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Abstract

We theoretically prove that the minimal rank of the interaction between two separated geometry blocks in an integral-equation based analysis of general three-dimensional objects, for a prescribed error bound, scales linearly with the electric size of the block diameter. We thus prove the existence of the error-bounded low-rank representation of both surface and volume based integral operators for electrodynamic analysis, irrespective of electric size and scatterer shape. The theoretical analysis developed in this work permits an analytical study of the minimal rank for a prescribed accuracy, for arbitrarily shaped objects with arbitrary electric sizes. Numerical experiments have verified its validity. This work provides a theoretical proof on why the low-rank matrix algebra can be employed to accelerate the computation of large-scale electrodynamic problems.

The rank studied in this paper is based on a singular value decomposition based minimal rank approximation of integral operators, which does not rely on the separation of observation and source coordinates. Methods that do not generate a minimal rank approximation for a prescribed accuracy can result in a rank that scales with electric size at a much higher rate.

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Wenwen Chai, *Student Member, IEEE* and Dan Jiao, *Senior Member, IEEE*

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Index Terms—Rank, Integral Operators, Electrodynamic Analysis, Three Dimensional, Theoretical Analysis

I. INTRODUCTION

DRIVEN BY the design of advanced engineering systems, there exists a continued need of reducing the complexity of computational electromagnetic methods. Recently, the \mathcal{H} - and \mathcal{H}^2 -matrix based mathematical framework [1-2] has been introduced and further developed to accelerate both iterative and direct solutions of the integral equation based analysis of electrodynamic problems [3-5]. The key technique in this mathematical framework is hierarchical low-rank matrix algebra that enables compact representation and efficient computation of dense matrices. The direct integral equation solver reported in [6] for solving large-scale electrodynamic problems can also be viewed in this mathematical framework. It successfully solved electrically large integral equations for problem sizes up to 1

M unknowns. In [4-5], the cost of an \mathcal{H} -matrix based direct computation is reduced for electrodynamic analysis. The resultant direct integral equation solver successfully solved electrodynamic problems of 96 wavelengths with more than 1 million unknowns in fast CPU time (less than 20 hours in LU factorization, 85 seconds in LU solution), modest memory consumption, and with the prescribed accuracy satisfied, on a single CPU running at 3 GHz.

Why the low-rank representation can be employed to accelerate the computation of electrodynamic problems? Does an error-bounded low-rank approximation of integral operators exist, regardless of electric size? In [5], through a singular value decomposition (SVD) based analysis, it is numerically shown, for large electric sizes (over 100 wavelengths) and various scatterers, the rank of a matrix block of size N formed between two geometrically separated groups, arising from the surface integral equation based electrodynamic analysis, scales as $O(N^{0.5})$. As a result, the block has a low rank. However, no theoretical proof has been developed to support this numerical finding.

The contribution of this work is a theoretical proof to the fact that the minimal rank of the interaction between two separated geometry blocks in an integral-equation based analysis of general three-dimensional objects, for a prescribed error bound, scales linearly with the electric size of the block diameter. This proof is applicable to various integral operators encountered in electrodynamic analysis such as electric field, magnetic field, combined field, surface-based, and volume-based integral operators. We have also derived an analytical error bound for the minimal rank approximation of the integral operator for electrodynamic analysis. Since the rank scales linearly with electric size of the block diameter, while the number of unknowns in a surface integral equation based analysis scales as electric size square, and that in a volume integral equation based analysis scales as electric size cube, we prove the existence of the error-bounded low-rank representation of both surface and volume integral operators for electrodynamic analysis, irrespective of electric size.

It is worth mentioning that the rank studied in this paper is the rank of a minimal rank approximation of the integral operator. It has been proven that given an accuracy requirement, the low-rank approximation generated from singular value decomposition (SVD) is the *minimal* rank approximation [1, pp. 63] for the given accuracy. Our numerical experiments show that methods that do not generate

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a minimal rank approximation such as interpolation [3], Taylor series expansion, and plane-wave expansion based methods can result in a rank that is much higher than the minimal rank required by accuracy. The rank also scales with electric size at a much higher rate.

