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USING FIBONACCI REPRESENTATIONS

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Robust Transmission of Unbounded Strings
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Abstract. Families of Fibonacci codes and Fibonacci representations are defined. Their main attributes are: (i) robustness, manifesting itself by the local containment of errors; (ii) simple encoding and decoding. The main application explored is the transmission of binary strings whose length is in an unknown range, using robust Fibonacci representations instead of the conventional error-sensitive logarithmic ramp representation. Though the former is asymptotically longer than the latter, the former is actually shorter for very large initial segments of integers.

Key words and phrases: Fibonacci codes, Fibonacci representations, Fibonacci systems of numeration, uniquely decipherable codes, prefix codes, universal representations, asymptotic length of Fibonacci codes.

1. Introduction

Efficient logarithmic ramp representations of binary strings of either unbounded length or a priori unknown length, have emerged some time ago in the somewhat related frameworks of data transmission \cite{5,15}, coding theory \cite{3} and unbounded searching \cite{1}. Logarithmic ramp representations rest on a simple idea: after writing the string $S$ — encoded in binary, say — the length of $S$, with leading 1, is similarly encoded and prefixed to $S$. The process of recursively placing the length of a string in front of that string is repeated until a short string, of length 3, say, is obtained. Since all strings representing lengths begin with a leading 1-bit, the bit 0 can be used to mark the end of the logarithmic ramp and the beginning of $S$.

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For example, the string \( S = 001011100 \) is represented as follows:

\[
\text{100 - 1001 - 0 - 001011100 .}
\]

The major disadvantage of this representation, however, lies in its vulnerability to errors. If an error occurs in the logarithmic ramp, then the decoding capability is lost and cannot, normally, be recovered.

The main contribution of this paper is to show that generalized Fibonacci systems of numeration [12, 6] can be exploited to construct binary uniquely decipherable (UD) codes which are robust and easy to encode and decode. They can, in particular, be exploited to represent unbounded strings efficiently.

The key idea lies in the following property of a Fibonacci numeration system of order \( m \) (\( m \geq 2 \)), denoted by \( \mathcal{F}(m) \) in the sequel: any positive integer \( N \) can be expressed uniquely as a sum of distinct \( m \)-th order Fibonacci numbers, provided that no \( m \) consecutive such numbers are used. In other words, the encoding of \( N \) in \( \mathcal{F}(m) \) is a binary encoding with the property that it contains no run of \( m \) or more consecutive 1-bits. A run of \( m \) consecutive 1-bits can thus be used as a comma, also called separator, separating consecutive codewords.

A representation is a bijection of a countable infinite set \( S_1 \) of strings onto a set \( S_2 \) of strings, such that any concatenation of the members of any subset of \( S_2 \) is UD [4, Ch. 4]. The set \( S_2 \) is called a code, and its members codewords. For example, the encoding of the positive integers using the standard binary numeration system \( \{1, 10, 11, 100, 101, \ldots \} \) is not a representation: the parses 1, 1 and 11 of the string 11 illustrate the problem. However, any prefix code is UD. (A prefix code is any code with the property that no codeword is a prefix of any other codeword.)

Let \( P = (a_1, \ldots, a_p) \) be an arbitrary binary string (the pattern). A pattern code (P-code) is a set \( T \) of binary strings, each of length \( \geq p \), such that for any \( z = z_1 z_2 \cdots z_{n+p} \in T \ (n \geq 0) \), \( P \) occurs in \( z \) precisely once, as a suffix. That is, \( z_{n+j} = a_j \) for \( j = 1, \ldots, p \), and there is no \( i \in [0, n-1] \) such that \( z_{i+j} = a_j \) for \( j = 1, \ldots, p \). Note that every P-code is a prefix code, and is thus UD.

A P-code is comma free or synchronizable (SP-code), if for any codeword \( z = z_1 z_2 \cdots z_n a_1 \cdots a_p \in T \), the pattern \( P \) does not appear as a block anywhere in \( a_2 \cdots a_p z_1 \cdots z_n a_1 \cdots a_{p-1} \). Thus for \( P = 0101 \), the string 11010101 is not in any P-code; 0110101 is in some P-code but not in any SP-code; and 10110101 is in some SP-code.

A receiver turned on in the midst of the transmission of an SP-code has only to identify \( P \) for unambiguous parsing of the code, which is not true for a general UD code. On the other hand, an SP-code is not in general complete. (A UD code is complete if addition of any codeword renders it non-UD.) However if \( P \) has autocorrelation \( PP = 10 \cdots 0 \) (see [10]), it is easy to see that the SP-code with pattern \( P \) can be completed. It is therefore not too surprising that the number
of codewords of an \( SP \)-code of fixed length \( N \) is maximized — for \( P \) of suitable length — when \( PP = 10 \cdots 0 \). This has been proved by Guibas and Odlyzko [10] for large \( n \) and \( p \) about \( \log_2 n \). It implies a conjecture of Gilbert [9].

A \( P \)-code with \( P = (01) \) has been considered in [16]. In this code the length of an encoded integer increases as the square root of the integer. For Fibonacci representations it increases only logarithmically. Some initial properties of Fibonacci fixed-length codes have been considered by Kautz [11]. The universality of \( P \)-codes was investigated by Lakshmanan [13].

Fibonacci numbers came up in previous work on \( P \)-codes as bounds for code lengths, etc., but not, it seems, as codewords in UD codes. The main new features of this work is the construction of robust codes based on the Fibonacci numeration system which are easy to encode and decode, the exploration of their properties, application to the robust transmission of strings of unknown sizes, and the “asymptotic efficiency” computation of this transmission.

In Section 2 we construct two basic Fibonacci representations, \( \varphi_1^{(m)} \) and \( \varphi_2^{(m)} \), based on a single \( P \)-code \( C_1^{(m)} \) derived from Fibonacci numbers of order \( m \) (\( m \geq 2 \)). The representation \( \varphi_1^{(m)} \) maps arbitrary binary integers onto \( C_1^{(m)} \). Here and in the sequel, a binary integer is a binary sequence with leading 1. A leading bit or leading string is the most significant (leftmost) bit or string of a binary string. The representation \( \varphi_2^{(m)} \) maps arbitrary binary strings, which may begin with leading 0, onto \( C_1^{(m)} \). We also give in Section 2 encoding and decoding algorithms for transforming standard binary encodings to the \( \varphi_1^{(m)} \) and \( \varphi_2^{(m)} \) representations and vice versa. We finally prove in Section 2, using the Kraft equality [8, Ch. 3], that \( C_1^{(m)} \) is complete.

