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Optimal Multi-Agent Coordination under Tree Formation Constraints

Wei Zhang and Jianghai Hu, *Member, IEEE*

Abstract

The optimal multi-agent coordination problem tries to find the motions for a group of agents that start from a set of initial positions and end at a set of destination positions with the least energy expenditure. Often times, the problem is formulated with constraints on the formation of the agents, namely, distances between certain pairs of agents need to be kept constant throughout the process. In this paper, the optimal multi-agent coordination problem is studied for the special case when the graph describing the formation constraint is a tree. The equations characterizing the optimal coordinated motions are derived in a suitably chosen coordinate system. The solutions to these equations, however, may fail to be optimal once extended beyond certain points called the conjugate points due to the failure of second order optimality condition. Two methods for computing the conjugate points are introduced. For an instance of the problem where the agents try to rotate around a center point, the conjugate points along a natural candidate solution are characterized analytically and verified through numerical simulations. Moreover, better solutions are found that consume less energy than the candidate solution after it is extended beyond its first conjugate point.

I. Introduction

Due to their diverse applications in engineering fields, multi-agent coordination problems have attracted increasing attention of the control community in the recent years. Their applications include, for example, air traffic management (ATM [9], [17], [30]), robotics ([4]), Unmanned Aerial Vehicle (UAVs [26], [31]), and spacecraft [13], [29], etc. In these applications, the system under study consists of a group of agents that can coordinate their motions to achieve a common goal or complete a common task, subject to some practical constraints. For example, the motions of the aircraft flying at the same altitude in a region of the airspace are coordinated by the air traffic controller for on-time arrivals and at the same time avoiding conflicts, namely, events when two aircraft are within 5 nautical miles horizontally in en-route airspace and 3 nautical miles close to airports. As another example, a team of mobile robots may coordinate their motions to carry a common object from one end of the room to the other end without dropping it or running into the obstacles. As an UAV example, a group of UAVs may need to fly in a certain formation to reduce their fuel expenditure and keep active communication links among them.

In all these examples, there are separation constraints among agent pairs due to safety, physical, or efficiency reasons. Such constraints can be classified as formation constraints. In this paper, we study a version of the optimal

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multi-agent coordination problem with formation constraints. More precisely, we aim to find the coordinated motions with the minimum energy cost that can move a group of agents from given initial positions to given destination positions within a certain time horizon, while at the same time satisfy some formation constraint, namely, the distances between certain pairs of agents need to be kept constant throughout the process.

Compared with the many existing works on multi-agent coordination, the focus of this paper has several distinctive features. First, we study the centralized coordination problem for multiple agents, rather than the decentralized ones studied in, for example, [14], [20], [22]. Second, many other works on formation-constrained multi-agent coordination focus on aspects such as stability (e.g. [6], [8], [10], [25], [28]), feasibility (e.g., [23], [27]), and consensus forming (e.g. [3], [21], [24]); whereas this paper is among the few (see, e.g., [7], [11], [12]) dealing with optimality of the coordinated motions. Furthermore, in this paper, simple kinetic models are used for the agent dynamics, and only holonomic constraints are considered in the formation constraints, as opposed to the numerous works dealing with nonholonomic constraints (e.g. [1], [16], [27]). Other related references can be found in the collection [18].

Although some of the above features are limiting in practice, they allow us to formulate and solve the optimal coordinated motion planning problem for multiple agents with formation constraints analytically, and to obtain higher order optimality conditions such as those characterized by conjugate points, thus making this paper one of the few contributions that can accomplish these two goals. Specifically, the main contributions of this paper consist of the following. By focusing on tree formation constraints, (geodesic) equations are derived to characterize the solutions to the optimal coordination problem. Since these equations are only necessary, their solutions may fail to be optimal once extended too long for various reasons, one of which is the occurrence of conjugate points. For an instance of the problem where the group of agents rotate around a common centroid, we derive the Jacobi equation that can be used to characterize the conjugate points along a candidate solution, and propose two methods for its solution. These methods are tested on a family of tree formation constraints, where we derive analytically the conjugate points, as well as the better solutions once the candidate solution is extended beyond its conjugate points.

The multi-agent coordination problem studied in this paper is originally proposed in its preliminary form in [11], and studied for a special case in [12]. This paper is a generalization of these works, and contains several significant improvements. First of all, this paper studies the much more general tree formation constraints, for which the snake formation studied in [12] is only a very special case. Second, this paper derives equations for the optimal solutions whose coefficients are matrices dependent on the graph describing the tree formation constraint, and in particular, for the case studied in Section III-C, optimality conditions on the initial positions of the agents in accordance with the graph and their mechanical interpretations. For the snake formation studied in [12], these conditions are trivially satisfied. More importantly, from a computational point of view, the Jacobi equation derived for the tree formation constraint in this paper is of the form $\ddot{x} + L_1\dot{x} + L_0 x = 0$ for some nonzero matrices L_1 and L_0, which necessitates the solution of a general second order matrix differential equation through matrix polynomial and latent roots/vectors analysis [19]; whereas in [12], the matrix L_1 is zero, and a simple eigenvalue/eigenvector analysis of

L_0 will suffice. Hence this paper deals with a much more challenging problem than the one studied in [12]. See Remark 3 in Section III-D for further comments on this aspect.

This paper is organized as follows. In Section II, the general formation constrained optimal multi-agent coordination problem is formulated. Then in Section III, we focus on the special case where the formation graph is a tree. First, the problem is reformulated as an optimal control problem in a more convenient new coordinate system (Section III-A). Then, first order optimality conditions are derived by using the optimal control theory (Section III-B). Next, for an important instance of the problem where the group of agents rotate around a common centroid, a natural candidate solution is proposed based on the first order conditions (Section III-C). For this candidate solution, second variation is then carried out to derive the Jacobi equation that characterizes the conjugate points beyond which the candidate solution will fail to be optimal (Section III-D). In Section IV, two methods are proposed to solve the Jacobi equation. These methods are illustrated in Section V, where for a family of tree formation constraints, we derive analytically the conjugate points along the candidate solution (Section V-A), and verify them numerically (Section V-B). Better solutions are also derived when the candidate solution is extended beyond its first conjugate point (Section V-C). Finally, some concluding remarks are given in Section VI.

II. Optimal Formation Constrained Multi-Agent Coordination

In this section, the problem of optimal multi-agent coordination under formation constraints is formulated. We first introduce some notations.

Consider $n+1$ agents moving on a plane $\mathbb{R}^2$. Their positions are denoted by the ordered $(n+1)$-tuple $\langle q_i \rangle_{i=0}^n = (q_0, \ldots, q_n)$, where $q_i \in \mathbb{R}^2$ is the position of agent i, $i = 0, \ldots, n$. A formation constraint on the locations of the $n+1$ agents can be described in terms of an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, whose set of vertices $\mathcal{V} = \{0, \ldots, n\}$ consists of $n+1$ nodes that correspond to the $n+1$ agents, and whose set of edges $\mathcal{E}$ is a subset of $\mathcal{V} \times \mathcal{V}$. An $(n+1)$-tuple $\langle q_i \rangle_{i=0}^n$ is said to satisfy the *$\mathcal{G}$-formation constraint* if and only if for each edge $(i,j) \in \mathcal{E}$, $0 \le i, j \le n$, the distance of agent i and agent j is at a prescribed value (say, unity):

$$\|q_i - q_j\| = 1 \text{ for each } (i,j) \in \mathcal{E}.$$

Note that in the above definition, if (i,j) is not an edge in $\mathcal{E}$, there is no constraint on the distance between agents i and j: $\|q_i - q_j\|$ can be either greater or smaller than 1.

Problem 1 (Optimal Formation Constrained Multi-Agent Coordination): Given a formation graph $\mathcal{G}$, and the starting position $\langle a_i \rangle_{i=0}^n$ and the destination position $\langle b_i \rangle_{i=0}^n$ of the $n+1$ agents, find the motions $\langle q_i(t) \rangle_{i=0}^n$ of the agents over a time interval $[0, t_f]$ so that

1) for each agent i, it starts from a_i at time 0 and ends at b_i at time t_f, i.e., $q_i(0) = a_i$, $q_i(t_f) = b_i$;
2) the locations $\langle q_i(t) \rangle_{i=0}^n$ of the $n+1$ agents satisfy the $\mathcal{G}$-formation constraint at all times t in $[0, t_f]$;
3) the total energy expenditure J is minimized, where J is defined by

$$J = \sum_{i=0}^{n} \int_0^{t_f} \|\dot{q}_i\|^2 \, dt. \tag{1}$$

Remark 1: A justification for choosing the energy expenditure J as the cost function for the problem can be found in [11]. Intuitively, J is the sum of $n+1$ terms; and minimizing each term $\int_0^{t_f} \|\dot{q}_i\|^2 \, dt$ will tend to make the motion $q_i(t)$ for agent i follow a straighter path with less speed variations. In comparison, if term $\int_0^{t_f} \|\dot{q}_i\| \, dt$ is used in the definition of J instead of $\int_0^{t_f} \|\dot{q}_i\|^2 \, dt$, then, as $\int_0^{t_f} \|\dot{q}_i\| \, dt$ is the length of the curve $q_i(t)$, $0 \le t \le t_f$, motions that follows the same path will have the same cost, even though some may be smoother than others. Obviously, in practice, smoother motions should be favored.

For brevity, in the rest of the paper, the above problem will be called the OFC problem. Thus the OFC problem tries to find the coordinated motions of the $n+1$ agents that can move them from $\langle a_i \rangle_{i=0}^n$ at time 0 to $\langle b_i \rangle_{i=0}^n$ at time t_f with the minimal energy expenditure, while at the same time maintaining the $\mathcal{G}$-formation constraint, namely, the distance between agents i and j is kept at the constant 1 at all times for $(i,j) \in \mathcal{E}$.

The starting position $\langle a_i \rangle_{i=0}^n$ and the destination position $\langle b_i \rangle_{i=0}^n$ are called *aligned* if they have the same centroid: $\frac{1}{n+1} \sum_{i=0}^n a_i = \frac{1}{n+1} \sum_{i=0}^n b_i = c$. In [11], it is proved that the general OFC problems can be reduced to the OFC problem where the initial and the destination positions are aligned. Hence without loss of generality we assume in the rest of this paper that $\langle a_i \rangle_{i=0}^n$ and $\langle b_i \rangle_{i=0}^n$ have a common centroid c, say, at the origin.

Assumption 1: Assume that the initial and the destination positions are aligned at the same centroid $c = 0$:

$$\frac{1}{n+1} \sum_{i=0}^n a_i = \frac{1}{n+1} \sum_{i=0}^n b_i = 0.$$

Under this assumption, the following result can be used to reduce the complexity of solving the OFC problem.

Lemma 1 ([11]): Suppose that in the OFC problem the starting position $\langle a_i \rangle_{i=0}^n$ and the destination position $\langle b_i \rangle_{i=0}^n$ are aligned at the common centroid $c = 0$. Then the optimal solutions $\langle q_i \rangle_{i=0}^n$ to the OFC problem under any formation constraint $\mathcal{G}$ satisfy

$$\frac{1}{n+1} \sum_{i=0}^n q_i(t) = 0, \quad \forall t \in [0, t_f].$$

In other words, the positions of the $n+1$ agents during the optimal coordinated motions are also centered at the origin under arbitrary formation constraints. This in effect reduces the dimension of the problem by two: the optimal solution $(q_0, \ldots, q_n)$ as a curve in $\mathbb{R}^{2(n+1)}$ lies in a subspace of codimension two.

III. OFC Problem under Tree Formation Constraint

In this paper, we focus on the OFC problem with a particular formation constraint structure, namely, when $\mathcal{G}$ is a tree. A connected undirected graph is called a *tree* if it has no loop. For a tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, typically a node, say, node 0, is identified as the root, and a layered structure can be established for the rest of the nodes according to their distances to the root. More precisely, a partial order $\preceq$ can be defined on $\mathcal{V}$ so that two nodes i and k satisfy $i \preceq k$ if and only if node i is a *predecessor* of node k, or equivalently, if and only if node i is on the shortest path between node k and the root. Otherwise, we write $i \not\preceq k$. Note that $i \preceq i$, and that it is possible that both $i \not\preceq k$ and $k \not\preceq i$ are true for certain nodes i and k. As an example, in the tree shown in Fig. 1, we have $1 \preceq 6$, while $3 \not\preceq 6$ and $6 \not\preceq 3$.

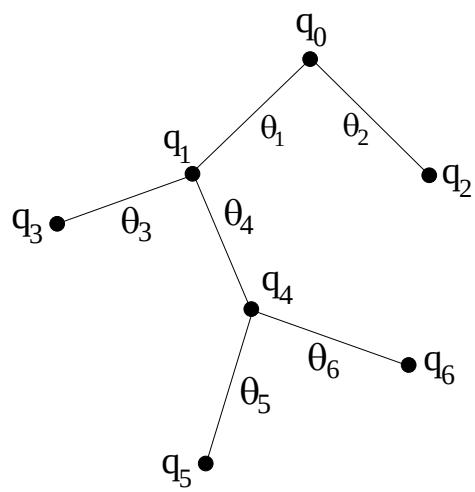


Fig. 1. An example of an $(n+1)$-tuple satisfying a tree formation constraint ($n = 6$).