II. THEORETICAL STUDY

A. Problem Description

The integral equation based analysis of electrodynamic problems results in a dense linear system of equations

$$\mathbf{Z}\mathbf{I} = \mathbf{V}. \quad (1)$$

Consider $\mathbf{Z}^{t,s}$, an arbitrary $m \times n$ off-diagonal block of the system matrix \mathbf{Z} , which describes the interaction between two separated groups (t and s) of the scatterer being analyzed. The objective of this work is to theoretically study whether there exists an error-bounded low-rank representation of $\mathbf{Z}^{t,s}$ irrespective of electric size and scatterer shape, and if such a representation exists, how the rank scales with electric size, and hence the number of unknowns N .

Given an accuracy requirement ε , as shown in [1, pp. 63], the rank- r representation (\mathbf{R}) generated from SVD is a minimal rank approximation of the original matrix \mathbf{M} that fulfils $\|\mathbf{M} - \mathbf{R}\|_2 \leq \varepsilon$. However, an SVD analysis is numerical, which makes it not feasible to find the actual rank required by accuracy for arbitrarily large electric sizes. As a result, an analytical approach, which is not restricted by computational resources and is valid for arbitrary shape, becomes necessary to develop a theoretical understanding on the rank's dependence with electric size. This paper provides such an analytical approach. In this approach, we are able to make a connection between an SVD analysis and a Fourier analysis. By utilizing the relationship between the two analyses in a linear and shift-invariant system, we succeed in analytically revealing the rank of the integral operators and its dependence with electric size.

B. Relationship between SVD and Fourier Analysis in a Linear Shift-Invariant System

A linear system can be modeled by:

$$b = \mathbf{H}f, \quad (2)$$

where f and b are vectors, and \mathbf{H} is a linear operator. We can perform SVD on \mathbf{H} to obtain

$$b = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^H f, \quad (3)$$

where superscript ' H ' denotes a complex conjugate transpose, $\mathbf{\Sigma}$ is the diagonal matrix comprising singular values, and \mathbf{V} and \mathbf{U} are matrices comprising singular vectors. Since \mathbf{V} and \mathbf{U} are both unitary, we have

$$\mathbf{V}^H b = \mathbf{\Sigma}(\mathbf{U}^H f). \quad (4)$$

which can be written compactly as

$$b^V = \mathbf{\Sigma}f^U, \quad (5)$$

where

$$b^V = \mathbf{V}^H b; f^U = \mathbf{U}^H f. \quad (6)$$

Multiplying a unitary matrix by a vector can be thought of as projecting this vector onto the orthonormal set defined by the

matrix. Thus, (5) can be viewed as representing the response b in the \mathbf{V} basis (b^V), the input f in the \mathbf{U} basis (f^U), and relating these two projections by a diagonal matrix ($\mathbf{\Sigma}$).

When the operator \mathbf{H} is both linear and shift invariant (LSIV), SVD turns to Fourier analysis [7]. More specifically, the singular vectors of an LSIV system are weighted Fourier basis functions (complex exponentials) and the singular values are the absolute values of the Fourier transform of the system's point spread function (impulse response function) [7, 8]. To see this more clearly, let's consider an LSIV system. Because an LSIV system operator is a convolution operator [7], the response b in space domain is a convolution of the input f with an impulse response h

$$b(\vec{r}) = f(\vec{r}) * h(\vec{r}), \quad (7)$$

in which \vec{r} denotes an arbitrary point in space. The above convolution can be converted to simple multiplication by Fourier analysis. Thus we have

$$\mathcal{F}(b(\vec{r})) = \mathcal{F}(h(\vec{r}))\mathcal{F}(f(\vec{r})), \quad (8)$$

where $\mathcal{F}(\cdot)$ denotes a Fourier transform. We can rewrite (8) as

$$b(r)^{FT} = \mathcal{F}(h(\vec{r}))f(r)^{FT}, \quad (9)$$

where $b(r)^{FT}$ is the representation of $b(r)$ in the Fourier basis, and $f(r)^{FT}$ is the representation of $f(r)$ in the Fourier basis. In other words, we represent the input in a unitary basis (Fourier basis), we also represent the response in a unitary basis (Fourier basis), and relate the two by $\mathcal{F}(h(\vec{r}))$. From (5) and (9), the relationship between SVD and Fourier analysis can be clearly seen. The Fourier bases may be different from the SVD-generated bases. However, if the system is linear and shift-invariant, the two bases are both Fourier bases [7]. Therefore, the Fourier analysis accomplishes the SVD analysis of a linear shift-invariant system.

