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A Comprehensive Theoretical Study on the Rank of the Integral Operators for Large-Scale Electrodynamic Analysis

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Abstract

We propose an analytical approach to study the rank of an integral operator, which is valid for an arbitrarily shaped object with an arbitrary electric size. With this analytical approach, we theoretically prove that for a prescribed error bound, the minimal rank of the interaction between two separated geometry blocks in an integral-equation operator, asymptotically, is a constant for 1-D distributions of source and observation points; grows very slowly with electric size as square root of the logarithm for 2-D distributions; and scales linearly with the electric size of the block diameter for 3-D problems. We thus prove the existence of an errorbounded low-rank representation of both surface- and volume-based integral operators for electromagnetic analysis, irrespective of electric size and object shape. Numerical experiments have validated the proposed analytical approach and its resultant findings on the rank of integral operators. This work provides a theoretical basis for employing and further developing low-rank matrix algebra for accelerating the computation of electrically large problems.

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Wenwen Chai, Student Member, IEEE and Dan Jiao, Senior Member, IEEE

Abstract—We propose an analytical approach to study the rank of an integral operator, which is valid for an arbitrarily shaped object with an arbitrary electric size. With this analytical approach, we theoretically prove that for a prescribed error bound, the minimal rank of the interaction between two separated geometry blocks in an integral-equation operator, asymptotically, is a constant for 1-D distributions of source and observation points; grows very slowly with electric size as square root of the logarithm for 2-D distributions; and scales linearly with the electric size of the block diameter for 3-D problems. We thus prove the existence of an error-bounded low-rank representation of both surface- and volume-based integral operators for electromagnetic analysis, irrespective of electric size and object shape. Numerical experiments have validated the proposed analytical approach and its resultant findings on the rank of integral operators. This work provides a theoretical basis for employing and further developing low-rank matrix algebra for accelerating the computation of electrically large problems.

Index Terms—Rank, Integral Operators, Electrodynamic Analysis, One-, Two-, and Three-Dimensional Analysis, Theoretical Analysis

I. INTRODUCTION

RIVEN BY the design of advanced engineering D systems, there exists a continued need of reducing the complexity of computational electromagnetic methods. Recently, the \mathcal{H} - and \mathcal{H}^2 -matrix based mathematical framework [1-2] has been introduced and further developed to accelerate both iterative and direct solutions of the integral equation based analysis of electrodynamic problems [3-5]. The resultant direct integral equation (IE) solver [4-5] successfully solved electrodynamic problems of 96 wavelengths with more than 1 million unknowns in fast CPU time (less than 20 hours in LU factorization, 85 seconds in LU solution), modest memory consumption, and with the prescribed accuracy satisfied, on a single CPU running at 3 GHz. The \mathcal{H} - and \mathcal{H}^2 -matrix based mathematical framework [1-2] encompasses a family of hierarchical low-rank matrix algebra that enables compact representation and efficient computation of dense matrices.

It has been acknowledged that low-rank methods [7, 10-12] are applicable to electrically small or moderate problems. However, why the low-rank property can also be explored to accelerate the computation of electrically very large problems? The ACA-based low-rank solutions have also solved electrically large integral equations with over 1 million unknowns [8-9]. Does an error-bounded low-rank approximation of integral operators exist, regardless of electric size?

It has been shown in [13] that electromagnetic fields, radiated or scattered by bounded sources, can be accurately represented by a finite number of samples, coincident with the number of degrees of freedom of the field, which is independent of the observation domain and depends only on the source geometry. This study is performed based on a representation that *separates* sources from observers. In [14], the section of theory based on [13] shows that the rank of the interaction between two separated blocks in a 3-D surface scatterer scales quadratically with the electric size of the block diameter. However, numerically by ACA and SVD, such a quadratic growth with electric size was not observed, as stated by the authors of [14].

Given an accuracy requirement ε , it has been proven that the rank-r representation (**R**) generated from singular value decomposition (SVD) is a minimal rank approximation of the original matrix **M** that fulfils $||\mathbf{M} - \mathbf{R}||_2 \le \varepsilon$ [15]. The SVD based minimal-rank approximation does not separate observation and source coordinates. It treats the entire matrix as a whole and finds a minimal number of vectors, and hence rank, to represent the matrix with prescribed accuracy. Our numerical experiments show that methods that do not generate a minimal rank approximation such as the interpolation [3], Taylor series expansion, and plane-wave expansion based separation of source and observation coordinates can result in a rank that is much higher than the minimal rank required by accuracy. The rank also scales with electric size at a rate higher than linear, as observed in existing fast IE solvers that rely on the separation of source and observation coordinates. To be more specific, in a source-observer separated representation of the integral operator, the Green's function $g(|\vec{r} - \vec{r}'|)$, which originally is the function of the *distance* between source \vec{r} ' and observer \vec{r} , becomes a function of the complete coordinates of \vec{r} ' and \vec{r} since it is approximated by a $f_1(\vec{r}) f_2(\vec{r})$ -type form to separate \vec{r} ' from \vec{r} .

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In [5], through a singular value decomposition (SVD) based analysis, it is numerically shown, for large electric sizes (over 100 wavelengths) and various scatterers, the rank of a matrix block of size N formed between two geometrically separated groups, arising from the surface integral equation based electrodynamic analysis, scales as $O(N^{0.5})$. As a result, the block has a low rank. However, no theoretical proof has been developed to support this numerical finding.

An SVD analysis is numerical, which makes it not feasible to find the actual rank required by accuracy for an arbitrarily large electric size. As a result, an analytical approach, which is not restricted by computational resources and is valid for arbitrary shape, becomes necessary to develop a theoretical understanding on the rank's dependence with electric size.

The contribution of this work is such an analytical approach. With his approach, we theoretically prove that the minimal rank of the interaction between two separated geometry blocks in an integral-equation based analysis of general 3-D objects, for a prescribed error bound, scales *linearly* with the electric size of the block diameter. For 2-D distributions of sources and observers, the minimal rank grows very slowly with electric size as square root of the logarithm of the electric size of the block diameter; for 1-D distributions, the minimal rank is a constant. These findings also agree with our finding on the rank of the inverse finite element matrix [18]. The proposed proof is applicable to various integral operators in electrodynamic analysis such as electric field, magnetic field, combined field, surface-, and volume-based integral operators. Since the rank scales linearly with the electric size of the block diameter, while the number of unknowns in a surface- and volume-IE based analysis scales with electric size in a quadratic, and cubic way respectively, we prove the existence of the error-bounded low-rank representation of both surface and volume integral operators for electromagnetic analysis, irrespective of electric size and problem shape.

II. THEORETICAL STUDY

A. Problem Description

The integral equation based analysis of electrodynamic problems results in a dense linear system of equations

$$\mathbf{Z}I = V \ . \tag{1}$$

Consider $\mathbb{Z}^{t,s}$, an arbitrary $m \times n$ off-diagonal block of the system matrix \mathbb{Z} , which describes the interaction between two separated groups (*t* and *s*) of the scatterer being analyzed. The objective of this work is to theoretically study whether there exists an error-bounded low-rank representation of $\mathbb{Z}^{t,s}$ irrespective of electric size and scatterer shape, and if such a representation exists, how the rank scales with electric size, and hence the number of unknowns *N*.

