ADIABATIC SHEARING OF INCOMPRESSIBLE NON-NEWTONIAN FLUIDS

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Abstract

We consider the analytic and numerical behavior of adiabatic shearing of incompressible non-Newtonian fluids. In this paper we investigatte the stability of the rigid motion under constant and oscillatory velocities and show that the rigid motion is independent of initial data.

1. INTRODUCTION

We consider an incompressible non-newtonian liquid with temperature dependent viscosity in an adiabatic shearing flow caused by a time dependent inertial force $f(t)$ or a steady boundary velocity $\bar{v}$. The fluid is confined in the strip between the planes $x = 0$ and $x = 1$. The body force, the flow and the velocity $v(x, t)$ are in the direction of the axis, y perpendicular to x.

Assuming the fluid has unit density and unit specific heat, the balance laws of momentum and energy read

$$v_t(x, t) = \sigma_x(x, t) + f(t), \tag{1.1}$$

or

$$v_t(x, t) = \sigma_x(x, t), \tag{1.1a}$$

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and

$$\theta_t(x,t) = \sigma(x,t)\, v_x(x,t), \tag{1.2}$$

where $\theta(x,t)$ the temperature and $\sigma(x,t)$ the shear stress, given by

$$\sigma(x,t) = \theta^{-\alpha}(x,t)\,|v_x|^{n-1}\, v_x, \quad 0 < \alpha < n \le 1. \tag{1.3}$$

The same equations are appropriate for a solid in the plastic region exhibiting thermal softening and strain rate sensitivity, but no strain hardening.

We specify initial conditions

$$v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \quad 0 \le x \le 1 \tag{1.4}$$

and consider the following two types of boundary conditions:

$$\sigma(0,t) = \sigma(1,t) = 0 \tag{1.5}$$

or

$$\sigma(0,t) = 0, \quad v(1,t) = \bar{v}, \tag{1.6}$$

for $t \in [0,\infty)$. The first correspond to the frictionless shearing and the second to the shearing under steady boundary velocity.

In this paper we are interested whether the dependence of viscosity in liquids (or the existence of thermal softening in solids) may assure the asymptotic stability of solution for the systems of equations (1.1), (1.2), (1.3), (1.4) and (1.5) or (1.1α), (1.2), (1.3), (1.4) and (1.6). Our objective is to investigate whether

(i) the rigid "oscillatory" motion

$$v_1(x,t) = v_1 + \int_0^t f(\tau)d\tau, \tag{1.7}$$

$$\theta_1(x,t) = \theta_1(x), \tag{1.8}$$

is asymptotically stable for (1.1), (1.2), (1.3), (1.4) and (1.5), i.e. whether the solution ($v(x,t)$, $\theta(x,t)$) of (1.1), (1.2), (1.3), (1.4) and (1.5) has, as $t \to \infty$, the asymptotic form (1.7) and (1.8) independently of the initial data and

(ii) the rigid motion under constant velocity

$$v_2(x,t) = \bar{v}, \tag{1.9}$$

$$\theta_2(x,t) = \theta_2(x), \tag{1.10}$$

is asymptotically stable for (1.1α), (1.2), (1.3), (1.4) and (1.6), independently of initial data.

Throughout this paper we assume that ($v(x,t)$, $\theta(x,t)$) is a solution of (1.1), (1.2), (1.3), (1.4) and (1.5) (or (1.1α), (1.2), (1.3), (1.4) and (1.6)) on $[0,1] \times [0,\infty)$ such that $v(.,t)$, $v_x(.,t)$, $v_t(.,t)$, $v_{xx}(.,t)$, $\theta(.,t)$, $\theta_x(.,t)$ are all in $C^0([0,\infty);\ L^2(0,1))$, $v_{xt}(.,t)$ is in $C^0((0,\infty);\ L^2(0,1))$ and $v_{tt}(.,t)$ is in $L^2_{loc}((0,\infty);\ L^2(0,1))$. K will denote a generic constant which can be estimated in terms of α, n, $v_0(x)$ and $\theta_0(x)$.

The physical situation is this: as the material is being sheared, energy is pumped into the system. Since the process is adiabatic, temperature will keep rising and will tend to infinity with time. Non uniform heating tends to create localization, thus, destabilization of the system. However, it turns out that the temperature tends to a constant in time function of x, uniformly in x on $[0,1]$ and the two types of flow, cited above, are asymptotically stable.