A. A New Coordinate System

It turns out that in studying the OFC problem with a tree formation constraint $\mathcal{G}$, it is more convenient to work in a different coordinate system than the canonical $(q_0, \ldots, q_n)$ for representing the agents' positions, as the formation constraint is not intrinsically encoded in the latter system. To see this, let $\langle q_i \rangle_{i=0}^n$ be an $(n+1)$-tuple satisfying the $\mathcal{G}$-formation constraint. For each edge $(i,k) \in \mathcal{E}$ where node i is an immediate predecessor of node k, we can associate an angle θ_k defined as the phase angle of the vector $q_k - q_i$. See Fig. 1 for an example when $n = 6$. Note that since $(i,k) \in \mathcal{E}$ and $\langle q_i \rangle_{i=0}^n$ satisfies the $\mathcal{G}$-formation constraint, we must have $\|q_i - q_k\| = 1$. Thus if we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, then from the above definition, we have $q_k - q_i = e^{j\theta_k}$ where $j = \sqrt{-1}$[1]. As a result, the position q_i of an arbitrary node i can be expressed in terms of the position q_0 of the root and the angles associated with all the edges on the shortest path from the root to node i as:

$$q_i = q_0 + \sum_{k \preceq i,\, k \neq 0} e^{j\theta_k}, \quad i = 0, \ldots, n. \tag{2}$$

Note that the summation in (2) is over all the predecessors of node i, including node i itself, except the root.

By Assumption 1 and Lemma 1, the optimal solutions $\langle q_i \rangle_{i=0}^n$ must satisfy $\sum_{i=0}^n q_i = 0$ at all times. Substituting in (2), we have

$$(n+1)q_0 + \sum_{i=1}^n \sum_{k \preceq i,\, k \neq 0} e^{j\theta_k} = 0,$$

or equivalently,

$$q_0 = -\frac{1}{n+1} \sum_{i=1}^n \sum_{k \preceq i,\, k \neq 0} e^{j\theta_k} = -\sum_{k=1}^n \frac{\chi_k}{n+1} e^{j\theta_k}. \tag{3}$$

[1]In this paper, j can either denote $\sqrt{-1}$ or be an integer index. There should be no ambiguity in its meaning as we have ensured that j with the two different interpretations will not appear in the same expression in the rest of the paper.

In the above equation, for an arbitrary node i other than the root, χ_i is an integer associated with node i defined as the number of successors of node i, including node i itself. More precisely,

$$\chi_i = \#\{k \neq 0 : \, i \preceq k\}. \tag{4}$$

As an example, in Fig. 1, we have $\chi_1 = 5$, $\chi_2 = \chi_3 = \chi_5 = \chi_6 = 1$, and $\chi_4 = 3$. Using (3), we can rewrite (2) as

$$q_i = \sum_{k=1,\, k \preceq i}^{n} \frac{n+1-\chi_k}{n+1} e^{j\theta_k} - \sum_{k=1,\, k \not\preceq i}^{n} \frac{\chi_k}{n+1} e^{j\theta_k}, \quad i = 0, \ldots, n. \tag{5}$$

Equation (5) defines a coordinate transformation between the canonical $(q_0, \ldots, q_n)$ and the new coordinates $(\theta_1, \ldots, \theta_n)$. Note that in the new coordinate system, the formation constraint, namely, $\|q_i - q_k\| = 1$ whenever $(i,k) \in \mathcal{E}$, is implicitly encoded.

We now derive the expression of the energy J defined in (1) in the new coordinate system. Differentiating (5) and taking the norm square, we have

$$\begin{aligned}
\|\dot{q}_i\|^2 &= \left\| \sum_{k \preceq i} \frac{n+1-\chi_k}{n+1} j e^{j\theta_k} \dot{\theta}_k - \sum_{k \not\preceq i} \frac{\chi_k}{n+1} j e^{j\theta_k} \dot{\theta}_k \right\|^2 \\
&= \left(\sum_{k \preceq i} \frac{n+1-\chi_k}{n+1} j e^{j\theta_k} \dot{\theta}_k - \sum_{k \not\preceq i} \frac{\chi_k}{n+1} j e^{j\theta_k} \dot{\theta}_k \right) \cdot \left(- \sum_{k \preceq i} \frac{n+1-\chi_k}{n+1} j e^{-j\theta_k} \dot{\theta}_k + \sum_{k \not\preceq i} \frac{\chi_k}{n+1} j e^{-j\theta_k} \dot{\theta}_k \right) \\
&= \sum_{k \preceq i} \frac{(n+1-\chi_k)^2}{(n+1)^2} \dot{\theta}_k^2 + \sum_{k_1 \preceq i,\, k_2 \preceq i,\, k_1 \neq k_2} \frac{(n+1-\chi_{k_1})(n+1-\chi_{k_2})}{(n+1)^2} \cos(\theta_{k_1} - \theta_{k_2}) \dot{\theta}_{k_1} \dot{\theta}_{k_2} \\
&\quad + \sum_{k \not\preceq i} \frac{\chi_k^2}{(n+1)^2} \dot{\theta}_k^2 + \sum_{k_1 \not\preceq i,\, k_2 \not\preceq i,\, k_1 \neq k_2} \frac{\chi_{k_1} \chi_{k_2}}{(n+1)^2} \cos(\theta_{k_1} - \theta_{k_2}) \dot{\theta}_{k_1} \dot{\theta}_{k_2} \\
&\quad - \sum_{k_1 \preceq i,\, k_2 \not\preceq i} \frac{(n+1-\chi_{k_1})\chi_{k_2}}{(n+1)^2} \cos(\theta_{k_1} - \theta_{k_2}) \dot{\theta}_{k_1} \dot{\theta}_{k_2},
\end{aligned}$$

for $i = 0, \ldots, n$. Note that the running indices k, k_1, k_2 in the above equation are all assumed implicitly to take values in the range $1, \ldots, n$. Moreover, the summations involving k_1 and k_2 are over ordered pairs: (k_1, k_2) and

(k_2, k_1) are counted separately. As a result, the energy J can be written as

$$\begin{aligned}
J &= \int_0^{t_f} \left(\sum_{i=0}^{n} \|\dot{q}_i\|^2 \right) dt \\
&= \int_0^{t_f} \Bigg\{ \sum_k \left[\frac{(n+1-\chi_k)^2}{(n+1)^2} \cdot \chi_k + \frac{\chi_k^2}{(n+1)^2} \cdot (n+1-\chi_k) \right] \dot{\theta}_k^2 \\
&\quad + \sum_{k_1 \neq k_2} \Bigg[\frac{(n+1-\chi_{k_1})(n+1-\chi_{k_2})}{(n+1)^2} \cdot \#\{i : k_1 \preceq i, k_2 \preceq i\} \\
&\qquad + \frac{\chi_{k_1}\chi_{k_2}}{(n+1)^2} \cdot \#\{i : k_1 \not\preceq i, k_2 \not\preceq i\} \\
&\qquad - \frac{(n+1-\chi_{k_1})\chi_{k_2}}{(n+1)^2} \cdot \#\{i : k_1 \preceq i, k_2 \not\preceq i\} \\
&\qquad - \frac{\chi_{k_1}(n+1-\chi_{k_2})}{(n+1)^2} \cdot \#\{i : k_1 \not\preceq i, k_2 \preceq i\} \Bigg] \cdot \cos(\theta_{k_1} - \theta_{k_2}) \dot{\theta}_{k_1} \dot{\theta}_{k_2} \Bigg\} dt \\
&= \int_0^{t_f} \left(\sum_{k_1, k_2 = 1}^{n} \Delta_{k_1 k_2} \cos(\theta_{k_1} - \theta_{k_2}) \dot{\theta}_{k_1} \dot{\theta}_{k_2} \right) dt. \qquad (6)
\end{aligned}$$

The constants $\Delta_{k_1 k_2}$ in equation (6) can be determined by comparing the coefficients with the previous equation. Specifically, for each $k = 1, \ldots, n$,

$$\Delta_{kk} = \frac{(n+1-\chi_k)^2}{(n+1)^2} \cdot \chi_k + \frac{\chi_k^2}{(n+1)^2} \cdot (n+1-\chi_k) = \frac{\chi_k(n+1-\chi_k)}{n+1}. \qquad (7)$$

For $k_1, k_2 = 1, \ldots, n$ with $k_1 \preceq k_2$,

$$\begin{aligned}
\Delta_{k_1 k_2} &= \frac{(n+1-\chi_{k_1})(n+1-\chi_{k_2})}{(n+1)^2} \cdot \chi_{k_2} + \frac{\chi_{k_1}\chi_{k_2}}{(n+1)^2} \cdot (n+1-\chi_{k_1}) - \frac{(n+1-\chi_{k_1})\chi_{k_2}}{(n+1)^2} \cdot (\chi_{k_1} - \chi_{k_2}) \\
&= \frac{(n+1-\chi_{k_1})\chi_{k_2}}{n+1}.
\end{aligned}$$

Similarly, for $k_1, k_2 = 1, \ldots, n$ with $k_2 \preceq k_1$,

$$\Delta_{k_1 k_2} = \frac{\chi_{k_1}(n+1-\chi_{k_2})}{n+1}.$$

On the other hand, if $k_1 \not\preceq k_2$ and $k_2 \not\preceq k_1$, then

$$\begin{aligned}
\Delta_{k_1 k_2} &= \frac{\chi_{k_1}\chi_{k_2}}{(n+1)^2} \cdot (n+1-\chi_{k_1}-\chi_{k_2}) - \frac{(n+1-\chi_{k_1})\chi_{k_2}}{(n+1)^2} \cdot \chi_{k_1} - \frac{\chi_{k_1}(n+1-\chi_{k_2})}{(n+1)^2} \cdot \chi_{k_2} \\
&= -\frac{\chi_{k_1}\chi_{k_2}}{n+1}.
\end{aligned}$$

To sum up, the energy expenditure J in the new coordinate system becomes

$$J = \int_0^{t_f} \left(\sum_{i,j=1}^{n} \Delta_{ij} \cos(\theta_i - \theta_j) \dot{\theta}_i \dot{\theta}_j \right) dt, \qquad (8)$$

where the constants Δ_{ij}, $1 \leq i, j \leq n$, are defined as

$$\Delta_{ij} = \begin{cases} \frac{(n+1-\chi_i)\chi_j}{n+1}, & \text{if } i \preceq j, \\ \frac{\chi_i(n+1-\chi_j)}{n+1}, & \text{if } j \preceq i, \\ -\frac{\chi_i\chi_j}{n+1}, & \text{otherwise.} \end{cases} \qquad (9)$$

Define the (column) vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and the matrix

$$G(\theta) = [g_{ij}(\theta)]_{1\le i,j\le n} \triangleq [\Delta_{ij}\cos(\theta_i - \theta_j)]_{1\le i,j\le n} \in \mathbb{R}^{n\times n}. \tag{10}$$

Then (8) can be simplified as

$$J = \int_0^{t_f} \dot{\theta}^T G(\theta)\dot{\theta}\, dt. \tag{11}$$

Thus the OFC problem is reduced to the following optimal control problem:

$$\text{Minimize } \int_0^{t_f} \frac{1}{2}u^T(t)G(\theta)u(t)\, dt \;\text{ subject to }\; u(t) = \dot{\theta}(t),\, t\in[0,t_f],\; \theta(0) = \theta_0,\, \theta(t_f) = \theta_f. \tag{12}$$

Here $\theta_0, \theta_f \in \mathbb{R}^n$ are chosen to match the initial position $\langle a_i\rangle_{i=0}^n$ and the final position $\langle b_i\rangle_{i=0}^n$, respectively.

Remark 2: The matrix $G(\theta)$ is indeed the Riemannian metric in the coordinate system $(\theta_1, \ldots, \theta_n)$ transformed from the canonical Euclidean metric on $\mathbb{R}^{2(n+1)}$ via the coordinate transformation (5), and J in (11) is the energy of the curve $\theta(t)$, $0 \le t \le t_f$, as measured by this metric. Thus the optimal control problem (12) is equivalent to finding the shortest distance curves from θ_0 to θ_f parameterized with constant speed [5].

B. Optimality Condition Obtained by First Variation

We now solve the optimal control problem (12). Define the Hamiltonian

$$H = \frac{1}{2}u^T G(\theta)u + \lambda^t u,$$

where $\lambda \in \mathbb{R}^n$ is the co-state. Then by the Maximum Principle [2], the optimal u is determined by

$$u = \text{argmin}_u H = \text{argmin}_u \left[\frac{1}{2}u^T G(\theta)u + \lambda^t u\right] \quad\Rightarrow\quad G(\theta)u + \lambda = 0,$$

while the dynamics of λ is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial\theta} = -\frac{1}{2}\frac{\partial}{\partial\theta}\left[u^T G(\theta)u\right].$$

Combining the above two equations, we obtain

$$-\frac{1}{2}\frac{\partial}{\partial\theta}\left[u^T G(\theta)u\right] = \dot{\lambda} = -G(\theta)\dot{u} - \frac{d}{dt}G(\theta)\cdot u.$$

Since $u = \dot{\theta}$, the above equation can be rewritten as

$$G(\theta)\ddot{\theta} = \frac{1}{2}\frac{\partial}{\partial\theta}\left[\dot{\theta}^T G(\theta)\dot{\theta}\right] - \frac{d}{dt}G(\theta)\cdot\dot{\theta}. \tag{13}$$

Equation (13) is called the *geodesic equation* and its solutions are called geodesics, as the optimal control problem (12) under study is an instance of the shortest distance problem under a suitable Riemannian metric [5] (see Remark 2).