Given an accuracy requirement ε , as shown in [15], the rank-*r* representation (**R**) generated from SVD is a minimal rank approximation of the original matrix **M** that fulfils ||**M** – **R** $||_2 \le \varepsilon$. However, an SVD analysis is numerical. Restricted by computational resources, it cannot be used to find the actual rank required by accuracy for an arbitrarily large electric size. As a result, an analytical approach, which is not limited by computational resources and is valid for an arbitrary shape, becomes necessary to develop a theoretical understanding on the rank's dependence with electric size. This paper provides

such an analytical approach. In this approach, we are able to make a connection between an SVD analysis and a Fourier analysis. By utilizing the relationship between the two analyses in a linear and shift-invariant system, we succeed in analytically revealing the rank of the integral operators and its dependence with electric size.

B. Relationship between SVD and Fourier Analysis in a Linear Shift-Invariant System

A linear system can be modeled by:

$$b = \mathbf{H}f , \qquad (2)$$

where f and b are vectors, and **H** is a linear operator. We can perform SVD on **H** to obtain

$$b = \mathbf{V} \boldsymbol{\Sigma} \mathbf{U}^H f , \qquad (3)$$

where superscript 'H' denotes a complex conjugate transpose, Σ is the diagonal matrix comprising singular values, and V and U are matrices comprising singular vectors. Since V and U are both unitary, we have

$$\mathbf{V}^{H}b = \mathbf{\Sigma} \left(\mathbf{U}^{H}f \right) \,. \tag{4}$$

which can be written compactly as

$$b^{V} = \Sigma f^{U} , \qquad (5)$$

$$b^{V} = \mathbf{V}^{H}b; \ f^{U} = \mathbf{U}^{H}f \ . \tag{6}$$

Multiplying a unitary matrix by a vector can be thought of as projecting this vector onto the orthonormal set defined by the matrix. Thus, (5) can be viewed as representing the response b in the **V** basis (b^{V}), the input f in the **U** basis (f^{U}), and relating these two projections by a diagonal matrix (Σ).

When the operator **H** is both linear and shift invariant (LSIV), SVD turns to Fourier analysis [16]. More specifically, the singular vectors of an LSIV system are weighted Fourier basis functions (complex exponentials) and the singular values are the absolute values of the Fourier transform of the system's point spread function (impulse response function) [16, 17]. To see this more clearly, let's consider an LSIV system. Because an LSIV system operator is a convolution operator [16], the response *b* in space domain is a convolution of the input *f* with an impulse response *h*

$$b(\vec{r}) = f(\vec{r}) * h(\vec{r})$$
, (7)

in which \vec{r} denotes an arbitrary point in space. The above convolution can be converted to a simple multiplication by Fourier analysis. Thus we have

$$\mathcal{F}(b(\vec{r})) = \mathcal{F}(h(\vec{r})) \mathcal{F}(f(\vec{r})) , \qquad (8)$$

where $\mathcal{F}(\)$ denotes a Fourier transform. We can rewrite (8) as

$$b(\vec{r})^{FT} = \mathcal{F}(h(\vec{r}))f(\vec{r})^{FT} , \qquad (9)$$

where $b(\vec{r})^{FT}$ is the representation of $b(\vec{r})$ in the Fourier basis, and $f(\vec{r})^{FT}$ is the representation of $f(\vec{r})$ in the Fourier basis. In other words, we represent the input in a unitary basis (Fourier basis), we also represent the response in a unitary basis (Fourier basis), and relate the two by $\mathcal{F}(h(\vec{r}))$. From (5) and (9), the relationship between SVD and Fourier analysis can be clearly seen. The Fourier bases may be different from the SVD-generated bases. However, if the system is linear and shift-invariant, the two bases are both Fourier bases [16]. Therefore, the Fourier analysis accomplishes the SVD analysis of a linear shift-invariant system.

C. Rank Revealing via Fourier Analysis of the Integral Operator

There exist many integral equation based formulations for analyzing 3-D electrodynamic problems. Examples are electric field integral equation, magnetic field integral equation, combined field integral equation, each of which can be formulated in a surface- or volume-based form. The underlying integral operators are all linear and shift invariant. Therefore, we can use Fourier analysis to analytically study the rank of the integral equation based system matrix.

The point-spread function in IE-based operators is Green's function. Without loss of generality, an integral equation based operator can be expressed as the convolution of a certain source f with Green's function g as the following:

$$b(\vec{r}) = \int g(|\vec{r} - \vec{r}'|) f(\vec{r}') d\vec{r}', \qquad (10)$$

where response *b* is the field at observation point \vec{r} , and \vec{r}' denotes a source point. The $d\vec{r}$ is a short notation of *dl* (line integral), *dS* (area integral), *dV* (volume integral) over the source domain, respectively, for one-, two-, and three-dimensional distribution of the sources.

Multiplying both sides of (10) by $e^{-j\vec{k}\cdot\vec{r}}$, and integrate over the observation domain \vec{r} , we obtain

$$\int b(\vec{r})e^{-j\vec{k}\cdot\vec{r}}d\vec{r} = \int \left[\int g(|\vec{r}-\vec{r}'|)f(\vec{r}')d\vec{r}'\right]e^{-j\vec{k}\cdot\vec{r}}d\vec{r}, \quad (11)$$

which can be further written as

 $\int b(\vec{r}) e^{-j\vec{k}\cdot\vec{r}} d\vec{r} =$ $\int f(\vec{r}') e^{-j\vec{k}\cdot\vec{r}'} d\vec{r}' \int g(|\vec{r}-\vec{r}'|) e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')} d(\vec{r}-\vec{r}')',$

and thereby

$$B(\vec{k}) = G(\vec{k})F(\vec{k}), \qquad (13)$$

(12)

$$B(k) = \int b(\vec{r}) e^{-j\vec{k}\cdot\vec{r}} d\vec{r}$$

$$F(\vec{k}) = \int f(\vec{r}') e^{-j\vec{k}\cdot\vec{r}'} d\vec{r}'$$

$$G(\vec{k}) = \int g(|\vec{r}-\vec{r}'|) e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')} d(\vec{r}-\vec{r}')$$
(14)

in which $\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$. The $B(\vec{k})$, $F(\vec{k})$, and $G(\vec{k})$ are the Fourier transforms of the sources *f*, observation fields *b*, and Green's function *g* respectively.

Eqn. (13), in a discrete form, can be written as:

$$\begin{cases}
B_1 \\
B_2 \\
\vdots \\
B_p
\end{cases} = \begin{pmatrix}
G_1 \\
G_2 \\
\vdots \\
G_p
\end{pmatrix} \begin{vmatrix}
F_1 \\
F_2 \\
\vdots \\
F_p
\end{vmatrix},$$
(15)

where G_i , F_i , and B_i are, respectively, $G(\vec{k})$, $F(\vec{k})$, and $B(\vec{k})$ at discrete \vec{k}_i (i = 0, 1, ...), and p denotes the number of frequency points in Fourier transform. Now consider an arbitrary source domain Ω_s that is geometrically disconnected from an arbitrary observation domain Ω_t . The number of degrees of freedom in the source domain is denoted by *n*, while that in the observation domain is denoted by *m*. The matrix block corresponding to the interaction between Ω_s and Ω_t is an off-diagonal block in the system matrix **Z** resulting from an IE-based analysis of an electromagnetic problem. Denote this block by **Z**^{*t,s*}. Thus, we have

$$b_{m\times 1} = (\mathbf{Z}^{t,s})_{m\times n} f_{n\times 1}, \qquad (16)$$

in which the subscripts denote the dimension of the corresponding vector or matrix. Given a prescribed accuracy, the minimal rank of $\mathbf{Z}^{t,s}$ can be numerically determined by SVD. Next, we show how to analyze the rank of $\mathbf{Z}^{t,s}$ analytically by the Fourier analysis of the integral operator.

