: fluctuating component of $q$
- $q_{in0}$: steady component of $q_{in}$
- $q_{inj}$: amplitude of the $j$th component of $q_{in}$
- $\psi_{inj}$: phase angle of the $j$th component of $q_{in}$
- $Q_0$: steady component of $q$
- $Q_j$: amplitude of the $j$th component of $q$
- $\psi_j$: phase angle of the $j$th component of $q$
\( p \)  dimensionless pressure in pipe
\( P_0 \)  steady component of \( p \)
\( p_j \)  amplitude of the \( j \)th component of \( p \)
\( \theta_j \)  phase angle of the \( j \)th component of \( p \)
\( \alpha \)  constant included in nonlinear damping
\( K_0 \)  equivalent linear coefficient of \(|q|q\) for the steady component of \( q \)
\( K_1 \)  equivalent linear coefficient of \(|q|q\) for the fluctuating component of \( q \)
\( \omega \)  dimensionless angular frequency
\( t \)  dimensionless time

In the following, all variables are described as if they had physical dimensions, but they are all treated as dimensionless variables.

INTRODUCTION

Pressure pulsations are frequently observed in pipelines connected to reciprocating compressors due to their periodic discharges or suction, and thereby harmful vibrations are generated in the piping systems. For the avoidance of such vibrations, so-called pulsation studies are required in the design stage of chemical plants including reciprocating compressors.

Methods are developed by Abe, et al \(^{(1)}\) and Sakai and Saeki \(^{(2)}\) to calculate natural frequencies of pressure pulsations in complicated piping systems by transfer matrix method. Hayama, et al \(^{(3)}\) have proposed a method to calculate resonant pressure amplitudes, in which nonlinear damping forces proportional to the square of fluid velocity are employed. Mohri and Hayama \(^{(4)}\) have generalized this method to calculate frequency responses of pressure pulsations, in which nonlinear damping forces are linearized by Fourier development method, assuming that the flow \( q \) in pipe is sinusoidal.

In practice, however, input flows to pipelines, generated by reciprocating compressors, are certainly periodic in time, but far from sinusoidal. Then, the pressure pulsations caused by them are also periodic, but not sinusoidal. In order to calculate such pressure pulsations in frequency domain by transfer matrix
method, the function \(|q|q\) which represents the non-linear damping, common in fluid oscillations, should be linearized, in some way, under that condition.

When \(q\) is periodic, the function \(|q|q\) is also periodic with the same period as \(q\), and then developed into Fourier series. So, the linearization coefficients are easily determined for each frequency component contained. It is, however, found that in some cases some of them happen to be negative, which means that the linearization method by Fourier development can produce unreasonable negative damping.

In this paper linearization methods for \(|q|q\), not producing negative damping, are presented. The author's idea is that, for the equivalent linearization of \(|q|q\) to be done correctly under the present condition, two coefficients should be determined, one for the steady component of \(q\) and the other for the fluctuating one including all frequency components.

To verify the idea to be reasonable for the problem under consideration, three linearization schemes based on this idea are tried. In the first scheme, the flow \(q\) is approximated by the steady component and a single sinusoidal wave with the double amplitude equal to the difference between the maximum and the minimum value during one period, and Fourier development method \(^{(5)}\) is applied. In the second, the derivative of the function \(|q|q\), which is proportional to a damping coefficient, is replaced by its mean value over one period. In the third, the least square method is applied.

By using these linearization schemes, pressure pulsations caused by nonsinusoidal, periodic input flows are calculated for a simple pipeline. The results obtained by the three schemes are all in good agreement with those by RKG (Runge-Kutta-Gill) integration method, which shows the authors' calculation methods to be reasonable.