The problem of shearing of a newtonian fluid under steady boundary velocities or an "oscillatory" inertial force was investigated in [1,2] and analogous asymptotic stability results were established there. The problem of shearing of a fluid of this type under steady boundary velocities was studied in [3] and the convergence to the uniform shearing was established there.

2. FRICTIONLESS SHEARING UNDER "OSCILLATORY" INERTIAL FORCE

Theorem 1 Assume $v_0(x) \in W^{2,2}(0,1)$, $\theta_0(x) \in W^{1,2}(0,1)$, $\theta_0(x) > 0$, $0 \le x \le 1$ and $f(t)$ is oscillatory, in the sense

$$\left|\int_0^t f(\tau)d\tau\right| \le K. \tag{1.11}$$

Then, there exists a unique classical solution of (1.1), (1.2), (1.3), (1.4) and (1.5) and, as $t \to \infty$,

$$v(x,t) = \int_0^t f(\tau)d\tau + \int_0^1 v_0(x)dx + 0(e^{-kt}), \tag{1.12}$$

$$\sigma(x,t) = 0(e^{-kt}), \tag{1.13}$$

$$\theta(x,t) \to \theta_1(x) > 0, \tag{1.14}$$

$$|\theta_x(x,t)| \le K, \tag{1.15}$$

uniformly in x on [0,1].

The proofs of these theorems, presented in section 2, are based on a priori estimates for solutions of (1.1)–(1.5) of (1.1α)–(1.6) under the assumed properties of $f(t)$, α and n. Essentially, we show that viscosity (or thermal softening of solids) in the momentum balance equation (1.1) or (1.1α) wins over the destabilizing effect of stress power in the energy balance equation (1.2) and enforces the decay of the velocity gradient to zero.

Lemma 2.1 Under conditions corresponding to Theorem 1,

$$\int_0^t \int_0^1 \sigma_x^2(x,\tau)dx \; d\tau \le K, \tag{2.1}$$

$$\int_0^1 \sigma^2(x,\tau)d\tau \le K, \tag{2.2}$$

for $x \in [0,1]$, $t \in [0,\infty)$.

Proof: We multiply (1.1) by σ_x, we integrate over $(0,1)\times(0,t)$, we integrate by parts with respect to x, we use (1.3), (1.4) and (1.5) and then we integrate by parts with respect to t and use (1.2) to deduce

$$\int_0^t\int_0^1 \sigma_x^2\, dxd\tau + \frac{1}{n+1}\int_0^1 \theta^{-\alpha}|v_x|^{n+1}\,dx + \frac{\alpha}{n+1}\int_0^t\int_0^1 \theta^{-2\alpha-1}|v_x|^{2(n+1)}dxd\tau$$
$$= \int_0^1 \theta^{-\alpha}|v_{0x}|^{n+1}\,dx. \tag{2.3}$$

Hence, (2.1) follows immediately from (2.3). In view of (1.5), (2.2) follows from (2.1) and Schwarz's inequality.

Lemma 2.2 Under conditions corresponding to Theorem 1,

$$\theta_0(x) \le \theta(x,t) \le K. \tag{2.4}$$

Proof: Using (1.3) and (1.2),

$$\theta^{\frac{n-\alpha}{n}}(x,t) = \theta_0^{\frac{n-\alpha}{n}}(x) + \frac{n-\alpha}{n}\int_0^t |\sigma(x,\tau)|^{\frac{1+n}{n}}\,d\tau. \tag{2.5}$$

Next, we multiply (1.1) by $|\alpha|^{\frac{1-n}{n}}\sigma_x$ and perform the same steps as for (2.3) to obtain

$$\int_0^t\int_0^1 |\sigma|^{\frac{1-n}{n}}\sigma_x^2\,dxd\tau + \frac{n}{2}\int_0^1 \theta^{-\frac{\alpha}{n}} v_x^2\,dx +$$
$$\frac{\alpha}{2}\int_0^t\int_0^1 \theta^{-\frac{\alpha}{n}-\alpha-1}|v_x|^{n+3}\,dxd\tau = \tag{2.6}$$
$$\frac{n}{2}\int_0^1 \theta_0^{-\frac{\alpha}{n}} v_{0x}^2\,dx.$$