From (14), the B vector in (15) can be written as:

$$\begin{vmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{vmatrix} = \mathbf{B}_{p \times m} \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{vmatrix} = \mathbf{B}_{p \times m} b_{m \times 1}$$
(17)

where $\mathbf{B}_{p \times m}$ is the $p \times m$ matrix that projects the observations b onto the space of Fourier modes. Its *ij*-th matrix element can be readily identified from (14) as

$$\mathbf{B}_{i,j} = \int_{j} e^{-j\vec{k}_{i}\cdot\vec{r}} d\vec{r}, \qquad 1 \le i \le p, \ 1 \le j \le m \ , \ (18)$$

where the integral is evaluated on the domain occupied by the *j*-th observer. For 1-, 2, and 3-D distribution of the observers, such a domain is a segment, an area, and a volume respectively. Similarly, from (14), the *F* vector in (15) can be written as:

Similarly, from (14), the F vector in (15) can be written as:

$$\begin{cases} F_1 \\ F_2 \\ \vdots \\ F_p \end{cases} = \mathbf{F}_{p \times n} \begin{cases} f_1 \\ f_2 \\ \vdots \\ f_n \end{cases} = \mathbf{F}_{p \times n} f_{n \times 1} ,$$
 (19)

where $\mathbf{F}_{p \times n}$ is the $p \times n$ matrix that projects the sources *f* onto the space of Fourier modes. Its *ij*-th matrix element can also be readily identified from (14) as

$$\mathbf{F}_{i,j} = \int_{j} e^{-jk_i \cdot \vec{r}} d\vec{r} , \qquad 1 \le i \le p, \ 1 \le j \le n \ , \ (20)$$

where the integral is evaluated on the domain occupied by the *j*-th source.

Substituting (17) and (19) into (15), we obtain

$$\mathbf{B}b = \mathbf{G}(\mathbf{F}f) \quad , \tag{21}$$

where **G** in boldface denotes the diagonal matrix composed of the Fourier coefficients of Green's function shown in (15). If **B** is unitary, then

$$b = (\mathbf{B}^{\mathrm{H}}\mathbf{G}\mathbf{F})f \quad . \tag{22}$$

In the context of matrix computation, the source and observation domain represented by an off-diagonal block are both finite. In addition, they may not span a period used for evaluating the discrete Fourier transform. Although Fourier bases are unitary, if a subset of these bases is chosen at selected source and observation points, neither **B** nor **F** is unitary. In this case, (21) can be written as

$$b = (\mathbf{B}^{\mathrm{H}}\mathbf{B})^{-1}(\mathbf{B}^{\mathrm{H}}\mathbf{G}\mathbf{F})f \quad . \tag{23}$$

With *p* chosen to be larger than *m*, $(\mathbf{B}^{H}\mathbf{B})$ is invertible. If $(\mathbf{B}^{H}\mathbf{B})$ is not invertible, we can also write **B** as

 $\mathbf{B} = \mathbf{U}_{B} \boldsymbol{\Sigma}_{B} \mathbf{V}_{B}^{H} , \qquad (24)$ which is the SVD of **B**. Then, we have

$$b = (\mathbf{V}_B \boldsymbol{\Sigma}_B^{-1} \mathbf{U}_B^H \mathbf{G} \mathbf{F}) f \quad . \tag{25}$$

Therefore, we obtain

$$\mathbf{Z}^{t,s} = \left(\mathbf{V}_{B}\boldsymbol{\Sigma}_{B}^{-1}\mathbf{U}_{B}^{H}\mathbf{GF}\right).$$
(26)

Thus, it is clear that if **B** and **F** are unitary, then the singular values of $\mathbf{Z}^{t,s}$ are nothing but the absolute values of **G**'s entries, which are the Fourier expansion coefficients of Green's function. In general cases where **B** and **F** may not be unitary, although the singular values are not the Fourier coefficients any more, the rank of $\mathbf{Z}^{t,s}$ is still bounded by the rank of diagonal matrix **G** since the rank of a matrix product is no greater than any of the matrices being multiplied. Therefore, we can analyze the Fourier transform of Green's function to analytically study the rank of $\mathbf{Z}^{t,s}$.

D. Rank Determined from an Analytical Fourier Analysis of the Green's Function

The Green's function for a general 3-D problem can be written as:

$$g(|\vec{r} - \vec{r}'|) = \frac{e^{-jk_0|\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|},$$
(27)

where \vec{r} denotes a source point, \vec{r} denotes an observation point, k_0 is the wave number corresponding to a frequency being studied. Let

$$\vec{R} = \vec{r} - \vec{r}' = R\hat{R} \tag{28}$$

with *R* being the magnitude of the distance vector \vec{R} and \hat{R} a unit vector along $\vec{r} - \vec{r}$ direction, (27) can be further written as

$$g(R) = \frac{e^{-j\kappa_0 \kappa}}{4\pi R} .$$
 (29)

The above Green's function satisfies the following partial differential equation in an infinite space

$$\nabla^2 g + k_0^2 g = \delta(\vec{r} - \vec{r}') .$$
 (30)

Its Fourier transform can be analytically obtained as the following.

First, we represent the right hand side of (30) by its Fourier transform

$$\delta(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \iiint e^{jk_x(x-x')} e^{jk_y(y-y')} e^{jk_z(z-z')} dk_x dk_y dk_z . (31)$$

Similarly, we write Green's function g as

$$g(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3} \iiint G(k) e^{jk_x(x-x')} e^{jk_y(y-y')} e^{jk_z(z-z')} dk_x dk_y dk_z$$
(32)

where G(k) is the Fourier transform of g. Substituting (32) into the left hand side of (30), we obtain

$$\nabla^{2}g + k_{0}^{2}g = \frac{1}{(2\pi)^{3}} \iiint G(k) \nabla^{2} [e^{jk_{x}(x-x)} e^{jk_{y}(y-y)} e^{jk_{z}(z-z)}] dk_{x} dk_{y} dk_{z}$$

+ $\frac{1}{(2\pi)^{3}} \iiint k_{0}^{2}G(k) e^{jk_{x}(x-x)} e^{jk_{y}(y-y)} e^{jk_{z}(z-z)} dk_{x} dk_{y} dk_{z}$
= $\frac{1}{(2\pi)^{3}} \iiint (-k^{2} + k_{0}^{2})G(k) e^{jk_{x}(x-x)} e^{jk_{y}(y-y)} e^{jk_{z}(z-z)} dk_{x} dk_{y} dk_{z}$
(33)

where

$$k^2 = k_x^2 + k_y^2 + k_z^2 . aga{34}$$

Since (33) is equal to (31), we obtain the Fourier transform of Green's function as the following

$$G(k) = \frac{1}{k_0^2 - k^2} \,. \tag{35}$$

The above approach was actually one of the methods used to derive Green's function in history, also known as Ohm-Rayleigh method [19, p. 30].