DIFFICULTIES ENCOUNTERED IN FOURIER DEVELOPMENT OF \(|q|q\)

When an input flow \(q_{in}\) is periodic in time with period \(2\pi/\omega\), the flow \(q\) caused by it in pipe is also periodic with the same period as \(q_{in}\), and then, it is expressed in Fourier series as Eq.(1),

\[
q = Q_0 + \sum_{j=1}^{\infty} Q_j \sin(j\omega t + \psi_j) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \ cd
\[ |q|q = K_0 Q_0 + \sum_{j=1}^{\infty} K_j F Q_j \cdot \sin(j\omega t + \psi_j) \]  \hspace{1cm} (2)

where the coefficients $K_0 F$, $K_j F$ are expressed as

\[
\begin{align*}
K_0 F &= \frac{\omega}{2\pi Q_0} \int_0^{2\pi/\omega} |q|qdt \\
K_j F &= \frac{\omega}{\pi Q_j} \int_0^{2\pi/\omega} |q|q \cdot \sin(j\omega t + \psi_j)dt \\
& \hspace{1cm} (j=1,2,\ldots) 
\end{align*}
\]  \hspace{1cm} (3)

In case where $Q_0$ vanishes, $K_0 F$ is determined as a limiting value when $Q_0$ tends to zero.

Eq. (2) shows a linear relationship between the nonlinear damping function $|q|q$ and frequency components contained in $q$. It is, then, possible to calculate pressure pulsations in frequency domain for each frequency component required.

It happens, however, that some of the coefficients $K_j F$ are negative for certain values of $Q_0$, $Q_j$, and $\psi_j$. Fig. 1 shows an example of such situations, where $K_1 F$, $K_2 F$ are plotted versus $Q_2$, as $Q_0$ being parameters, and letting $Q_1=1$, $\psi_1=0$, $\psi_2=\pi/2$. As seen from Fig. 1, $K_1 F$ is positive, but $K_2 F$ becomes negative for smaller values of $Q_2$. By the way, $K_1 F$ has been positive for all other parameters employed for calculations.

Negative $K_j F$ produces unreasonable negative damping for the $j$th component and makes it unstable. This fact shows that Fourier development method fails to give the correct linearization of the function $|q|q$ in case of non sinusoidal flow $q$ and is not applicable to the present problem.

EQUIVALENT LINEARIZATION OF $|q|q$, NOT PRODUCING NEGATIVE DAMPING

For the equivalent linearization of the function $|q|q$ under the condition that $q$ is periodic in time, but not sinusoidal, it seems not to be correct to determine linearization coefficients for all components contained in $q$, as done in Fourier development method.

After considerations to solve the difficulties, the authors have come to the idea that, for the correct linearization of the function $|q|q$ to be done under the present condition, the flow $q$ should be expressed as the sum of the steady component $Q_0$ and the fluctuating
one $\mathcal{Q}$, and two coefficients, $K_0$ for $Q_0$ and $K_1$ for $\mathcal{Q}$, should be determined, in such a way as written in Eqs. (4) and (5),

$$\mathcal{Q} = Q_0 + \mathcal{q}$$ ..............................(4)

$$|\mathcal{q}| q = K_0 Q_0 + K_1 \mathcal{q}$$ ..............................(5)

where the zero mean value is assumed for $\mathcal{q}$.

Based on this idea, three schemes are introduced for the equivalent linearization of the function $|\mathcal{q}| q$, not producing unreasonable negative damping.

**Scheme 1 (Single Sinusoidal Approximation)** In this scheme the fluctuating component $\mathcal{q}$ in each pipe element is approximated by a single sinusoidal wave with double amplitude $2Q_{pp}$ equal to the difference between the maximum value of $\mathcal{q}$ and the minimum during one period, and then $K_0$ and $K_1$ are obtained by Fourier development method. The results are already published and $K_0$ and $K_1$ of this scheme are plotted in Fig.2 by broken lines versus $\beta (=Q_0/Q_{pp})$.

**Scheme 2 (Mean Derivative Approximation)** It is well known that flow oscillations in pipe are simulated by electric lattice circuits. An example of a straight pipe with an input flow $q_{in}$ at one end is shown in Fig.3, where the pipe is divided into $N$ pipe elements with the same length $\ell$, $m$ and $c$ are equivalent inductance and capacitance of the pipe element, determined by its dimensions and fluid properties. $q_i q_{il}$ represent nonlinear resistance in the ith pipe element, and $a_{ex} |q_{ex}|$ is the exit nonlinear resistance determined by the boundary condition.

