

8-2-2011

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TR-ECE-11-14

August 2, 2011

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– This work was supported by a grant from Intel Corporation, a grant from Office of Naval Research under award N00014-10-1-0482, and a grant from NSF under award 0747578.

Abstract

Full-wave solutions of Maxwell's equations break down at low frequencies. Existing methods for solving this problem either are inaccurate or incur additional computational cost. In this work, a fast full-wave finite-element based solution is developed to eliminate the low frequency breakdown problem in a reduced system of order one. It is applicable to general 3-D problems involving ideal conductors as well as non-ideal conductors immersed in inhomogeneous, lossless, lossy, and dispersive materials. The proposed method retains the rigor of the theoretically rigorous full-wave solution recently developed for solving the low-frequency breakdown problem, while eliminating the need for an eigenvalue solution. Instead of introducing additional computational overhead to fix the low-frequency breakdown problem, the proposed method significantly speeds up the low-frequency computation with its $O(1)$ solution.

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Abstract—Full-wave solutions of Maxwell’s equations break down at low frequencies. Existing methods for solving this problem either are inaccurate or incur additional computational cost. In this work, a fast full-wave finite-element based solution is developed to eliminate the low frequency breakdown problem in a reduced system of order one. It is applicable to general 3-D problems involving ideal conductors as well as non-ideal conductors immersed in inhomogeneous, lossless, lossy, and dispersive materials. The proposed method retains the rigor of the theoretically rigorous full-wave solution recently developed for solving the low-frequency breakdown problem, while eliminating the need for an eigenvalue solution. Instead of introducing additional computational cost to fix the low-frequency breakdown problem, the proposed method significantly speeds up the low-frequency computation with its $O(1)$ solution.

Index Terms— Low-frequency breakdown, finite element methods, electromagnetic analysis, full-wave analysis, fast solution, null space

I. INTRODUCTION

IT has been observed that a full-wave based solution of Maxwell’s equations breaks down at low frequencies. Such a problem is especially severe in digital and mixed-signal integrated circuit applications in which signals have a broad bandwidth from DC to about the third harmonic frequency. In these applications, full-wave solvers typically break down at and below tens of MHz, which are right in the range of circuit operating frequencies. Inaccurate low-frequency models can lead to erroneous and misleading results in the analysis and design of integrated circuits (ICs) and systems. The inaccuracy at low frequencies is also found to be the major contributor to the violation of passivity, stability, and causality in existing frequency-domain models, which leads to divergence in time-domain simulation. Therefore, there is a critical need to solve the low-frequency breakdown problem.

Manuscript received August 1, 2011. This work was supported by a grant from Intel Corporation, a grant from Office of Naval Research under award N00014-10-1-0482, and a grant from NSF under award 0747578.

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Existing solutions to the low-frequency breakdown problem can be categorized into two classes. One class is to stitch a static- or quasi-static based electromagnetic solver with a full-wave based electromagnetic solver. This approach is inaccurate because static/quasi-static solvers involve fundamental approximations such as decoupled \mathbf{E} and \mathbf{H} , which is only true at DC. Moreover, at which frequency to switch between different solvers is an issue. As often seen in practice, the stitched results may not reach a consensus at their interfaces. Engineers usually have to employ an approximation-based model to achieve a smooth transition between static, quasi-static, and full-wave solvers. Besides the issue of accuracy, such an artificially created model often violates passivity, stability, and causality. The other class of methods for solving the low-frequency breakdown problem is to extend the validity of full-wave solvers to low frequencies. Existing approaches that belong to this category include introducing the loop-tree and loop-star basis functions to achieve a natural Helmholtz decomposition of the current in integral-equation-based methods [2]; using the tree-cotree splitting to provide an approximate Helmholtz decomposition for edge elements in finite-element-based methods (FEM) [3]; formulating current-charge integral equations and the augmented electric field integral equation [4-5], and developing Calderon preconditioner based methods [6-7]. These methods have been successful in extending the capability of existing full-wave solvers to low frequencies that cannot be solved previously. They have also suggested a number of new research questions to be considered. For example, if a method does not provide a universal solution of Maxwell’s equations from high frequencies down to any low frequency, then at which frequency, one should switch to a different solution method? If a method utilizes certain low-frequency approximations, when these approximations are valid and valid to which degree of accuracy? When a full-wave solution breaks down, is it true that natural or approximate Helmholtz decomposition can be used to produce accurate results? In other words, does a range of frequencies exist, in which neither traditional full-wave solvers (due to breakdown) nor natural or approximate Helmholtz decomposition (due to low-frequency approximations) can produce accurate results?

To address the aforementioned questions, it becomes necessary to know the true solution of full-wave Maxwell’s equations with \mathbf{E} and \mathbf{H} coupled from DC to high frequencies. Such a solution, which is a continuous function of frequency in a

full electromagnetic spectrum, was derived in [1, 8] without making use of theoretical approximations. In the two papers, it is shown that the root cause of low-frequency breakdown problem is finite machine precision. To bypass the barrier of finite machine precision, the method proposed in [1, 8] transformed the original frequency-dependent deterministic problem to a generalized eigenvalue problem, the nonzero eigenvalues of which are always within machine precision. With the inexact zero eigenvalues fixed to be zero, the method successfully bypassed the barrier of finite machine precision, and solved the low-frequency breakdown problem with a universal solution that is valid at both high and low frequencies. Moreover, the frequency dependence of the solution to Maxwell's equations is explicitly revealed from DC to high frequencies. One can use the resultant analytical model of frequency dependence to develop a theoretical understanding on how the field solution should scale with frequency in both low and high frequency regimes; at which frequency full-wave effects become important; at which frequency static assumptions yield good accuracy; etc.

The problem considered in [1, 8] is a purely lossless problem containing ideal dielectrics and perfect conductors or a purely lossy problem consisting of good conductors only. The low-frequency breakdown problem in general cases that involve both inhomogeneous lossless/lossy dielectrics and non-ideal conductors is significantly complicated by the frequency-dependent coupling between dielectrics and non-ideal conductors. In addition, the matrix resulting from the analysis of the metal-dielectric composite is highly unbalanced, which further complicates the low-frequency breakdown problem. In [9-10], this problem was successfully solved and a theoretically rigorous solution to Maxwell's equations was derived for general problems with inhomogeneous lossless and/or lossy dielectrics and non-ideal conductors from DC to high frequencies. In [13], it is further shown that the method developed in [1, 8-10] for a finite element based analysis is equally applicable to solving the low-frequency breakdown in an electric field integral equation.

The methods developed in [1, 8-10, 13] require an eigenvalue solution of a large-scale system of $O(N)$ with N being the problem size. Although with advanced techniques, the eigenvalue solutions can also be found in linear complexity [11-12], the resultant computational cost of solving the low frequency problem is still not desirable. Additional computational cost is also observed in other existing methods for solving the low-frequency breakdown problem.

In this work, without compromising accuracy, we propose a fast solution to eliminate the low frequency breakdown problem in a full-wave solver. Such a fast solution is, in fact, a direct outcome of the theoretical model developed in [1, 8-10] that explicitly reveals the frequency dependence of the solution to Maxwell's equations from DC to high frequencies. Such a theoretical model suggests that one can use one solution vector obtained from the traditional full-wave solver to reduce the original system of $O(N)$ to be a system of $O(1)$, and then fix the low-frequency breakdown problem readily in the reduced $O(1)$ system. In this way, we equally bypass the barrier of finite machine precision, preserve the theoretical rigor of the solution developed in [1, 8-10], while obtaining the field solution at low frequencies including DC without introducing any additional

computational cost. Instead, we accelerate the low-frequency computation by obtaining the field solution in $O(1)$ complexity.