If both source points \vec{r} and observation points \vec{r} are distributed in a 2-D domain, without loss of generality, assume z = z'. From (33), since the term operated on by ∇^2 is $e^{jk_x(x-x)}e^{jk_y(y-y')} \cdot 1$, we obtain

$$k^2 = k_x^2 + k_y^2 . (36)$$

Notice that the above does not suggest that $\partial/\partial z = 0$ because when the right hand side of (32) is integrated out, we obtain $\delta(z-z')$. It simply means for 2-D distribution of observers and sources, there is no need to introduce Fourier modes in the third dimension to represent Green's function. If both source points \vec{r} and observation points \vec{r} are distributed in a 1-D domain satisfying z = z' and y = y', from (33), we have

$$k^2 = k_x^2 . (37)$$

Again, the above does not suggest that $\partial/\partial y = \partial/\partial z = 0$ because when (32) is integrated out, we obtain $\delta(y-y')\delta(z-z')$. As can be seen from the above derivation, the Fourier transform of Green's function for 1-, 2-, and 3-D distributions of source points and observation points has the same form as that shown in (35). The only difference is the difference in k^2 .

Now, we are ready to determine the rank of **G** in (26). The **G** is the diagonal matrix shown in (15), the entries of which are given in (14), which are the Fourier coefficients of Green's function

$$G(k_i) = \int g(|\vec{r} - \vec{r}'|) e^{-j\vec{k}_i \cdot (\vec{r} - \vec{r}')} d(\vec{r} - \vec{r}') .$$
(38)

For a finite source-observation domain, the geometrical identity defined by $(\vec{r} - \vec{r}')$ is finite. Take a 3-D $(\vec{r} - \vec{r}')$ domain as an example, (38) can be explicitly written as

$$G(k_i) = \int_{\varphi_i}^{\varphi_2} \int_{\theta_i}^{\theta_2} \int_{R_i}^{R_2} \frac{e^{-jk_0R}}{4\pi R} e^{-j\vec{k}_i \cdot \vec{R}} R^2 \sin\theta dr d\theta d\varphi , \qquad (39)$$

where the upper and lower limits describe the region that $(\vec{r} - \vec{r}')$ occupies. If $(\vec{r} - \vec{r}')$ has multiple disconnected regions, then the matrix block corresponding to such a source-observation interaction is the union of the matrix block in each separated region. Then for each separated region, we can analyze the rank of the corresponding matrix block via (39).

(45)

The sum of the rank of the matrix block for each separated region is the upper bound of the rank of the entire matrix block associated with the interaction between \vec{r} and \vec{r} .

Let $(\varphi_1, \varphi_2) = (0, 2\pi)$, $(\theta_1, \theta_2) = (0, \pi)$, and $R_1 \rightarrow 0, R_2 \rightarrow \infty$. Then (39) becomes (35), and hence

$$G(k_i) = \frac{1}{k_0^2 - (k_{xi}^2 + k_{yi}^2 + k_{zi}^2)},$$
(40)

where

$$k_{xi} = m \frac{2\pi}{D} = \frac{m\pi}{a}$$

$$k_{yi} = n \frac{2\pi}{D} = \frac{n\pi}{a}$$

$$k_{yi} = p \frac{2\pi}{D} = \frac{p\pi}{a}$$
(41)

in which *m*, *n*, *p* are integer numbers, *D* is the maximal size of the problem along *x*-, *y*-, and *z*-direction, and *a* is half of *D*. In what follows, we use (40) and its corresponding 2-D and 1-D forms to analytically analyze the rank of **G** because the rank of a smaller $(\vec{r} - \vec{r}')$ domain determined by (39) is bounded by the rank dictated by (40).

Without performing a detailed quantitative analysis, from (40), we already can predict the existence of a low-rank representation of Green's function. The reason is straightforward. Given a k_0^2 , not all of the Fourier modes have a large Fourier coefficient, only those whose wave number square (k_i^2) are the closest to k_0^2 have the largest Fourier coefficients, while others can be truncated based on the magnitude of their Fourier coefficients and a prescribed accuracy. The total number of Fourier modes representing a function defined on a surface and that defined in a volume is, respectively, proportional to (electric size)² and (electric size)³. Thus, the total number of Fourier modes is linearly proportional to N. However, Green's function is different from an arbitrary function that depends on x, y, and z, due to its R-only dependence, its Fourier transform has a special form shown in (40). As a result, only a subset of Fourier modes needs to be used to represent the Green's function for a given accuracy, while the rest can be discarded without sacrificing the prescribed accuracy. Hence the rank of **G** is less than *N*, thus being low rank. In addition, (40) also reveals why the rank of a 2-D problem is, in general, less than that of a 3-D distribution of sources and observers. This is because in the former, the Fourier modes are distributed on a 2-D grid as can be seen from (36) and (41), while in the latter; the Fourier modes are distributed on a 3-D lattice. Thus, the number of Fourier modes satisfying a prescribed accuracy in a 2-D case is smaller than that in a 3-D case. The above analysis is conceptual. Next, we provide a quantitative analysis of the rank of **G** and its dependence with electric size.

Given an accuracy requirement ε , the rank of diagonal matrix **G** is the number of Fourier coefficients $G_i = G(k_i)$ satisfying the following criterion

$$\frac{G_i}{\max\{G_i\}} = \frac{1/|k_i^2 - k_0^2|}{\max\{1/|k_i^2 - k_0^2|\}_i} \ge \varepsilon \cdot$$
(42)

Since the maximum of $1/|k_i^2 - k_0^2|$ occurs at the minimum of $|k_i^2 - k_0^2|$, (42) can be written as

$$\frac{1/\left|k_{i}^{2}-k_{0}^{2}\right|}{1/\left|k_{i}^{2}-k_{0}^{2}\right|_{\min}} \ge \varepsilon \cdot$$
(43)

Let

$$\Delta_{\min} = 1 / \left| k_i^2 - k_0^2 \right|_{\min}, \qquad (44)$$

We have

with

$$k_i^2 = \left[(m\pi)^2 + (n\pi)^2 + (p\pi)^2 \right] / a^2$$
(46)

$$k_i^2 = \left\lfloor (m\pi)^2 + (n\pi)^2 \right\rfloor / a^2 \tag{47}$$

$$k_i^2 = \lfloor (m\pi)^2 \rfloor / a^2 \tag{48}$$

for 3-, 2-, and 1-D distribution of sources and observers respectively.

 $\left|k_{i}^{2}-k_{0}^{2}\right|\leq\Delta_{\min}/\varepsilon,$

To determine the rank from (45), we can find out the maximum displacement $\Delta_k > 0$ satisfying

$$\left(k_0 + \Delta_k\right)^2 - k_0^2 \le \Delta_{\min} / \varepsilon, \qquad (49)$$

and then compute the number of modes that can exist between k_0 and $k_0 + \Delta_k$. For the modes satisfying (45) and having k_i^2 smaller than k_0^2 , a similar analysis can be performed.

