

11-22-2011

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Liu, Haixin and Jiao, Dan, "A Theoretical Study on the Rank's Dependence with Electric Size of the Inverse Finite Element Matrix for Large--Scale Electrodynamical Analysis" (2011). *ECE Technical Reports*. Paper 425.  
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TR-ECE-11-20

November 22, 2011

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– This work was supported in part by supported by NSF under award No. 0747578 and No. 0702567.

## Abstract

The rank of the inverse finite-element matrix is theoretically studied for 1-D, 2-D, and 3-D electrodynamic problems. We find that the rank of the inverse finite-element matrix is a constant irrespective of electric size for 1-D electrodynamic problems. For 2-D electrodynamic problems, the rank grows very slowly with electric size as square root of the logarithm of the electric size of the problem. For 3-D electrodynamic problems, the rank scales linearly with the electric size. The findings of this work are both theoretically proved and numerically verified. They are applicable to problems with inhomogeneous materials and arbitrarily shaped structures.

# A Theoretical Study on the Rank's Dependence with Electric Size of the Inverse Finite Element Matrix for Large-Scale Electrodynamical Analysis

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**Abstract**— The rank of the inverse finite-element matrix is theoretically studied for 1-D, 2-D, and 3-D electrodynamic problems. We find that the rank of the inverse finite-element matrix is a constant irrespective of electric size for 1-D electrodynamic problems. For 2-D electrodynamic problems, the rank grows very slowly with electric size as *square root of the logarithm* of the electric size of the problem. For 3-D electrodynamic problems, the rank scales linearly with the electric size. The findings of this work are both theoretically proved and numerically verified. They are applicable to problems with inhomogeneous materials and arbitrarily shaped structures.

**Index Terms**—Rank, Finite Element Methods, Electrodynamic Analysis, Inverse Matrix

## I. INTRODUCTION

**D**RIVEN BY the design of advanced engineering systems, there exists a continued need of reducing the complexity of computational electromagnetic methods. Recently, the  $\mathcal{H}$ -matrix based mathematical framework was introduced to accelerate the direct solution of the finite element based analysis of electrodynamic problems [1]. In an  $\mathcal{H}$  matrix [3], if  $\mathbf{C}$  is an  $m \times n$  off-diagonal block that describes the interaction between two well separated subsets of the entire unknown set, it can be written as  $\mathbf{C} = \mathbf{A}\mathbf{B}^T$  where  $\mathbf{A}$  is of dimension  $m \times k$ ,  $\mathbf{B}$  is of dimension  $n \times k$ , and  $k$  denotes the rank of  $\mathbf{C}$  with  $k < m$  and  $k < n$ . Besides a hierarchical low-rank representation, the  $\mathcal{H}$ -matrix framework [3] also encompasses a system of fast arithmetics that permits compact storage and efficient computation of dense matrices. In [1], it was proved that the sparse matrix resulting from a finite-element-based analysis of electrodynamic problems can be represented by an  $\mathcal{H}$  matrix without any approximation, and the inverse of this sparse matrix has a data-sparse  $\mathcal{H}$ -matrix approximation with error well controlled. Based on the proof, an  $\mathcal{H}$ -matrix-based direct finite-element solver of  $O(kN \log N)$  memory cost and  $O(k^2 N \log^2 N)$  time cost was developed for solving

electrodynamic problems, where  $k$  is the rank that is adaptively determined based on accuracy requirements, and  $N$  is the matrix size. To satisfy a prescribed accuracy, the rank  $k$  required by a static analysis is shown to be a constant. However, for electrodynamic analysis, to achieve a constant order of accuracy at each frequency, in general,  $k$  is required to increase with electric size. To understand the actual computational complexity of an  $\mathcal{H}$ -matrix based method for large-scale electrodynamic analysis, it becomes important to develop a theoretical understanding of the rank's dependence with electric size. The contribution of this work is a theoretical analysis on the rank's dependence with electric size of the inverse finite element matrix for 1-D, 2-D, and 3-D electrodynamic problems, which is, also, numerically verified.