In what follows, we first state the low frequency breakdown problem; then present the proposed fast low-frequency full-wave solution. We will start with the proposed solution for solving problems with ideal conductors; then proceed to the solution for solving problems with non-ideal conductors. The dielectrics can be inhomogeneous, lossless, lossy, or dispersive.

II. THE LOW-FREQUENCY BREAK-DOWN PROBLEM

Consider a general 3-D electromagnetic problem that involves both inhomogeneous dielectric materials and non-ideal conductors. A full-wave FEM-based analysis of such a problem results in the following matrix equation in frequency domain

$$\mathbf{A}(\omega)x(\omega) = b(\omega), \quad (1)$$

where ω is angular frequency, and

$$\mathbf{A}(\omega) = \mathbf{S} - \omega^2 \mathbf{T} + j\omega \mathbf{R}, \quad (2)$$

in which stiffness matrix \mathbf{S} , mass matrix \mathbf{T} , and conductivity related mass matrix \mathbf{R} are assembled from their elemental contributions as follows:

$$\begin{aligned} \mathbf{S}_{ij}^e &= \iiint_{V^e} \mu_r^{-1} (\nabla \times \mathbf{N}_i) \cdot (\nabla \times \mathbf{N}_j) dV, \\ \mathbf{T}_{ij}^e &= \iiint_{V^e} \frac{\epsilon_r}{c^2} \mathbf{N}_i \cdot \mathbf{N}_j dV, \\ \mathbf{R}_{ij}^e &= \iiint_{V^e} \mu_0 \sigma \mathbf{N}_i \cdot \mathbf{N}_j dV + \frac{1}{c} \iint_{S_o} (\hat{n} \times \mathbf{N}_i) \cdot (\hat{n} \times \mathbf{N}_j) dS \\ b_i^e &= -j\omega \mu_0 \iiint_{V^e} \mathbf{N}_i \cdot \mathbf{J} dV. \end{aligned} \quad (3)$$

In (3), c is the speed of light in free space, σ is conductivity, ϵ_r is relative permittivity, \mathbf{J} represents a current source, \mathbf{N} is the normalized vector basis function used to expand \mathbf{E} field, and superscript e denotes the contribution from an element.

The solution of (1) breaks down at low frequencies. In [1, 8], the root cause has been found to be finite machine precision, and a detailed analysis of the root cause has been given. Here, we give a brief summary. When frequency is low enough that the contribution of frequency dependent terms in (2), either $\omega^2 \mathbf{T}$ or $j\omega \mathbf{R}$, is lost due to finite machine precision, breakdown occurs. When this happens, the solution of (1) can be completely wrong because stiffness matrix \mathbf{S} is not invertible. To be specific, the value of \mathbf{S}_{ij} is $O(l)$ because $\nabla \times \mathbf{N}$ is proportional to $1/l$, and the value of \mathbf{T}_{ij} is proportional to $10^{-17} l^3$, where l is the average edge length used in a 3-D discretization of an electromagnetic structure. In state-of-the-art VLSI circuits, l is at the level of $1 \mu\text{m}$. Hence, the ratio of \mathbf{T} 's norm over \mathbf{S} 's norm is in the order of 10^{-29} , which is significantly smaller than that in a microwave or millimeter wave circuit. Since the norm of \mathbf{T} is 10^{-29} smaller than the norm of \mathbf{S} in a VLSI circuit, at low frequencies where $\omega^2 \mathbf{T}$ is sixteen orders of magnitude smaller than \mathbf{S} , even one uses double-precision computing, the mass matrix \mathbf{T} is essentially treated as zero by computers when performing the addition of $\omega^2 \mathbf{T}$ and \mathbf{S} . The same analysis applies to the lossy system where \mathbf{R} exists.

III. PROPOSED FAST LOW-FREQUENCY FULL-WAVE SOLUTION FOR PROBLEMS WITH IDEAL CONDUCTORS

A. Theoretical Basis of the Proposed Fast Solution

The theoretical model of the true solution to Maxwell's equations from DC to high frequencies developed in [1, 8] has provided a theoretical basis for the proposed fast solution. Next, we will use the lossless case that only involves ideal dielectrics and perfect conductors as an example to introduce this theoretical model. In a lossless case, (2) becomes

$$\mathbf{A}(\omega) = \mathbf{S} - \omega^2 \mathbf{T}. \quad (4)$$

From [1, 8], the solution of (4) can be obtained by solving the following generalized eigenvalue problem that is frequency independent

$$\mathbf{S}v = \lambda \mathbf{T}v, \quad (5)$$

where λ is eigenvalue and v is eigenvector. Denoting the diagonal matrix formed by all the eigenvalues by Λ , and the matrix formed by all the eigenvectors by V , the inverse of (4) can be written as

$$[\mathbf{A}(\omega)]^{-1} = V(\Lambda - \omega^2 \mathbf{I})^{-1} V^T, \quad (6)$$

where \mathbf{I} is an identity matrix. We also point out in [1] that the eigenvalues of (5) can be divided into two groups: one group is associated with physical DC modes as well as nonphysical ones originated from the null space of \mathbf{S} , and the other is associated with the nonzero resonance frequencies of the 3-D structure. The first group has zero eigenvalues. However, numerically they cannot be computed as exact zeros due to finite machine precision. Thus, we need to correct the inexact zeros to be exact zeros. With that, (6) becomes

$$\begin{aligned} [\mathbf{A}(\omega)]^{-1} &= (V_0 \ V_h) \begin{bmatrix} -\omega^2 \mathbf{I} & 0 \\ 0 & \Lambda_h - \omega^2 \mathbf{I} \end{bmatrix}^{-1} (V_0 \ V_h)^T, \\ &= -\frac{1}{\omega^2} V_0 V_0^T + V_h [\Lambda_h - \omega^2 \mathbf{I}]^{-1} V_h^T \end{aligned}, \quad (7)$$

where V_0 denotes the eigenvectors corresponding to zero eigenvalues, V_h and Λ_h denote the eigenvectors and eigenvalues corresponding to nonzero eigenvalues, i.e. higher order modes. As can be seen from (7), the frequency dependence of the solution to the full-wave FEM system matrix is explicitly derived. Except for ω , all the other terms are frequency independent. With such a continuous function of frequency, one can rigorously obtain the field solution from high frequencies down to any low frequency including DC without suffering from low-frequency breakdown.

To obtain a solution shown in (7) that is free of low-frequency breakdown, apparently, one has to first solve a generalized eigenvalue problem shown in (5), the computation of which could be nontrivial. In fact, with its analytical model of the frequency dependence, (7) already suggests a fast yet rigorous low-frequency full-wave solution that avoids the eigenvalue solution, which can be seamlessly incorporated into existing full-wave solvers to fix the breakdown problem readily. The details are given below.

From (7), it can be seen clearly that given any frequency ω , the field solution is the superposition of a number of 3-D eigenmodes. For a DC eigenmode, i.e. an eigenvector

corresponding to zero eigenvalue, its weight in the field solution is proportional to $\frac{1}{\omega^2}$; for a higher order eigenmode,

its weight in the field solution is proportional to $\frac{1}{\lambda_i - \omega^2}$, where

λ_i is the corresponding eigenvalue. At low frequencies where the weight of the higher order eigenmodes is significantly smaller than that of the DC eigenmodes, the contribution of the higher order eigenmodes in the field solution is negligible. As a result, (7) can be written as

$$[\mathbf{A}(\omega)]^{-1} \approx \left[-\frac{1}{\omega^2} V_0 V_0^T \right], \quad (8)$$

the accuracy of which, ε , can be controlled to any desired order by choosing an ω that is low.