In 1-D cases, since $k_i = \frac{m\pi}{a}$, the distance between two adjacent wave number k_i is a constant. The number of Fourier modes between k_0 and $k_0 + \Delta_k$ is thus proportional to Δ_k . Therefore, the rank k in 1-D cases can be written as:

$$k\big|_{1D} \sim \Delta_k \,. \tag{50}$$

In 2-D cases, since the number of Fourier modes having a wave number between k_0 and $k_0 + \Delta_k$ is proportional to the area of a ring with inner radius of k_0 and outer radius of $k_0 + \Delta_k$. Thus, the rank k in 2-D cases can be written as:

$$k\big|_{2D} \sim \left(k_0 + \Delta_k\right)^2 - k_0^2 = 2k_0 \Delta_k + \Delta_k^2 \,. \tag{51}$$

In 3-D cases, the number of Fourier modes having a wave number between k_0 and $k_0 + \Delta_k$ is proportional to the volume of a spherical ring with inner radius of k_0 and outer radius of $k_0 + \Delta_k$. Thus,

$$k\big|_{3D} \sim \left(k_0 + \Delta_k\right)^3 - k_0^3 = 3k_0^2 \Delta_k + 3k_0 \Delta_k^2 + \Delta_k^3.$$
 (52)

From the above, it can be seen that the rank's dependence with electric size is determined by the Δ_k 's dependence with electric size. This question has been thoroughly studied in [18]. In fact, the Fourier transform of Green's function has a direct relationship with the inverse of the finite element matrix by comparing (35) to the inverse of the finite element matrix shown in [18]. It is proved in [18] that Δ_k for 1-, 2-, and 3-D modes satisfying (49), for a given ε , scale with frequency in the following way:

$$\Delta_k \Big|_{1D} \sim O(1) \,. \tag{53}$$

$$\Delta_k \Big|_{2D} \sim O(\sqrt{\log k_0} \,/\, k_0) \tag{54}$$

$$\Delta_k \Big|_{3D} \sim O(1/k_0) \,. \tag{55}$$

(56)

Substituting them into (50-52), we obtain

$$Rank_{1D} = constant$$
.

$$Rank\Big|_{2D} \sim O(\sqrt{\log k_0}) \tag{57}$$

$$Rank\Big|_{3D} \sim O(k_0) \,. \tag{58}$$

Thus, for 1-D problems, for a prescribed error bound, the rank is a constant; for 2-D problems, the scaling rate of the rank is less than linear; while for 3-D problems, the rank increases linearly with the electric size of the problem. The theoretical results shown in (56-58) have also been numerically verified by finding out the number of modes having wave numbers shown in (46-48) and meanwhile satisfying (45) [18].

E. Implication on the Complexity of IE-Based Computation

The rank's scaling rate with electric size has a direct impact on the complexity of low-rank based methods in electrodynamic computation. Take the \mathcal{H}^2 matrix based method as an example [2-3], due to its *nested* low-rank representation, with the rank scaling with electric size linearly in a general 3-D problem, the storage and the matrix-vector multiplication of a dense matrix, resulting from a surface IE-based analysis, both have a complexity of $O(N\log N)$, with N being matrix size. The detailed complexity analysis is given below.

In an \mathcal{H}^2 -based representation of a dense matrix **Z**, each block that characterizes the interaction between a source domain that is geometrically separated from an observation domain is called an admissible block. Consider an arbitrary admissible block $\mathbf{Z}^{t,s}$. It is represented by a factorized low-rank form $\mathbf{V}^{t}\mathbf{S}^{t,s}\mathbf{V}^{sT}$ with **V** being nested in an \mathcal{H}^{2} -representation. The entire unknown set of \mathbf{Z} is partitioned into two subsets level by level until the leaf level is reached based on a predetermined constant *leafsize*. Each node in the resulting binary tree is called a cluster. Since V is nested, we only need to store $V^{\#t \times k}$ at leaf clusters, and for each nonleaf cluster, we store transfer matrices $\mathbf{E}^{k \times k}$. Here, #*t* denotes the number of unknowns in cluster *t*, and k is the rank. The coupling matrix $\mathbf{S}^{k \times k}$ is stored for each admissible block at each tree level. The storage of an \mathcal{H}^2 -based representation of Z, which is also the cost of a matrix-vector multiplication, hence can be evaluated as the following:

$$Storage(\mathbf{Z}) = Cost(\mathbf{Z}v) = \sum_{l=0}^{r} \left[\left(kb_{l} \right)^{2} \cdot \left(2^{l} + nb_{l} \right) \right]$$
$$\leq 2 \cdot \sum_{l=0}^{p} \left[O\left(\left(\sqrt{2^{-l}} \right)^{2} \right) \cdot N \cdot 2^{l} \right] \leq 2N \cdot \sum_{l=0}^{p} \left[O\left(2^{-l} \right) \cdot 2^{l} \right]$$
(59)
$$= O(N \log N)$$

In the above, *v* denotes an arbitrary vector, *P* is the tree depth, kb_l is the rank of the admissible blocks at tree level *l*, and nb_l is the number of admissible blocks at tree level *l*. Due to a binary tree, the number of clusters is 2^l at level *l*, where l = 0represents the root level of the inverted binary tree. In the first row of (59), $(kb_l)^2 2^l$ is the cost of storing the transfer matrix $\mathbf{E}^{k \times k}$ of each nonleaf cluster at level *l*, where $k = kb_l$, while the $(kb_l)^2 nb_l$ is the cost of storing the coupling matrix $\mathbf{S}^{k\times k}$ for each admissible block at level *l*. In the second row of (59), we utilize the fact that the matrix size at level *l* is $2^{-l}N$, and the rank scales as the square root of it, thus $kb_l = \sqrt{2^{-l}N}$. We also utilize the fact that the number of admissible blocks at level *l* is proportional to the number of nodes (clusters) at this level, hence $nb_l = O(2^l)$. In the first row of (59), we add the cost of storing both the transfer matrices **E** and the coupling matrices **S** to compute the total cost. As for the cost of storing $\mathbf{V}^{\#t\times k}$ at leaf clusters, it is linear since there are O(N) leaf clusters, and each leaf $\mathbf{V}^{\#t\times k}$ has a cost of $O(leafsize \times leafsize)$, which is constant.

To summarize, when ascending an inverted binary tree, at each tree level, the number of matrix blocks is reduced by half, and the cost of storage as well as a matrix-vector multiplication is doubled since the cost scales as the square of the rank, and the rank scales as the square root of the matrix size in a surface IE based analysis. As a result, the computational complexity is the same O(N) at each tree level. Since there are $\log N$ levels, the total complexity is O(NlogN) for electrodynamic computation. This is the complexity of a conventional \mathcal{H}^2 -based method for electrodynamic computation. A further acceleration of the \mathcal{H}^2 -based method is possible with the understanding of the rank's actual growth rate with electric size. In addition, neither prevailing fast multipole based nor FFT-based methods have utilized the low-rank property of the electrodynamic kernels. If this property is utilized, these methods may be further accelerated.

III. NUMERICAL VALIDATION OF THE PROPOSED ANALYTICAL APPROACH FOR RANK STUDY

We first *quantitatively* validate the proposed analytical approach for analyzing the rank of the IE operators.

A. Example 1

The first example has a 1-D distribution of sources and observers. The source domain is in the range of $x' \in (-1.5A, -0.5A)$, while the observation domain is located at $x \in (0.5A, 1.5A)$. It is clear that $\vec{r} - \vec{r}' = (x - x', 0, 0)$ with $x - x' \in (A, 3A)$. The wavelength $\lambda = 1$ m, thus $k_0 = 2\pi$ in Green's function. The A is chosen as 4λ . A uniform discretization along x is used with a space step of $\Delta = 1/50\lambda$.

The Fourier transforms shown in (14) for sources, observations, and Green's function are performed in the same range of (a_1, a_2) with $a_1 = -3A$ and $a_2 = 3A$. Hence, we have

$$B(k_{i}) = \int_{a_{1}}^{a_{2}} b(x)e^{-jk_{i}x}dx$$

$$F(k_{i}) = \int_{a_{1}}^{a_{2}} f(x')e^{-jk_{i}x'}dx'$$

$$G(k_{i}) = \int_{a_{1}}^{a_{2}} g(|x-x'|)e^{-jk_{i}(x-x')}d(x-x')$$
(60)

with

$$k_i = i \frac{2\pi}{(a_2 - a_1)}, \quad i = 0, \pm 1, \pm 2, \dots$$
 (61)



Fig. 1. Comparison between the matrix generated from the proposed approach and the original matrix for a line example. (a) Real part of the first column of \mathbf{Z} . (b) Imaginary part.