In Section 3 we construct an alternate UD code \( C_2^{(m)} \), based on Fibonacci numbers of order \( m \), and a natural representation \( \varphi_3^{(m)} \) which maps the positive integers onto \( C_2^{(m)} \). The main difference between \( C_1^{(m)} \) and \( C_2^{(m)} \) is that \( C_2^{(m)} \) contains binary integers only. It is possible to construct many other Fibonacci codes. Some variants of interest are investigated in [7], where also the robustness of Fibonacci codes is examined in greater detail. Using again the Kraft equality, we show that also \( C_2^{(m)} \) is complete, and we compare the densities of \( C_1^{(m)} \) and \( C_2^{(m)} \).

In the main Section 4 we apply Fibonacci representations to the problem of the robust transmission of binary strings in an unknown range. We show that the logarithmic ramp representation is asymptotically shorter than any Fibonacci representation, but that, nevertheless, integers in a very large initial range have shorter Fibonacci representations, depending on the order \( m \) of the underlying Fibonacci numeration system. The “transition point” for \( \varphi_1^{(m)} \) is \( F_2^{(m)} - 1 = \)}
For $m = 2$, it is

$$\frac{1}{2} (F_{53}^{(2)} + F_{61}^{(2)} - 1) = 34, 696; 689; 675; 649; 696 \approx 3.470 \times 10^{16},$$

and for $m = 4$,

$$\frac{1}{3} (F_{231}^{(4)} + 2F_{229}^{(4)} + F_{228}^{(4)} - 1) \approx 4.194 \times 10^{65}.$$

These computations are based on a list of higher-order Fibonacci numbers which Gerald Bergum has kindly prepared for us.

We point out that every Fibonacci code is a fixed infinite set, independent of the probability distribution of any given source. In particular, the code does not have to be constructed anew for every probability distribution, as, for example, a Huffman code. On the other hand, the independence of probability implies that Fibonacci codes, unlike Huffman codes, are not generally optimal. In the final Section 5 we show, however, that a very broad family of Fibonacci representations, including $\varphi_1^{(m)}$, $\varphi_2^{(m)}$ and $\varphi_3^{(m)}$, is universal in the sense of Elias [3]. That is, the expected representation lengths lie within a constant multiple of the optimal entropy lower bound.

2. Two Basic Fibonacci Representations

Fibonacci numbers of order $m \geq 2$ are defined by the recurrence

$$F_n^{(m)} = F_{n-1}^{(m)} + F_{n-2}^{(m)} + \cdots + F_{n-m}^{(m)} \quad (n \geq 1), \quad (1)$$

where $F_{-m+1}^{(m)} = F_{-m+2}^{(m)} = \cdots = F_{-2}^{(m)} = 0$, $F_{-1}^{(m)} = F_0^{(m)} = 1$.

Thus $F_1^{(m)} = 2$ for all $m \geq 2$, $F_2^{(3)} = 4$, $F_3^{(3)} = 7$, $F_4^{(3)} = 13$.

In the sequel we often write $F_i$ for $F_i^{(m)}$ when an arbitrary but fixed $m$ is the underlying order of $F_i$.

Every nonnegative integer $N$ has precisely one binary encoding of the form

$$N = \sum_{i=0}^{k} d_i F_i \quad (d_i \in \{0, 1\}, 0 \leq i \leq k),$$

such that there is no run of $m$ consecutive Fibonacci numbers of order $m$ in the summation. This is the $\mathcal{F}^{(m)}$-numeration system [12, 6]. The encoding of the
Table 1: The $\mathcal{F}^{(3)}$-encoding, sequence of messages $M$, $C_1^{(3)}$-code and mappings $\varphi_1^{(3)}$ and $\varphi_2^{(3)}$.

<table>
<thead>
<tr>
<th>Message $M$</th>
<th>Code $C_1^{(3)}$</th>
<th>$\mathcal{F}^{(3)}$ Encoding</th>
<th>Integer $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>111</td>
<td>01</td>
<td>1</td>
</tr>
<tr>
<td>00</td>
<td>0111</td>
<td>110</td>
<td>2</td>
</tr>
<tr>
<td>000</td>
<td>00111</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>10111</td>
<td>100</td>
<td>4</td>
</tr>
<tr>
<td>0000</td>
<td>000111</td>
<td>101</td>
<td>5</td>
</tr>
<tr>
<td>01</td>
<td>010111</td>
<td>110</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>100111</td>
<td>1000</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>110111</td>
<td>1001</td>
<td>8</td>
</tr>
<tr>
<td>00000</td>
<td>0000111</td>
<td>1010</td>
<td>9</td>
</tr>
<tr>
<td>001</td>
<td>0010111</td>
<td>1011</td>
<td>10</td>
</tr>
<tr>
<td>02</td>
<td>0100111</td>
<td>1100</td>
<td>11</td>
</tr>
<tr>
<td>03</td>
<td>0110111</td>
<td>1101</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>1000111</td>
<td>1000</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>1010111</td>
<td>1001</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>1100111</td>
<td>1010</td>
<td>15</td>
</tr>
<tr>
<td>000000</td>
<td>00000111</td>
<td>10011</td>
<td>16</td>
</tr>
</tbody>
</table>

The first few nonnegative integers in $\mathcal{F}^{(3)}$ is shown in the two right-hand columns of Table 1.

For any $i \geq 1$, let $I_i$ denote the string of $i$ consecutive 1-bits, $0I_i$ the string $I_i$ prefixed by 0, and $I_i0$ the string $I_i$ postfixed by 0. Similarly, $0_i$ denotes the string of $i$ consecutive 0-bits.