Using

$$|\sigma(x,t)|^{\frac{1+n}{n}} = \int_1^x |\sigma(\xi,t)|^{\frac{1}{n}}(sgn\,\sigma)\sigma_x\,d\xi. \tag{2.7}$$

Schwarz's inequality and $|\sigma|^{\frac{1}{n}} = |\sigma|^{\frac{1-n}{2n}}\ |\sigma|^{\frac{1+n}{2n}}$, we obtain

$$\int_0^1 |\sigma|^{\frac{1+n}{n}}\ dx \le K \int_0^1 |\sigma|^{\frac{1-n}{n}}\ \sigma_x^2 dx + \frac{1}{K}\int_0^1 |\sigma|^{\frac{1+n}{n}}\ dx, \tag{2.8}$$

whence

$$\int_0^1 |\sigma|^{\frac{1+n}{n}}\ dx \le K \int_0^1 |\sigma|^{\frac{1-n}{n}}\ \sigma_x^2 dx. \tag{2.9}$$

Hence, using (2.7), Schwarz's inequality and (2.9), integrating over $(,t)$ and taking on account (2.6),

$$\int_0^t |\sigma|^{\frac{1+n}{n}}\ d\tau \le K. \tag{2.10}$$

Therefore, (2.4) follows directly from (2.5) and (2.10).

Lemma 2.3 Under conditions corresponding to Theorem 1,

$$\int_0^1 \sigma^2(x,t)dx \le K. \tag{2.11}$$

Proof: Using (1.3) and (2.4),

$$\sigma^2 = \theta^{-2\alpha}\ |v_x|^{2n} \le \begin{cases} \dfrac{1}{\min\limits_{x\in[0,1]} \theta^{\alpha}}\ \theta^{-\alpha}\ |v_x|^{n-1}, & \text{if } |v_x| > 1 \\ \dfrac{1}{K}, & \text{if } |v_x| \le 1. \end{cases} \tag{2.12}$$

Therefore, using (2.12) and (2.3) we obtain (2.11).

Lemma 2.4 Under conditions corresponding to Theorem 1,

$$\int_0^1 \sigma_x^2 (x,t)dx < Ke^{-kt}. \tag{2.13}$$

Proof: Multiplying (1.1) by σ_{xt}, integrating with respect to x over (0,1) and taking on account (1.3), (1.4) and (1.5) we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \sigma_x^2 dx + n\int_0^1 \theta^{-\alpha}\,|v_x|^{n-1}\,v_{xt}^2 dx = \alpha\int_0^1 \theta^{-2\alpha-1}\,|v_x|^{2n}\,v_x\,v_{xt}\,dx, \tag{2.14}$$

whence, by Cauchy inequality and (1.3),

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int_0^1 \sigma_x^2 dx + \frac{n}{K}\int 0^1\,\theta^{-\alpha}\,|v_x|^{n-1}\,v_{xt}^2 dx \le K\int_0^1 \theta^{-3\alpha-2}|v_x|^{3(n+1)}dx = \\ K\int_0^1 \theta^{\frac{3\alpha-2n}{n}}\,|\sigma|^{\frac{3(n+1)}{n}}\,dx.\end{gathered} \tag{2.15}$$

Integrating (2.15) over t and using (2.6), $\alpha < n$ and (2.4),

$$\begin{gathered}\frac{1}{2}\int_0^1 \sigma_x^2 dx \le K\int_0^t\int_0^1 \theta^{-\frac{\alpha}{n}-\alpha-1}\,|v_x|^{3(n+1)}\,\min_{x\in[0,1]}{}^{-(2\alpha+1-\frac{\alpha}{n})}\,dx + \\ \frac{1}{2}\int_0^1 \sigma_{0x}^2 dx \le K.\end{gathered} \tag{2.16}$$