## II. THEORETICAL ANALYSIS

In [1], it is shown that the inverse of the finite element matrix  $\mathbf{Y}$  can be written as

$$\mathbf{Y}^{-1} = \mathbf{V}(\Lambda - k_0^2 \mathbf{I})^{-1} \mathbf{V}^T, \quad (1)$$

where  $k_0$  is free-space wave number,  $\mathbf{I}$  is an identity matrix,  $\Lambda$  is a diagonal matrix comprised of the eigenvalues, and  $\mathbf{V}$  is the eigenvector matrix of the following frequency-independent eigenvalue problem:

$$\mathbf{S}\mathbf{v} = \lambda \mathbf{T}\mathbf{v}, \quad (2)$$

in which  $\mathbf{S}$  is a stiffness matrix, and  $\mathbf{T}$  is a mass matrix.

From (1), it can be seen that the weight of each eigenvector  $\mathbf{V}_i$  in the field solution is proportional to  $1/|\lambda_i - k_0^2|$ . Given an accuracy requirement  $\varepsilon$ , we can keep the eigenvectors whose eigenvalues satisfy the following condition

$$|\lambda_i - k_0^2| \leq \min\{|\lambda_i - k_0^2|, i=1, 2, \dots, N\} / \varepsilon = \Delta_{\min} / \varepsilon, \quad (3)$$

in which  $\Delta_{\min}$  is the smallest distance between the eigenvalues and  $k_0^2$ , while ignoring the other eigenvectors without sacrificing the prescribed accuracy. Hence, we have

$$\mathbf{Y}^{-1} = \tilde{\mathbf{V}}_{N \times k} (\tilde{\Lambda}_{k \times k} - k_0^2 \mathbf{I}_{k \times k})^{-1} (\tilde{\mathbf{V}}_{N \times k})^T, \quad (4)$$

where  $\tilde{\mathbf{V}}_{N \times k}$  is a subset of  $\mathbf{V}$ , which includes the  $k$  eigenvectors that satisfy (3), and  $\tilde{\Lambda}_{k \times k}$  is the diagonal matrix composed of corresponding eigenvalues. From (4), it is clear that for a given accuracy  $\varepsilon$ , the rank of the inverse finite-element matrix is nothing but  $k$ , which is the number of

Manuscript received September 22, 2011. This work was supported in part by supported by NSF under award No. 0747578 and No. 0702567. Haixin Liu and Dan Jiao are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, USA.

the eigenvalues  $\lambda_i$  satisfying (3). From (4), it can also be seen that if the frequency is extremely close to the square root of one of the eigenvalues, the rank of the inverse finite element matrix is equal to 1 irrespective of the electric size. However, when sweeping over a range of frequencies, one can encounter all kinds of frequencies. The objective of this paper is thus to theoretically examine the maximum rank one can possibly encounter when sweeping from small to very large frequencies.

To determine the rank of (4), we will first compute the maximum displacement,  $\Delta_k > 0$ , that satisfies

$$(k_0 + \Delta_k)^2 - k_0^2 \leq \Delta_{\min} / \varepsilon. \quad (5)$$

We then compute the number of modes that can exist between  $k_0$  and  $k_0 + \Delta_k$ , which yields the rank of (4), together with the number of modes satisfying (3) and having eigenvalues smaller than  $k_0^2$ . Since the latter is in the same order as the former, in the following, we only consider the former. The inequality (5) results in the upper bound of  $\Delta_k$ :

$$\Delta_k \leq \sqrt{k_0^2 + \Delta_{\min} / \varepsilon} - k_0 = \frac{\Delta_{\min} / \varepsilon}{\sqrt{k_0^2 + \Delta_{\min} / \varepsilon} + k_0}. \quad (6)$$

Next, to facilitate an efficient analytical analysis, we consider a cubic computational domain filled by a uniform material. Later, we will show that the rank's dependence with electric size developed based on such a simplified model is equally applicable to problems with inhomogeneous materials, arbitrary structures, and an irregular domain. In addition, the boundary condition considered is either a Dirichlet- or a Neumann-type boundary condition. Such boundary conditions are natural for closed region problems. For open region problems, the Dirichlet-type boundary condition constructed from the exact field solution, also, constitutes an exact absorbing boundary condition. In the model considered here, the space is filled by a single material, thus the field solution on the truncation boundary is analytically known.