The code $C_1^{(m)} = C_1$ is a $P$-code with pattern $P = 1_m$ defined as follows. The first two codes are $1_m$ and $01_m$ of length $m$ and $m + 1$ respectively. The codes of length $m + n$ ($n \geq 2$) each consist of the suffix $01_m$ and a prefix of length $n - 1$. These prefixes are the first few $\mathcal{F}^{(m)}$-encodings of the nonnegative integers, in increasing size, which can be encoded by at most $n - 1$ bits. Leading 0-bits are
prefixed, where necessary, to complete the length to \( m+n \). The first 16 codewords of \( C_1^{(2)} \) appear in the middle column of Table 1.

The representation \( \varphi_1^{(m)} = \varphi_1 \) maps the set of positive integers \( \mathbb{Z}^+ \) bijectively onto \( C_1 \), such that if \( N_1 < N_2 \), then \( \varphi_1(N_1) \) is lexicographically smaller than \( \varphi_1(N_2) \) (see the three right-hand columns of Table 1 for \( m = 3 \)).

Below we give some basic properties of \( C_1 \) and \( \varphi_1 \). We remark that since each codeword in \( C_1 \) ends in 1\( m \), \( C_1 \) is a prefix code. The length of a codeword is the number of binary bits it comprises. Regarding lengths of codewords in \( C_1 \) we have,

**Lemma 1.** The code \( C_1 \) contains precisely \( F_{n-1} \) codewords of length \( m+n \), which are partitioned as follows: \( F_{n-2} \) with leading 0, \( F_{n-3} \) with leading 10, \( F_{n-4} \) with leading 110, \( \ldots \), \( F_{n-m-1} \) with leading 1\( m \)-10 (\( n \geq 0 \)).

**Proof.** Induction on \( n \). Clear for \( n = 0, 1 \) and 2. Suppose the result holds for \( n \). The definition of \( C_1 \) implies that the codewords of length \( m+n+1 \) can be produced from those of length \( m+n \) by prefixing 0 to all the latter, and by prefixing 1 to all of them except to the \( F_{n-m-1} \) codewords with leading 1\( m \)-0. This gives \( F_{n-1} \) codewords with leading 0, \( F_{n-2} \) with leading 10, \( F_{n-3} \) with leading 110, \( \ldots \), \( F_{n-m} \) with leading 1\( m \)-10 — a total of \( F_n \) codewords of length \( m+n+1 \). □

**Corollary 1.** Let \( S_n^{(m)} = S_n = \sum_{i=-1}^{n} F_i^{(m)} \) (\( n \geq -1 \)) and \( S_n^{(m)} = 0 \) for \( n < -1 \). Then all and only all the \( F_{n-1} \) integers in the interval \( I_{n-1} = [S_{n-2} + 1, S_{n-1}] \) have \( \varphi_1 \)-representation length \( m+n \) (\( n \geq 0 \)).

**Proof.** The integer 1 is in \( I_{-1} \) which has representation length \( m \). Next \( 2 \in I_0 \) of length \( |\varphi_1(2)| = m+1 \). By Lemma 1, the next \( F_1 = 2 \) integers, those in \( [F_0 + F_1 + 1, F_0 + F_1] \) have length \( m+2 \), that is, \( |\varphi_1(3)| = |\varphi_1(4)| = m+2 \). The proof is completed by induction on \( n \). □

We thus have,

**Corollary 2.** If \( |\varphi_1(k)| \leq m+n \), then \( k \) is in the interval \( [1, S_{n-1}] \) (\( n \geq 0 \)). □

For the encoding and decoding processes, it is useful to compute \( S_n \) efficiently.

**Lemma 2.** Let \( S_n = \sum_{i=-1}^{n} F_i \) (\( n \geq -1 \)). Then

\[
S_n = \frac{1}{m-1} \left( F_{n+2} + \sum_{i=0}^{m-3} (m - 2 - i)F_{n-i} - 1 \right) \quad (n \geq -1, m \geq 2). \tag{2}
\]

**Proof.** Induction on \( n \) for arbitrary but fixed \( m \). For \( n = -1 \), the right-hand-side of (2) becomes

\[
\frac{1}{m-1} (F_1 + (m - 2)F_{-1} - 1) = 1 = F_{-1} = S_{-1}.
\]

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If the assertion is true for \( n \), then
\[
S_{n+1} = S_n + F_{n+1}
\]
\[
= \frac{1}{m-1} (F_{n+2} + (m-1)F_{n+1} + \sum_{i=0}^{m-5} (m-2-i)F_{n-i} - 1)
\]
\[
= \frac{1}{m-1} (F_{n+3} + \sum_{i=0}^{m-3} (m-2-i)F_{n+1-i} - 1),
\]
where we used the recurrence (1). 

**Encoding Algorithm.** Given a positive integer \( N \), compute \( \varphi_1(N) \).

(i) If \( N = 1 \), then \( \varphi_1(N) = 1_m \). End. If \( N = 2 \), then \( \varphi_1(N) = 01_m \). End.

(ii) Find \( n \) such that \( S_{n-2} < N \leq S_{n-1} \). Let \( Q = N - S_{n-2} - 1 \) and encode \( Q \) in \( F^{(m)} \). The approximate value of \( n \) can be computed from the asymptotic result \( N \sim \lambda w^n \), see Theorem 4, Section 4 below. Then \( S_{n-2} \) can be computed using Lemma 2.

(iii) Adjoin \( 01_m \) as suffix to the \( F^{(m)} \)-encoding of \( Q \). Adjoin leading 0-bits, if necessary, to make \( \varphi_1(N) \) of length \( m + n \). End.

**Example.** Let \( m = 3 \), \( N = 11 \). Since \( S_2(3) = 8 < N = 11 < S_3(3) \), we have \( n = 4 \) and \( Q = 2 \). Hence \( \varphi_1(11) = 0100111 \).

**Decoding Algorithm.** Given \( \varphi_1(N) \), compute \( N \).

(i) Remove the suffix \( 1_m \).

(ii) If the remaining prefix is empty, then \( N = 1 \). End. If it is \( \{0\} \), then \( N = 2 \). End.

(iii) Remove the suffix \( 0 \).

(iv) The remaining prefix is an \( F^{(m)} \)-encoding. Transform it into standard binary encoding, say \( b \) (for example, by using a stored table of \( m \)-th order Fibonacci numbers, or by computing them using (1)). Then \( N = b + S_{n-2} + 1 \) (where \( n = |\varphi_1(N)| - m \), and \( S_{n-2} \) is computed using Lemma 2).