C. Rank Revealing via an Analytical Fourier Analysis of the Integral Operator

There exist many integral equation based formulations for analyzing 3-D electrodynamic problems. Examples are electric field integral equation, magnetic field integral equation, combined field integral equation, each of which can be formulated in a surface-based or volume-based form. The underlying integral operators are all linear and shift invariant. Therefore, we can use Fourier analysis to analytically study the rank of the integral equation based system matrix.

The point-spread function in integral equation based operators is Green's function and its variants. The Green's function for a 3-D problem can be written as:

$$g(\vec{r}) = \frac{e^{-jk|\vec{r}|}}{4\pi|\vec{r}|} = \frac{e^{-jkR}}{4\pi R}. \quad (10)$$

Without loss of generality, an integral equation based operator can be expressed as the convolution of a certain source f with Green's function as the following:

$$b(\vec{r}) = \int g(|\vec{r} - \vec{r}'|)f(\vec{r}')d\vec{r}', \quad (11)$$

where the integral could be one-, two-, or three-dimensional. The Fourier analysis of the above results in:

$$B(\tilde{k}) = \mathbf{G}(\tilde{k})F(\tilde{k}), \quad (12)$$

which, in a discrete form, can be written as:

$$\begin{Bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_{n-1} \end{Bmatrix} = \begin{pmatrix} \mathbf{G}_0 & & & \\ & \mathbf{G}_1 & & \\ & & \mathbf{G}_2 & \\ & & & \ddots \\ & & & & \mathbf{G}_{n-1} \end{pmatrix} \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{Bmatrix}, \quad (13)$$

where \mathbf{G}_i , F_i , and B_i are, respectively, $\mathbf{G}(\tilde{k}_i)$, $F(\tilde{k}_i)$, and $B(\tilde{k}_i)$ at discrete \tilde{k}_i ($i = 0, 1, \dots$).

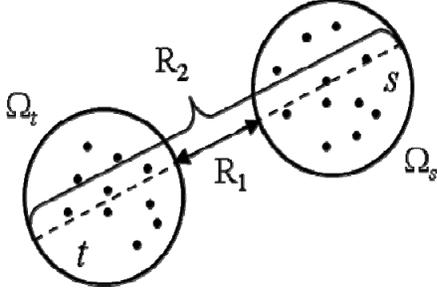


Fig. 1. Illustration of two separated groups t and s .

Consider two separated groups t and s of the entire unknown set of a general 3-D object, as illustrated in Fig. 1. The corresponding domains are Ω_t and Ω_s . The minimum distance between the two groups is denoted by R_1 , and the maximum distance is R_2 :

$$\begin{aligned} R_1 &= \min(|\bar{r} - \bar{r}'|), \\ R_2 &= \max(|\bar{r} - \bar{r}'|) \end{aligned} \quad (14)$$

where, $\bar{r} \in \Omega_t$, $\bar{r}' \in \Omega_s$.

The strong η -admissibility condition is usually used in the low-rank matrix algebra to quantify the separation between two groups [1-2]:

$$a = \max\{\text{diam}(\Omega_t), \text{diam}(\Omega_s)\} \leq \eta \text{dist}(\Omega_t, \Omega_s), \quad (15)$$

in which $\text{diam}(\cdot)$ is the Euclidean diameter of a group, $\text{dist}(\cdot, \cdot)$ is the Euclidean distance between two groups, a is the maximum group diameter, and η is a positive parameter. It is not difficult to find the relationship between R_1 and R_2 in (14) and η and the block diameter a in (15) as:

$$\begin{aligned} R_1 &= \frac{a}{\eta} \\ R_2 &= R_1 + c_0 a, \quad 1 < c_0 \leq 2 \end{aligned} \quad (16)$$

where c_0 is a constant coefficient between 1 and 2.