Now, regarding $q_i$ as a current through the ith pipe element and $p_i$ as a voltage across the ith capacitance, we have the following relations

$$m \dot{q}_i = p_{i-1} - p_i - a_i q_i |q_i|, \quad (i=1,2,\ldots,N)$$

$$c \dot{p}_i = q_i - q_{i+1}, \quad (i=1,2,\ldots,N-1)$$

$$c \dot{p}_0 = 2(q_{in} - q_1), \quad c \dot{p}_N = 2(q_N - q_{ex})$$ ..............................(6)

where $\cdot$ denotes the derivative with respect to time.

Elimination of $p_{i-1}$ and $p_i$ from Eqs.(6) yields Eq.(7),

$$m \ddot{q}_i + 2a_i q_i \dot{q}_i + 2q_i/c = q_{i-1}/c + q_{i+1}/c \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdOTS
\documentclass{article}
\usepackage{amsmath}
\begin{document}
\begin{equation}
\begin{pmatrix}
\mathcal{Q} = Q_0 + \mathcal{q} \\
\mathcal{q} = K_0 Q_0 + K_1 \mathcal{q}
\end{pmatrix}
\end{equation}

\end{document}
for \( q_i \). Though the absolute value \(|q_i|\) is not constant, it is replaced, in the second approximation, by its mean value during one period. Then, the equivalent linear coefficient \( K_1 \) is determined for each pipe element as, dropping the suffix \( i \),

\[
K_1 = \frac{2\pi/\omega}{1} \int_0^{2\pi/\omega} |q| \, dt
\]

(8)

\( K_0 \) or \( K_0Q_0 \) is determined from the steady component of \(|q|q\), as seen from Eq.(5).

When the fluctuating component \( q \) is sinusoidal, the integration in Eq.(8) is easily evaluated and \( K_1 \) is expressed as, putting \( \beta = Q_0/Q_1 \),

\[
K_1/Q_1 = \begin{cases} 
(4/\pi)(\beta \sin^{-1} \beta + \sqrt{1-\beta^2}) & \text{for } 0 \leq \beta < 1 \\
2\beta & \text{for } \beta \geq 1
\end{cases}
\]

(9)

The values of \( K_1/Q_1 \) are plotted versus \( \beta = Q_0/Q_1 \) by a solid line in Fig.2. \( K_0/Q_1 \) is the same as \( K_0/Q_{pp} \) in the first scheme.

**Scheme 3 (Least Square Method)** The right-hand side of Eq.(5) is, in general, not an exact expression for \(|q|q\). So, there exists an error \( \varepsilon \), as expressed in Eq.(10),

\[
\varepsilon = |q|q - K_0Q_0 - K_1\bar{q}
\]

(10)

In the third scheme the coefficients \( K_0 \) and \( K_1 \) are determined, so as to minimize the integration of square of the error over one period. This is satisfied by the following relations,

\[
\int_0^{2\pi/\omega} \varepsilon^2 \, dt = 0, \quad \int_0^{2\pi/\omega} \frac{\partial \varepsilon}{\partial K_0} \, dt = 0, \quad \int_0^{2\pi/\omega} \frac{\partial \varepsilon}{\partial K_1} \, dt = 0
\]

(11)

Then, \( K_0 \) and \( K_1 \) in this scheme are expressed as,

\[
K_0 = \frac{\omega}{2\pi Q_0} \int_0^{2\pi/\omega} |q|q \, dt
\]

(12)

\[
K_1 = \int_0^{2\pi/\omega} |q|q \, \bar{q} \, dt / \int_0^{2\pi/\omega} \bar{q}^2 \, dt
\]

\( K_0 \) in Eq.(12) is the same as \( K_{0F} \) in Eq.(3).