**Example.** Let \( m = 3 \), \( \varphi_1(11) = 1010111 \). Then \( n = 4 \) and \( S_{n-2} = S_2 = 8 \). The prefix 101 in \( F^{(3)} \) remaining at the end of step (iii) transforms into itself in standard binary encoding. Thus \( N = 8 + 5 + 1 = 14 \) in decimal, that is, \( N \) is the binary string 1110.

We shall now address ourselves to the problem of representing arbitrary binary strings which are not necessarily binary integers.

Towards this end, let \( M = Z^0 \cup O Z^0 \), where \( Z^0 \) is the set of nonnegative integers, and \( O Z^0 \) the set of all nonnegative integers with leading binary zeros. The bijection \( \varphi_2^{(m)} = \varphi_2 \) maps \( M \) onto \( C_1 \). The subset \( Z^0 \) is mapped onto the subset of integers of \( C_1 \), that is, onto the subset of codewords with leading 1-bit,
such that if $N_1 < N_2$ with $N_1, N_2 \in \mathbb{Z}^+$, then $\varphi_2(N_1)$ is lexicographically less than $\varphi_2(N_2)$. The subset $OZ^0$ is mapped onto the subset of $C$, with leading 0:

Let $R = 0, N \in OZ^0$, where $N \in \mathbb{Z}^+$. Then $\varphi_2(R) = 0, \varphi_2(N)$. This mapping for $m = 3$ can be observed in the two left-hand columns of Table 1.

Encoding and decoding are even simpler than for $\varphi_1$. The essence is that if $N \in \mathbb{Z}^+$, then the $f^{(m)}$-encoding of $N$, with $0_1$ postfixed, gives $\varphi_2(N)$. The process is reversed for decoding.

For later reference we record the following

**Lemma 3.** If $k \in \mathbb{Z}^+$ and $|\varphi_2(k)| \leq m + n + 1$, then $k \in [1, F_n - 1]$, that is, precisely the first $F_n - 1$ positive integers have $\varphi_2$-representations of lengths up to $m + n + 1$ ($n \geq 1$).

**Proof.** The definition of $\varphi_2$ implies that for $k \in \mathbb{Z}^+$, $\varphi_2(k)$ is the $f^{(m)}$ encoding $E(k)$ of $k$, postfixed by $0_1$. For $k = F_n - 1$ we clearly have $|E(k)| = n$, and for $k = F_n, |E(k)| = n + 1$. 

Our final result in this section is

**Theorem 1.** The code $C_1$ is complete.

**Proof.** Any countable UD code $C$ satisfies the Kraft inequality

$$\sum_{c \in C} 2^{-|c|} \leq 1$$

(see e.g. [8, Ch. 3; Ch. 9, Ex. 3.7]). It follows that if

$$\sum_{c \in C} 2^{-|c|} = 1$$

(the Kraft equality), then $C$ is complete.

Let $\sigma_1^{(m)} = \sigma_1 = \sum_{c \in C_1} 2^{-|c|}$. By Corollary 1, 

$$\sigma_1 = \sum_{n=0}^{\infty} 2^{-m-n} F_{n-1}.$$ 

Thus by (1),

$$\sigma_1 - (2^{-m} + 2^{-m-1}) = \sum_{n=2}^{\infty} 2^{-m-n} (F_{n-2} + F_{n-3} + \cdots + F_{n-m-1})
= \frac{1}{2} \sum_{n=1}^{\infty} 2^{-m-n} F_{n-1} + \frac{1}{2^2} \sum_{n=0}^{\infty} 2^{-m-n} F_{n-1}
+ \frac{1}{2^2} \sum_{n=-1}^{\infty} 2^{-m-n} F_{n-1} + \cdots
+ \frac{1}{2^m} \sum_{n=-m}^{\infty} 2^{-m-n} F_{n-1}
= \frac{1}{2} \sigma_1 - 2^{-m}
+ \frac{1}{2^2} \sigma_1 + \frac{1}{2^3} \sigma_1 + \cdots + \frac{1}{2^m} \sigma_1.$$ 

Thus, $\sigma_1 - 2^{-m} = \sigma_1 \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^m} \right) = \sigma_1 \left( 1 - \frac{1}{2^m} \right)$, so $\sigma_1 = 1$. 

**Notes.** (i) The P-code $C_1$ is not an SP-code, since $1_m \in C_1$. Deleting $1_m$ makes it into an SP-code with $P = 0_1$, but then it is not complete. It can be completed, but then it will contain runs of length $\geq m$ of 1-bits, and the decoding
may be harder than for \( C_1 \). The Lakshmanan codes [13] are all incomplete, since they do not contain \( P \).

(ii) If \( b_n \) denotes the number of codes of length \( m + n \) in a \( P \)-code with \( |P| = m \), then for \( C_1 \), Lemma 2 says that \( b_n = F_{n-1} \) \((n \geq 0)\). Lakshmanan [13] showed that \( b_n \leq F_n \) \((n \geq 1)\). The bound \( b_n \leq F_n \) cannot, however, be assumed for all \( n \geq 1 \), since, as in the proof of Theorem 1, it can be seen easily that \( \sum_{n=1}^{\infty} 2^{-m-n} F_n = 2 - 3 \cdot 2^{-m} > 1 \).

(iii) It is not hard to construct \( P \)-codes for which the bound \( F_n \) is assumed for small \( n \). But for larger \( n \) the inequality \( b_n \leq F_n \) is then strict. The decoding of such codes may be harder than for \( C_1 \).

3. An Alternate Fibonacci Code and Representation

The code we define now is conceptually simpler than \( C_1 \). For \( m \geq 2 \), \( C_2^{(m)} = C_2 \) consists of the codeword \( 1_{m-1} \) and the \( F^{(m)} \)-encodings of the positive integers in increasing order, the latter postfixed by \( 01_{m-1} \).

The representation \( \varphi_2^{(m)} = \varphi_3 \) maps the positive integers bijectively onto \( C_2 \) such that if \( N_1, N_2 \in \mathbb{Z}^+ \) with \( N_1 < N_2 \), then \( \varphi_3(N_1) \) is lexicographically smaller than \( \varphi_3(N_2) \). The first few integers represented by \( \varphi_3^{(2)} \) and \( \varphi_3^{(3)} \) are shown in Table 2.