The interval for integration (a_1, a_2) is chosen to carry out the discrete Fourier transforms of observations *b*, sources *f*, and Green's function *g* in a common range. The *f*, *b*, and *g* are padded with zeros in the range beyond where they are originally defined.

From (60), the diagonal matrix **G** is obtained with Fourier mode index $i \in (-ns/2+1, ns/2)$, where *ns* is the number of sampling points along *x*, which is $ns = (a_2 - a_1)/\Delta + 1$. The **B** matrix and **F** matrix are constructed based on (18) and (20). Specifically, their *ij*-th element for this example is

$$\mathbf{B}_{i,j} = e^{-jk_i x_j} \Delta, \quad \mathbf{F}_{i,j} = e^{-jk_i x_j'} \Delta, \tag{62}$$

where x_j and x'_j are, respectively, the *j*-th observation and source point, while *i* denotes the index of the Fourier mode. With **G**, **B**, **F** obtained, we construct **Z**^{*t*,*s*} based on (23), thus

$$\mathbf{Z}^{t,s} = (\mathbf{B}^{\mathrm{H}}\mathbf{B})^{-1}(\mathbf{B}^{\mathrm{H}}\mathbf{G}\mathbf{F}).$$
(63)

To assess the accuracy of the proposed approach, we compare (63) with the original matrix that is directly constructed from the following



Fig. 2. Comparison between the matrix generated from the proposed approach and the original matrix for a plate example. (a) Real part of the 998-th column of \mathbf{Z} . (b) Imaginary part.

$$\mathbf{Z}_{orig,ij}^{t,s} = \frac{e^{-jk_0|x_i - x'_j|}}{4\pi |x_i - x'_j|} \Delta.$$
(64)

In Fig. 1, we plot the first column of the matrix obtained from (63) in comparison with that of the original matrix shown in (64). An excellent agreement can be observed. The same agreement is observed in all the other columns of the $\mathbf{Z}^{t,s}$ matrix. To assess the entire matrix error of (63), we evaluate the following

$$error = \frac{\left\|\mathbf{Z}_{orig}^{t,s} - \mathbf{Z}^{t,s}\right\|}{\left\|\mathbf{Z}_{orig}^{t,s}\right\|} .$$
(65)

in which 2-norm is used. The error is shown to be 0.1%. Hence, the accuracy of the factorized form shown in (63) is validated. As a result, the rank of $\mathbf{Z}^{t,s}$ is bounded from above by the rank of diagonal matrix **G**.

B. Example 2

The second example is two separated plates that are horizontally displaced. One is located at $(0 \le x' \le A, 0 \le y' \le A, z'=0)$, the other at $(3A \le x \le 4A, 0 \le y \le A, z=0)$. It is clear that $\vec{r} - \vec{r}' = (x - x', y - y', 0)$ with $(x - x') \in (2A, 4A)$ and $(y - y') \in (-A, A)$. The wavelength is 1 m, thus $k_0 = 2\pi$ in Green's function. The A is chosen as 1 wavelength. A uniform discretization along both x and y is used with a space step $\Delta = 1/40 \lambda$.

The Fourier transforms shown in (14) for sources, observations, and Green's function are performed in a common range of $(a_1, a_2) = (0, 4A)$, and $(b_1, b_2) = (-A, A)$. Thus, we have

$$B(k_{i}) = \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} b(x, y) e^{-jk_{xi}x} e^{-jk_{yi}y} dx dy$$

$$F(k_{i}) = \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} f(x', y') e^{-jk_{xi}x'} e^{-jk_{yi}y'} dx' dy'$$

$$G(k_{i}) = ,$$

$$\int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} g(|x-x'|, |y-y'|) e^{-jk_{xi}(x-x')} e^{-jk_{yi}(y-y')} d(x-x') d(y-y')$$
(66)

where

$$k_{xi} = m \frac{2\pi}{(a_2 - a_1)}, \ k_{yi} = n \frac{2\pi}{(b_2 - b_1)}, \ m, n = 0, \pm 1, \pm 2, \dots.$$
 (67)

Based on (66), the diagonal matrix **G** is obtained with Fourier mode index $m \in (-nx/2+1, nx/2)$, and $n \in (-ny/2+1, ny/2)$, where nx is the number of sampling points along x which is $nx = (a_2 - a_1)/\Delta + 1$, and ny is the number of sampling points along y which is $ny = (b_2 - b_1)/\Delta + 1$. The **B** matrix and **F** matrix are constructed based on (18) and (20). Specifically, their *ij*-th element for this example is

$$\mathbf{B}_{i,j} = e^{-jk_{xi}x_j - jk_{yi}y_j} \Delta^2, \quad \mathbf{F}_{i,j} = e^{-jk_{xi}x_j' - jk_{yi}y_j'} \Delta^2.$$
(68)

We then construct $\mathbf{Z}^{t,s}$ based on (23), thus

$$\mathbf{Z}^{t,s} = (\mathbf{B}^{\mathsf{H}}\mathbf{B})^{-1}(\mathbf{B}^{\mathsf{H}}\mathbf{G}\mathbf{F}).$$

To assess the accuracy of the proposed approach, we compare the above with the original matrix that is directly constructed as the following

$$\mathbf{Z}_{orig,ij}^{t,s} = \frac{e^{-jk_0\sqrt{(x_i - x'_j)^2 + (y_i - y'_j)^2}}}{4\pi\sqrt{(x_i - x'_j)^2 + (y_i - y'_j)^2}} \,\Delta^2 \,. \tag{69}$$

In Fig. 1, we plot a randomly selected column (column 998) of the matrix obtained from the proposed approach in comparison with that of the original matrix shown in (69), an excellent agreement can be observed. In addition, we compute (65) to assess the entire matrix error, which is shown to be 2.3%.

C. Example 3

We also consider the same two plates as simulated in the above example but displaced normally by 2*A*. Thus, one plate is located at $(0 \le x' \le A, 0 \le y' \le A, z'=0)$, while the other is at $(0 \le x \le A, 0 \le y \le A, z=2A)$. This is a 3-D configuration of sources and observers. It is clear that $\vec{r} - \vec{r}' = (x - x', y - y', h)$ with $(x - x') \in (-A, A)$ and $(y - y') \in (-A, A)$. The other parameters are the same as used in the above example.

The Fourier transforms shown in (14) for sources, observations, and Green's function are performed in a common range of $(a_1, a_2) = (-A, A)$, and $(b_1, b_2) = (-A, A)$, and (0, h=2A). Hence, we have

$$B(k_{i}) = \int_{0}^{h} \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} b(x, y) \delta(z-h) e^{-jk_{xi}x} e^{-jk_{yi}y} e^{-jk_{zi}z} dxdydz$$

$$F(k_{i}) = \int_{0}^{h} \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} f(x', y') \delta(z-0) e^{-jk_{xi}x'} e^{-jk_{yi}y'} e^{-jk_{zi}z'} dx'dy'dz'$$

$$G(k_{i}) = \int_{0}^{h} \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} \frac{e^{-jk_{0}\sqrt{(x-x')^{2} + (y-y')^{2} + (z-z')^{2}}}}{\sqrt{(x-x')^{2} + (y-y')^{2} + (z-z')^{2}}} \delta(z-z'-h) \times$$

$$e^{-jk_{xi}(x-x')} e^{-jk_{yi}(y-y')} e^{-jk_{zi}(z-z')} d(x-x')d(y-y')d(z-z')$$
(70)