When flow \( q \) consists of the steady component \( Q_0 \) and a sinusoidal wave \( Q_1 \sin(\omega t + \psi_1) \), \( K_0 \) and \( K_1 \) in Eq.(12) coincide with the results in reference (5), which are shown in Fig.2 by broken lines in the form of \( K_0/Q_1 \) and \( K_1/Q_1 \).
CALCULATION PROCEDURES

Applying assumed input flows with higher harmonics up to the n-th order, as expressed in Eq.(13), to one end of a simple pipe shown in Fig.3,

\[ q_{in} = Q_{in0} + \sum_{j=1}^{n} Q_{ijn} \cdot \sin(j\omega t + \psi_{ijn}) \]  \hspace{1cm} (13)

pressure pulsations are calculated under the boundary condition

\[ p_N = \alpha_{ex} |q_{ex}| q_{ex} \]  \hspace{1cm} (14)

The results are compared with those calculated by RKG integration method. The calculation procedures by means of the above linearization schemes are as follows.

Step 1 Write the input flow \( q_{in} \), the flow \( q_i \), and the pressure \( p_i \) in complex form as,

\[ q_{in} = Q_{in0} + \sum_{j=1}^{n} Q_{ijn} \cdot \exp(j(j\omega t + \psi_{ijn})) \]  \hspace{1cm} (15)
\[ q_i = Q_{i0} + \sum_{j=1}^{n} Q_{ij} \cdot \exp(j(j\omega t + \psi_{ij})) \]  \hspace{1cm} (15)
\[ p_i = P_{i0} + \sum_{j=1}^{n} P_{ij} \cdot \exp(j(j\omega t + \psi_{ij})) \]  \hspace{1cm} (15)

where \( Q_{ij} \) and \( P_{ij} \) are complex. Linearize \( |q_i|q_i \) and \( |q_{ex}|q_{ex} \) as shown in Eq.(5),

\[ |q_i|q_i = K_{i0}Q_{i0} + K_{ij} \sum_{j=1}^{n} Q_{ij} \cdot \exp(j(j\omega t + \psi_{ij})) \]  \hspace{1cm} (15)
\[ |q_{ex}|q_{ex} = K_{ex0}Q_{ex0} + K_{ex} \sum_{j=1}^{n} Q_{exj} \cdot \exp(j(j\omega t + \psi_{exj})) \]  \hspace{1cm} (15)

Then, we have the following relations for steady components and j-th harmonics, respectively.