From (8), it is clear that at low frequencies where the contribution from higher order eigenmodes is negligible, the space where the field solution resides is the union of the DC eigenmodes V_0 . In other words, the field solution resides in the null space of stiffness matrix \mathbf{S} . In (8), all the null-space vectors should be included because they are linearly independent with each other and each of them is indispensable in building a complete null space. Given a 3-D structure, even though the number of physical DC modes could be a few, the null space is mixed with both physical DC modes and non-physical ones. A linear combination of these two still resides in the same null space. As a result, solely from null-space vectors, one cannot distinguish physical DC modes from non-physical ones. Moreover, one cannot discard a subset of null-space vectors to reduce the size of null space since the remaining ones are not complete. Given an excitation vector, it can have a projection onto all of the null-space vectors, and hence each of the null-space vectors can have a contribution to the field solution. However, if one keeps all the null-space vectors, the resultant computational cost is high because the null space of stiffness matrix \mathbf{S} is large and, also, grows with matrix size N linearly. Therefore, how to handle the increased size of the null space becomes critical in developing a fast low-frequency solution.

Our solution to this problem is to utilize the fact that all the null-space vectors share the same zero eigenvalue in common although their eigenvectors are completely different. Based on this fact, we propose to use the right hand side vector (excitation vector) to shrink the dimension of the space where the field solution resides. To explain, in a deterministic solution, the right hand side is always known. In other words, we solve (1) for a given right hand side $b(\omega)$. With $b(\omega)$, effectively, all the null-space vectors are grouped together and the contribution from all the null-space vectors can be represented by a single vector w_0 as shown below:

$$x(\omega) = -\frac{1}{\omega^2} V_0 V_0^T b(\omega) = w_0. \quad (9)$$

In other words, the field solution vectors obtained at different frequencies are linearly dependent with each other, and hence representing the same solution space. A grouping like (9) would not be possible if the eigenvectors do not share the same eigenvalue in common, which is the case for higher order eigenvectors V_h . As can be seen from (7), even by right

multiplying with a right hand side vector $b(\omega)$, the eigenvectors V_h cannot be grouped together and represented by a one-vector based space. This is because the contribution from each V_h is different at different frequencies in the field solution owing to the difference in eigenvalues. By right multiplying with $b(\omega)$, although the contribution from all the higher order eigenvectors also becomes a single vector, this vector can be linearly independent with each other at different frequencies.

What is implied by (9) is significant: given a right hand side, only one vector is adequate to span the low frequency solution.

Although the above analysis is developed based on a lossless system shown in (4), in which both dielectrics and conductors are lossless, the finding that the field solutions at low frequencies can be fully represented by a reduced space of $O(1)$ is equally applicable to problems with dispersive and lossy dielectrics since the field solution still resides in the same null-space of stiffness matrix \mathbf{S} . To be specific, for problems with inhomogeneous lossless and/or lossy dielectrics, \mathbf{T} in (4) and (5) becomes a complex-valued matrix because of complex permittivity. At low frequencies where the contribution from higher order modes can be neglected, based on the analysis given in [10], (9) becomes

$$x(\omega) = -\frac{1}{\omega^2} V_0 (V^T \mathbf{T} V)^{-1} (V_0, V_h)^T b(\omega).$$

Since $(V^T \mathbf{T} V)^{-1}$ and V_h do not depend on frequency, again the above can be represented by a single-vector based space at different frequencies. For problems filled by a dispersive material, \mathbf{T} in (4) and (5) becomes frequency dependent because of frequency dependent permittivity. In this case, the $V_0 V_0^T$ in (9) will be scaled by a frequency dependent coefficient associated with relative permittivity, while the space represented by (9) is still of dimension 1 at different frequencies.

B. Proposed Fast Low-Frequency Full-Wave FEM Solution

Eqn. (9) serves as a theoretical basis for the proposed fast low-frequency full-wave solution of $O(1)$. As long as we can find the single vector w_0 that forms the $O(1)$ space in which all the low-frequency solutions reside, given a frequency regardless of how low it is, we can expand the field solution in this $O(1)$ space, and transform the original system of $O(N)$ shown in (1) to an $O(1)$ system, from which the low-frequency breakdown problem can be readily fixed.

To obtain w_0 and also avoid solving the generalized eigenvalue problem shown in (5), we developed the following approach. As can be seen from (8) and (9), at a low frequency where the contribution from higher order eigenmodes is negligible, the field solution $x(\omega)$ is in the space formed by a single vector w_0 . Therefore, we can use one solution vector obtained at such a frequency as a complete and accurate representation of the space formed by w_0 , i.e. the space where all the low-frequency solutions reside. Denoting such a frequency by f_{ref} , we solve the original system (1) as it is and

obtain a single solution vector, which is denoted by x_{ref} .

With x_{ref} , given any low frequency ω , we can expand the solution of the FEM-based system equation by using

$$x(\omega) = x_{ref} y, \quad (10)$$

with unknown coefficient y solved as the following:

$$x_{ref}^T (\mathbf{S} - \omega^2 \mathbf{T}) x_{ref} y = x_{ref}^T b(\omega). \quad (11)$$

As a result, the system is reduced to a one by one system. However, the low frequency breakdown problem still remains in the reduced $O(1)$ system when the term associated with ω^2 is neglected due to finite machine precision. To fix this problem, as the theoretically rigorous solution developed in [1], we utilize the fact that $\mathbf{S} x_{ref} = 0$ to vanish $\mathbf{S} x_{ref}$. This can be done because, as can be seen from (9), x_{ref} is a null-space vector that satisfies $\mathbf{S} V_0 = 0$. Therefore, (11) becomes:

$$x_{ref}^T (-\omega^2 \mathbf{T}) x_{ref} y = x_{ref}^T b(\omega), \quad (12)$$

which can be solved at any low frequency without breakdown. With unknown coefficient y solved from (12), the field solution can be recovered from (10). In this way, we can rapidly fix the low frequency breakdown problem, and meanwhile retaining the theoretical rigor of the low-frequency solution developed in [1, 8].

The aforementioned solution is applicable to problems with inhomogeneous lossless and/or lossy dielectrics, as well as problems filled with a dispersive material. For the latter, at low frequencies, although the field solution still resides in the null space of \mathbf{S} , the solution could scale with frequency in a complicated way since \mathbf{T} now becomes frequency dependent.

The remaining problem is whether we can always find an appropriate f_{ref} . This is discussed in the following section.

C. Existence of f_{ref} and Its Choice

The choice of f_{ref} is subject to two requirements. First, since we need to solve (1) at f_{ref} to obtain the field solution, the f_{ref} should be chosen at a frequency where the full-wave solution does not break down yet. Second, since we use the solution vector obtained at f_{ref} to represent the $O(1)$ space formed by w_0 , the field solution at f_{ref} should have a form shown in (9). In other words, the field solution at f_{ref} should be dominated by DC eigenmodes, with the contribution from higher order eigenmodes being negligible. To summarize, the f_{ref} should be a frequency at which the field solution is dominated by DC eigenmodes and meanwhile the full-wave solution does not break down yet. To choose f_{ref} , the first question we need to answer is whether such a frequency exists or not. To examine the existence of f_{ref} , we need to take a look at the relative relationship between the breakdown frequency, zero eigenvalues, smallest nonzero eigenvalue, and the largest eigenvalue of (5).

In lossless cases, the eigenvalue λ_i of (5) corresponds to one resonant frequency f_i of the 3-D structure being simulated. The f_i and λ_i have the following relationship:

$$f_i = \sqrt{\lambda_i} / 2\pi. \quad (13)$$

In lossy cases, although the resonance frequency becomes complex, it has the same relationship with the eigenvalue of the numerical system.