We note that \( C_2 \) is not a prefix code: Table 2 shows that \( \varphi_3^{(3)}(1) \) is a prefix of, say, \( \varphi_3^{(3)}(4) \), and \( \varphi_3^{(3)}(2) \) is a prefix of, say, \( \varphi_3^{(3)}(11) \). Of course \( \varphi_3^{(2)}(1) \) is a prefix of \( \varphi_3^{(2)}(N) \) for every \( N > 1 \). However, \( C_2 \) is a UD code, because the concatenation of any two codewords generates the separator \( 01_m \) between them. When parsing a concatenation of \( C_2 \)-codewords, the comma is placed just in front of the last 1-bit of any \( 1_m \) encountered.

The general length-distribution of codewords in \( C_2 \) is given by

**Lemma 4.** For \( C_2 \) there are \( F_{n-1} - F_{n-2} \) codewords of length \( m + n - 1 \) which can be partitioned as follows: \( F_{n-5} \) with leading \( .10 \), \( F_{n-4} \) with leading \( 110 \), \( \ldots, F_{n-m-1} \) with leading \( 1_{m-1}0 \) \((n \geq 0)\).

**Proof.** Note that \( C_2 \) is the same as \( C_1 \) except for two changes: (i) all codewords with leading 0 are omitted; (ii) the common suffix is \( 01_{m-1} \) instead of \( 01_m \). Now apply Lemma 1. □

**Corollary 3.** The code \( C_2 \) contains precisely \( F_n \) codewords of length not exceeding \( m + n \) \((n \geq 0)\).

**Proof.** There is one codeword of length \( m - 1 \), \( F_1 - F_0 \) of length \( m + 1 \), \( F_2 - F_1 \) of length \( m + 2 \), \ldots, \( F_n - F_{n-1} \) of length \( m + n \) \((n \geq 0)\). Adding gives the
Table 2: The codes $C_2^{(2)}$, $C_2^{(3)}$ and the mappings $\varphi_2^{(2)}$, $\varphi_2^{(3)}$.

<table>
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<tr>
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<th>$C_2^{(3)}$</th>
<th>$1385321$</th>
<th>$C_2^{(2)}$</th>
<th>$N$</th>
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<td>16</td>
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</tbody>
</table>

Encoding and decoding for $\varphi_2$ is very simple: the encoding of 1 is $1_{m-1}$. For $N \in \mathbb{Z}^+$, $N > 1$, the $f^{(m)}$-encoding of $N-1$ with $01_{m-1}$ postfixed is $\varphi_2(N)$. The procedure is reversed for decoding.

We now prove,

**THEOREM 2.** The code $C_2$ is complete.

**PROOF.** It suffices to show that the Kraft equality $\sum_{c \in C_2} 2^{-|c|} = 1$ holds.

Let $\sigma_2^{(m)} = \sigma_2 = \sum_{c \in C_2} 2^{-|c|}$. By Lemma 4,

$$\sigma_2 = \sum_{n=0}^{\infty} (F_{n-1} - F_{n-2}) 2^{-m-n+1}.$$

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Let \( \sigma_3 = \sum_{n=0}^{\infty} F_{n-1}2^{-m-n+1} \), \( \sigma_4 = \sum_{n=0}^{\infty} F_{n-2}2^{-m-n+1} \). Then

\[
\sigma_4 = \frac{1}{2} \sum_{n=0}^{\infty} F_{n-1}2^{-m-n+1} = \frac{1}{2} \sigma_3.
\]

Thus

\[
\sigma_2 = \sigma_3 - \frac{1}{2} \sigma_3 = \frac{1}{2} \sigma_3 = \sum_{n=0}^{\infty} F_{n-1}2^{-m-n} = 1,
\]

as we established in the proof of Theorem 1.

Theorems 1 and 2 assert that no codeword can be adjoined to either \( C_1 \) or \( C_2 \) without losing their UD property. This does not imply that they necessarily have the same density, however. In fact, \( C_2 \) contains one code of length \( m - 1 \), whereas the minimum length of the \( C_1 \)-codewords is \( m \). Since both \( C_1 \) and \( C_2 \) satisfy the Kraft equality, the density of \( C_1 \) must be larger than that of \( C_2 \) for some codelengths. We shall see that this is in fact the case everywhere except for small codelengths. This may at first seem counterintuitive, since the separator \( 01_{m-1} \) of \( C_2 \) is shorter than the separator \( 01_m \) of \( C_1 \). Note, however, that if we rotate the leading 1-bit of every codeword of \( C_2 \) to its right-hand end, the resulting code — which is a prefix code! — contains words with leading 0-bits and every word ends in \( 01_m \). It is not, however, identical to \( C_1 \): In the latter there are codewords with leading \( 1_{m-1} \), which do not exist in the former.

Corollary 2 implies that precisely the first \( S_{n-1} \) codewords of \( C_1 \) have lengths \( \leq m + n \) \( (n \geq 0) \). By Corollary 3, the first \( F_n \) codewords of \( C_2 \) have lengths \( \leq m + n \) \( (n \geq 0) \). Thus the quantity

\[
D_n^{(m)} = D_n = S_{n-1} - F_n \quad (n \geq 0)
\]

measures the density difference between the two codes \( C_1 \) and \( C_2 \).

**Theorem 3.** The density difference between \( C_1 \) and \( C_2 \) is \( D_{-1} = -1 \), \( D_n = S_{n-m-1} \) for \( n \geq 0 \); thus \( D_n = 0 \) for \( 0 \leq n < m \), and \( D_n > 0 \) for \( n \geq m \).

**Proof.** We have \( D_0 = S_{-1} - F_0 = 0 \). For \( n > 0 \), \( D_n = S_{n-1} - F_n = \sum_{i=-1}^{n-1} F_i - (F_{n-1} + \cdots + F_{n-m}) = S_{n-m-1} \).

4. Transmission of Binary Strings in an Unknown Range

Transmitting a binary string whose length is unbounded or lies in an unknown range by means of a \( P \)-code or a Fibonacci code such as \( C_2 \) clearly has the advantage that any error such as a transmission error, will be locally contained, because of the solid separator \( P \) which terminates each codeword of the string (except the
last, for $C_2$). This is in contrast to the logarithmic ramp representation, where any error in the logarithmic ramp section can play total havoc with the decoding efforts.