Evaluating the k-th component of the above vector equation (13), we obtain

$$\begin{aligned}
\left[G(\theta)\ddot{\theta}\right]_k &= \frac{1}{2}\frac{\partial}{\partial\theta_k}\left[\dot{\theta}^T G(\theta)\dot{\theta}\right] - \left[\frac{d}{dt}G(\theta)\cdot\dot{\theta}\right]_k \\
&= \frac{1}{2}\frac{\partial}{\partial\theta_k}\left[\sum_{i,j=1}^{n}\Delta_{ij}\cos(\theta_i-\theta_j)\dot{\theta}_i\dot{\theta}_j\right] + \sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)(\dot{\theta}_k-\dot{\theta}_j)\dot{\theta}_j \\
&= \frac{1}{2}\left[-\sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)\dot{\theta}_k\dot{\theta}_j + \sum_{1\le j\le n}\Delta_{jk}\sin(\theta_j-\theta_k)\dot{\theta}_j\dot{\theta}_k\right] \\
&\qquad + \sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)(\dot{\theta}_k-\dot{\theta}_j)\dot{\theta}_j \\
&= -\sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)\dot{\theta}_k\dot{\theta}_j + \sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)(\dot{\theta}_k-\dot{\theta}_j)\dot{\theta}_j \\
&= -\sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)\dot{\theta}_j^2,
\end{aligned}$$

where $[\cdot]_k$ denotes the k-th component of a vector. Therefore, the geodesic equation (13) is equivalent to

$$\left[G(\theta)\ddot{\theta}\right]_k = -\sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)\dot{\theta}_j^2, \quad k=1,\ldots,n. \tag{14}$$

From the standard result of optimal control theory, equation (14) gives a necessary condition for a curve $\theta(t)$ to be a solution to the OFC problem. Conversely, a curve $\theta(t)$, $t\in[0,t_f]$, satisfying the geodesic equation (14) is a solution to the OFC problem if t_f is sufficiently small [2], [5]. However, for large t_f, the optimality of $\theta(t)$ may be lost due to various reasons. Later in this section, we will study one of them, namely, the occurrence of conjugate points.

C. A Special Instance of the OFC Problem and Its Candidate Solutions

Instead of studying the general solutions to the geodesic equation (14), we consider a special case. Suppose that the initial position $\langle a_i\rangle_{i=0}^n$ and the destination position $\langle b_i\rangle_{i=0}^n$ of the $n+1$ agents are not only aligned at the common centroid 0, but they can be obtained from each other by a rotation around the origin. More precisely, we assume that $b_i = R_{t_f}(a_i)$, $i=0,\ldots,n$, where for each $\alpha\in\mathbb{R}$, R_α denotes the rotation operation around the origin by the angle α counterclockwise.

Under this assumption, a natural candidate solution $q^* = \langle q_i^*\rangle_{i=0}^n$ to the OFC problem under any formation constraint $\mathcal{G}$ can be described as follows: $q_i^*(t) = R_t(a_i)$, $t\in[0,t_f]$, $i=0,\ldots,n$. In other words, all the $n+1$ agents rotate at unit angular velocity counterclockwise around the origin from $\langle a_i\rangle_{i=0}^n$ at time 0 to $\langle b_i\rangle_{i=0}^n$ at time t_f. In the coordinate system $\theta=(\theta_1,\ldots,\theta_n)$ constructed in Section III-A, this candidate solution q^* corresponds to

$$\theta^* = \theta_0 + t\cdot\mathbf{1}, \tag{15}$$

where $\theta_0 = (\theta_1^0,\ldots,\theta_n^0)$ is the new coordinate corresponding to the initial position $\langle a_i\rangle_{i=0}^n$, and $\mathbf{1}\in\mathbb{R}^n$ is the vector whose components are all 1's.

In order for q^* to be a solution to the OFC problem, θ^* must satisfy the geodesic equation (14). Since $\ddot{\theta}^* \equiv 0$, $\dot{\theta}^*_j \equiv 1$, and $\theta^*_k - \theta^*_j \equiv \theta^0_k - \theta^0_j$ are constant for $1 \le j, k \le n$, we must have

$$\sum_{1 \le j \le n} \Delta_{kj} \sin(\theta^0_k - \theta^0_j) = 0, \quad k = 1, \ldots, n. \tag{16}$$

Equation (16) is the condition on θ_0 so that starting from the corresponding initial position $\langle a_i \rangle_{i=0}^n$, the coordinated motion q^* obtained by rotating around the origin satisfies the first order optimality condition (13).

1) A Mechanical Interpretation of Condition (16): To better understand the implication of condition (16), we next develop a mechanical analogy. To this purpose, we first prove a useful identity. Suppose that $(\theta_1, \ldots, \theta_n)$ is the new coordinate for an $(n+1)$-tuple $\langle q_i \rangle_{i=0}^n$ satisfying the $\mathcal{G}$ formation constraint under the coordinate transformation (5) described in Section III-A, and let Δ_{ij} be defined as in (9).

Lemma 2: For each $l = 1, \ldots, n$, we have

$$\sum_{\{i:\, l \preceq i\}} q_i = \sum_{1 \le k \le n} \Delta_{lk}\, e^{j\theta_k}. \tag{17}$$

Proof: Substituting in (5), we have

$$\begin{aligned}
\sum_{\{i:\, l \preceq i\}} q_i &= \sum_{\{i:\, l \preceq i\}} \left(\sum_{k \preceq i} \frac{n+1-\chi_k}{n+1} e^{j\theta_k} - \sum_{k \not\preceq i} \frac{\chi_k}{n+1} e^{j\theta_k} \right) \\
&= \sum_{\{k:\, l \preceq k\}} \left[\frac{n+1-\chi_k}{n+1} \cdot \chi_k - \frac{\chi_k}{n+1} \cdot (\chi_l - \chi_k) \right] e^{j\theta_k} \\
&\quad + \sum_{\{k:\, k \preceq l\}} \frac{n+1-\chi_k}{n+1} \cdot \chi_l\, e^{j\theta_k} - \sum_{\{k:\, k \not\preceq l,\, l \not\preceq k\}} \frac{\chi_k}{n+1} \cdot \chi_l\, e^{j\theta_k} \\
&= \sum_{\{k:\, l \preceq k\}} \frac{(n+1-\chi_l)\chi_k}{n+1} e^{j\theta_k} + \sum_{\{k:\, k \preceq l\}} \frac{\chi_l(n+1-\chi_k)}{n+1} e^{j\theta_k} + \sum_{\{k:\, k \not\preceq l,\, l \not\preceq k\}} \frac{-\chi_l \chi_k}{n+1} e^{j\theta_k}.
\end{aligned}$$

The desired conclusion can be obtained by comparing the coefficients in the last equation with (9). ■

Now suppose $\theta_0 = (\theta^0_1, \ldots, \theta^0_n)$ is the new coordinate corresponding to the initial position $\langle a_i \rangle_{i=0}^n$, and that it satisfies condition (16). By Lemma 2, for each $l = 1, \ldots, n$,

$$\sum_{\{i:\, l \preceq i\}} a_i = \sum_{1 \le k \le n} \Delta_{lk}\, e^{j\theta^0_k}.$$

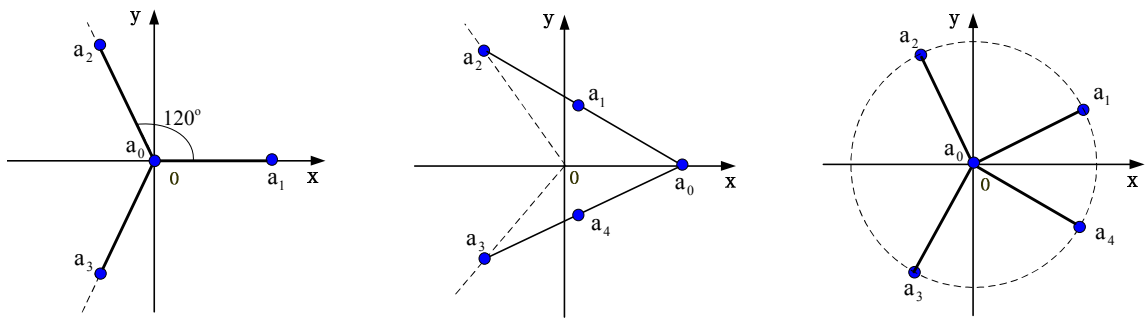
Thus, if $\mathrm{Im}[\cdot]$ denotes the imaginary part of complex numbers, then

$$\mathrm{Im}\left[\left(\sum_{\{i:\, l \preceq i\}} a_i \right) \cdot e^{-j\theta^0_l} \right] = \mathrm{Im}\left[\sum_{1 \le k \le n} \Delta_{lk}\, e^{j(\theta^0_k - \theta^0_l)} \right] = \sum_{1 \le k \le n} \Delta_{lk} \sin(\theta^0_k - \theta^0_l) = 0, \tag{18}$$

where the last equality follows from condition (16). Since every step of the above derivation is reversible, we conclude that condition (16) is equivalent to the following:

$$\text{the vector } \sum_{\{i:\, l \preceq i\}} a_i \text{ is of the same direction as } e^{j\theta^0_l}, \text{ for } l = 1, \ldots, n. \tag{19}$$

Note that for each $l = 1, \ldots, n$, since node l is not the root, it has an immediate predecessor, say, node i. By the definition of θ^0_l, $e^{j\theta^0_l}$ is the direction of the vector $a_l - a_i$. Hence condition (19) says that the sum of the positions



(a) (b) (c)

Fig. 2. Examples of different initial positions: (a) and (c) satisfy condition (19) while (b) does not. However, (c) does not satisfy Assumption 1.

of all successors of node l, including node l itself, is of the same direction as the vector pointing from the position of the immediate predecessor of node l to the position of node l. In particular, if node l is a *leaf*, namely, a node with no other successors other than itself, then condition (19) says that its immediate predecessor must be located on the line connecting a_l and the origin. For example, the agent a_2 in Fig. 2-(b) is a leaf, but its predecessor a_1 is not on the line connecting a_2 and the origin; thus the initial position in Fig. 2-(b) does not satisfy condition (19).

A mechanical interpretation of condition (19) can be given as follows. Corresponding to the $n+1$ agents, there are $n+1$ unit point masses located at $a_0, \ldots, a_n$. For each edge $(i,j) \in \mathcal{E}$, there is a rigid rod of zero mass and unit length connecting the i-th and the j-th agents. Moreover, for each agent i, there is a centrifugal force pointing from the origin to a_i with the strength $\|a_i\|$. Then it can be easily verified that the initial condition $\langle a_i \rangle_{i=0}^n$ satisfies condition (19) if and only if the mechanical system described above is in equilibrium, namely, all the forces acting on each agent, including the centrifugal force and the forces by the rods connecting to it, add up to zero.

By noting the constraint that $\sum_{i=0}^n a_i = 0$, an alternative mechanical interpretation can be given as follows. Instead of the centrifugal forces, assume that in the $(n+1)$-point mass system connected by rods described above, for each pair $0 \leq i, j \leq n$, the point mass located at a_i has a repulsive force of $a_j - a_i$ acting on the point mass located at a_j. Then it can also be verified that condition (19) is equivalent to that this mechanical system is in a balance of forces.

If the initial position satisfies condition (19), or equivalently, condition (16), then θ^* defined in (15) is a solution to equation (13), hence is a candidate solution to the corresponding OFC problem. The above mechanical analogies provide much easier-to-use criteria for checking whether θ^* with a given initial position is a candidate solution. As examples, one can immediately see that θ^* is a candidate solution for the initial position shown in Fig. 2-(a), but not for (b) and (c). The reason is that the point-mass system corresponding to (b) is not in a balance of forces, while although (c) satisfies condition (19), it violates Assumption 1.

D. Optimality Condition Obtained by Second Variation

Consider the special instance of the OFC problem described in Section III-C. Suppose that $\langle a_i \rangle_{i=0}^n$ satisfies condition (16). Then the candidate solution θ^* defined in (15) satisfies the first order optimality condition (14), and,

as a result, is an optimal solution to the OFC problem if t_f is sufficiently small.

However, as the time horizon t_f increases, θ^* may fail to be an optimal solution to the OFC problem. One possible reason is the occurrence of conjugate points [15], namely, the failure of θ^* to meet the second order optimality condition. Intuitively, a conjugate point is encountered along a geodesic when infinitesimally there are more than one geodesic connecting its two end points. For the simplest example of conjugate points, consider a sphere. Geodesics on the sphere are great circles. A great circle emitting from the south pole will no longer be distance-minimizing between its two end points after passing its first conjugate point, namely, the north pole. The reason that the north pole is a conjugate point is because there is more than one great circle connecting it to the south pole.