Theoretically speaking, a 3-D structure can have an infinitely large number of resonant frequencies. In reality, the number of resonant frequencies that can be numerically found is limited because of finite mesh size. Let the smallest nonzero resonance frequency be f_{\min} and the largest be f_{\max} , with their corresponding eigenvalues being λ_{\min} , and λ_{\max} respectively. The f_{\min} is determined by the largest physical dimension of the structure; while f_{\max} that can be numerically identified is determined by the smallest mesh size. Therefore, the ratio between f_{\max} and f_{\min} is proportional to the ratio between the largest physical dimension of the structure and the smallest mesh size, in other words, the aspect ratio of the problem being considered. Since the eigenvalue λ_i is the square of the resonance frequency as can be seen from (13), the ratio of λ_{\max} to λ_{\min} is the square of the aspect ratio. We denote the distance between λ_{\max} and λ_{\min} in terms of orders of magnitude by m . Their relative locations are illustrated in Fig. 1 along the axis of eigenvalue λ . Besides nonzero eigenvalues from λ_{\min} to λ_{\max} , (5) has a large null space, the eigenvalues of which are analytically known to be zero.

Next, we examine the relationship between the breakdown frequency, 0, λ_{\min} , and λ_{\max} . Since the root cause of

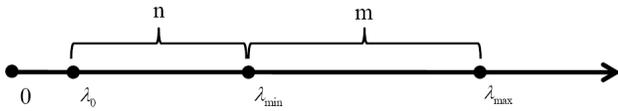


Fig. 1. Illustration of eigenvalues along the axis of λ . (λ_{\min} is the smallest nonzero eigenvalue, λ_{\max} is the largest eigenvalue, λ_0 is the angular frequency square corresponding to breakdown frequency.)

low-frequency breakdown problem is finite machine precision, at the frequency where a full-wave solution breaks down, the corresponding $\omega^2 \mathbf{T}$ should be beyond what can be captured by machine precision with respect to \mathbf{S} . In double precision computing, such an ω^2 should be 16 orders of magnitude smaller than $\|\mathbf{T}^{-1}\mathbf{S}\|$, and hence λ_{\max} . We denote such a breakdown ω^2 by $\lambda_0 = \omega^2$. Thus, if the distance between λ_0 and λ_{\min} is n , then $n = 16 - m$, as illustrated in Fig. 1. The frequency corresponding to λ_0 is denoted by f_0 , at and below of which the breakdown occurs. The n is always greater than 0 since m is less than 16. This is because as long as one can mesh the structure with a computer having finite precision, the difference between λ_{\max} and λ_{\min} is within machine precision.

Now it is ready to examine the existence of f_{ref} . From Fig. 1, it can be seen that f_{ref} should be above f_0 so that the full-wave solution does not break down yet and well below f_{\min} so that

the nonzero higher order eigenmodes can be neglected without losing accuracy. In other words, f_{ref} should fall into the range between f_0 and f_{\min} . Therefore, the angular frequency square corresponding to f_{ref} , $\lambda_{ref} = (2\pi f_{ref})^2$, should be between λ_0 and λ_{\min} . To ensure good accuracy, λ_{ref} should be chosen at least 2 orders magnitude smaller than λ_{\min} to obtain better than 1% accuracy. As a result, for f_{ref} , and hence λ_{ref} to exist, n shown in Fig. 1 should be no less than 2.

The condition of $n \geq 2$ is well satisfied in today's technology. We can use an integrated circuit as an example to quantitatively examine n . Driven by Moore's law, the smallest feature size of integrated circuits has been kept pushing down to the nanometer regime. Compared with the aspect ratio encountered in other engineering systems, the difference between the largest geometrical scale and the smallest scale present in today's integrated circuits can be viewed the largest. This is also the major reason why the low-frequency breakdown problem is found to be most critical in integrated circuit problems. In these problems, the ratio between the largest and the smallest feature size is approximately 1 cm



Fig. 2. Illustration of the possible range for λ_{ref}

versus 10 nm, which is 10^6 . Thus, the ratio of f_{\max} to f_{\min} is 10^6 , and hence the ratio of λ_{\max} to λ_{\min} is 10^{12} . Therefore, $m = 12$, and hence $n > 2$. As a result, as can be seen from the grey region in Fig. 2, there is a range between λ_0 and $(\lambda_{ref})_{\max}$, from which we can select any frequency to serve as f_{ref} with good accuracy achieved. Here, the $(\lambda_{ref})_{\max}$ is the largest λ_{ref} that can be chosen based on required accuracy.

It is worth mentioning that in future technologies in which the smallest feature size will be pushed further down, for example, to 2 orders of magnitude smaller than currently available while the largest feature size remains similar, then λ_{\max} will be pushed 4 orders of magnitude higher along the axis of λ with λ_{\min} remained almost the same as before. In this case, $n < 2$ can happen. Then we cannot find a frequency at which the field solution is dominated by DC eigenmodes while the full-wave solution has not broken down yet. In other words, when the full-wave solution breaks down due to finite machine precision, some higher order eigenmodes will also make important contributions to the field solution. For this case, the theoretically rigorous method for handling the low-frequency problem developed in [1, 8] is equally valid. As for the proposed fast low-frequency solution, in addition to the one vector shown in (9) that covers the contribution from all the DC eigenmodes for a given excitation, we can extend the algorithm to cover a few other vectors that characterize higher order eigenmodes. This will be considered in the future when there is a practical need for such a solution.

Next, we show how to develop a fast low-frequency full-wave solution in an $O(1)$ system for problems that involve

non-ideal conductors. Rigorously speaking, when low frequencies are considered, conductors cannot be treated as perfect conductors because fields penetrate into conductors at low frequencies.

IV. PROPOSED FAST LOW-FREQUENCY FULL-WAVE SOLUTION FOR PROBLEMS WITH NON-IDEAL CONDUCTORS

A. Theoretical Basis of the Proposed Fast Solution

Consider a problem that involves both inhomogeneous dielectric materials and non-ideal conductors. We divide field unknowns x in (1) into two groups: unknowns outside conductors x_o and unknowns inside conductors x_i . For unknowns that reside on the conducting surface, we categorize them into x_s . The FEM based system matrix $\mathbf{A}(\omega)$ shown in (1) can be correspondingly cast into the following form:

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{A}_{oo}(\omega) & \mathbf{A}_{oi}(\omega) \\ \mathbf{A}_{io}(\omega) & \mathbf{A}_{ii}(\omega) \end{bmatrix}, \quad (14)$$

where

$$\begin{aligned} \mathbf{A}_{oo}(\omega) &= \mathbf{S}_{oo} - \omega^2 \mathbf{T}_{oo} \\ \mathbf{A}_{oi}(\omega) &= \mathbf{S}_{oi} - \omega^2 \mathbf{T}_{oi} \\ \mathbf{A}_{io}(\omega) &= \mathbf{S}_{io} - \omega^2 \mathbf{T}_{io} \\ \mathbf{A}_{ii}(\omega) &= \mathbf{S}_{ii} + j\omega \mathbf{R}_{ii} - \omega^2 \mathbf{T}_{ii}. \end{aligned} \quad (15)$$

In [9, 10], a rigorous solution of $\mathbf{A}(\omega)$'s inverse is explicitly derived, which is applicable from high frequencies down to any low frequency including DC. With a common excitation used in the FEM-based analysis, which is a current source launched outside conductors, the right hand side vector of (1) can be written as

$$b(\omega) = \{-j\omega I \quad 0\}_i^T, \quad (16)$$

where the first row corresponds to x_o , and the second row corresponds to x_i . At low frequencies where the contribution from higher order eigenmodes can be neglected without losing accuracy, it is shown in [9, 10] that the field solution inside conductors (x_i) and that outside conductors (x_o) can be explicitly written as

$$\begin{aligned} x_o &= \left(-\frac{1}{\omega^2} V_o V_o^T + \frac{1}{j\omega} \mathbf{Q} V_{ii,0} V_{ii,0}^T \mathbf{Q}^T \right) (-j\omega I) \\ x_i &= -\frac{1}{j\omega} V_{ii,0} V_{ii,0}^T \mathbf{Q}^T (-j\omega I) \end{aligned} \quad (17)$$

where

$$\mathbf{Q} = V_o V_o^T \mathbf{T}_{oi} + V_h (\Lambda_h)^{-1} V_h^T \mathbf{S}_{oi}; \quad (18)$$

V_o and V_h respectively represent the null-space eigenvectors and higher order eigenvectors of the system outside conductors; and $V_{ii,0}$ denotes the null-space eigenvectors of the system inside conductors.