If the transmission is restricted to integers, one of the representations $\varphi_1$ or $\varphi_3$ can be used. For arbitrary strings which are not necessarily integers, $\varphi_2$ is employed.

It is natural to inquire about the asymptotic length of Fibonacci representations. How does it compare with the length of the logarithmic representation? What size should $m$ be? These are the kind of questions we address ourselves to in this section.

We show that $F_n \sim \lambda u^n$ ($\sim$ denotes “asymptotic to”), where $u^{(m)} = u$ is a Pisot-Vijayaraghavan (PV) number, that is, an algebraic integer $>1$ all of whose conjugates other than $u$ itself lie in the open unit circle $|z| < 1$ (see e.g. Cassels [2, Ch. 8]) and $\lambda^{(m)} = \lambda$ a positive number. We also give a sharp estimate of $u$. These facts give us a good handle on estimating the asymptotic length of Fibonacci representations.

**Lemma 5.** For all $m \geq 2$, the polynomial

$$f(z) = z^m - (z^{m-1} + z^{m-2} + \cdots + z + 1)$$

has $m$ distinct roots, one of which is a PV-number satisfying

$$2 - 2^{-m+1} < u < 2 - 2^{-m}.$$ 

**Proof.** The first part has been proved by Miles [14]. See also Knuth [12, Sect. 5.4.2, Ex. 5]. For proving the second part, note that

$$f(z) = \frac{z^{m+1} - 2z^m + 1}{z - 1} = \frac{z^m(z - 2) + 1}{z - 1}.$$ 

Let $p(z) = 1 - (2 - z)z^m$. Then

$$p(2 - 1/2^m) = 1 - 2^{-m}(2 - 2^{-m})^m = 1 - (1 - 2^{-(m+1)})^m > 0.$$ 

On the other hand, using the binomial expansion, we get

$$p(2 - 1/2^{m-1}) = 1 - 2^{-(m-1)}(2 - 2^{-(m-1)})^m = 1 - 2(1 - 2^{-m})^m$$

$$< 1 - 2(1 - m2^{-m}) = 1 + m2^{-(m-1)} \leq 0$$

for $m \geq 0$. 

We remark that with a little more effort, the interval for $u$ can be narrowed further.
THEOREM 4. If $F_n$ is the $n$-th term of the $m$-order Fibonacci sequence defined by the recurrence (1), then $F_n \sim \lambda u^n$ for large $n$ (fixed $m$), where

$$\lambda = \lambda_1 = \frac{(2 - u_2)(2 - u_3) \cdots (2 - u_m)}{(u_1 - u_2)(u_1 - u_3) \cdots (u_1 - u_m)} > 0,$$

and $u_i^{(m)} = u_i$ ($i > 1$) are the conjugates of the PV-number $u = u_1$ in the polynomial (3). Moreover, $F_n \sim u^n$ for large $n$ and large $m$.

PROOF. The solution of the recurrence

$$F_n - F_{n-1} - \cdots - F_{n-m} = 0$$

is clearly

$$F_n = \lambda_1 u_1^n + \lambda_2 u_2^n + \cdots + \lambda_m u_m^n,$$

where $u = u_1, u_2, \ldots, u_m$ are the roots of the polynomial (3), and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are suitable constants. Since $u$ is a PV-number, we have $|u_i| < 1$ for $i > 1$. Hence

$$\lambda_1 u_1^n + \lambda_2 u_2^n + \cdots + \lambda_m u_m^n \sim \lambda_1 u_1^n$$

for large $n$.

The recurrence (1) implies $F_i = 2^i$ for $0 \leq i < m$. Hence,

$$V \lambda_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & u_2 \\ \vdots & \vdots & \vdots \\ 1 & u_1^{m-1} & u_2^{m-1} & u_m^{m-1} \end{bmatrix} = \prod_{k \geq \ell} (u_k - u_{\ell})$$

where

$$V = \begin{bmatrix} 1 & 1 & 1 \\ u_1 & u_2 & \cdots & u_m \\ \vdots & \vdots & \vdots \\ 1 & u_1^{m-1} & u_2^{m-1} & u_m^{m-1} \end{bmatrix} = \prod_{k \geq \ell} (u_k - u_{\ell})$$

is the Vandermonde determinant. This implies

$$\lambda_1 = \lambda = \frac{(2 - u_2)(2 - u_3) \cdots (2 - u_m)}{(u_1 - u_2)(u_1 - u_3) \cdots (u_1 - u_m)}.$$

By the Symmetric Polynomial Theorem it follows that $\lambda$ is real. Since $F_n \sim \lambda u^n$ and $F_n$ and $u$ are positive, we have in fact $\lambda > 0$. 

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In view of Lemma 5 we can write $u_1 = u = 2 - \delta/2^{m-1}$, where $\delta$ is a suitable number in the range $1/2 < \delta < 1$. Thus

$$
\lambda = \frac{(u_1-u_2+\delta2^{-2(m-1)})\cdots(u_1-u_m+\delta2^{-(m-1)})}{(u_1-u_2)(u_1-u_3)\cdots(u_1-u_m)}
\cdot
\left[1 + \frac{\delta}{2^{m-1}(u_1-u_2)}\right]\cdots\left[1 + \frac{\delta}{2^{m-1}(u_1-u_m)}\right].
$$

Let $\epsilon > 0$. For $m$ sufficiently large

$$
\frac{\delta}{2^{m-1}|u_1 - u_i|} < \frac{\epsilon}{m} \quad (i = 2, \ldots, m).
$$

Hence $|\lambda| < \left(1+\frac{\epsilon}{m}\right)^m \to \epsilon^m$, which can be made arbitrarily close to 1 for sufficiently large $m$. Thus $\lambda \to 1$ as $m \to \infty$. \qed

Theorem 4 enables us to give an asymptotic estimate of the length of any Fibonacci representation. We carry this out below for $\varphi_3$, but it is not much different for $\varphi_1$ and $\varphi_2$ (for which we get a slightly smaller asymptotic length).