In order to characterize the conjugate points along θ^*, we need to find the variation of θ in equation (14) around the nominal solution θ^*. To this purpose, let $\delta\theta(t) \in \mathbb{R}^n$ be a variation of $\theta^*(t)$ for $t \in [0, t_f]$. Since the two end points of θ^* are fixed, the variation must be proper, i.e., $\delta\theta(0) = \delta\theta(t_f) = 0$. Taking the variation $\delta\theta$ in (14) along the solution θ^*, we obtain

$$\left[G(\theta^*)\cdot\ddot{\delta\theta} + \delta G(\theta^*)\cdot\ddot{\theta}^*\right]_k = -\sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^*-\theta_j^*)(\delta\theta_k-\delta\theta_j)(\dot{\theta}_j^*)^2 - \sum_{1\le j\le n}2\Delta_{kj}\sin(\theta_k^*-\theta_j^*)\,\dot{\theta}_j^*\,\dot{\delta\theta}_j,$$

for $k = 1, \ldots, n$. Note that $\ddot{\theta}^* \equiv 0$, $\dot{\theta}_j^* \equiv 1$, and $\theta_k^* - \theta_j^* \equiv \theta_k^0 - \theta_j^0$. Thus, the above equation is reduced to

$$\begin{aligned}\left[G(\theta^*)\cdot\ddot{\delta\theta}\right]_k &= -\sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^0-\theta_j^0)(\delta\theta_k-\delta\theta_j) - \sum_{1\le j\le n}2\Delta_{kj}\sin(\theta_k^0-\theta_j^0)\dot{\delta\theta}_j\\ &= -\left[\sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^0-\theta_j^0)\right]\delta\theta_k + \sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^0-\theta_j^0)\delta\theta_j\\ &\quad - \sum_{1\le j\le n}2\Delta_{kj}\sin(\theta_k^0-\theta_j^0)\dot{\delta\theta}_j\,. \end{aligned} \tag{20}$$

To simplify the above equation, define the constants

$$\mu_k = \sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^0-\theta_j^0), \quad k = 1, \ldots, n. \tag{21}$$

Define the matrices Λ and $G_s(\theta^*) \in \mathbb{R}^{n\times n}$ as

$$\Lambda = \text{diag}\,(\mu_1, \ldots, \mu_n), \quad G_s(\theta^*) = \left[\Delta_{ij}\sin(\theta_i^0-\theta_j^0)\right]_{1\le i,j\le n}. \tag{22}$$

Then equation (20) can be written in matrix form as

$$G(\theta^*)\,\ddot{\delta\theta} = -\Lambda\,\delta\theta + G(\theta^*)\,\delta\theta - 2G_s(\theta^*)\,\dot{\delta\theta},$$

or equivalently,

$$\ddot{\delta\theta} = -2G(\theta^*)^{-1}G_s(\theta^*)\,\dot{\delta\theta} - \left[G(\theta^*)^{-1}\Lambda - I\right]\delta\theta. \tag{23}$$

Equation (23) is called the *Jacobi equation*. A conjugate point along θ^* (or along q^* in the canonical coordinates) occurs at time τ if there is a nontrivial solution to the Jacobi equation that vanishes at both time 0 and time τ, i.e., if there is a solution $\delta\theta(t)$ not identically zero for $t \in [0, \tau]$ with $\delta\theta(0) = \delta\theta(\tau) = 0$. By the standard result of

optimal control theory [2], [15], once $t_f > \tau$, θ^* will no longer be optimal, or equivalently, the corresponding q^* will no longer be the optimal coordinated motion of the $n+1$ agents from $\langle a_i \rangle_{i=0}^n$ at time 0 to $\langle b_i \rangle_{i=0}^n$ at time t_f.

In the next section, we will find the conjugate points along θ^* by solving the Jacobi equation. From the above discussions, this will give us upper bounds on t_f for the optimality of θ^*.

Remark 3: In theory, the Jacobi equation (23) can also be derived through a purely differential geometric approach by computing the curvature tensors of the Riemannian metric $G(\theta)$, similar to the one adopted in [12] in studying the special snake formation case. However, such an approach is almost infeasible here due to the increased problem complexity caused by the general tree formation structure. Hence the direct variational approach is adopted here.

Remark 4: As another remark, although in this section we only carry out the second variation for the special instance of the OFC problem described in Section III-C, Jacobi equation similar to equation (23) can be derived for an arbitrary instance of the OFC problem around any candidate solution satisfying equation (14) by using a similar approach. In these general cases, however, the coefficient matrices in the Jacobi equation are no longer constant, but time-varying instead. Thus the solution of the Jacobi equation becomes a much more challenging task.

IV. Solution of the Jacobi Equation

To solve the Jacobi equation (23), we first write it in the generic form

$$L_2 \ddot{x} + L_1 \dot{x} + L_0 x = 0, \tag{24}$$

where $x = \delta\theta$, and L_0, L_1, and L_2 are constant matrices defined by

$$L_2 = I, \quad L_1 = 2G(\theta^*)^{-1} G_s(\theta^*), \quad L_0 = G(\theta^*)^{-1}\Lambda - I.$$

From the discussions at the end of Section III-D, in order to find the conjugate points along θ^*, we need to solve equation (24). In particular, we need to find those nontrivial solutions $x(t)$ that vanish at both time 0 and a positive time τ. For each such solution, a corresponding conjugate point is located at $\theta^*(\tau)$. In this section, two methods are proposed to solve equation (24): a direct method suitable for analytical calculation and an indirect method suitable for numerical verification.

A. Direct Method

The equation (24) is a second order homogeneous matrix differential equation (MDE) with constant coefficient matrices $L_0, L_1, L_2 \in \mathbb{R}^{n\times n}$. A standard way of solving a general MDE is described in [19] and can be adopted to solve our problem. In the following, we review some of the important results. The interested reader can refer to [19] for proofs and other details.

Definition 1: Consider the homogeneous constant-coefficient MDE of order l given by

$$L_l x^{(l)}(t) + L_{l-1} x^{(l-1)}(t) + \cdots + L_1 x^{(1)}(t) + L_0 x(t) = 0, \tag{25}$$

where $L_0, L_1, \ldots, L_l \in \mathbb{C}^{n\times n}$, $\det(L_l) \neq 0$, and $x : \mathbb{R}_+ \to \mathbb{C}^n$ is l-time differentiable. Define the *matrix polynomial* as: $L(\lambda) = \sum_{j=0}^{l} \lambda^j L_j$ for $\lambda \in \mathbb{C}$.

1) λ_0 is a *latent root* of $L(\lambda)$ if $\det(L(\lambda_0)) = 0$;
2) A nonzero vector $v_0 \in \mathbb{C}^n$ is a *latent vector* of $L(\lambda)$ associated with a latent root λ_0 if $L(\lambda_0)v_0 = 0$;
3) A sequence of vectors, $v_0, v_1, \ldots, v_{k-1} \in \mathbb{C}^n$ with $v_0 \neq 0$, is a *Jordan chain* of length k for $L(\lambda)$ corresponding to the latent root λ_0 if the following k relations hold:

$$\begin{aligned} &L(\lambda_0)v_0 = 0, \\ &L(\lambda_0)v_1 + \frac{1}{1!}L^{(1)}(\lambda_0)v_0 = 0, \\ &\quad\vdots \\ &L(\lambda_0)v_{k-1} + \frac{1}{1!}L^{(1)}(\lambda_0)v_{k-2} + \cdots + \frac{1}{(k-1)!}L^{(k-1)}(\lambda_0)v_0 = 0. \end{aligned} \tag{26}$$

Here $L^{(j)}(\lambda)$ denotes the j-th order derivative of $L(\lambda)$ with respect to λ.

The above definitions for a matrix polynomial is conceptually similar to the eigenpairs and the Jordan chains of a numerical matrix except that the number of the latent roots of $L(\lambda)$ (counting multiplicity) is $l \cdot n$, and that the latent vectors associated with different latent roots are not necessarily linearly independent. Vectors in the same Jordan chain of $L(\lambda)$ could also be linearly dependent. On the other hand, most other facts about the Jordan chains of numerical matrices still apply here. For example, if λ_i is a latent root of $L(\lambda)$ with algebraic multiplicity n_i and geometric multiplicity m_i, then there are m_i sets of Jordan chains associated with λ_i and the numbers of vectors in these Jordan chains sum up to n_i. See [19] for more details.

With the above definitions, the following lemma describes the relationship between the Jordan chains of $L(\lambda)$ and the primitive solutions of the MDE (25).

Lemma 3: Let $L(\lambda)$ be the matrix polynomial associated with MDE (25).

1) If v_0 and v_1 are two latent vectors of $L(\lambda)$ associated with latent roots λ_0 and λ_1 ($\lambda_0 \neq \lambda_1$), respectively, then $x_0(t) = v_0\, e^{\lambda_0 t}$ and $x_1(t) = v_1\, e^{\lambda_1 t}$ are two linearly independent solutions of the MDE (25).
2) If $v_0, v_1, \ldots, v_{k-1}$ is a Jordan chain of length k for $L(\lambda)$ corresponding to the latent root λ_0, then

$$\begin{aligned} x_0(t) =& v_0\, e^{\lambda_0 t}, \\ x_1(t) =& (tv_0 + v_1)\, e^{\lambda_0 t}, \\ &\vdots \\ x_{k-1}(t) =& \left(\sum_{j=0}^{k-1} \frac{t^j}{j!} v_{k-1-j} \right) e^{\lambda_0 t} \end{aligned} \tag{27}$$

are k linearly independent solutions of the MDE (25).
3) Solutions of the form (27) but belonging to different Jordan chains of $L(\lambda)$ are linearly independent.

MDE (25) has an $l \cdot n$-dimensional solution space. By Lemma 3, a Jordan chain of length k for $L(\lambda)$ can provide exactly k independent solutions. Thus the whole solution space of the MDE (25) is fully characterized by the Jordan chains of the corresponding matrix polynomial $L(\lambda)$, as is summarized in the following theorem.

Theorem 1: Let $L(\lambda)$ be the matrix polynomial associated with the MDE (25), which has a total of s different sets of Jordan chains of sizes $n_1, \ldots, n_s$. Let X be an $n \times ln$ matrix whose columns consist of all the vectors in these Jordan chains. Corresponding to these Jordan chains, we define a block diagonal matrix $J_b = diag\{J_1, \ldots, J_s\}$, where J_j is a Jordan block of size n_j, $j = 1, \ldots, s$. Then (X, J_b) is called a *Jordan pair* of $L(\lambda)$, and every solution of MDE (25) has the form

$$x(t) = Xe^{J_b t}z \tag{28}$$

for some $z \in \mathbb{C}^{ln}$.

The conjugate points along θ^* can be located by finding particular solutions $x(t)$ to the equation (24) that start from zero and come back to zero at a later time. Since equation (24) is just a second order MDE, by Theorem 1, its $2n$-dimensional solution space is spanned by linearly independent vector functions defined in terms of the latent roots and the corresponding Jordan chains of $L(\lambda)$. Therefore, the problem of finding the conjugate points of θ^* can be transformed to computing the latent roots and the Jordan chains of $L(\lambda)$.

B. Indirect Method

Another way of solving equation (24) is to transform it into a first-order MDE. Denote $y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \in \mathbb{C}^{2n}$. Then equation (24) is equivalent to

$$\dot{y} = \begin{bmatrix} 0 & I \\ -L_2^{-1}L_0 & -L_2^{-1}L_1 \end{bmatrix} y \triangleq Ay. \tag{29}$$

Hence, by the standard result of linear system theory, the solution to equation (29) is

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} \triangleq \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}. \tag{30}$$

We are interested in those non-trivial solutions $x(t)$ that start from 0 at time 0 and come back to 0 at some finite time $\tau > 0$, i.e., $x(0) = x(\tau) = 0$. Since $x(0) = 0$, by (30), such a solution is of the form

$$x(t) = M_{12}(t) \cdot \dot{x}(0).$$

For non-trivial solutions, $\dot{x}(0) \neq 0$. In order for $x(\tau)$ to be zero, $M_{12}(\tau)$ must be singular. Thus, we can find the conjugate points of θ^* by looking for those $\tau > 0$ at which $M_{12}(\tau)$ is singular.

The indirect method is conceptually much simpler than the direct method. However, in our case, it is difficult to analytically compute the matrix exponential e^{At} and thus in turn the conjugate points. On the other hand, under tree formation constraints, the matrices L_0, L_1, and L_2 usually have some special structures that one can employ to analytically characterize the latent roots, latent vectors, and in turn the solutions to the Jacobi equation (24). In the next section, we will illustrate this by using first the direct method to analytically compute the conjugate points of θ^* for a particular class of tree formation constraints and then the indirect method to numerically verify the results.

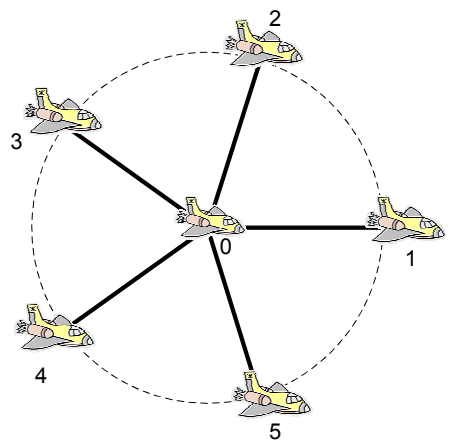


Fig. 3. Initial position for an example formation of agents with $n = 5$.