In an integral equation based system matrix \mathbf{Z} , the interaction between two separated groups t and s is characterized by matrix block $\mathbf{Z}^{t,s}$. Given a prescribed accuracy, its minimal rank can be numerically determined by SVD. From the analysis given in Section II.B, this rank is the same as the rank of the diagonal matrix \mathbf{G} in (13), the entries of which are the Fourier transform of Green's function at discrete frequency points. Thus, next, we perform a Fourier

analysis of the Green's function to analytically determine the rank of $\mathbf{Z}^{t,s}$.

The Green's function, which is the kernel function of (11), apparently, depends on both observation point \bar{r} and source point \bar{r}' . However, the Green's function does not depend on the shape of the scatterer. Instead, it is only determined by the distance between the observation and source points, $|\bar{r} - \bar{r}'|$, thus defined over a finite one-dimensional range of (R_1, R_2) . Therefore, we perform a Fourier analysis of (10) in the same range as the following

$$g(R) = \sum_i \mathbf{G}_i e^{j2\pi \frac{i}{(R_2-R_1)} R}, \quad R \in (R_1, R_2), \quad (17)$$

where discrete $\mathbf{G}_i = \mathbf{G}(\tilde{k}_i)$ can be evaluated from

$$\mathbf{G}_i = \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} g(R) e^{-j2\pi \frac{i}{R_2-R_1} R} dR, \quad i = 0, 1, \dots \quad (18)$$

Substituting (10) into (18), we can analytically evaluate (18) as:

$$\mathbf{G}_i = \frac{1}{4\pi} \sum_{n=1}^{\infty} (j)^n (n-1)! \frac{1}{(k + \tilde{k}_i)^n} \left[\frac{e^{-j(k+\tilde{k}_i)R_2}}{R_2^n} - \frac{e^{-j(k+\tilde{k}_i)R_1}}{R_1^n} \right], \quad (19)$$

where

$$\tilde{k}_i = 2\pi \frac{i}{R_2 - R_1}, \quad i = 0, 1, \dots \quad (20)$$

Substituting (16) into (20), we obtain

$$\tilde{k}_i = 2\pi \frac{i}{c_0 \eta R_1} = 2\pi \frac{i}{c_0 a}. \quad (21)$$

Using (21) and (16), (19) becomes

$$\mathbf{G}_i = \frac{1}{4\pi} \times \sum_{n=1}^{\infty} (j)^n (n-1)! \frac{c_0^n \eta^n}{(c_0 k a + 2\pi i)^n} \left[\frac{e^{-j(1+c_0\eta) \frac{(c_0 k a + 2\pi i)}{c_0 \eta}}}{(1+c_0\eta)^n} - \frac{e^{-j \frac{(c_0 k a + 2\pi i)}{c_0 \eta}}}{1} \right] \quad (22)$$

The maximum of $|\mathbf{G}_i|$ ($i = 0, 1, \dots$) is $|\mathbf{G}_0|$, with corresponding index i being 0. Given an accuracy requirement ε , the rank of the diagonal matrix \mathbf{G} in (13) can be determined from the number of \mathbf{G}_i s that satisfy the following condition

$$\frac{|\mathbf{G}_i|}{|\mathbf{G}_0|} > \varepsilon. \quad (23)$$

Before generating a quantitative plot of (22) to examine the rank, we can conduct a quick analytical analysis. For electrically large problems, (22) can be approximated by

$$|\mathbf{G}_i| \sim \frac{1}{4\pi} \frac{c_0 \eta}{(c_0 k a + 2\pi i)} \quad (24)$$

as the other higher order terms decay much faster. Substituting (24) into (23), we obtain