A careful examination of (17) reveals that the low frequency solution can be expanded by two groups of vectors

$$\Phi_1 = \begin{pmatrix} -\mathbf{Q} V_{ii,0} \\ V_{ii,0} \end{pmatrix} \text{ and } \Phi_2 = \begin{pmatrix} V_o \\ 0 \end{pmatrix}, \quad (19)$$

which span the real and imaginary part of the solution, respectively. The number of column vectors in Φ_1 is equal to the number of DC modes of the system inside conductors; the number of column vectors in Φ_2 is equal to the number of DC modes for what's outside the conductor. The total number of vectors in $[\Phi_1, \Phi_2]$ is the same as that of the DC modes of the entire stiffness matrix \mathbf{S} , i.e. the size of the null space of \mathbf{S} . In addition, all the vectors in $[\Phi_1, \Phi_2]$ are linearly independent with each other, which can be easily proved because V_o and $V_{ii,0}$ each contains linearly independent eigenvectors, and the two eigenvector sets are also mutually independent. In addition, $[\Phi_1, \Phi_2]$ resides in the null space of stiffness matrix \mathbf{S} . In other words, $[\Phi_1, \Phi_2]$ satisfies

$$\mathbf{S}[\Phi_1, \Phi_2] = 0. \quad (20)$$

This can be proved as the following:

$$\begin{aligned} \mathbf{S}[\Phi_1, \Phi_2] &= \begin{pmatrix} \mathbf{S}_{oo} & \mathbf{S}_{oi} \\ \mathbf{S}_{io} & \mathbf{S}_{ii} \end{pmatrix} \begin{pmatrix} -\mathbf{Q} V_{ii,0} & V_o \\ V_{ii,0} & 0 \end{pmatrix}, \\ &= \begin{pmatrix} (\mathbf{S}_{oi} - \mathbf{S}_{oo} \mathbf{Q}) V_{ii,0} & \mathbf{S}_{oo} V_o \\ (\mathbf{S}_{ii} - \mathbf{S}_{io} \mathbf{Q}) V_{ii,0} & \mathbf{S}_{io} V_o \end{pmatrix} \end{aligned} \quad (21)$$

which is zero. To see this clearly, let's analyze the matrix entries one by one. Based on the rigorous solution developed in [9, 10], the second-column entries in the last matrix shown in (21), $\mathbf{S}_{oo} V_o$ and $\mathbf{S}_{io} V_o$, can be immediately recognized as zero because V_o , the null-space eigenvectors of the system outside conductors, represents a gradient-type field solution that satisfies $\nabla \times \mathbf{E} = 0$. In the first-column entries, $(\mathbf{S}_{ii} - \mathbf{S}_{io} \mathbf{Q}) V_{ii,0}$ is zero because $V_{ii,0}$ is the null space of $(\mathbf{S}_{ii} - \mathbf{S}_{io} \mathbf{Q})$ [9, 10]; $(\mathbf{S}_{oi} - \mathbf{S}_{oo} \mathbf{Q}) V_{ii,0}$ is zero because it is $(\mathbf{A}_{oi} - \mathbf{A}_{oo} \mathbf{Q}) V_{ii,0}$ at DC, and \mathbf{Q} , as shown in (18), is nothing but $\mathbf{A}_{oo}^{-1} \mathbf{A}_{oi}$ at DC. Hence, we prove that (21) is zero, thus $[\Phi_1, \Phi_2]$ belongs to the null space of \mathbf{S} . Since the dimension of $[\Phi_1, \Phi_2]$ is the same as the null space of stiffness matrix \mathbf{S} , and (20) holds true, we conclude that $[\Phi_1, \Phi_2]$ constitutes a complete null space of stiffness matrix \mathbf{S} .

Like in cases with ideal conductors, the dimension of the null space encountered in cases with non-ideal conductors also linearly grows with matrix size N . To shrink the dimension of the space where the field solution resides for constructing a fast low-frequency solution, a method similar to that developed for cases with ideal conductors can be developed. The details are as follows. With right hand side $b(\omega)$ known in (16), it can be seen from (17) that the real part of the low-frequency solution x , $re(x)$, is nothing but a superposition of the vectors in Φ_1 ; the imaginary part of the low-frequency solution x , $im(x)$, is nothing but a superposition of the vectors in Φ_2 . With $b(\omega)$, all the null-space vectors in Φ_1, Φ_2 are grouped together, yielding a single vector wr -based representation of $re(x)$ as shown in the following:

$$re(x) = \begin{pmatrix} re(x_o) \\ re(x_i) \end{pmatrix} = \begin{pmatrix} -\mathbf{Q}V_{ii,0}V_{ii,0}^T\mathbf{Q}^T I \\ V_{ii,0}V_{ii,0}^T\mathbf{Q}^T I \end{pmatrix} = wr. \quad (22)$$

Similarly, all the null-space vectors in Φ_2 are grouped together via $b(\omega)$, yielding a single vector wi - based representation of $im(x)$ as shown in the following:

$$im(x) = \begin{pmatrix} im(x_o) \\ im(x_i) \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}V_0V_0^T I \\ 0 \end{pmatrix} = wi. \quad (23)$$

The two vectors wr and wi form a complete space for representing the field solution of (2) at low frequencies. This finding again holds true for problems with inhomogeneous lossless and/or lossy dielectrics as well as problems filled with a dispersive material.

B. Proposed Fast Low-Frequency Full-Wave FEM Solution

To obtain the reduced space of $O(1)$ composed of wr and wi and also avoid solving the eigenvalue problem, similar to the approach developed for cases with ideal conductors, at one frequency f_{ref} whose corresponding angular frequency square is between λ_0 and $(\lambda_{ref})_{max}$, we solve the original system (1) to obtain the field solution x_{ref} . Different from cases with ideal conductors, we separate this solution into two vectors, namely, the real part x_{re} and the imaginary part x_{im} . The x_{re} constitutes the same $O(1)$ space as that formed by wr , while x_{im} represents the same $O(1)$ space formed by wi .