V. An Example

Consider the tree formation pattern $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the root node 0 has n ($n \geq 2$) immediate successors, i.e., $\mathcal{V} = \{0, 1, \ldots, n\}$ and $\mathcal{E}= \{(0, i) : i = 1, \ldots, n\}$. Suppose that the starting positions $\langle a_i \rangle_{i=0}^n$ of the $n + 1$ agents are give by: $a_0 = 0$, $a_i = (\cos((i-1)\phi_n), \sin((i-1)\phi_n))$, $i = 1, \ldots, n$, where

$$\phi_n = \frac{2\pi}{n}. \tag{31}$$

In other words, at time $t = 0$, agent 0 is located at the origin while all the other agents are evenly distributed on the unit circle. Thus in the new coordinate system, $\theta_0 = \langle \theta_i^0 \rangle_{i=1}^n = \langle (i-1)\phi_n \rangle_{i=1}^n$. See Fig. 3 for an example when $n = 5$. Suppose that the destination positions are $\langle b_i \rangle_{i=0}^n = \langle R_{t_f}(a_i) \rangle_{i=0}^n$. As in Section III-C, we consider the candidate solution $q^*(t) = \langle q_i^*(t) \rangle_{i=0}^n = \langle R_t(a_i) \rangle_{i=0}^n$, $0 \leq t \leq t_f$, which in the new coordinate system is given by θ^* defined in (15). It is easy to see that $\langle a_i \rangle_{i=0}^n$ satisfies condition (19), hence condition (16). Thus θ^* is a solution to the geodesic equation (13). As a result, if t_f is sufficiently small, the coordinated motion described by q^*, namely, agent 0 remains at the origin while all the other agents rotate around it at constant unit angular velocity, is a solution to the OFC problem. However, the first conjugate point along θ^* occurs at $\theta^*(\tau_n)$ for some positive time τ_n dependent on n, implying that θ^*, hence q^*, is no longer optimal if $t_f > \tau_n$. In this section, we shall use the two methods in Section IV to derive τ_n.

A. Analytical Solution

The analytical computation of the conjugate points along θ^* can be broken up into three steps. First, we compute the constant coefficient matrices in the Jacobi equation for this particular example. To solve the Jacobi equation, we next find all the latent roots and latent vectors of the corresponding matrix polynomial. Using these latent roots/vectors, we can then obtain the general solutions of the Jacobi equation, as well as the conjugate points.

1) Computation of the Coefficient Matrices in the Jacobi Equation: To find the conjugate points along θ^*, we need to solve the Jacobi equation (24). First we compute the coefficient matrices L_0, L_1, and L_2. Under the formation pattern $\mathcal{G}$, we have $\chi_0 = n+1$, and $\chi_1 = \cdots = \chi_n = 1$. Thus Δ_{ij} defined in (9) becomes

$$\Delta_{ij} = \begin{cases} \frac{n}{n+1}, & \text{if } i = j\ , \\ \frac{-1}{n+1}, & \text{if } i \neq j\ , \end{cases} \quad 1 \leq i, j \leq n, \tag{32}$$

and the matrix $G(\theta)$ defined in (10) evaluated along θ^* can be simplified to

$$G(\theta^*) = [g_{ij}]_{1\leq i,j\leq n} = [\Delta_{ij}\cos(\theta_i^0 - \theta_j^0)]_{1\leq i,j\leq n} = [\Delta_{ij}\cos((i-j)\phi_n)]_{1\leq i,j\leq n}\,. \tag{33}$$

Similarly, the matrix $G_s(\theta^*)$ defined in (22) becomes

$$G_s(\theta^*) = [\Delta_{ij}\sin(\theta_i^0 - \theta_j^0)]_{1\leq i,j\leq n} = [\Delta_{ij}\sin((i-j)\phi_n)]_{1\leq i,j\leq n} = \left[-\frac{1}{n+1}\sin((i-j)\phi_n)\right]_{1\leq i,j\leq n}.$$

The following lemma is needed for computing the inverse matrix $G(\theta^*)^{-1}$.

Lemma 4: Let m be an integer that is not an integer multiple of n, i.e., $m \neq l\cdot n$, $\forall l \in \mathbb{Z}$. Let $\beta \in \mathbb{R}$ be arbitrary. Then the following relations hold:

$$\sum_{k=0}^{n-1}\cos(km\phi_n + \beta) = 0, \quad \sum_{k=0}^{n-1}\sin(km\phi_n + \beta) = 0.$$

Proof: Since m is not an integer multiple of n, we have $e^{jm\phi_n} = e^{j\frac{2m\pi}{n}} \neq 1$. Thus,

$$\sum_{k=0}^{n-1} e^{j(km\phi_n+\beta)} = e^{j\beta}\sum_{k=0}^{n-1}\left(e^{jm\phi_n}\right)^k = e^{j\beta}\frac{1-\left(e^{jm\phi_n}\right)^n}{1-e^{jm\phi_n}} = e^{j\beta}\frac{1-e^{jmn\phi_n}}{1-e^{jm\phi_n}} = 0.$$

Taking the real and imaginary parts of the above identity will give us the desired conclusions. ■

Using Lemma 4, we can derive the following results.

Lemma 5: For $n \geq 3$, the inverse of $G(\theta^*)$ defined in (33), denoted by $G(\theta^*)^{-1} = [g^{ij}]_{1\leq i,j\leq n}$, is given by

$$g^{ij} = \begin{cases} \frac{n+4}{n+2}, & i = j, \\ \frac{2}{n+2}\cos\left((i-j)\phi_n\right), & i \neq j. \end{cases} \tag{34}$$

Proof: For each $i = 1, \ldots, n$,

$$\begin{aligned}\sum_{k=1}^{n} g_{ik}g^{ki} &= \sum_{k\neq i}\frac{-1}{n+1}\cos((i-k)\phi_n)\cdot\frac{2}{n+2}\cos((k-i)\phi_n) + \frac{n}{n+1}\cdot\frac{n+4}{n+2} \\ &= \sum_{k\neq i}\frac{-2}{(n+1)(n+2)}\cdot\frac{1+\cos(2(i-k)\phi_n)}{2} + \frac{n(n+4)}{(n+1)(n+2)} \\ &= \frac{-(n-1)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} + \frac{n(n+4)}{(n+1)(n+2)} = 1,\end{aligned}$$

where in deriving the second last equality we have used Lemma 4 with $m = 2$. On the other hand, if $i \neq j$,

$$\begin{aligned}
\sum_{k=1}^{n} g_{ik} g^{kj} &= \sum_{k\neq i,\, k\neq j} \frac{-1}{n+1}\cos((i-k)\phi_n)\cdot\frac{2}{n+2}\cos((k-j)\phi_n) \\
&\quad + \frac{n}{n+1}\cdot\frac{2}{n+2}\cos((i-j)\phi_n) + \frac{-1}{n+1}\cos((i-j)\phi_n)\cdot\frac{n+4}{n+2} \\
&= \sum_{k=1}^{n} \frac{-2}{(n+1)(n+2)}\cdot\frac{\cos((i-j)\phi_n)+\cos((i+j-2k)\phi_n)}{2} \\
&\quad + \left[\frac{4}{(n+1)(n+2)} + \frac{2n}{(n+1)(n+2)} - \frac{n+4}{(n+1)(n+2)}\right]\cos((i-j)\phi_n) \\
&= \left[\frac{-n}{(n+1)(n+2)} + \frac{4}{(n+1)(n+2)} + \frac{2n}{(n+1)(n+2)} - \frac{n+4}{(n+1)(n+2)}\right]\cos((i-j)\phi_n) = 0.
\end{aligned}$$

This completes the proof of the lemma. ■

Remark 5: Lemma 5 is only valid when $n \geq 3$, as can be seen from its proof. In the subsequent discussions, we shall assume $n \geq 3$ temporarily, and then deal with the $n = 2$ case separately.

Furthermore, using Lemma 4, we can compute the matrix $\Lambda = \text{diag}\,(\mu_1, \ldots, \mu_n)$ defined in (22). For each $k = 1, \ldots, n$, μ_k defined in (21) is simplified to

$$\mu_k = \sum_{1\leq j\leq n} \Delta_{kj}\cos((k-j)\phi_n) = \sum_{j\neq k}\frac{-1}{n+1}\cos((k-j)\phi_n) + \frac{n}{n+1} = \frac{1}{n+1} + \frac{n}{n+1} = 1.$$

Therefore, $\Lambda = \text{diag}\,(\mu_1, \ldots, \mu_n) = I$ is simply the identity matrix.

Given the simplified $G(\theta^*)^{-1}$ and Λ, the coefficient matrices in equation (24) can now be computed as

$$L_2 = I, \qquad [L_0]_{ij} = \left[G(\theta^*)^{-1}\Lambda - I\right]_{ij} = \frac{2}{n+2}\cos((i-j)\phi_n),$$

and, by applying Lemma 4 again,

$$\begin{aligned}
[L_1]_{ij} &= \left[2G(\theta^*)^{-1}G_s(\theta^*)\right]_{ij} = 2\sum_{k=1}^{n} g^{ik}[G_s(\theta^*)]_{kj} \\
&= 2\cdot\frac{-1}{n+1}\sin((i-j)\phi_n) + 2\sum_{k=1}^{n}\frac{2}{n+2}\cos((i-k)\phi_n)\cdot\frac{-1}{n+1}\sin((k-j)\phi_n) \\
&= \frac{-1}{n+1}\sin((i-j)\phi_n) + \sum_{k=1}^{n}\frac{-2}{(n+1)(n+2)}\left[\sin((i-j)\phi_n) - \sin((2k-i-j)\phi_n)\right] \\
&= \frac{-4}{n+2}\sin((i-j)\phi_n).
\end{aligned}$$

As a result, the matrix polynomial associated with the MDE (24) becomes, for $1 \leq i, j \leq n$,

$$[L(\lambda)]_{ij} = \left[\sum_{k=0}^{2}\lambda^k L_k\right]_{ij} = \begin{cases} \lambda^2 + \frac{2}{n+2}, & i = j, \\ \frac{-4}{n+2}\sin((i-j)\phi_n)\lambda + \frac{2}{n+2}\cos((i-j)\phi_n), & i \neq j. \end{cases} \tag{35}$$

2) Latent Roots and Latent Vectors of the Matrix Polynomial $L(\lambda)$: By Theorem 1, solutions to equation (24) can be expressed in terms of the latent roots and the Jordan chains of $L(\lambda)$. We now compute these for the matrix polynomial $L(\lambda)$ defined in (35). We will show that $L(\lambda)$ has a zero latent root with multiplicity $2(n-1)$, associated

with which there are $n-1$ Jordan chains of length 2, as well as four distinct nonzero latent roots of multiplicity one.

First observe that each row of $L(0) = L_0 = [\frac{2}{n+2}\cos((i-j)\phi_n]_{1\le i,j\le n}$ sums up to zero. This indicates that $\lambda = 0$ is a latent root of $L(\lambda)$ with a corresponding latent vector $\mathbf{1} = (1,\ldots,1)^T$. However, this is only part of the latent vectors associated with the latent root 0. To find the other latent vectors, observe that the rows of $L(0)$ are cyclic permutations of the same vector that can be thought of as the real part of a basis of the Discrete Fourier Transform (DFT). In light of the orthogonality of the DFT bases, it is possible to find the other latent vectors corresponding to the latent root $\lambda = 0$ from the set of DFT bases defined as follows.

Lemma 6: Define a set of vectors $v_k^c, v_k^s \in \mathbb{R}^n$, $k = 0,\ldots,n$, as

$$v_k^c = [cos(k\phi_n), \cos(2k\phi_n),\ldots,\cos(nk\phi_n)]^T, \qquad v_k^s = [sin(k\phi_n), \sin(2k\phi_n),\ldots,\sin(nk\phi_n)]^T.$$

Let $V \triangleq V^c \cup V^s \triangleq \{v_k^c : 0 \le k \le [\frac{n}{2}]\} \cup \{v_k^s : 1 \le k < \frac{n}{2}\}$, where $[\cdot]$ denotes the integer part of a real number. Then V is an orthogonal basis of $\mathbb{R}^n$.

The proof of Lemma 6 is a simple application of Lemma 4; hence it is omitted here.

Remark 6: v_0^s and $v_{\frac{n}{2}}^s$ (when n is even) are zero vectors; thus they are not included in V^s. It is easily seen that V defined above contains exactly n linearly independent vectors of $\mathbb{R}^n$, for any $n \ge 2$.

Using Lemma 6, we are able to characterize all the Jordan chains of $L(\lambda)$ associated with $\lambda = 0$.

Proposition 1: $\lambda_0 = 0$ is a latent root of the matrix polynomial $L(\lambda)$ defined in (35), and for each $v \in V\backslash\{v_1^c, v_1^s\}$, the pair of vectors $\{v,\, v\}$ constitute a Jordan chain of $L(\lambda)$ corresponding to λ_0 of length 2.