For a cubic computational domain of side length  $a$ , the eigenvalues of (2) are analytically known to be

$$\lambda_i = K^2 = [(m\pi)^2 + (n\pi)^2 + (p\pi)^2] / a^2. \quad (7)$$

where  $K$  denotes a resonance wave number,  $m$ ,  $n$ , and  $p$  are integer numbers. In 1-D cases, since  $K^2 = (m\pi)^2 / a^2$ , the number of modes between  $k_0$  and  $k_0 + \Delta_k$  is proportional to  $\Delta_k$ . Therefore, the rank  $k$  in 1-D cases can be written as:

$$k|_{1D} \sim \Delta_k. \quad (8)$$

In 2-D cases, we have  $K^2 = [(m\pi)^2 + (n\pi)^2] / a^2$ , so the number of modes, which corresponds to the rank, between  $k_0$  and  $k_0 + \Delta_k$  is proportional to the area of a ring with inner radius of  $k_0$  and outer radius of  $k_0 + \Delta_k$ . Thus, the rank  $k$  in 2-D cases can be written as:

$$k|_{2D} \sim (k_0 + \Delta_k)^2 - k_0^2 = 2k_0\Delta_k + \Delta_k^2. \quad (9)$$

In 3-D cases, the number of modes between  $k_0$  and  $k_0 + \Delta_k$  is proportional to the volume of a spherical ring with inner radius of  $k_0$  and outer radius of  $k_0 + \Delta_k$ . Thus,

$$k|_{3D} \sim (k_0 + \Delta_k)^3 - k_0^3 = 3k_0^2\Delta_k + 3k_0\Delta_k^2 + \Delta_k^3. \quad (10)$$

Given a  $k_0$ ,  $\Delta_{\min}$ , which is the smallest distance between the eigenvalues and  $k_0^2$ , is bounded from above by the difference between two adjacent eigenvalues that enclose  $k_0$ . Let the two eigenvalues be  $K_1^2$  and  $K_2^2$ , with  $K_1^2 < K_2^2$ . For 1-D cases, the square root of adjacent eigenvalues differ by  $\pi/a$ , so

$$\Delta_{\min}|_{1D} < K_2^2 - K_1^2 \sim O(K_1^2) \sim O(k_0). \quad (11)$$

For 2-D and 3-D cases, the derivation is more involving. Let

$$S = (aK/\pi)^2, \quad (12)$$

(7) can be rewritten as

$$S = m^2 + n^2 + p^2. \quad (13)$$

Since  $m$ ,  $n$  and  $p$  are all integers,  $S$  is also an integer. All the integers  $S$  satisfying (13) can be written as a series in an ascending order with degenerate eigenvalues excluded:  $S_1, S_2, \dots, S_i, \dots$ , and  $S_i < S_{i+1}$ . Note that this series only includes distinct eigenvalues. Define

$$Q(x) = \sum_{S_i < x} 1 \quad (14)$$

as the function that counts the number of integers satisfying (13) and smaller than  $x$ . In 1908, E. Landau [2] deduced the asymptotic formulas of  $Q(x)$  for 2-D and 3-D problems as the following:

$$Q(x)|_{2D} \sim O(x/\sqrt{\log x}) \quad (15)$$

$$Q(x)|_{3D} \sim O(x). \quad (16)$$

Different from (9) and (10), (15) and (16) count the number of distinct eigenvalues instead of the number of modes since the latter can have degenerate eigenvalues. From (15) and (16), the difference between two adjacent eigenvalues for 2-D and 3-D problems, and hence the upper bound of  $\Delta_{\min}$ , can be calculated as the following:

$$\Delta_{\min}|_{2D} < (K_2^2 - K_1^2)|_{2D} = \left(\frac{\pi}{a}\right)^2 (S_{i+1} - S_i)|_{2D} \sim \left(\frac{\pi}{a}\right)^2 \frac{dx}{dQ}|_{2D} \Big|_{x=K_1^2} \quad (17)$$