Hence

$$\int_0^1 \sigma_x^2(x,t)dx \le K, \quad t\in[0,\infty). \tag{2.17}$$

Now, combining (2.15), (2.17), (2.4),

$$\sigma^2 \le \int_0^1 \sigma_x^2\,dx, \tag{2.18}$$

$$\sigma_x^2 \le \int_0^1 \sigma_{xx}^2 dx = \int_0^1 v_{xt}^2 dt, \tag{2.19}$$

$n \le 1$ and

$$|v_x| \leq \frac{1}{K}\, |\sigma|^{\frac{1}{n}}, \tag{2.20}$$

we obtain the differential inequality

$$\frac{d\psi(t)}{dt} + K\ \psi(t) \leq K\ \psi^{\frac{3(n-1)}{2n}}(t), \tag{2.21}$$

where

$$\psi(t) = \int_0^1 \sigma_x^2(x, t)dx. \tag{2.22}$$

We note that, by account of (2.1) and (2.17),

$$\int_0^t \psi(\tau)d\tau \leq K, \tag{2.23}$$

and

$$\psi(t) \leq K. \tag{2.24}$$

By the procedure used in [2], we can prove (2.13).

We are now ready to establish (1.12), (1.13), (1.14) and (1.15). Using (2.18) and (2.13), we obtain (1.13). On account of (1.3), (1.13) and (2.4),

$$v_x^2(x, t) < Ke^{-kt}. \tag{2.25}$$

Hence, as $t \to \infty$,

$$v(x, t) = v_\infty(t) + 0(e^{-kt}). \tag{2.26}$$

Integrating (1.1) over $x = [0, 1]$ and using (1.5), (1.4) and (2.26) we arrive at (1.12). Next, differentiating (2.5) with respect to x, we obtain

$$(\frac{n-\alpha}{n})\,\theta^{-\frac{\alpha}{n}}\,\theta_x = (\frac{n-\alpha}{n})\,\theta_0^{-\frac{\alpha}{n}}\,\theta_{0x} + \frac{(n-\alpha)(1+n)}{n^2}\int_0^t |\sigma|^{\frac{1}{n}}\,(sgn\,\sigma)\sigma_x\,d\tau, \qquad (2.27)$$

whence, using (2.4), (1.13), Schwarz's inequality,

$$\sigma_x^2(x,t) = \int_0^1 \sigma_x^2(x,t)dx + 2\int_0^1\int_y^x \sigma_x(\xi,t)\sigma_{xx}(\xi,t)d\xi dy,$$

(2.13) and (2.15), we obtain (1.15). Since, for every $0 \le x \le 1$, $\theta(x,t)$ is an increasing function of t, $\theta(x,t)$ converges as $t \to \infty$, to $\theta_1(x) < \infty$. Furthermore, by (1.15) and the Arzela-Ascoli theorem, the convergence is uniform in x. This proves (1.4).

3. SHEARING UNDER STEADY BOUNDARY VELOCITY

Theorem 2 Assume $v_0(x) \in W^{2,2}(0,1)$, $\theta_0(x) \in W^{1,2}(0,1)$, $\theta_0(x) > 0$, $0 \le x \le 1$. Then, there exists a unique classical solution of (1.1α), (1.2), (1.3), (1.4) and (1.6) and, as $t \to \infty$,

$$v(x,t) = \bar{v} + 0(e^{-kt}), \qquad (1.16)$$

$$\sigma(x,t) = 0(e^{-kt}), \qquad (1.17)$$

$$\theta(x,t) \to \theta_2(x) > 0, \qquad (1.18)$$

$$|\sigma_x(x,t)| \le K, \qquad (1.19)$$

uniformly in x on [0,1].

Lemma 3.1 Under conditions corresponding to Theorem 2 we obtain (2.1) and (2.2).

Proof: We multiply (1.1α) by σ_x and perform the same steps as in Lemma 2.1 to obtain again (2.3).

Lemma 3.2 Under conditions corresponding to Theorem 2, (2.4) holds.

Proof: We multiply (1.1α) by $|\sigma|^{\frac{1-n}{n}}\,\sigma_x$ and perform the same steps as in Lemma 2.5 to obtain again (2.6). The proof follows the same lines as in Lemma 2.2, since $\sigma(0,t)=0$ in both cases.

Following exactly the same steps as in the case of shearing under oscillatory inertial force, we obtain the estimates (1.16), (1.17), (1.18) and (1.19).

We have thus, established a priori the estimates in our Theorems. The proof that a solution exists can be obtained by a routine procedure of mathematical analysis [1,4,5].

4. NUMERICAL RESULTS

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