The above can be further evaluated as

$$B(k_{i}) = e^{-jk_{ij}h} \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} b(x, y) e^{-jk_{ii}x} e^{-jk_{yi}y} dxdy$$

$$F(k_{i}) = \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} f(x', y') e^{-jk_{ii}x'} e^{-jk_{yi}y'} dx'dy'$$

$$G(k_{i}) = e^{-jk_{ij}h} \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} \frac{e^{-jk_{0}\sqrt{(x-x')^{2} + (y-y')^{2} + h^{2}}}}{\sqrt{(x-x')^{2} + (y-y')^{2} + h^{2}}} \times$$

$$e^{-jk_{xi}(x-x')} e^{-jk_{yi}(y-y')} d(x-x')d(y-y')$$
(71)

Different from (69), now the original matrix becomes

$$\mathbf{Z}_{orig,ij}^{t,s} = \frac{e^{-jk_0}\sqrt{(x_i - x'_j)^2 + (y_i - y'_j)^2 + h^2}}{4\pi\sqrt{(x_i - x'_j)^2 + (y_i - y'_j)^2 + h^2}} \Delta^2 .$$
(72)



Fig. 3. Comparison between the matrix generated from the proposed approach and the original matrix for the second plate example. (a) Real part of the 800-th column of \mathbf{Z} . (b) Imaginary part the 800-th column of \mathbf{Z} .

In Fig. 3, we plot a randomly selected column (column 800) of the matrix obtained from the proposed approach in comparison with that of the original matrix shown in (72). An excellent agreement can be observed. In addition, we compute (65) to assess the entire matrix error, which is shown to be 2.4%.

In addition to the above three examples, we have also tested many other examples. They all demonstrate the correctness of the factorized form shown in (23) thus (26) obtained via a Fourier transform of the integral operator. In fact, performing a Fourier transform on a convolution integral is the technique underlying existing FFT-based IE solvers. Therefore, in addition to a theoretical proof developed in Section II.C, we also numerically prove that one can use **G**'s rank to analytically analyze the rank of an IE operator irrespective of the operator kind, scatterer shape, and electric size.

IV. NUMERICAL VALIDATION OF THE RANK'S DEPENDENCE WITH ELECTRIC SIZE

In Section II.D, we theoretically deduce the rank's growth rate with electric size from the Fourier transform of Green's function. In this section, we numerically validate our theoretical findings of the rank by performing an SVD to find out the minimal rank required by a given accuracy.

A. Two separated lines

The first example simulated has a 1-D distribution of sources and observers. It is the same as the example described in Section III.A, but the side length A is increased from 1λ to 100λ to study the rank's dependence with electric size. The dense matrix that characterizes the interaction between the source line and the observation line has the following elements

$$\mathbf{Z}_{ij} = \frac{e^{-jk_0|x_i - x'_j|}}{|x_i - x'_j|} .$$
(73)

The mesh density chosen is 10 segments per wavelength. After constructing **Z** based on (73), we perform an SVD on **Z**. In Fig. 4(a), we plot the normalized singular values sorted in a descending order obtained from 1λ to 100λ versus singular value index. There are 100 lines in this figure. However, they all overlap with each other above 10^{-14} accuracy. The singular values below 10^{-14} are more than 14 orders of magnitude smaller than the largest singular value. Due to machine precision, these singular values cannot be accurately obtained by computers, thus they differ from one simulation to the other simulation, and hence cannot be used to study the rank's growth with electric size.

In Fig. 4(b), we plot the rank of \mathbf{Z} versus electric size for two different accuracy settings. The rank is determined by the number of singular values that satisfy the following criterion

$$\frac{\sigma_i}{\sigma_1} \ge \varepsilon \quad . \tag{74}$$

where σ_i is the i-th singular value, σ_1 is the largest singular value, and ε is the accuracy requirement, which is chosen as 10^{-4} , and 10^{-8} respectively. It is clear that the rank is a constant regardless of electric size. This is not a surprising result since it is already shown by Fig. 4(a). Since the singular-value lines for



Fig. 4. Rank study of the interaction between two separated lines. (a) Singular value distributions for 100 different electric sizes from 1 to 100 wavelengths. (b) Rank for two accuracy requirements versus electric size.

different electric sizes are all on top of each other, for a given accuracy, the resulting horizontal index, thus rank is the same for all electric sizes. Thus, the theoretical result shown in (56) is verified.

B. Two Configurations of a Plate-Plate Interaction

In the second example, we consider two separated plates in two configurations. In one configuration, the two plates are located in the same plane; while in the other configuration; one plate is normally displaced from the other plate. The two examples are the same as the two example described in Section III.B and Section III.C respectively. The only difference is that instead of having a fixed A, we increase A from 1λ to 60λ to study the rank's dependence with electric size.

We first study the effect of mesh density on the rank's growth with electric size. Since SVD is computationally intensive, it does not permit a fine discretization for studying a large electric size. If the effect of mesh density on the rank's growth rate is little, we can use a coarser mesh and thereby a smaller matrix to study the rank for the same electric size. In Fig. 5, we plot the rank determined with $\mathcal{E} = 10^{-4}$ for this example versus electric size for three different mesh densities:



Fig. 5. Rank versus electric size generated with three different mesh densities.

 $\lambda/2$, $\lambda/3$, and $\lambda/5$ respectively. As can be seen from this figure, the three lines are almost on top of each other. Therefore, in this and the example shown in next subsection, we use $\lambda/2$ as the mesh criterion so that larger electric sizes can be studied with SVD.

In Fig. 6(a), we plot the normalized singular values obtained from 1λ to 40λ versus singular value index. There are two sets of lines in this figure. The solid red lines correspond to the in-plane configuration of the two plates, while the dashed blue lines are the singular values of the normally displaced plate configuration. Each set has 20 lines representing singular values from 1 λ , 3 λ , 5 λ , ..., to 39 λ respectively from left to right. It can be seen clearly that different from the 1-D case shown in the first example, when electric size increases, the entire singular value distribution is expanded to the right in both plate configurations, thus requiring more singular values and hence a larger rank to reach the same accuracy. However, for any given accuracy within machine precision, the rank for both configurations is shown to be less than the matrix size, which is the largest singular value index, as can be seen from Fig. 6(a). Therefore, the matrix has a low-rank property.

In Fig. 6(b), we plot the rank versus electric size of A from 1λ to 60λ for both configurations of the plates. Case 1 represents the case where the two plates are on the same plane, while Case 2 is the other configuration. For Case 2, we plot the rank versus electric size for four different accuracy settings from $\mathcal{E} = 10^{-12}$, 10^{-10} , 10^{-8} , to 10^{-4} . The scaling of the rank is much closer to the linear scaling than to the quadratic scaling, both of which are plotted in Fig. 6(b) for reference. It can also be seen that the scaling rate for a lower-order accuracy setting is larger than that of a higher-order accuracy setting. As for Case 1, the rank is shown to grow slowly with electric size. The growth rate is less than linear. It is clear that the rank required by Case 2 is larger than that in Case 1 for the same accuracy. This can be easily understood by comparing (72) with (69). The representation of Green's function in (72) requires more Fourier modes than that in (69) because h is involved, and its electric size increases. In Case 1, the Green's function for the two-plate interaction is solely determined by the 2-D x-y plane information, while in



(b)

Fig. 6. Rank study of the interaction between two separated plates in two configurations. (a) Singular value distributions for 20 different electric sizes from 1 to 40 wavelengths with a step of 2 wavelengths (Red solid: Case 1; Blue dashed: Case 2). (b) Rank versus electric size for both configurations required by different accuracy criteria.