\[ P_{i0} = P_{i-10} - \alpha K_{i0}Q_{i0}, \quad (i=1,2,\ldots,N) \] \hspace{1cm} (17)
\[ Q_{i0} = Q_{in0}, \quad (i=1,2,\ldots,N), \quad Q_{ex0} = Q_{in0} \] \hspace{1cm} (17)
\[ P_{ij} = P_{i-1j} - (gj\omega m + \alpha K_{ij})Q_{ij}, \quad (i=1,2,\ldots,N) \] \hspace{1cm} (18)
\[ Q_{ij} = Q_{ij} - gj\omega cP_{ij}/2 \] \hspace{1cm} (18)
\[ Q_{i+1j} = Q_{ij} - gj\omega cP_{ij}, \quad (i=1,2,\ldots,N-1) \] \hspace{1cm} (18)
\[ Q_{exj} = Q_{Nj} - gj\omega cP_{Nj}/2 \] \hspace{1cm} (18)
The boundary condition is also linearized as,
\[
P_{NO} = a_{ex} e^{K_{ex} t} 0_{ex} 0 \quad \{ \quad \}
\]
\[
P_{Nj} = a_{ex} e^{K_{ex} t} 0_{ex} j \quad \{ \quad \}
\]
\[ \text{Step 2 Assume } P_{00} \text{ and } P_{0j}. \text{ Then, first of all, } Q_{1j} \text{ are determined from the second equation of Eqs.}(18), \text{ because } Q_{inj} \text{ is known, and the response curve of } q_1 \text{ in time domain is easily obtained, taking the absolute value and the phase angle of } Q_{1j} \text{ into consideration and summing up to the } n \text{th order together with } Q_{10}. \text{ So, the linearization coefficients } K_{10} \text{ and } K_{11} \text{ for the first pipe element are determined using one of the three linearization schemes described above, by which we get } P_{10} \text{ from the first equation of Eqs.}(17) \text{ and } P_{1j} \text{ from the first one of Eqs.}(18), \text{ respectively. The relations between } (P_{1j}, Q_{1j}) \text{ and } (P_{0j}, Q_{inj}) \text{ are expressed in matrix form as,}
\]
\[
\begin{bmatrix}
P_{1j} \\
Q_{1j}
\end{bmatrix} =
\begin{bmatrix}
1 - (j\omega)^2 c/2 + j\omega K_{11} c/2 & -\alpha K_{11} - j\omega m \\
-j\omega c/2 & 1
\end{bmatrix}
\begin{bmatrix}
P_{0j} \\
Q_{inj}
\end{bmatrix}
\]
\[ \text{(20)} \]
\[ \text{As } P_{1j} \text{ and } Q_{1j} \text{ are now known, } Q_{2j} \text{ are known from the third equation of Eqs.}(18) \text{ and the response curve of } q_2 \text{ in time domain is obtained in the same way as stated above, from which the linearization coefficients } K_{20} \text{ and } K_{21} \text{ for the second pipe element are determined using one of the three linearization schemes. Then, } P_{20} \text{ and } P_{2j} \text{ are determined from the first equation of Eqs.}(17) \text{ and the first one of Eqs.}(18), \text{ respectively. This procedure continues to the } N \text{th pipe element and the following relations are obtained.}
\]
\[
\begin{bmatrix}
P_{i+1j} \\
Q_{i+1j}
\end{bmatrix} =
\begin{bmatrix}
1 - (j\omega)^2 c + j\omega K_{11} c & -\alpha K_{11} - j\omega m \\
-j\omega c & 1
\end{bmatrix}
\begin{bmatrix}
P_{ij} \\
Q_{ij}
\end{bmatrix}
\]
\[ \text{(21)} \quad (i=1,2,\ldots,N-1) \]
\[ \text{Finally, } Q_{exj} \text{ are obtained by } P_{Nj} \text{ and } Q_{Nj}. \text{ This is expressed in matrix form as}
\]
\[
\begin{bmatrix}
P_{Nj} \\
Q_{exj}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
-j\omega c/2 & 1
\end{bmatrix}
\begin{bmatrix}
P_{Nj} \\
Q_{Nj}
\end{bmatrix}
\]
\[ \text{(22)} \]
\[ \text{Step 3 From the matrix equations (20), (21), and (22), the following expression is obtained, multiplying successively,}
\]
\[
\begin{bmatrix}
P_{Nj} \\
Q_{exj}
\end{bmatrix} =
\begin{bmatrix}
T_{11j} & T_{12j} \\
T_{21j} & T_{22j}
\end{bmatrix}
\begin{bmatrix}
P_{0j} \\
Q_{inj}
\end{bmatrix}
\]
\[ \text{(23)} \]
The steady component \( P_{NO} \) is written as,

\[
P_{NO} = P_{00} - \left( \sum_{i=1}^{N} \alpha K_{i0} \right) Q_{in0} \quad \cdots \cdots \cdots \cdots (24)
\]

\( P_{NO}, P_{NJ}, \) and \( Q_{exj} \) must satisfy the boundary conditions expressed in Eqs.(19). However, \( P_{NO}, P_{NJ}, \) and \( Q_{exj} \) just obtained by the assumed \( P_{00} \) and \( P_{0j} \), in general, cannot satisfy the boundary conditions in Eqs.(19). Then, by equating \( P_{NO} \) in Eqs.(19) to that in Eq.(24), a renewed \( P_{00} \) is obtained as,

\[
P_{00} = \left( \alpha_{ex} K_{ex0} + \sum_{i=1}^{N} \alpha K_{i0} \right) Q_{in0} \quad \cdots \cdots \cdots \cdots (25)
\]