Corollary 3 implies that the largest integer representable by $\varphi_3$ with $m+n$ bits is $F_n$ ($n \geq 0$). If $k$ is the number of bits in the standard binary numeration system necessary to encode $F_n$, then $2^{k-1} \leq F_n < 2^k$. For large $n$ we have by Theorem 4, $2^{k-1} \leq \lambda u^n < 2^k$, where $u = 2 - \delta2^{-(m+1)}$, $\delta$ a suitable real number satisfying $1/2 < \delta < 1$. Then $k-1 \leq n\lg u + \lg \lambda < k$, where $\lg$ denotes $\log$ to the base 2.

Expanding $\lg u$ into a Taylor series,

$$
\lg u = \lg(2 - \delta 2^{-m+1}) = 1 + \lg(1 - \delta 2^{-m})
= 1 - (\delta 2^{-m} + \delta^2 2^{-(2m+1)} + \cdots)\lg e.
$$

Thus

$$
k \approx n\left(1 - (\delta 2^{-m} + \delta^2 2^{-(2m+1)})\lg e\right) + \lg \lambda. \quad (4)
$$

In the logarithmic ramp representation $R(m)(F_n^{(m)}) = R(F_n)$ of $F_n$, an extra 0-bit prefixes the string itself. Therefore,

$$
|R(F_n)| = k + 1 + [\lg(k+1)] + [\lg[\lg(k+1)] + 1] + \cdots + 3,
$$

where the last $\lg \lg \ldots$ term is 3. Using the approximation $k = n(1 - \delta 2^{-m}\lg e) + \lg \lambda$, the lengths difference is

$$
\Delta = m - 1 + \frac{n\delta}{2^m}\lg e - \lg \lambda - [\lg(k+1)] - [\lg[\lg(k+1)] + 1] - \cdots - 3,
$$

where $k$ is given by (4).
This formula shows that for every fixed \( m, \Delta > 0 \) if \( n \) is sufficiently large, so ultimately the logarithmic ramp representation is shorter than any Fibonacci representation. But the crossover point depends exponentially on \( m \). In fact, for very large initial values, the latter representations are in fact shorter than the former. The following computational results for \( m = 2, 3 \) and 4 refer to \( \phi_1 \) and \( C_1 \).

We have \( |R^{(2)}(n)| = 4 \) bits and \( |\phi_1^{(2)}(n)| = 5 \) bits for \( n = 5, 6, 7 \). But \( |\phi_1^{(2)}(n)| \leq |R^{(2)}(n)| \) for all integers \( n \) in the range

\[
8 \leq n \leq F_{27}^{(2)} - 1 = 514,228.
\]

Beyond this point, the representation \( \phi_1^{(2)}(n) \) becomes slowly larger than \( R^{(2)}(n) \).

For \( m = 3 \), \( |R^{(3)}(n)| < |\phi_1^{(3)}(n)| \) for \( 3 \leq n \leq 7 \). But \( |\phi_1^{(3)}(n)| \leq |R^{(3)}(n)| \) for all

\[
8 \leq n \leq \frac{1}{2} \left( F_{63}^{(3)} + F_{61}^{(3)} - 1 \right) = 34,696,689,675,849,696 \approx 3.470 \times 10^{16}.
\]

For larger \( n \), \( |\phi_1^{(3)}(n)| \) becomes slowly larger than \( |R^{(3)}(n)| \). Thus for \( n = \frac{1}{2} \left( F_{80}^{(3)} + F_{78}^{(3)} - 1 \right) \approx 1.095 \times 10^{21} \) we have \( |\phi_1^{(3)}(n)| - |R^{(3)}(n)| = 1 \), and this difference is 5, for example at \( n = \frac{1}{2} \left( F_{146}^{(3)} + F_{144}^{(3)} \right) \approx 3.208 \times 10^{38} \). Incidentally, the difference \( \Delta \) does not increase monotonically; it usually decreases at points where \( R(n) \) picks up a new \( log \) \ldots \ term on its logarithmic ramp.

We have \( |R^{(4)}(n)| < |\phi_1^{(4)}(n)| \) for \( 2 \leq n \leq 7 \), \( |\phi_1^{(4)}(n)| \leq |R^{(4)}(n)| \) for \( 8 \leq n \leq 116 \), and \( |\phi_1^{(4)}(n)| - |R^{(4)}(n)| = 1 \) for \( 117 \leq n \leq 127 \). But \( |\phi_1^{(4)}(n)| \leq |R^{(4)}(n)| \) for all

\[
128 \leq n \leq \frac{1}{3} \left( F_{231}^{(4)} + 2F_{229}^{(4)} + F_{228}^{(4)} - 1 \right) \approx 4.194 \times 10^{65}.
\]

Beyond this point, the representation \( \phi_1(n) \) becomes very slowly larger than \( R^{(4)}(n) \).

These computational results and the asymptotic formula for \( \Delta \) (valid for \( \phi_2 \)), both indicate that \( |\phi_1^{(m)}(n)| \leq |R^{(m)}(n)| \) for exponentially larger \( n \) as \( m \) increases. Hence if we expect many of the transmitted strings to be very large, it may be advantageous to select a larger value of \( m \) than for the transmission of shorter strings.

5. Universality of Fibonacci Codes and Representations

Let \( C \) be a countably infinite UD binary code, and \( M = \{m(1), m(2), \ldots \} \supset Z^+ \) a countable set of messages. Let \( S = \{(1, p_1), (2, p_2), \ldots, (n, p_n)\} \) be a source
of the first \( n \) positive integers, with associated positive probabilities \( p_1 \geq p_2 \geq \cdots \geq p_n \) \( (\sum_{i=1}^{n} p_i \leq 1) \). The source may also include some noninteger messages with their associated probabilities. The entropy of the integers in the source is

\[ H(P) = -\sum_{i=1}^{n} p_i \log p_i \].

Let \( \varphi : \mathbb{Z}^+ \to \mathbb{C}^+ \) be a binary representation of the positive integers such that \( |\varphi(i)| \leq |\varphi(i+1)| \) \( (i \geq 1) \). Then \( \varphi \) is called universal if

\[ \frac{E_P(L)}{\max\{1, H(P)\}} \leq K \],

where \( E_P(L) = \sum_{i=1}^{n} p_i|\varphi(i)| \) is the expected codeword length of the representations of the integers in the source, and \( K \) is a positive constant independent of the probability distribution \( P = \{p_1, p_2, \ldots, p_n\} \). This definition reduces to Elias' universality definition when there are no noninteger messages.