$$i < \frac{c_0}{2\pi} \left(\frac{1}{\varepsilon} - 1 \right) k a. \quad (25)$$

As a result, we have

$$\text{rank} = O(ka). \quad (26)$$

Hence, we prove that the rank of \mathbf{G} , and thereby the rank of $\mathbf{Z}^{t,s}$, which is the interaction between two separated geometry blocks, is linearly proportional to the electric size of the maximum block diameter. In a surface integral equation based solver, the row and column dimension of $\mathbf{Z}^{t,s}$ is proportional to the *square* of the electric size of the maximum block diameter. In a volume integral equation based solver, the row and column dimension of $\mathbf{Z}^{t,s}$ is proportional to the *cube* of the electric size. In either case, $\mathbf{Z}^{t,s}$ is low rank irrespective of electric size. Moreover, it can be seen from the aforementioned theoretical analysis, as long as R_1 (minimum distance between the two groups) is greater than zero, i.e. the two groups are geometrically separated (non-overlapping), the error bounded low-rank representation exists, regardless of electric size. It can also be seen that if the two groups overlap, thus the corresponding matrix block contains diagonal elements; the error-controlled low-rank representation does not exist.

In Fig. 2(a) and (b), based on (22), we plot $|\mathbf{G}_i / \mathbf{G}_0|$ with \mathbf{G}_i sorted in a descending order, for electric size $ka=10, 20, 40, 80, 160, 320, 640,$ and 1280 respectively. Without the theoretical approach developed in this work, it would not be feasible to obtain the rank for such large electric sizes. From the relationship between an SVD analysis and a Fourier analysis in an LSIV system, $|\mathbf{G}_i / \mathbf{G}_0|$ is nothing but normalized singular values. From Fig. 2, it is clear that given an accuracy requirement, the rank of \mathbf{G} increases with electric size. The quantitative relationship can be seen from Table I, where we list the rank with respect to electric size required for achieving accuracy $\varepsilon=0.05$, and $\varepsilon=0.01$ respectively. It is clear that the rank scales linearly with the electric size of the block diameter. When generating Fig. 2 and Table I, $\eta=0.5$ and $c_0=2$ are used in the calculation of (22).

D. Analytical Error Bound of the SVD-Based Minimal Rank Approximation of Integral Operators for Large-Scale Electrodynamics Analysis

The Fourier coefficient of the Green's function shown in (22) also reveals the error bound of the minimal-rank representation of the integral equation based operator for large-scale electrodynamic analysis. With the rank of the interaction between two separated geometry blocks chosen as r , the error is bounded by

$$\frac{\|\mathbf{Z}^{t,s} - (\tilde{\mathbf{Z}}^{t,s})_{\text{rank } r}\|}{\|\mathbf{Z}^{t,s}\|} = \frac{\|\mathbf{G} - \tilde{\mathbf{G}}_{\text{rank } r}\|}{\|\mathbf{G}\|} \leq \frac{|\mathbf{G}_r|}{|\mathbf{G}_0|}, \quad (27)$$

where \mathbf{G}_r is the r -th diagonal entry of \mathbf{G} , assuming the diagonal elements are sorted in a descending order from \mathbf{G}_0 to \mathbf{G}_{n-1} . The equality in (27) is due to the relationship between an SVD analysis and a Fourier analysis for a linear and shift-invariant operator as analyzed in Section II.B. The inequality in (27) is because \mathbf{G} is diagonal. As can be seen from (27) and

(18), the center-point based approximation of Green's function only captures \mathbf{G}_0 , thus being a rank-1 approximation.

For electrically large problems, from (22), we obtain

$$\frac{|\mathbf{G}_r|}{|\mathbf{G}_0|} \stackrel{\text{large } ka}{\approx} \frac{1}{\left[1 + \frac{2\pi}{c_0} \frac{r}{k\eta R_1}\right]} = \frac{1}{\left[1 + \frac{2\pi}{c_0} \frac{r}{ka}\right]}. \quad (28)$$

It is clear that any desired order of accuracy can be achieved via rank r irrespective of electric size ka . In addition, given a required order of accuracy, when electric size ka increases by a certain ratio, the error of the rank- r approximation can be kept to the desired order by increasing rank r by the same ratio.