Next, we use x_{re} and x_{im} to form space

$$z = [x_{re}, x_{im}]. \quad (24)$$

Expanding field solution in this space, and testing the system by the same space, we obtain a reduced system of order one as follows

$$z^T(\mathbf{S} - \omega^2\mathbf{T} + j\omega\mathbf{R})z = z^T b(\omega). \quad (25)$$

Again, this reduced system still experiences the low-frequency breakdown problem because of finite machine precision. To overcome this problem, we vanish $z^T\mathbf{S}z$ based on the fact shown in (20). Thus, we have

$$z^T(-\omega^2\mathbf{T} + j\omega\mathbf{R})z = z^T b(\omega). \quad (26)$$

The left-hand-side matrix of (26) is a 2×2 matrix. Apparently, the solution of (26) can still break down at low frequencies since \mathbf{T} - and \mathbf{R} -related terms have different frequency dependence. However, this problem does not exist in (26) because (26) is a diagonal matrix (the proof is given in the Appendix) as shown in the following:

$$\begin{pmatrix} -\omega^2\tilde{\mathbf{T}} & 0 \\ 0 & j\omega\tilde{\mathbf{R}} \end{pmatrix} y = z^T b(\omega) \quad (27)$$

where

$$\tilde{\mathbf{R}} = x_{re}^T \mathbf{R} x_{re} = x_{re,i}^T \mathbf{R}_{ii} x_{re,i},$$

$$\tilde{\mathbf{T}} = x_{im}^T \mathbf{T} x_{im} = x_{im,o}^T \mathbf{T}_{oo} x_{im,o}. \quad (28)$$

After the diagonal 2×2 system shown in (27) is solved, y is known. The solution of the original system (1) can then be readily obtained from

$$x(\omega) = z y. \quad (29)$$

By doing so, we obtain the field solution at any low frequency including DC. As can be seen from the above procedure, instead of introducing additional computational cost into a full-wave solver, the proposed method accelerates the full-wave computation at low frequencies by an $O(1)$ solution.

V. NUMERICAL RESULTS

In order to validate the proposed method, we simulated a number of on-chip and package examples.

The first example is a 3-D on-chip interconnect embedded in inhomogeneous materials shown in Fig. 3. In this figure, the detailed geometrical and material parameters are given. The structure is of length $2000 \mu\text{m}$ into the paper. Along the length direction, the front and the back end each is attached to an air layer, which is then truncated by a Neumann-type boundary condition. The top and bottom planes shown in Fig. 3 are backed by a PEC (perfect electric conducting) boundary condition. The left and right boundary conditions are Neumann-type boundary conditions. The shaded region is occupied by conductors. To validate the proposed fast solution for cases with ideal conductors, the conductor is assumed to be perfect. The cases with conductor loss will be considered in the third example. A current source of 1 A is launched from the bottom plane to the center conductor in the $M2$ layer. The smallest mesh size is $0.1 \mu\text{m}$. For this example, a traditional full-wave solver breaks down at $\sim 10\text{MHz}$. In our simulation, we choose $f_{ref} = 100\text{MHz}$ (the reason is given later in this section) and solve the original system (1) at this frequency to obtain x_{ref} . The field solution at any lower frequency including DC is then solved from (10) and (12). In Fig. 4 (a), we plot the electric field distribution at 10^{-32}Hz in the transverse plane of the 3-D interconnect simulated by the proposed method. In Fig. 4 (b), we plot the electric field distribution simulated by a conventional full-wave FEM solver. Clearly, the proposed method produces an accurate electric field distribution, whereas the traditional solver breaks down. In Table I, we compare the results generated by the proposed solution and those obtained from the rigorous solution developed in [1, 8] that solved a generalized eigenvalue problem shown in (5). The capacitances extracted by these two solutions agree very well with each other. The relative error of the proposed solution is shown to be very small compared to the rigorous solution. It is clear that the proposed fast low-frequency solution preserves the accuracy of the theoretically rigorous solution in [1, 8] while eliminating the need for solving an eigenvalue problem.

Since the proposed solution utilizes the solution vector obtained at one frequency, f_{ref} , to obtain the field solution at any low frequency where a traditional full-wave solver would break down, one might be interested to know how the f_{ref} is

determined in this example. The f_{ref} is analytically estimated from the geometrical and mesh data based on the theoretical analysis given in section III.C. First, we analytically estimate f_{min} , f_{max} and f_0 , which are found to be $f_{min} \sim 3 \times 10^{10}$ Hz, $f_{max} \sim 6.7 \times 10^{14}$ Hz, and $f_0 \sim 1 \times 10^7$ Hz. In our estimation, a uniform material with an effective permittivity is used. These estimation results agree very well with numerical data, in which f_{min} and f_{max} are shown to be 3.8×10^{10} Hz and 1×10^{15} Hz respectively. As mentioned, the conventional full-wave solver breaks down at ~ 10 MHz. This agrees with our analytical prediction since the square of this breakdown frequency is 16 orders of magnitude smaller than λ_{max} . From the estimated f_{min} and f_0 , we know that f_{ref} can be arbitrarily chosen between 1×10^7 Hz and 3×10^9 Hz with good accuracy. This range is above f_0 and one order of magnitude smaller than f_{min} so that the resultant λ_{ref} is at least two orders of magnitude smaller than λ_{min} . This is how $f_{ref} = 100$ MHz is determined.

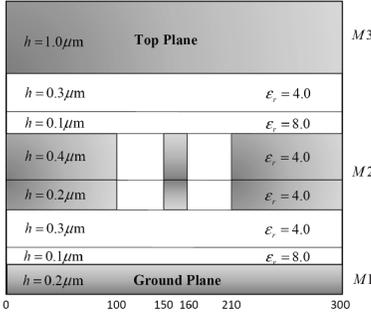


Fig. 3. Illustration of an on-chip 3-D interconnect.

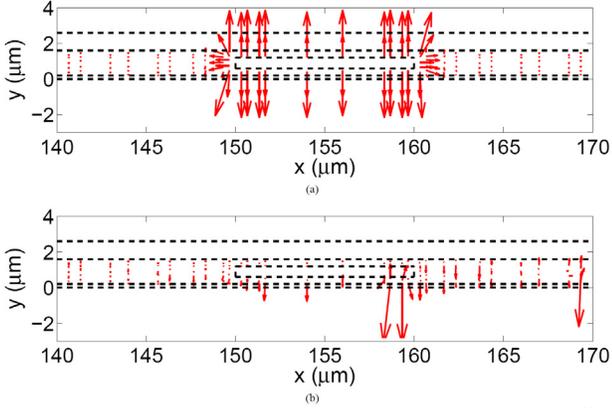


Fig. 4. E field distribution of a 3-D on-chip interconnect at 10^{-32} Hz. (a) Proposed method. (b) Conventional full-wave FEM method.

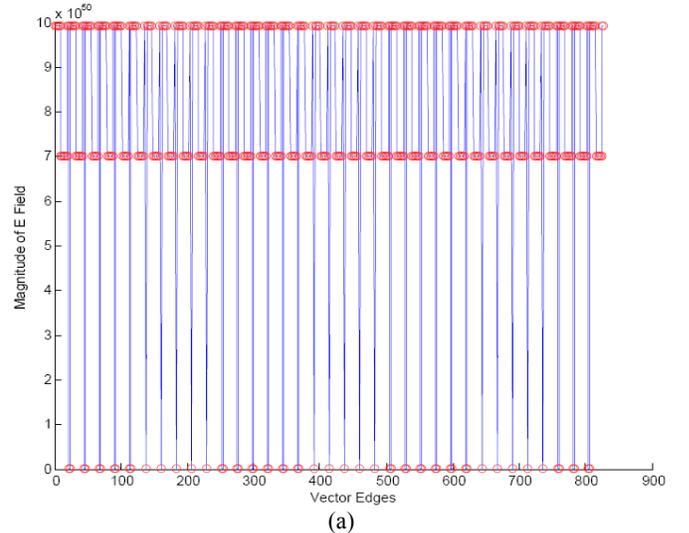
Table I. Comparison of the Capacitance Results of a 3-D On-Chip Interconnect Structure.