$$\sim O(\sqrt{\log x}) \sim O(\sqrt{\log K_1^2}) \sim O(\sqrt{\log k_0})$$

$$\Delta_{\min}|_{3D} < (K_2^2 - K_1^2)|_{3D} = \left(\frac{\pi}{a}\right)^2 (S_{i+1} - S_i)|_{3D} \sim \left(\frac{\pi}{a}\right)^2 \frac{dx}{dQ}|_{3D} \Big|_{x=K_1^2} = O(1) \quad (18)$$

Substituting  $\Delta_{\min}$  in (11), (17), and (18) for 1-D, 2-D, and 3-D cases, respectively, into (6), we obtain

$$\Delta_k|_{1D} \leq O(1). \quad (19)$$

$$\Delta_k|_{2D} \leq O(\sqrt{\log k_0} / k_0) \quad (20)$$

$$\Delta_k|_{3D} \leq O(1/k_0). \quad (21)$$

Now we are ready to obtain the dependence of the rank with electric size. For 1-D problems, substituting (19) into (8), we have

$$k|_{1D} = \text{constant}. \quad (22)$$

For 2-D problems, substituting (20) into (9), we obtain

$$k|_{2D} \sim O(\sqrt{\log k_0}) \quad (23)$$

for electrically large problems in which  $k_0$  is large. For 3-D problems, substituting (21) into (10), we obtain

$$k_{|_{3D}}^1 \sim O(k_0). \quad (24)$$

As a result, we discover that for 1-D problems, the rank of the inverse finite-element matrix is a constant; for 2-D problems, the rank increases slowly in the order of the square root of  $\log k_0$ , while for 3-D problems, the rank increases linearly with the electric size of the problem.

For different absorbing boundary conditions, if they are imposed correctly, the resultant field solution, and hence  $\mathbf{Y}^{-1}$ , and thereby its rank behavior, should agree well with that obtained in the above with an exact absorbing boundary condition. For a problem having non-uniform dielectric materials and arbitrarily shaped conductors, it can always be decomposed into multiple subdomains, with each subdomain filled by a uniform material. Consider a matrix block  $\mathbf{Y}_{22}$  that is assembled from a subdomain filled by a uniform material. The finite element matrix can be cast into the following form

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix}, \quad (25)$$

where  $\mathbf{Y}_{11}$  is the diagonal block assembled from the rest of the problem that is inhomogeneous. The inverse of (25) can be performed by first eliminating unknowns associated with the 11-block, and then invert the Schur complement of the  $\mathbf{Y}_{22}$  block, which is  $\mathbf{Y}_{22} - \mathbf{Y}_{21}\mathbf{Y}_{11}^{-1}\mathbf{Y}_{12}$ . This can be computed by using the inverse of the sum of two matrices. As a result, we obtain

$$\begin{aligned} (\mathbf{Y}_{22} - \mathbf{Y}_{21}\mathbf{Y}_{11}^{-1}\mathbf{Y}_{12})^{-1} &= (\mathbf{Y}_{22} + \mathbf{Y}_{21}(-\mathbf{Y}_{11}^{-1})\mathbf{Y}_{12})^{-1} \\ &= \mathbf{Y}_{22}^{-1}(\mathbf{I} - \mathbf{Y}_{21}(\mathbf{I} + (-\mathbf{Y}_{11}^{-1})\mathbf{Y}_{12}\mathbf{Y}_{21})^{-1}(-\mathbf{Y}_{11}^{-1})\mathbf{Y}_{12}\mathbf{Y}_{22}^{-1}) \end{aligned} \quad (26)$$

Since the rank of the product of two matrices is smaller or equal to the minimum rank of the two matrices being multiplied, if the rank of  $\mathbf{Y}_{22}^{-1}$  is  $k$ , the rank of (26) is also bounded by  $k$ . Therefore, the rank's dependence with electric size identified from a problem with a uniform material is equally applicable to problems with inhomogeneous materials.