Renewed \( P_{0j} \) are obtained, by equating \( P_{NJ} \) in Eqs.(19) to those in Eqs.(23), as,

\[
P_{0j} = -\frac{T_{12j} - \alpha_{ex} K_{exj} T_{22j}}{T_{11j} - \alpha_{ex} K_{exj} T_{21j}} Q_{inj} \quad \cdots \cdots \cdots \cdots (26)
\]

Using these renewed \( P_{00} \) and \( P_{0j} \), the calculation procedures after Step 2 are iterated until \( P_{00} \) and \( P_{0j} \) converge within 1% relative error, and \( K_{i0}, K_{i1}, Q_{ij}, \) and \( P_{ij} \) are fixed.

Then, the time history of the inlet pressure pulsation \( p_{0} \) over one period is obtained, by summing up \( P_{00} \) and \( P_{0j} \) in the form

\[
p_{0} = P_{00} + \sum_{j=1}^{\infty} \left| P_{0j} \right| \sin(j\omega t + \psi_{inj} + \theta_{0j}) \right) \quad \cdots \cdots \cdots \cdots (27)
\]

\[
\theta_{0j} = \tan^{-1}\left\{ \delta m(P_{0j})/\Re(P_{0j}) \right\}
\]

RESULTS OF CALCULATIONS

The wave forms of inlet pressure pulsation \( p_{0} \) over one period calculated by each linearization scheme and RKG method are shown in Figs.4, 5 and 6, where broken lines represent the results obtained by RKG method, and solid lines correspond to those by scheme 1, 2, and 3 from above, respectively. In the calculations the number of division \( N=10 \) is chosen. The values of equivalent inductance \( m \) and capacitance \( c \) are set equal to \( 1/N \). The constant \( \alpha =0.002 \) is chosen and the constant \( \alpha_{ex} \) is chosen as \( \alpha_{ex}=0.05 \) for positive \( q_{ex} \) and \( \alpha_{ex}=0.075 \) for negative \( q_{ex} \). Input flows used for calculations are listed in Tables 1, 2, and 3, together with the fundamental forcing angular frequency \( \omega \).

The steady components \( P_{00} \) and the amplitudes \( \left| P_{0j} \right| \) of jth harmonics obtained by each calculation method are also tabulated in Tables 1, 2, and 3, respectively. Those values by RKG method are obtained by Fourier
development of the calculated time history of pressure $p_0$. In the calculation by RKG method, the time interval is chosen as $1/(16 \times N)$ of one period of the highest harmonic contained in the input flow. $K_{10}$ and $K_{11}$ are calculated by Simpson's integration formula, using the time interval of $1/16$ of the one period of the highest harmonic contained in the input flow.

It is found from the results shown in Figs. 4, 5, and 6, and tables 1, 2, and 3 that the pressure pulsations calculated by the proposed equivalent linearization schemes for $|q|_q$ are all in good agreement with those by RKG method. This shows that authors' idea for the equivalent linearization of $|q|_q$ in case of non-sinusoidal flows and the calculation procedures are reasonable.

By the way, a personal computer is used for the above calculations. The computer times in calculations by the three linearization schemes, on an average, are reduced to $1/30$ to $1/10$ of those by RKG method.

CONCLUDING REMARKS

To calculate pressure pulsations in pipelines caused by non-sinusoidal, periodic input flows in frequency domain by transfer matrix method, it is necessary to linearize the nonlinear damping function $|q|_q$. It is found that in some cases the linearization by Fourier development method produces negative coefficients for higher harmonics, leading to negative damping.

The idea is proposed that for the equivalent linearization of $|q|_q$ in case of non-sinusoidal input flow, two coefficients should be determined, one for the steady component and the other for the fluctuating one. To verify this idea, three linearization schemes based on this idea are introduced and, applying input flows with several harmonics, pressure pulsations are calculated for a simple pipe.