Frequency (Hz)	Capacitance (F)		Solution relative error
	The rigorous solution [1,8]	The proposed fast solution	
1e8	4.4852e-12	4.4853e-12	8.9415e-04
1e5	4.4851e-12	4.4853e-12	8.9169e-04
1e3	4.4851e-12	4.4853e-12	8.9169e-04
1e-1	4.4851e-12	4.4853e-12	8.9169e-04
1e-16	4.4851e-12	4.4853e-12	8.9169e-04
1e-32	4.4851e-12	4.4853e-12	8.9169e-04

Next, in order to demonstrate the capability of the proposed solver in solving problems with a dispersive material, we consider a parallel plate structure filled with a material with complicated frequency dependence. The width, height, and length of the structure are set to be $10 \mu\text{m}$, $1 \mu\text{m}$, and $35 \mu\text{m}$, respectively in accordance with the typical dimensions of on-chip circuits. The dielectric material between two PEC plates is FR4, which is modeled by the following dielectric loss model [14]

$$\epsilon_r(\omega) = \epsilon'_\infty + \frac{\Delta\epsilon'}{1 + j\frac{\omega}{\omega_0}} \quad (19)$$

where $\epsilon'_\infty = 4.9$, $\Delta\epsilon' = 0.28$, and $\omega_0 = 2 \times 10^6 \text{ s}^{-1}$. A current source of 1 A is injected from the bottom plane to the top plane. The smallest edge length used in discretization is $1 \mu\text{m}$. We analytically estimate $f_0 \sim 1 \times 10^6$ Hz, $f_{min} \sim 1.9 \times 10^{12}$ Hz, and $f_{max} \sim 6.7 \times 10^{13}$ Hz. These data are in good agreement with the actual data, which is shown to be 1.9×10^{12} Hz for f_{min} and 1×10^{14} Hz for f_{max} . In addition, we examine the 1-norm of $\omega^2 \mathbf{T}$ over that of \mathbf{S} , we find that it is larger than machine precision when frequency is higher than 6×10^6 Hz, which agrees with the fact that the conventional full-wave FEM solver breaks down at ~ 1 MHz. Based on these analytical estimations, we choose 100 MHz as f_{ref} in this simulation. In Fig. 5(a), we plot the electric field at each edge in the computational domain at 10^{-32} Hz simulated by the proposed method in comparison with that obtained from the rigorous method developed in [9, 10]. Two results agree very well with each other and both exhibit an open circuit phenomenon. In contrast, the traditional full-wave FEM solver gives very small field values, which is wrong, as shown in Fig. 5(b). In Table II, we compare the admittances simulated using the proposed method, the rigorous solution [9, 10], and a conventional FEM solver. It is clear that the proposed solution agrees very well with the rigorous solution, whereas the conventional FEM solver is totally wrong at low frequencies.



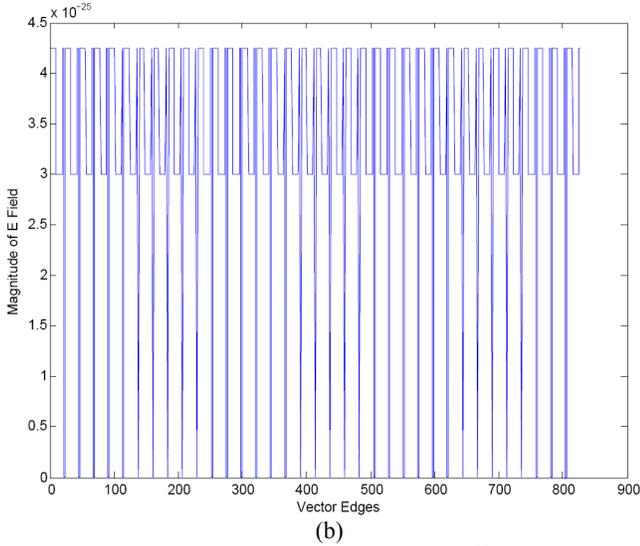


Fig. 5. Electric field simulated at each edge at 10^{-32} Hz. (a) Proposed method (in red) and rigorous method [9, 10] (in blue). (b) Conventional full-wave FEM method.

impedance is extracted between one part of the inductor and the bottom reference ground with the other port left open. In Table V, ‘open’ means open circuit. Moreover, in order to verify our theoretical analysis of $[\Phi_1, \Phi_2]$, we checked the number of DC modes for the system inside and outside the conductor. One is 356 and the other is 365. Their addition is 731, which is exactly the number of DC modes of the entire S matrix.

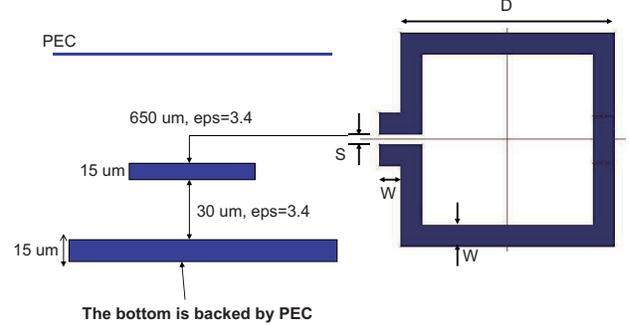


Fig. 6. The geometry and material of a 3-D spiral inductor.

Table II: Admittance extracted by three methods

Frequency (Hz)	Real Part of the Admittance (1/Ω)			Imaginary Part of the Admittance (1/Ω)		
	Proposed Solution	Rigorous Solution [9,10]	Conventional Full-wave Method	Proposed Solution	Rigorous Solution [9,10]	Conventional Full-wave Method
10^8	2.7581597e-18	2.75815973e-18	2.758159737e-18	1.5163937646e-14	1.516393756e-14	1.516349335e-14
10^7	2.75539591e-17	2.7553958e-17	2.755395883e-17	1.5164805934e-14	1.5164805852e-14	1.51510056e-14
10^6	2.50443535e-16	2.50443533e-16	2.477684473e-16	1.5243647519e-14	1.5243647435e-14	1.18176313e-14
10^5	2.47768450e-16	2.477684473e-16	2.504435330e-16	1.595260031e-14	1.5952600245e-14	- 3.7530252e-13
10^3	2.72219529e-18	2.72219526e-18	2.722195262e-18	1.6030430535e-14	1.6030430446e-14	- 3.7530252e-09
10^{-1}	2.72222216e-22	2.72222212e-22	0	1.603043909e-14	1.6030438998e-14	- 0.3753025
10^{-16}	2.72222216e-37	2.72222216e-37	2.72222212e-37	1.603043909e-14	1.6030438998e-14	- 3.7530252e+29
10^{-32}	2.72222216e-53	2.72222212e-53	2.722222129e-53	1.603043909e-14	1.6030438998e-14	- 3.7530252e+61

The last example involves both inhomogeneous dielectrics and non-ideal conductors. It is a 3D spiral inductor residing on a package. The geometry of the spiral inductor is shown in Fig. 6. Its diameter (D) is 1000 μm . The metallic wire is 100 μm wide and 15 μm thick. The metal conductivity is 5.8×10^7 . The port separation (S) is 50 μm . The inductor is backed by two package planes. The backplane is 15 μm thick. In this simulation, the smallest mesh size is 10 μm in dielectric regions. Based on an analytical estimation, f_{\min} and f_{\max} are found to be ~ 15 GHz and $\sim 1.5 \times 10^4$ GHz, respectively. Moreover, we can estimate that f_0 is between 0.1 MHz and 1 MHz, which is also verified by the simulation based on the conventional full-wave solver. Based on f_{\min} and f_0 , we chose 10 MHz as f_{ref} in this simulation.