### III. VALIDATION

We first simulated a parallel plate example that has analytical solutions. The length, width and height of the structure were 0.4 m, 0.4 m, and 0.02 m respectively. The frequency simulated was 1,493,800 KHz. The mesh size was chosen as 0.1 wavelengths. There were 2799 unknowns. We computed the inverse of the FEM matrix and used the singular value decomposition (SVD) to determine the inverse's rank based on an accuracy criterion of 1%. The rank was shown to be 26. It is clear that the inverse is low rank. Moreover, the rank numerically determined from SVD agrees very well with that analytically determined from (7) and (3). Therefore, in the following, we use the analytical formulas (7) and (3) to determine the rank since it is not feasible to use SVD to numerically determine the rank for very large electric sizes.

Based on (7), we plotted the upper bound of  $\Delta_{\min}$ ,  $K_2^2 - K_1^2$ ,

for 1-D, 2-D, and 3-D problems respectively from small to very large electric sizes such as 7000 wavelengths in Fig.1 (a), (c), and (e). As can be seen, the dependence of  $\Delta_{\min}$  with electric size agrees very well with the theoretical analysis. We then computed the rank by counting the number of eigenvalues satisfying (3) for a given accuracy requirement  $\varepsilon$ , which is chosen as 1% here. The  $k_0^2$  is chosen at the center point of two adjacent eigenvalues to examine the maximum possible rank  $k$ . In Fig. 1(b), (d), and (f), we plot the rank  $k$  for 1-D, 2-D, and 3-D problems with respect to electric size. Again, the rank's dependence with electric size agrees very well with our theoretical analysis. In addition, from these figures, it can be seen clearly that the rank of the inverse finite element matrix is much less than matrix dimension  $N$ .

### IV. CONCLUSIONS

The rank's dependence with electric size of the inverse finite element matrix is theoretically studied for electrodynamic analysis. The findings provide a theoretical basis for employing and further developing the low-rank matrix algebra for accelerating computational electromagnetic methods.

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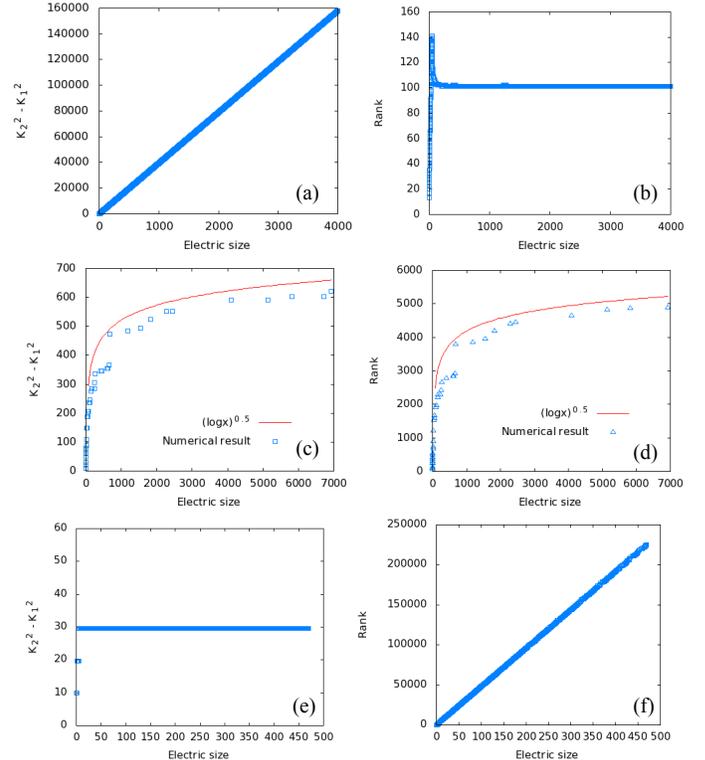


Fig. 1. The upper bound of  $\Delta_{\min}$  ( $K_2^2 - K_1^2$ ) and rank  $k$  with respect to electric size. (a-b) 1-D problems; (c-d) 2-D problems; (e-f) 3-D problems.