Table I. Rank's dependence with electric size for achieving required accuracy for 8 different electric sizes from $ka=10$ to $ka=1280$.

	ka	10	20	40	80
$\varepsilon = 0.05$	Rank	46	86	178	336
$\varepsilon = 0.01$	Rank	244	454	944	1782

	ka	160	320	640	1280
$\varepsilon = 0.05$	Rank	680	1352	2700	5372
$\varepsilon = 0.01$	Rank	3600	7072	13014	24398

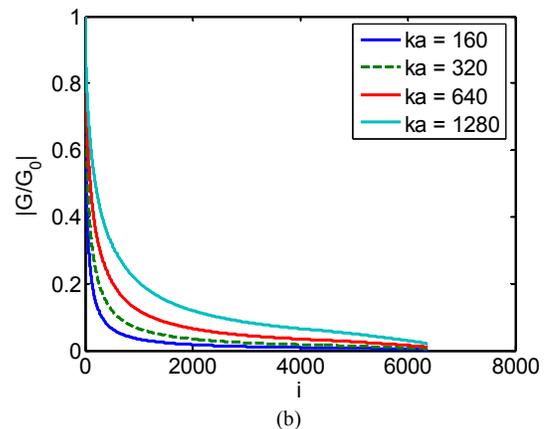
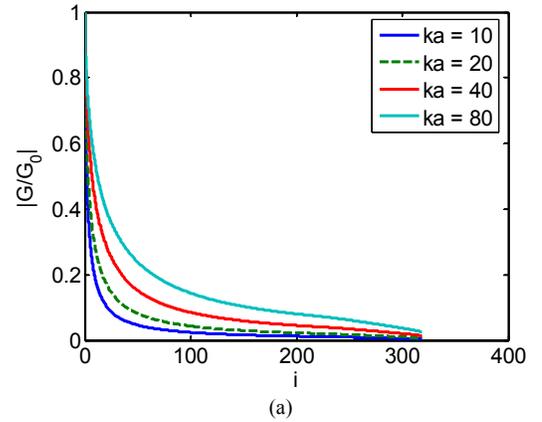


Fig. 2. Normalized Fourier expansion coefficient of Green's function in a descending order for eight different electric sizes from $ka=10$ to $ka=1280$.