Proof: We need to verify that $L(0)v = 0$ and that $L(0)v + L^{(1)}(0)v = 0$, where $L^{(1)}(0)$ is the derivative of $L(\lambda)$ evaluated at $\lambda = 0$. By (35), $L(0) = L_0 = [\frac{2}{n+2}\cos((i-j)\phi_n)]_{1\le i,j\le n}$ and $L^{(1)}(0) = L_1 = [\frac{-4}{n+2}\sin((i-j)\phi_n)]_{1\le i,j\le n}$. Each $v \in V \setminus \{v_1^c, v_1^s\}$ is of the form $v = v_k^c$ for some $0 \le k \le [\frac{n}{2}]$ with $k \ne 1$, or $v = v_k^s$ for some $2 \le k \le \frac{n}{2}$. In the former case,

$$L(0)v = L_0 v = \sum_{j=1}^{n} \frac{2}{n+2}\cos((i-j)\phi_n)\cos(jk\phi_n) = \sum_{j=1}^{n} \frac{\cos((i-j+jk)\phi_n) + \cos((i-j-jk)\phi_n)}{n+2} = 0,$$

where we have used Lemma 4 and the fact that $k \pm 1$ is not an integer multiple of n. In an entirely similar way, we can prove that $L^{(1)}(0)v = 0$, and that for $v = v_k^s$ for some $2 \le k \le \frac{n}{2}$, $L(0)v = L^{(1)}(0)v = 0$. Together, this completes the proof of the lemma. ■

Since $V \setminus \{v_1^c, v_1^s\}$ has $n-2$ vectors, Proposition 1 describes $n-2$ Jordan chains, each of which is of length 2. Together, by Lemma 3, these Jordan chains characterize $2n-4$ independent solutions of the Jacobi equation (24). Indeed, let u_i, $i = 1,\ldots,n-2$, be an enumeration of the vectors in $V \setminus \{v_1^c, v_1^s\}$, and define

$$U_0(t) \triangleq [u_1, tu_1, \ldots, u_{n-2}, tu_{n-2}]. \tag{36}$$

Then any linear combination of the columns of $U_0(t)$ is a solution to equation (24). On the other hand, equation (24) should have $2n$ independent solutions in total. It turns out that the four missing independent solutions are provided by the latent vectors associated with the nonzero latent roots of $L(\lambda)$.

Proposition 2: Define the complex vector $v_1 = v_1^c + jv_1^s$, where $j = \sqrt{-1}$. Its conjugate is $\bar{v}_1 = v_1^c - jv_1^s$. Define

$$\omega_1 = \frac{-n + \sqrt{2n(n+1)}}{n+2}, \qquad \omega_2 = \frac{-n - \sqrt{2n(n+1)}}{n+2}. \tag{37}$$

Then the nonzero latent roots of $L(\lambda)$ and their corresponding latent vectors can be characterized as follows:

1) $\lambda_1 = j\omega_1$ and $\lambda_2 = -j\omega_1$ are two latent roots of $L(\lambda)$ with latent vectors v_1 and $\bar{v}_1$, respectively;
2) $\lambda_3 = j\omega_2$ and $\lambda_4 = -j\omega_2$ are two latent roots of $L(\lambda)$ with latent vectors v_1 and $\bar{v}_1$, respectively.

Proof: Note that $v_1 = v_1^c + jv_1^s = [\cos(i\phi_n) + j\sin(i\phi_n)]_{1\le i\le n} = [e^{ji\phi_n}]_{1\le i\le n}$. Thus, for each $i = 1, \ldots, n$,

$$[L_0 v_1]_i = \sum_{k=1}^{n} \frac{2}{n+2}\cos((i-k)\phi_n)e^{jk\phi_n} = \sum_{k=1}^{n} \frac{1}{n+2}[e^{j(i-k)\phi_n} + e^{-j(i-k)\phi_n}]e^{jk\phi_n} = \frac{n}{n+2}e^{ji\phi_n},$$
$$[L_1 v_1]_i = \sum_{k=1}^{n} \frac{-4}{n+2}\sin((i-k)\phi_n)e^{jk\phi_n} = \sum_{k=1}^{n} \frac{j2}{n+2}[e^{j(i-k)\phi_n} + e^{-j(i-k)\phi_n}]e^{jk\phi_n} = \frac{j2n}{n+2}e^{ji\phi_n},$$

or equivalently,

$$L_0 v_1 = \frac{n}{n+2}v_1, \quad L_1 v_1 = \frac{j2n}{n+2}v_1. \tag{38}$$

In other words, v_1 is an eigenvector of L_0 and L_1 corresponding to the eigenvalue $\frac{n}{n+2}$ and $j\frac{2n}{n+2}$, respectively. Since $L_2 = I$, we have

$$L(\lambda)v_1 = (L_0 + \lambda L_1 + \lambda^2 L_2)v_1 = \left(\frac{n}{n+2} + \frac{j2n}{n+2}\lambda + \lambda^2\right)v_1 = \frac{(n+2)\lambda^2 + j2n\lambda + n}{n+2}v_1.$$

It is easy to verify that λ_1 and λ_3 are two roots of the quadratic equation $(n+2)\lambda^2 + j2n\lambda + n = 0$. Hence, from the above equation, $L(\lambda_1)v_1 = 0$ and $L(\lambda_3)v_1 = 0$, which implies that v_1 is a (common) latent vector associated with the two latent roots λ_1 and λ_3 of $L(\lambda)$. By taking the complex conjugate, we conclude that $\lambda_2 = \bar{\lambda}_1$ and $\lambda_4 = \bar{\lambda}_3$ are two latent roots of $L(\lambda)$ with a common latent vector $\bar{v}_1$. ■

By Lemma 3, there are four independent solutions to the MDE (24) corresponding to the four distinct nonzero latent roots λ_i, $i = 1, \ldots, 4$. We arrange these solutions into the columns of a matrix defined by

$$U_{\bar{0}}(t) = [e^{j\omega_1 t}v_1,\ e^{-j\omega_1 t}\bar{v}_1,\ e^{j\omega_2 t}v_1,\ e^{-j\omega_2 t}\bar{v}_1] = V_1 \cdot \begin{bmatrix} e^{\Omega_1 t} & e^{\Omega_2 t} \end{bmatrix}, \tag{39}$$

where V_1, Ω_1, and Ω_2 are complex matrices defined by

$$V_1 = \begin{bmatrix} v_1 & \bar{v}_1 \end{bmatrix}, \qquad \Omega_1 = \begin{bmatrix} j\omega_1 & 0 \\ 0 & -j\omega_1 \end{bmatrix}, \qquad \Omega_2 = \begin{bmatrix} j\omega_2 & 0 \\ 0 & -j\omega_2 \end{bmatrix}. \tag{40}$$

Any complex linear combination of the columns of $U_{\bar{0}}(t)$ of the form $x(t) = U_{\bar{0}}(t)z$ for some $z = [z_1, z_2, z_3, z_4]^T$ in $\mathbb{C}^4$ is a solution to equation (24). However, we are only interested in those real solutions. For this purpose, z must be chosen so that $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Plugging into $x(t) = U_{\bar{0}}(t)z$, we conclude that the four dimensional real solution space of equation (24) corresponding to the nonzero latent roots of $L(\lambda)$ is spanned by the columns

of two real matrices in $\mathbb{R}^{n\times 2}$:

$$U_1^{Re}(t) = [\text{Re}(e^{j\omega_1 t}v_1),\, \text{Im}(e^{j\omega_1 t}v_1)] = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) \\ -\sin(\omega_1 t) & \cos(\omega_1 t) \end{bmatrix}, \tag{41}$$

$$U_2^{Re}(t) = [\text{Re}(e^{j\omega_2 t}v_1),\, \text{Im}(e^{j\omega_2 t}v_1)] = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} \cos(\omega_2 t) & \sin(\omega_2 t) \\ -\sin(\omega_2 t) & \cos(\omega_2 t) \end{bmatrix}. \tag{42}$$

To sum up the results in this section, we can now characterize all the real solutions to equation (24).

Proposition 3: Every real solution to the MDE (24) is of the form

$$x(t) = U_0(t)c_0 + U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2, \tag{43}$$

for some constants $c_0 \in \mathbb{R}^{2(n-2)}$, $c_1 \in \mathbb{R}^2$, and $c_2 \in \mathbb{R}^2$. Here the matrices $U_0(t)$, $U_1^{Re}(t)$ and $U_2^{Re}(t)$ are defined in (36), (41), and (42), respectively.

3) Conjugate Points Along θ^*: To find the conjugate points along θ^*, we need to look for those nontrivial real solutions $x(t)$ to the Jacobi equation (24) that start from 0 and return to 0 at some positive time τ, i.e., $x(0) = 0$, and $x(\tau) = 0$. Let $x(t)$ be one such solution. By Proposition 3, $x(t)$ is of the form $x(t) = U_0(t)c_0 + U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2$ for some constants c_0, c_1 and c_2. Observe that the first term $U_0(t)c_0$ is an affine function of time t, and always lies in the subspace spanned by the $n-2$ basis vectors in $V \setminus \{v_1^c, v_1^s\}$, where V is the set of bases given in Lemma 6. On the other hand, the second and the third terms $U_1^{Re}(t)c_1$ and $U_2^{Re}(t)c_2$ always lie in the 2-dimensional subspace spanned by the other two basis vectors v_1^c and v_1^s. Hence, in order for $x(0) = x(\tau) = 0$ to hold for some $\tau > 0$, we must have $c_0 = 0$. Thus $x(t) = U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2$.

By (41) and (42), at time 0, $U_1^{Re}(0) = U_2^{Re}(0) = [v_1^c,\, v_1^s]$. Hence to satisfy $x(0) = 0$, we must have

$$x(0) = U_1^{Re}(0)c_1 + U_2^{Re}(0)c_2 = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} (c_1 + c_2) = 0,$$

which implies that $c_2 = -c_1$ by the linear independence of v_1^c and v_1^s. Thus, if we write $c_1 = [a, b]^T$, then

$$\begin{aligned} x(t) &= U_1^{Re}(t)c_1 + U_2^{Re}(t))(-c_1) = [U_1^{Re}(t) - U_2^{Re}(t)]c_1 \\ &= \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} a(\cos(\omega_1 t) - \cos(\omega_2 t)) + b(\sin(\omega_1 t) - \sin(\omega_2 t)) \\ -a(\sin(\omega_1 t) - \sin(\omega_2 t)) + b(\cos(\omega_1 t) - \cos(\omega_2 t)) \end{bmatrix} \\ &= 2\sin(\omega_- t) \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} -a\sin(\omega_+ t) + b\cos(\omega_+ t) \\ -a\cos(\omega_+ t) - b\sin(\omega_+ t) \end{bmatrix}, \end{aligned} \tag{44}$$

where ω_+ and ω_- are constants defined by

$$\omega_+ = \frac{\omega_1 + \omega_2}{2} = -\frac{n}{n+2}, \qquad \omega_- = \frac{\omega_1 - \omega_2}{2} = \frac{\sqrt{2n(n+1)}}{n+2}. \tag{45}$$

Note that a and b can not be zero at the same time (otherwise $x(t) \equiv 0$ is trivial). Under this constraint, it can be easily checked that the two entries of the last factor in (44), $-a\sin(\omega_+ t) + b\cos(\omega_+ t)$ and $-a\cos(\omega_+ t) - b\sin(\omega_+ t)$, can not be zero at the same time. Thus, in order to satisfy $x(\tau) = 0$ for some $\tau > 0$, we must have $\sin(\omega_- \tau) = 0$, i.e., $\tau = \frac{k\pi}{\omega_-}$ for some $k = 1, 2, \ldots$. This gives us the times when conjugate points along θ^* are encountered.

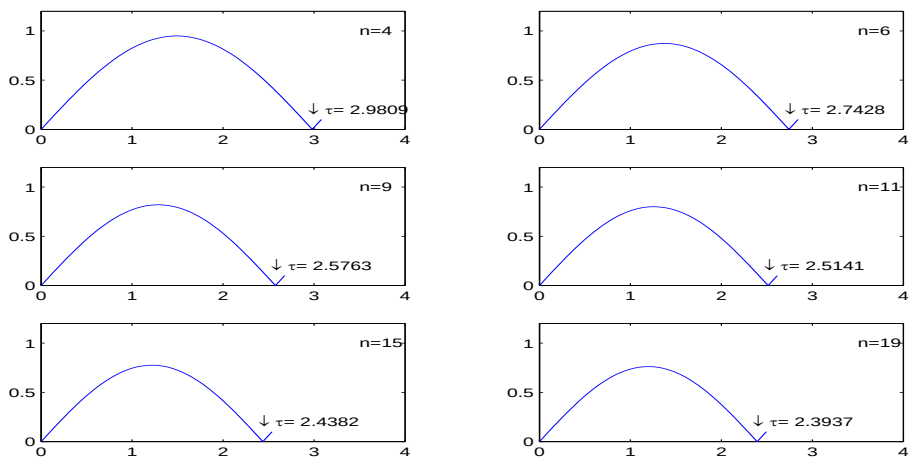


Fig. 4. The smallest singular value of the matrix $M_{12}(t)$ as a function of t for different n.