Let \( f = \{f_i(x)\}_{i=1}^{\infty} \) be a finite sequence of polynomials with \( \deg(f_1) \geq \deg(f_j) > 0 \) for all \( j \) satisfying \( 1 \leq j \leq k \). All polynomial coefficients are constants which may depend on \( m \). For simplicity we assume that all coefficients of \( f_1 \) are nonnegative.

A more general notion of Fibonacci representation than used above will now be introduced.

A Fibonacci representation of a set \( M \supset \mathbb{Z}^+ \) is any binary representation \( \psi \) such that all and only all the first \( \mathcal{L}(F_{f(\ell)}) + d \) positive integers have representations of length up to \( \ell \), where \( \mathcal{L} \) denotes a finite linear combination: \( \mathcal{L}(F_{f(\ell)}) = \sum c_i F_{f_i(\ell)} \), where the \( c_i \) and \( d \) are constants which may depend on \( m \), and \( c_1 > 0 \).

By applying the definition successively to \( \ell_{\min} = |\psi(1)|, \ell_{\min} + 1, \ell_{\min} + 2, \ldots \), it follows that \( \psi \) is a representation of \( M \) if and only if all the positive integers \( n \) in the interval

\[ [\mathcal{L}(F_{f(\ell-1)}) + d + 1, \mathcal{L}(F_{f(\ell)}) + d] \]

have representation length \( |\psi(n)| = \ell \) for all \( \ell \geq \ell_{\min} \).

Note that \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) are representations also according to the new definition: By Corollary 2, the \( \varphi_1 \)-representations of precisely the first \( S_{n-1} \) positive integers have length up to \( m + n \). Furthermore, we have

\[ S_{n-1} = \mathcal{L}(F_{f(m+n)}) + d = \frac{1}{m-1} \left( F_{n+1} + \sum_{i=0}^{m-3} (m-2-i)F_{n-1-i} - 1 \right), \]

where

\[ f_1(n+m) = n + m - (m-1), \quad c_1 = \frac{1}{m-1}, \quad d = -\frac{1}{m-1}, \]

\[ f_i(n+m) = n + m - (m - 1 + i), \quad c_i = \frac{m-i}{m-1} \quad (i = 2, \ldots, m-1). \]
Let $f = \{f_1\}$. For $f_1(m+n+1) = m+n+1-(m+1), c_1 = 1, d = -1$, Lemma 3 implies that precisely the first $L(F_f(m+n+1)) + d = F_n - 1$ positive integers have $\varphi_2$-representations of length not exceeding $m+n+1$. For $f_1(m+n) = m+n-(m), c_1 = 1, d = 0$, Corollary 3 implies that precisely the first $L(F_f(m+n)) + d = F_n$ positive integers have $\varphi_3$-representations of length not exceeding $m+n$.

**Lemma 6.** Let $\psi$ be a binary representation such that $|\psi(k)| \leq c_1 + c_2 \log k$ ($k \in \mathbb{Z}^+$), where $c_1$ and $c_2$ are constants and $c_2 > 0$. Let $p_k = p(k)$ be the probability of $k$. If $p_1 \geq p_2 \geq \cdots \geq p_n, \sum_{i=1}^n p_i \leq 1$, then $\psi$ is universal.

**Proof.** For $1 \leq j \leq n$ we have $1 \geq \sum_{i=1}^j p_i \geq j p_j$, so $\log j \leq -\log p_j$.

Hence

$$\sum_{j=1}^n \log j \leq -\sum_{j=1}^n \log p_j = H(P),$$

and

$$E_P(L) = \sum_{i=1}^n p_i |\psi(i)| \leq \sum_{i=1}^n p_i (c_1 + c_2 \log i) \leq c_1 + c_2 \sum_{i=1}^n p_i \log i \leq c_1 + c_2 H(P).$$

Thus

$$\frac{E_P(L)}{\max\{1, H(P)\}} \leq \begin{cases} c_1 + c_2 & \text{for } H(P) \leq 1 \\ c_1 + c_2 < c_1 + c_2 & \text{for } H(P) > 1. \end{cases}$$

**Theorem 5.** Any Fibonacci representation is universal.

**Proof.** If $\psi$ is a Fibonacci representation, then any integer $n \in \left[\sum_{i=0}^\ell F_i + d + 1, \sum_{i=0}^\ell F_i + d\right]$ has representation length $|\psi(n)| = \ell$ for all $\ell \geq \ell_{\min}$. Thus

$$n \geq \sum_{i=0}^\ell F_i + d + 1 \geq c_1 F_1 + d + d \geq c_1 F_{a_1} + d,$$

where $a_1$ is the leading coefficient of $f_1$. By Theorem 4, $F_k \sim \lambda u^k$ where $\lambda > 0$, so $F_k > K u^k$ for all $k \geq 0$, where $K > 0$ is a suitable constant. Thus

$$n \geq c_1 K u^a + d,$$

so

$$\ell \leq \frac{1}{a_1} \left(\log \frac{n-d}{c_1 K}\right) + 1 = \frac{1}{a_1} \left(\log \left(\frac{n+K}{c_1 K}\right) - \log \left(c_1 K\right)\right) + 1,$$

where $K = -d$ and $n+K > 0$.

If $K > 0$, then $\log \left(c_1 K\right) > \log \left(\frac{n+K}{c_1 K}\right)$. So assume $K > 0$.

If $n \geq K$, then $\log \left(\frac{n+K}{c_1 K}\right) \leq \log \left(2n\right) = \log_2 2 + \log_2 n$. 

- 17 -
If \( n < K_1 \), then \( \log_2(n + K_1) < \log_2(2K_1) \). Thus in all cases \( \log_2(n + K_1) \leq K_2 + \log_2 n \) for a suitable constant \( K_2 > 0 \). Thus

\[
|\psi(n)| = t \leq K_3 + K_4 \log n
\]

for suitable constants \( K_3 \) and \( K_4 > 0 \). Now apply Lemma 6.

References