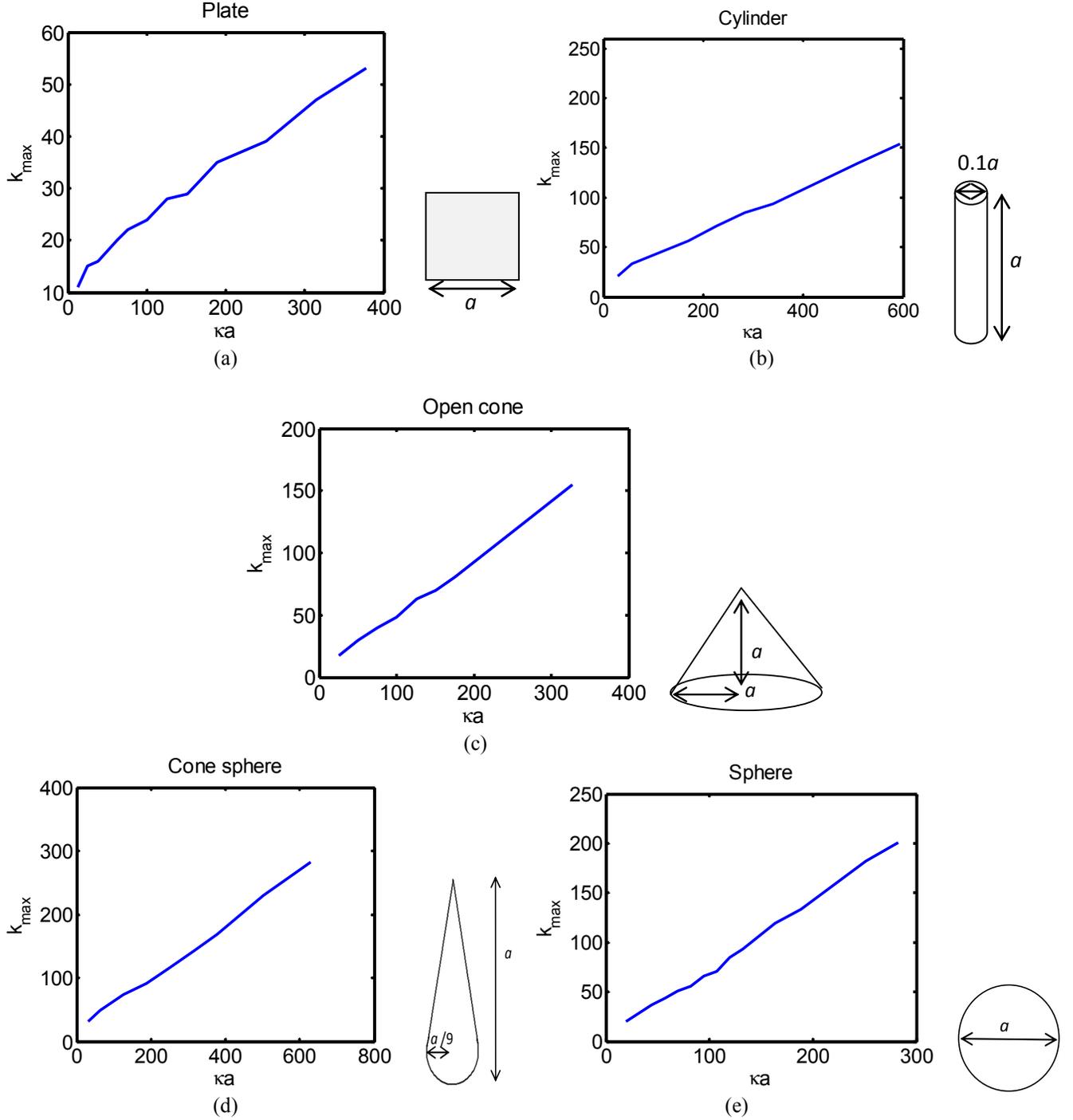


Fig. 3. Rank generated by ACA+ [9, 1] and SVD with respect to electric size for a variety of scatterer shapes. (a) Plate. (b) Cylinder. (c) Open cone. (d) Cone sphere. (e) Sphere.

III. NUMERICAL VERIFICATION

To further verify the proposed theoretical analysis, we numerically determined the rank of a plate, cylinder, open cone, cone sphere, and sphere, resulting from a surface-based electric field integral operator, by ACA+ [9, 1] and SVD from small to very large electric sizes. In Fig. 3, we plot the maximal rank k_{\max} among all the off-diagonal blocks of the system matrix versus electric size for all of the five different

scatterers. It is clear that k_{\max} is $O(ka)$. Thus it verified the proposed theoretical analysis.

IV. CONCLUSION

A theoretical study is conducted in this work to analyze the minimal rank of integral operators encountered in electrodynamic analysis and its dependence with electric size for a prescribed error bound. We show that the rank generated by singular value decomposition is the minimal rank required

by accuracy because the rank- r approximation produced by SVD is proven to be the minimal rank approximation for a prescribed accuracy. The SVD-based low-rank approximation does not rely on the separation of observation and source coordinates for separated geometry blocks, while methods that separate observation and source coordinates such as interpolation and plane wave expansion based methods do not lead to a minimal rank approximation of the electrodynamic kernel. As a result, the rank obtained from these methods is observed to scale with electric size at a much higher rate.

The SVD analysis is numerical, which prevents an analytical study. By recognizing the relationship between an SVD analysis and a Fourier analysis in a linear and shift-invariant system, we successfully derived an analytical error bound of the SVD-based minimal rank approximation of the integral operator, and revealed the relationship between the rank and the electric size for satisfying a prescribed accuracy. The rank of the interaction between two separated geometry blocks is shown to scale linearly with the electric size of the block diameter. We thus theoretically proved the existence of an error bounded low-rank representation of electrodynamic integral operators irrespective of electric size and scatterer shape. Numerical experiments further verified this theoretical finding.

The theoretical proof developed in this work provides a theoretical basis for employing and further developing low-rank matrix algebra for accelerating the integral equation based computation of electrodynamic problems.

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