Theorem 2: For the particular OFC problem studied in this section, the set of conjugate points along the candidate solution θ^* is given by

$$\left\{\theta^*(\tau) : \tau = \frac{k(n+2)\pi}{\sqrt{2n(n+1)}},\ k = 1, 2, \ldots\right\}.$$

As a result, the first conjugate point along θ^* occurs at time

$$\tau_n = \frac{\pi}{\omega_-} = \frac{(n+2)\pi}{\sqrt{2n(n+1)}}. \tag{46}$$

From the expression (46), we can see that τ_n decreases as n increases, and $\tau_n \to \frac{\pi}{\sqrt{2}}$ as $n \to \infty$.

Since a geodesic is no longer distance-minimizing beyond its first conjugate point, we have

Theorem 3: θ^* is not an optimal solution to the OFC problem if $t_f > \tau_n$.

As mentioned in Remark 5, the above derivations are valid only under the condition $n \geq 3$. When $n = 2$, the coefficient matrices L_0, L_1, and L_2 in equation (24) are all 2×2. Hence, analytical solution can be obtained much more easily compared with the general n case. After some careful computation, the first conjugate point along θ^* occurs at the time $\tau_2 = \frac{\pi}{\sqrt{2}}$ for $n = 2$, which, interestingly, is exactly the limit of τ_n as $n \to \infty$.

Remark 7: In the $n = 2$ case, there are three agents, with the root agent at the middle of the other two. Thus the example tree formation degenerates into the snake formation studied in [12]. The time τ_2 that the first conjugate point occurs computed above is consistent with the result obtained in [12] using a different approach.

B. Numerical Verification

We now use the indirect method discussed in Section IV-B to numerically verify our results in Theorem 2.

Our approach is as follows. Given the agent number $n+1$, we first compute the coefficient matrices L_0, L_1 and L_2 in equation (24), and assemble them into the matrix A according to (29). Then we compute the exponential matrix e^{At} for t starting from 0 and increasing at a certain step size. For each t, we partition e^{At} into blocks as defined in (30) and compute the singular values of $M_{12}(t)$. If the smallest singular value of $M_{12}(t)$ is sufficiently

n	4	6	9	11	15	19
τ_n (analytical)	2.9804	2.7422	2.5758	2.5136	2.4377	2.3931
τ_n (numerical)	2.9809	2.7428	2.5763	2.5141	2.4382	2.3937

TABLE I

COMPARISON OF THE NUMERICAL AND THE ANALYTICAL RESULTS ON THE TIME EPOCH τ_n WHEN THE FIRST CONJUGATE POINT ALONG θ^* OCCURS.

close to 0, then we declare that $M_{12}(t)$ is singular, and that the first conjugate point is found. See Section IV-B for the detailed explanation of the above procedures.

To illustrate our results, the above procedures are carried out for six integers selected randomly between 3 and 20. In Fig. 4, for each n, we plot the smallest singular value of $M_{12}(t)$ as a function of the time t and annotate the smallest t such that $M_{12}(t)$ is singular. In Table I, the numerical results are compared with the analytical results obtained according to (46) and rounded to 4 decimal digits. As can be seen from the table, considering the numerical errors, the numerical results agree very well with the analytical ones.

C. Better Solutions Beyond the Conjugate Points

Once extended beyond its first conjugate point, θ^* will no longer be an optimal solution to the OFC problem. The reason is that, although it still satisfies the first order optimality condition, namely, the geodesic equation (13), it fails to meet the second order optimality condition, and better solutions can be obtained by infinitesimal perturbations around θ^*. We will illustrate these for the example studied in this section.

We first derive the second-order variation of the cost function $J(\theta)$ at θ^*. Recall that $J(\theta)$ as defined in (11) can be written as

$$J(\theta) = \int_0^{t_f} L(\theta, \dot{\theta})\, dt,$$

where $L(\theta, \dot{\theta}) = \dot{\theta}^T G(\theta)\dot{\theta}$. Let $x = \delta\theta$ be a proper variation of θ^* over $[0, t_f]$. Then $x(0) = x(t_f) = 0$. The first variation of J around θ^*, $\frac{d}{d\epsilon}\big|_{\epsilon=0} J(\theta^* + \epsilon x)$, is zero since θ^* satisfies the geodesic equation (13). The second variation of J around θ^* can be computed as

$$\delta^2 J(\theta^*; x) = \frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} J(\theta^* + \epsilon x) = \int_0^{t_f} \left(x^T L_{\theta\theta} x + 2x^T L_{\theta\dot{\theta}} \dot{x} + \dot{x}^T L_{\dot{\theta}\dot{\theta}} \dot{x} \right) dt,$$

where $L_{\theta\theta}$, $L_{\theta\dot{\theta}}$, and $L_{\dot{\theta}\dot{\theta}}$ are $n \times n$ matrices with entries defined by

$$[L_{\theta\theta}]_{ij} = \frac{\partial^2 L(\theta, \dot{\theta})}{\partial\theta_i \partial\theta_j}\bigg|_{\theta=\theta^*, \dot{\theta}=\dot{\theta}^*}, \quad [L_{\theta\dot{\theta}}]_{ij} = \frac{\partial^2 L(\theta, \dot{\theta})}{\partial\theta_i \partial\dot{\theta}_j}\bigg|_{\theta=\theta^*, \dot{\theta}=\dot{\theta}^*}, \quad [L_{\dot{\theta}\dot{\theta}}]_{ij} = \frac{\partial^2 L(\theta, \dot{\theta})}{\partial\dot{\theta}_i \partial\dot{\theta}_j}\bigg|_{\theta=\theta^*, \dot{\theta}=\dot{\theta}^*}. \tag{47}$$

Recall that $\theta^* = \theta_0 + t\mathbf{1}$; thus $\dot{\theta}^* = \mathbf{1}$. From $L(\theta, \dot{\theta}) = \dot{\theta}^T G(\theta)\dot{\theta}$, we can compute

$$L_{\theta\theta} = 2G(\theta^*) - 2\Lambda, \quad L_{\theta\dot{\theta}} = -2G_s(\theta^*), \quad L_{\dot{\theta}\dot{\theta}} = 2G(\theta^*). \tag{48}$$

These computations are mostly straightforward, except for the diagonal entries of $L_{\theta\dot{\theta}}$. Hence we only list the detailed derivation of these entries below: for each $i = 1, \ldots, n$,

$$\left[L_{\theta\dot{\theta}}\right]_{ii} = \left.\frac{\partial^2 L(\theta, \dot{\theta})}{\partial\theta_i \partial\dot{\theta}_i}\right|_{\theta=\theta^*, \dot{\theta}=\dot{\theta}^*} = \left.\frac{\partial}{\partial\theta_i}\right|_{\theta=\theta^*} \sum_{j=1}^{n} 2[G(\theta)]_{ij}\dot{\theta}_j^* = -2\sum_{j=1}^{m} \Delta_{ij}\sin(\theta_i^* - \theta_j^*) = 0.$$

Here we have used the fact that $[G(\theta)]_{ij} = \Delta_{ij}\cos(\theta_i - \theta_j)$ and that $\dot{\theta}_j^* = 1$ for all $j = 1, \ldots, n$. In addition, the last equality follows from condition (18), which is satisfied by θ^* since it is a geodesic.

Using (48), the second-order variation of J can be simplified to

$$\delta^2 J(\theta^*; x) = 2\int_0^{t_f} \left[x^T(G(\theta^*) - \Lambda)x - 2x^T G_s(\theta^*)\dot{x} + \dot{x}^T G(\theta^*)\dot{x}\right] dt. \tag{49}$$

Integrating by part and using the fact that $G(\theta^*)$ is a constant matrix, the above equation reduces to

$$\begin{aligned}\delta^2 J(\theta^*; x) &= 2\int_0^{t_f} \left[x^T(G(\theta^*) - \Lambda)x - 2x^T G_s(\theta^*)\dot{x}\right] dt + 2x^T G(\theta^*)\dot{x}\Big|_0^{t_f} - 2\int_0^{t_f} x^T G(\theta^*)\ddot{x}\, dt \\ &= 2\int_0^{t_f} x^T\left[(G(\theta^*) - \Lambda)x - 2G_s(\theta^*)\dot{x} - G(\theta^*)\ddot{x}\right] dt \\ &= -2\int_0^{t_f} x^T G(\theta^*)\left(\ddot{x} + L_1\dot{x} + L_0 x\right) dt. \end{aligned} \tag{50}$$

Note that the last factor of the integrand is exactly the left hand side of the Jacobi equation (24).

Having derived the second variation of J around θ^* with respect to the proper variation x in (50), we now show that, if $n > 2$ and $t_f > \tau_n$, a proper variation x_f can be constructed such that $\delta^2 J(\theta^*; x_f) < 0$. As a result, for ϵ small enough, the perturbation of θ along x_f, $\theta^*(t) + \epsilon \cdot x_f(t)$ for $t \in [0, t_f]$, will correspond to a smaller cost function J, leading to a better solution than θ^*.

Assume $t_f > \tau_n$. Consider x_f of the form

$$x_f = V_1\left(e^{\Omega_f t} - e^{\Omega_2 t}\right) z, \tag{51}$$

where $z = \begin{bmatrix} z_1 & \bar{z}_1 \end{bmatrix}^T \in \mathbb{C}^2$ for some $z_1 \in \mathbb{C}$; V_1 and Ω_2 are defined in (40); and Ω_f is defined by

$$\Omega_f = \begin{bmatrix} j\omega_f & 0 \\ 0 & -j\omega_f \end{bmatrix} \text{ with } \omega_f = \omega_2 + \frac{2\pi}{t_f}. \tag{52}$$

It is easy to check that x_f defined above is a real function satisfying $x_f(0) = x_f(t_f) = 0$. Thus x_f is a proper variation over $[0, t_f]$. In the Appendix, we compute that

$$\delta^2 J(\theta^*; x_f) = \frac{16n^2\pi^2(t_f - \tau_n)\|z_1\|^2}{(n+1)\tau_n t_f^2 \omega_+}\int_0^{t_f} \sin^2(\pi t/t_f)\, dt. \tag{53}$$

By (45), $\omega_+ < 0$. Since the integral in (53) is positive and $t_f > \tau_n$, we have $\delta^2 J(\theta^*; x_f) < 0$.

As a result, if $t_f > \tau_n$, a solution θ_f^* better than θ^* can be obtained by an infinitesimal perturbation of θ^* along the x_f direction: $\theta_f^* = \theta^* + \epsilon \cdot x_f$ for ϵ small enough. In Fig. 5, we illustrate the coordinated motions corresponding to θ^* and θ_f^* by plotting the snapshots of these two motions at six evenly spaced time epochs between 0 and $t_f = 2\pi > \tau_3$ for $n = 3$: $t = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$. For illustration purpose, a relatively large ϵ is chosen in the plots to better render the difference between the two motions. In Fig. 6, the phase angles that the three outer agents

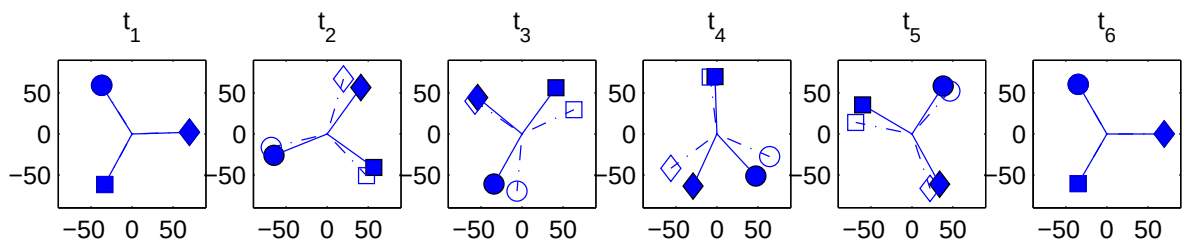


Fig. 5. Snapshots of the coordinated motions corresponding to θ^* (dashed) and θ_f^* (shaded), for $n = 3$.
(Diamond: Agent 1; Circle: Agent 2; Square: Agent 3)

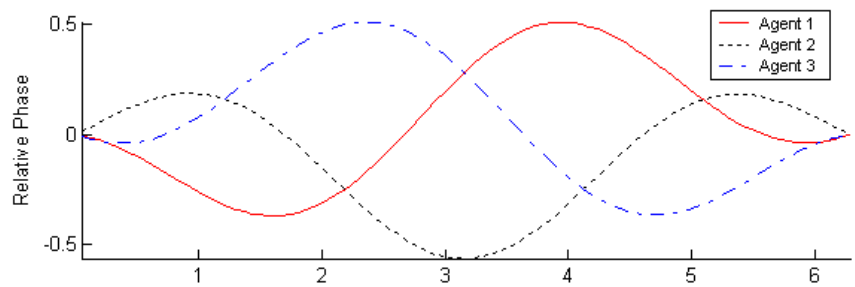


Fig. 6. Comparison of the phase angles that agent 1, 2, and 3 make with respect to agent 0 in the motions corresponding to θ_f^* and θ^*, $n = 3$.

make with respect to the center one are compared for θ^* and θ_f^*: their differences $[\theta_f^* - \theta^*]_i = \epsilon \cdot [x_f]_i$ are plotted over $[0, t_f]$ for $i = 1, 2, 3$.