Case 2, the Green's function is contributed by the third dimension. Therefore, the growth rate of the rank with electric size for Case 1 is still governed by a 2-D based growth rate which is less than linear, while the rank of Case 2 is closer to a 3-D based rank.

Discussion: From Fig. 6(a), for Case 2, it can be seen that there is fairly wide a range of index *i* within which the normalized singular values are quite flat. After this range, the normalized singular values drop more rapidly. This phenomenon is what is exactly predicted by (35). The wavenumbers closest to k_0^2 have the largest singular values, and these wavenumbers distribute themselves on a spherical shell. If one stops at this range to observe the rank, he will get a

quadratic growth with the electric size. For 2-D distributions, if



Fig. 7. Rank study of the interaction between two separated spheres. (a) Singular value distributions for 20 different electric sizes from 1 to 40 wavelengths with a spacing of 2 wavelengths. (b) Rank for four accuracy requirements versus electric size. (c) Rank for four accuracy requirements versus N.

one stops at the flat range to observe the rank, he will get a linear scaling. However, the resulting representation cannot be used because the error is too large. In other words, by only keeping Fourier modes distributed on a spherical shell (in 3-D cases) or circle (in 2-D cases) closest to k_0^2 , the resultant error is too large to use. Therefore, one has to incorporate also those modes whose wavenumbers are away from k_0^2 by a certain distance, i.e. inside a volume of a spherical ring with inner radius of k_0 and outer radius of $k_0 + \Delta_k$ as shown by (52), to obtain an accurate representation of the integral operator. The height of this volume, Δ_k , is inversely proportional to frequency asymptotically in 3-D distributions. That is why the resultant rank is linearly proportional to frequency. In addition, one may observe the growth rate with electric size changes if different accuracy requirements are set, as shown also by Fig. 6(b). It is also higher than linear when the accuracy setting is low for 3-D distributions. That is because the growth rate has not converged yet. One can increase the accuracy setting until the growth rate does not increase any more. Upon convergence, the growth rate is linear, which is proved by the theoretical bound of Δ_{μ} .

C. Two separated spheres

In the third example, we consider two separated spheres. One sphere is centered at the origin with diameter *A*, and the other is centered at (2A, 0, 0) with the same diameter. The λ is 1 m, and *A* is increased from 1λ to 40λ . The mesh density is $\lambda/2$. The sources and observers are located on the spherical surface. The matrix corresponding to the source-observation interaction has the following element

$$\mathbf{Z}_{ij} = \frac{e^{-jk_0|\vec{r}_i - \vec{r}'_j|}}{|\vec{r}_i - \vec{r}'_j|} \ . \tag{75}$$

The matrix size, which is the number of sources (column dimension of the matrix) as well as the number of observers (row dimension of the matrix), ranges from 13, 315, 1018, 2124, to 17204 when the electric size of the sphere diameter A increases from 1λ to 40λ . The SVD is then used to compute the rank of matrix \mathbf{Z} for a given accuracy. In Fig. 7(a), we plot the normalized singular values obtained from 1λ to 40λ with a spacing of 2λ versus singular value index. There are 20 lines in this figure. The singular value lines are shown to expand to the right when electric size increases. In Fig. 7(b), we plot the rank of Z versus electric size for four different accuracy settings. The linear scaling line is also plotted for reference. As can be seen, the growth rate of the rank with electric size agrees very well with linear scaling. In Fig. 7(c), we plot the rank of Zobtained with four different accuracy settings versus matrix size N. It is clear that the rank scales with N as $N^{0.5}$. This is because the rank scales linearly with the electric size, while N of a surface distribution of sources and observers scales with electric size quadratically.



Fig. 8. Rank generated by ACA+ and SVD with respect to electric size for a variety of scatterer shapes. (a) Plate. (b) Cylinder. (c) Open cone. (d) Cone sphere. (e) Sphere.

D. A suite of electrically large examples

To further verify the proposed theoretical analysis, we numerically determined the rank of a plate, cylinder, open cone, cone sphere, and sphere, resulting from a surface-based electric field integral operator (EFIE) by ACA+ [1, 6] and SVD from small to very large electric sizes. A detailed description of this scheme can be found from [4] and Section IV.A in [5]. Basically, we first use ACA+ to obtain a factorized low-rank form, and then perform an SVD on the factorized form to find out the minimal rank required by accuracy. The ACA+ is used here because a direct SVD is very expensive when matrix size is large. For all these examples at all the electric sizes we simulate, a mesh size of 0.1 λ is used. By an \mathcal{H} -matrix partition scheme (Section II.D in [5]), we partition the dense system matrix into admissible blocks and inadmissible blocks level by level. The admissible blocks are blocks that satisfy max{diam(Ω_t), diam(Ω_s)} $\leq \eta$ dist(Ω_t, Ω_s), where $\eta = 1$ is used. In an \mathcal{H} matrix, the admissible blocks are represented by low-rank matrices, while inadmissible blocks are stored in a full matrix format. The *leafsize* used for the multilevel tree construction in the \mathcal{H} -matrix partition is 32. The error used in ACA+ and SVD truncation is 10⁻⁴. We then find the maximal rank k_{max} among all the admissible blocks at all tree levels for each example simulated. It is clear that k_{max} corresponds to the rank of the matrix block that has the largest electric size in each example. In Fig. 8, we plot the k_{max} versus electric size for all of the five different scatterers. As can be seen clearly, k_{max} is O(ka). Thus it verified the proposed theoretical analysis.

V. CONCLUSION

A theoretical study is conducted in this work to analyze the minimal rank of integral operators encountered in electrodynamic analysis and its dependence with electric size for a prescribed error bound. We highlight the fact that the rank generated by singular value decomposition is the minimal rank required by accuracy. The SVD-based low-rank approximation does not rely on the separation of observation and source coordinates for separated geometry blocks, while methods that separate observation and source coordinates such as interpolation and plane wave expansion based methods do not lead to a minimal rank approximation of the electrodynamic kernel. As a result, the rank obtained from these methods is observed to scale with electric size at a much higher rate.

The SVD analysis is numerical, which prevents a study of the rank for an arbitrarily large electric size. By recognizing the relationship between an SVD analysis and a Fourier analysis in a linear and shift-invariant system, we successfully develop an analytical approach to analyze the rank of an integral operator, and reveal the relationship between the rank and the electric size for satisfying a prescribed accuracy. The rank of the interaction between two separated geometry blocks is shown to scale linearly with the electric size of the block diameter in a general 3-D problem. As long as the rank is smaller than the matrix dimension, the matrix is called a low-rank matrix. We thus theoretically prove the existence of an error bounded low-rank representation of electrodynamic integral operators irrespective of electric size and object shape. Moreover, numerous results are generated to validate both the proposed analytical approach for analyzing the rank and the findings of this work on the rank's asymptotic dependence with electric size. The implication of this work on the complexity of IE-based electrodynamic computation using low-rank methods is also discussed.

The theoretical proof developed in this work provides a theoretical basis for employing and further developing low-rank matrix algebra for accelerating the integral equation based computation of electrically large problems.

Moreover, through the proposed theoretical study of the rank of the integral operator, we have found that the Fourier transform of Green's function has a direct relationship with the singular values of the inverse of the finite element matrix. As a result, the low-rank property of the IE operator, the inverse of IE operator, and the inverse finite element matrix is demonstrated by a single proof.

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