In Table IV, we compare the input impedance simulated by three solutions at low frequencies: the proposed solution, the rigorous solution [9, 10], and the conventional full-wave FEM solution. It is clear that among the three solutions, the proposed solution is in an excellent agreement with the rigorous solution, both of which can generate correct frequency dependence for real and imaginary parts. It is worth mentioning that the input

VI. CONCLUSIONS

It has been observed that a full-wave solution of Maxwell’s equations breaks down at low frequencies. In order to efficiently eliminate the low-frequency breakdown problem, this work presents a fast low frequency full-wave finite-element based solution, for both problems involving ideal conductors and problems with non-ideal conductors immersed in inhomogeneous, lossless, lossy, and dispersive materials. It retains the theoretical rigor of the solution developed in [1, 8-10], while eliminating the need for an eigenvalue solution. We have identified that the low frequency solution is dominated by the null space of the stiffness matrix. Although the dimension of the null space grows linearly with the problem size, we show that a single solution vector obtained at one low frequency serves as a complete space for representing the contribution from all the null-space vectors for a given excitation. Therefore, utilizing one such vector, we reduce the original system of $O(N)$ to an $O(1)$ system. By dropping the resultant stiffness matrix rigorously based on the fact that the field solution is in the null-space of the stiffness matrix, we successfully bypass the barrier of finite machine precision which is the root cause of low-frequency breakdown

Table IV: Input Impedance Comparison

Frequency (Hz)	Real Part of the Input Impedance (Ω)			Imaginary Part of the Input Impedance (Ω)		
	Proposed Solution	Rigorous Solution [9,10]	Conventional Full-wave Method	Proposed Solution	Rigorous Solution [9,10]	Conventional Full-wave Method
10^7	2.7484e-1	2.7413e-1	2.7413e-1	-1.6252e4	-1.6252e4	-1.6252e4
10^5	2.7484e-1	2.7300e-1	2.4058e-1	-1.6252e6	-1.6252e6	-1.6412e6
10^3	2.7484e-1	2.7300e-1	1.9373e8	-1.6252e8	-1.6252e8	3.2457e8
10^{-1}	2.7484e-1	2.7300e-1	-23.6	-1.6252e12	-1.6252e12	349.0
10^{-16}	2.7484e-1	2.7300e-1	-6.051e-10	-1.6252e27	-1.6252e27	-2.638e-11
10^{-32}	2.7484e-1	2.7300e-1	-5.000e-40	-1.6252e43	-1.6252e43	-7.8598e-22
0	2.7484e-1	2.7300e-1	0	Open	Open	0

and, also, solve the breakdown problem efficiently. Instead of introducing additional computational cost to fix the low-frequency breakdown problem, the proposed method significantly speeds up the low-frequency computation with its $O(1)$ solution. The proposed method can be used to capture complicated frequency dependence at low frequencies due to material dispersion and conductor loss.

Moreover, the reduced space of $O(1)$ identified in this work serves as a complete representation of the contribution from all the null-space vectors, i.e. DC eigenmodes, for a given excitation. Such an $O(1)$ space not only can be used to rapidly fix the low-frequency breakdown problem in finite element based methods, but also can be employed by other frequency-domain and time-domain methods for fast and accurate low-frequency analysis. In addition, the proposed $O(1)$ space effectively shrinks the dimension of the original null space that grows linearly with the problem size, and hence can be used in other applications where null-space vectors are required.

We have also theoretically analyzed the relationship between zero frequency, breakdown frequency, the first nonzero eigenvalue, and the highest eigenvalue of the numerical system; from which we demonstrated the validity of the proposed $O(1)$ solution in technologies that are available today. For future technologies or applications in which not only DC eigenmodes but also higher order eigenmodes contribute to the solution at the breakdown frequency, the proposed $O(1)$ space can be flexibly expanded to cover a few other vectors that characterize nonzero higher order modes in addition to the single vector that represents the contribution from all DC modes, with the total cost still minimized to be negligible.

A large part of this paper is devoted to derivations that serve as the theoretical basis of the proposed fast solution. For a quick reference, readers can refer to section III.B and section IV.B, which is the outcome of the proposed research. As can be seen from these two sections, the implementation of the proposed fast low-frequency full-wave solution is user friendly.

Appendix

Here, we prove that (27) is a diagonal matrix.

Based on (22), the x_{re} in (24) can be compactly written as

$$x_{re} = \begin{pmatrix} -\mathbf{Q}V_{ii,0}y_i \\ V_{ii,0}y_i \end{pmatrix}, \quad (\text{A.1})$$

where $y_i = V_{ii,0}^T \mathbf{Q}^T I$ denotes the coefficient vector that carries the weight of each null-space eigenvector in field solution. Similarly, based on (23), the x_{im} in (24) can be compactly written as

$$x_{im} = \begin{pmatrix} V_0 y_o \\ 0 \end{pmatrix}, \quad (\text{A.2})$$

where $y_o = V_0^T I / \omega$. By using (A.1) and (A.2), we have

$$z^T (-\omega^2 \mathbf{T} + j\omega \mathbf{R}) z = \begin{pmatrix} -y_i^T (\mathbf{Q}V_{ii,0})^T & y_i^T (V_{ii,0})^T \\ y_o^T V_0^T & 0 \end{pmatrix} \times \begin{pmatrix} -\omega^2 \mathbf{T}_{oo} & -\omega^2 \mathbf{T}_{oi} \\ -\omega^2 \mathbf{T}_{io} & j\omega \mathbf{R}_{ii} \end{pmatrix} \begin{pmatrix} -\mathbf{Q}V_{ii,0}y_i & V_0 y_o \\ V_{ii,0}y_i & 0 \end{pmatrix} \quad (\text{A.3})$$

By utilizing the following fact:

$$\mathbf{T}_{oo} \mathbf{Q}V_{ii,0}y_i - \mathbf{T}_{oi}V_{ii,0}y_i = 0, \quad (\text{A.4})$$

it can be readily derived that the off-diagonal terms of (A.3) are zero. Next, we show why (A.4) holds true.

Since x_{re} and x_{im} are obtained from a field solution x_{ref} that satisfies (1), from (14) and (16), the x_{ref} 's components x_o and x_i satisfy

$$\mathbf{A}_{oo}(\omega)x_o + \mathbf{A}_{oi}(\omega)x_i = -j\omega I. \quad (\text{A.5})$$

Thus

$$\mathbf{A}_{oo}(\omega)x_{re,o} + \mathbf{A}_{oi}(\omega)x_{re,i} = 0, \quad (\text{A.6})$$

where $x_{re,o}$ is the real part of x_o , and $x_{re,i}$ is the real part of x_i . Since $x_{re,o} = -\mathbf{Q}V_{ii,0}y_i$ and $x_{re,i} = V_{ii,0}y_i$, as can be seen from (A.1), we have

$$(\mathbf{S}_{oo} - \omega^2 \mathbf{T}_{oo})(-\mathbf{Q}V_{ii,0}y_i) + (\mathbf{S}_{oi} - \omega^2 \mathbf{T}_{oi})(V_{ii,0}y_i) = 0, \quad (\text{A.7})$$

which can be further written as:

$$(\mathbf{S}_{oi} - \mathbf{S}_{oo} \mathbf{Q})V_{ii,0}y_i + \omega^2 (\mathbf{T}_{oo} \mathbf{Q}V_{ii,0}y_i - \mathbf{T}_{oi}V_{ii,0}y_i) = 0. \quad (\text{A.8})$$

Since $(\mathbf{S}_{oi} - \mathbf{S}_{oo} \mathbf{Q})V_{ii,0} = 0$ as shown in Section IV.A, (A.4) is obtained.

In addition to recognizing that (A.3) is diagonal, the derivation of (27) also utilizes the fact that the displacement

current can be neglected inside conductors compared to conduction current from DC to very high frequencies.

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