As the results, it is found that the wave forms calculated by three schemes are all in good agreement with those by RKG method, which shows the authors' idea for the equivalent linearization of $|q|_q$ in case of non-sinusoidal flow input and calculation procedures to be reasonable.

REFERENCE


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678.

Fig. 1 The coefficients $K_1F$ and $K_2F$ calculated by Eq. (3). $Q_1=1$, $\psi_1=0$, and $\psi_2=\pi/2$. 953
Fig. 2 Equivalent linearization coefficients, in case where flow \( q \) consists of the steady component and a single sinusoidal wave.

Fig. 3 Equivalent lattice circuit of a straight pipe.
Fig. 4 Calculated wave forms of the inlet pressure pulsation $p_0$ caused by $q_{in}$ given in Table 1.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Input Flow $q_{in}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.314</td>
<td>$q_{in} = 1 + \sin(4\omega t) + 0.5\sin(5\omega t) + 1.5\sin(6\omega t)$</td>
</tr>
</tbody>
</table>

| Method      | $|P_{00}|$ | $|P_{04}|$ | $|P_{05}|$ | $|P_{06}|$ |
|-------------|--------|--------|--------|--------|
| RKG         | 0.25   | 2.14   | 1.13   | 3.12   |
| Scheme 1    | 0.60   | 2.10   | 1.37   | 3.05   |
| Scheme 2    | 0.25   | 2.20   | 1.50   | 3.19   |
| Scheme 3    | 0.25   | 2.23   | 1.55   | 3.25   |

Table 1 Fundamental forcing angular frequency $\omega$, input flow $q_{in}$, and the steady component $P_{00}$ and the amplitudes of $j$th harmonic $|P_{0j}|$.  

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Fig. 5 Calculated wave forms of the inlet pressure pulsation $p_0$ caused by $q_{in}$ given in Table 2.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Input Flow $q_{in}$</th>
<th>Responses of jth harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.47</td>
<td>$q_{in} = 2\sin(9\omega t) + 1.5\sin(10\omega t) + \sin(11\omega t)$</td>
<td></td>
</tr>
</tbody>
</table>

| Method   | $P_{00}$ | $|P_{09}|$ | $|P_{010}|$ | $|P_{011}|$ |
|----------|----------|----------|-----------|-----------|
| RKG      | -0.15    | 3.28     | 3.56      | 1.42      |
| Scheme 1 | 0.0      | 3.15     | 3.33      | 1.44      |
| Scheme 2 | -0.18    | 3.34     | 3.85      | 1.51      |
| Scheme 3 | -0.17    | 3.29     | 3.70      | 1.50      |

Table 2 Fundamental forcing angular frequency $\omega$, input flow $q_{in}$, and the steady component $P_{00}$ and the amplitudes of jth harmonic $|P_{0j}|$. 
Fig. 6 Calculated wave forms of the inlet pressure pulsation $P_0$ caused by $q_{in}$ given in Table 3.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Input Flow $q_{in}$</th>
<th>Responses of jth harmonics</th>
</tr>
</thead>
</table>
| 1.57     | $q_{in} = 3 + \sin(\omega t) + 0.333\sin(3\omega t) + 0.2\sin(5\omega t) + 0.147\sin(7\omega t)$ | Method | $P_{00}$ | $|P_{01}|$ | $|P_{03}|$ | $|P_{05}|$ | $|P_{07}|$
|          |                     | RKG | 0.90    | 2.80   | 0.95   | 0.52   | 0.22   |
|          |                     | Scheme 1 | 0.88    | 2.85   | 0.98   | 0.55   | 0.22   |
|          |                     | Scheme 2 | 0.88    | 2.85   | 0.98   | 0.55   | 0.22   |
|          |                     | Scheme 3 | 0.90    | 2.85   | 0.98   | 0.55   | 0.22   |

Table 3 Fundamental forcing angular frequency $\omega$, input flow $q_{in}$, and the steady component $P_{00}$ and the amplitudes of jth harmonic $|P_{0j}|$. 

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