VI. Conclusion

In this paper, the problem of optimal multi-agent coordination under tree formation constraint is formulated. The geodesic equation characterizing the optimal coordinated motions is derived in a suitably chosen coordinate system. For a special instance of the problem when the group of agents rotate around a common centroid, optimality conditions of a natural candidate solution are studied. In particular, we conclude that, under certain condition on the initial position, the candidate solution is optimal when short enough, and is no longer optimal after surpassing its first conjugate point. To compute the conjugate points, two methods, one analytical and one numerical, are proposed to solve the corresponding Jacobi equation. These methods are illustrated through an example where the first method is applied to obtain the analytical expressions of all the conjugate points, and the second method is used to verify the results. Furthermore, better solutions after the candidate solution is extended beyond its first conjugate point are also derived for this example.

The approaches adopted in this paper are general enough to make the results meaningful in a variety of applications involving optimal multi-agent coordination. As extensions, our future research will focus on the OFC problem with general formation constraints that are not necessarily described by trees.

APPENDIX
DERIVATION OF EQUATION (53)

In this appendix, we will derive the results in equation (53). First, we introduce a useful lemma.

Lemma 7: Given the matrices $G_s(\theta^*)$ and V_1 as defined in (22) and (52), respectively, we have

$$G_s(\theta^*)V_1 = V_1 \begin{bmatrix} \frac{n}{2(n+1)}j & 0 \\ 0 & \frac{-n}{2(n+1)}j \end{bmatrix}. \tag{54}$$

Lemma 7 can be easily verified using Lemma 4; hence we omit its proof here.

Let V_1, Ω_1, Ω_2 be defined as in (40). For arbitrary $z \in \mathbb{C}^2$, by the discussion after equation (40), $x^{(i)} \triangleq V_1 e^{\Omega_i t} z$, $i = 1, 2$, are both solutions to the Jacobi equation (24). Substituting $x^{(i)}$ into (24), we have

$$\ddot{x}^{(i)} + L_1 \dot{x}^{(i)} + L_0 x^{(i)} = \left(V_1 \Omega_i^2 + L_1 V_1 \Omega_i + L_0 V_1\right) e^{\Omega_i t} z = 0. \tag{55}$$

Since the above equality holds for arbitrary z and $e^{\Omega_i t}$ is nonsingular, we must have

$$V_1 \Omega_i^2 + L_1 V_1 \Omega_i + L_0 V_1 = 0, \quad i = 1, 2. \tag{56}$$

Taking the difference of (56) for $i = 2$ and $i = 1$, we get

$$L_1 V_1 (\Omega_2 - \Omega_1) = -V_1(\Omega_2^2 - \Omega_1^2) = (\omega_2^2 - \omega_1^2) V_1. \tag{57}$$

Define $\bar{\Omega} = \begin{bmatrix} j\bar{\omega} & 0 \\ 0 & -j\bar{\omega} \end{bmatrix}$ where $\bar{\omega} = \frac{2\pi}{t_f}$. Then Ω_f in (52) is $\Omega_f = \Omega_2 + \bar{\Omega}$, and x_f defined in (51) can be written as $x_f = V_1 e^{\Omega_f t} z - V_1 e^{\Omega_2 t} z = x_f^{(1)} - x^{(2)}$ where $x_f^{(1)} \triangleq V_1 e^{\Omega_f t} z$. Since $x^{(2)}$ satisfies (55), we have

$$\begin{aligned}
\ddot{x}_f + L_1 \dot{x}_f + L_0 x_f &= \ddot{x}_f^{(1)} + L_1 \dot{x}_f^{(1)} + L_0 x_f^{(1)} \\
&= [V_1(\Omega_2 + \bar{\Omega})^2 + L_1 V_1 (\Omega_2 + \bar{\Omega}) + L_0 V_1] e^{\Omega_f t} z \\
&= [V_1(2\Omega_2 \bar{\Omega} + \bar{\Omega}^2) + L_1 V_1 \bar{\Omega}] e^{\Omega_f t} z \qquad \text{(by equation (56))} \\
&= [-(2\omega_2 \bar{\omega} + \bar{\omega}^2) V_1 + L_1 V_1 \bar{\Omega}] e^{\Omega_f t} z \\
&= L_1 V_1 \left[-\frac{2\omega_2 \bar{\omega} + \bar{\omega}^2}{\omega_2^2 - \omega_1^2} (\Omega_2 - \Omega_1) + \bar{\Omega} \right] e^{\Omega_f t} z \qquad \text{(by equation (57))} \\
&= L_1 V_1 \begin{bmatrix} j\omega_m & 0 \\ 0 & -j\omega_m \end{bmatrix} e^{\Omega_f t} z,
\end{aligned} \tag{58}$$

where ω_m is given by

$$\omega_m = -\frac{2\omega_2 \bar{\omega} + \bar{\omega}^2}{\omega_2^2 - \omega_1^2} (\omega_2 - \omega_1) + \bar{\omega} = \frac{\bar{\omega}(\omega_1 - \omega_2 - \bar{\omega})}{\omega_1 + \omega_2} = \frac{\bar{\omega}(\omega_- - \bar{\omega}/2)}{\omega_+} = \frac{\bar{\omega}(\frac{\pi}{\tau_n} - \frac{\pi}{t_f})}{\omega_+} = \frac{2\pi^2 (t_f - \tau_n)}{\tau_n t_f^2 \omega_+}. \tag{59}$$

Since $t_f > \tau_n$ and ω_+ defined in (45) satisfies $\omega_+ < 0$, we have $\omega_m < 0$.

Recall that $L_1 = 2G(\theta^*)^{-1}G_s(\theta^*)$. Therefore, plugging (58) into (50), we have

$$\begin{aligned}\delta^2 J(\theta^*, x_f) &= -2\int_0^{t_f} x_f^T G(\theta^*)\cdot 2G(\theta^*)^{-1}G_s(\theta^*)V_1 \begin{bmatrix} j\omega_m & 0 \\ 0 & -j\omega_m \end{bmatrix} e^{\Omega_f t} z\, dt \\ &= -4\int_0^{t_f} x_f^T G_s(\theta^*)V_1 \begin{bmatrix} j\omega_m & 0 \\ 0 & -j\omega_m \end{bmatrix} e^{\Omega_f t} z\, dt \\ &= \frac{2n\omega_m}{n+1}\int_0^{t_f} x_f^T V_1 e^{\Omega_f t} z\, dt, \end{aligned} \tag{60}$$

where we have used Lemma 7 in deriving the last equation. If by assumption, we choose $z = \begin{bmatrix} z_1 & \bar{z}_1 \end{bmatrix}^T$ for some $z_1 \in \mathbb{C}$, then $x_f = V_1(e^{\Omega_f t} - e^{\Omega_2 t})z$ is real. Hence, $x_f^T = x_f^H = z^H(e^{-\Omega_f t} - e^{-\Omega_2 t})V_1^H$ where the superscript H denotes conjugate transpose. Moreover, it is easily verified that $v_1^H v_1 = \|v_1\|^2 = n$ and, for $n > 2$, $v_1^H \bar{v}_1 = \sum_{i=1}^n e^{-j2i\phi_n} = 0$. Thus,

$$V_1^H V_1 = \begin{bmatrix} v_1^H v_1 & v_1^H \bar{v}_1 \\ \bar{v}_1^H v_1 & \bar{v}_1^H \bar{v}_1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} = nI.$$

With the above results, equation (60) can be simplified to

$$\begin{aligned}\delta^2 J(\theta^*, x_f) &= \frac{2n\omega_m}{n+1}\int_0^{t_f} z^H(e^{-\Omega_f t} - e^{-\Omega_2 t})V_1^H V_1 e^{\Omega_f t} z\, dt \\ &= \frac{2n^2\omega_m}{n+1}\int_0^{t_f} z^H(I - e^{\bar{\Omega} t})z\, dt \\ &= \frac{2n^2\omega_m}{n+1}\int_0^{t_f} \begin{bmatrix} z_1^H \\ \bar{z}_1^H \end{bmatrix} \begin{bmatrix} 1 - e^{j\bar{\omega}t} & 0 \\ 0 & 1 - e^{-j\bar{\omega}t} \end{bmatrix} \begin{bmatrix} z_1 & \bar{z}_1 \end{bmatrix} dt \\ &= \frac{2n^2\omega_m}{n+1}\int_0^{t_f} (2 - e^{j\bar{\omega}t} - e^{-j\bar{\omega}t})\|z_1\|^2\, dt \\ &= \frac{8n^2\omega_m\|z_1\|^2}{n+1}\int_0^{t_f} \sin^2(\bar{\omega}t/2)\, dt. \\ &= \frac{16n^2\pi^2(t_f - \tau_n)\|z_1\|^2}{(n+1)\tau_n t_f^2\omega_+}\int_0^{t_f} \sin^2(\pi t/t_f)\, dt. \end{aligned}$$

REFERENCES

[1] A. Bicchi and L. Pallottino. Optimal planning for coordinated vehicles with bounded curvature. In *Algorithmic and Computational Robotics: New Directions (Hanover, NH, 2000)*, pages 181–189. A K Peters, Natick, MA, 2001.

[2] A. E. Bryson and Y. C. Ho. *Applied Optimal Control*. Taylor and Francis, 1975.

[3] J. Cortes, S. Martinez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Trans. Automatic Control*, 51(7), 2006. To appear.

[4] J. P. Desai and V. Kumar. Nonholonomic motion planning for multiple mobile manipulators. In *Proc. IEEE Int. Conference on Robotics and Automation*, volume 4, pages 3409–3414, Albuquerque, NM, April 1997.

[5] M. P. do Carmo. *Riemannian Geometry*. Birkhäuser Boston Inc., Boston, MA, 1992.

[6] M. Egerstedt and X. Hu. Formation constrained multi-agent control. *IEEE Trans. Robotics and Automation*, 17(6):947–951, Dec. 2001.

[7] J. A. Fax. *Optimal and Cooperative Control of Vehicle Formations*. PhD thesis, California Institute of Technology, 2002.

[8] J. A. Fax and R. M. Murray. Graph laplacians and stabilization of vehicle formations. In *Proc. 15th IFAC Congress*, Barcelona, Spain, 2002.

[9] E. Frazzoli, Z.-H. Mao, J.-H. Oh, and E. Feron. Resolution of conflicts involving many aircraft via semidefinite programming. *J. of Guidance, Control, and Dynamics*, 24(1):79–86, 2001.

[10] V. Gazi and K. M. Passino. Stability analysis of swarms. *IEEE Trans. Automatic Control*, 48(4):692–697, Apr. 2003.

[11] J. Hu, M. Prandini, and S. Sastry. Optimal coordinated motions of multiple agents moving on a plane. *SIAM J. Control and Optimization*, 42(2):637–668, 2003.

[12] J. Hu, M. Prandini, and C. Tomlin. Conjugate points in formation constrained optimal multi-agent coordination: a case study. *Submitted to SIAM J. Control and Optimization*, 2006. In revision.

[13] I. I. Hussein, A. M. Bloch, D. J. Scheeres, D. C. Hyland, and N. H. McClamroch. Optimal fuel-image motion planning for a class of dual spacecraft formations. In *Proc. American Control Conference*, pages 2405–2410, Portland, OR, Jun. 2005.

[14] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of group of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automatic Control*, 48(6):988–1001, June 2003.

[15] Jürgen Jost. *Riemannian Geometry and Geometric Analysis*. Springer-Verlag, Berlin, 3rd edition, 2002.

[16] P. S. Krishnaprasad. Eulerian many-body problems. In *Dynamics and Control of Multibody Systems (Brunswick, ME, 1988)*, pages 187–208. Amer. Math. Soc., Providence, RI, 1989.

[17] J. Kuchar and L. C. Yang. Survey of conflict detection and resolution modeling methods. *IEEE Trans. on Intelligent Transportation Systems*, 1(4):179–189, 2000.

[18] V. Kumar, N. Leonard, and A. S. Morse, editors. *Cooperative Control*, volume 309 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 2004.

[19] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press Inc., Orlando, FL, 1985.

[20] N. E. Leonard and E. Fiorelli. Virtual leaders, artificial potentials and coordinated control of groups. In *Proc. 40th IEEE Conference on Decision and Control*, volume 3, pages 2968–2973, Orlando, FL, 2001.

[21] J. Lin, A. S. Morse, and B. D. O. Anderson. The multi-agent rendezvous problem - the asynchronous case. In *Proc. 43th IEEE Conf. on Decision and Control*, volume 2, pages 1926–1931, Dec. 2004.

[22] Z. Lin, M. Brouche, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Trans. Automatic Control*, 49(4):622–629, Apr. 2004.

[23] Z. Lin, B. Francic, and M. Maggiore. Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Trans. Automatic Control*, 50(1):121–127, January 2005.

[24] W. Ren, R. W. Beard, and E. M. Atkins. A survey of consensus problems in multi-agent coordination. In *Proc. American Control Conference*, volume 3, pages 1859–1864, Portland, OR, Jun. 2005.

[25] R. O. Saber, W. B. Dunbar, and R. M. Murray. Cooperative control of multi-vehicle systems using cost graphs and optimization. In *Proc. American Control Conference*, Denver, CO, Jun. 2003.

[26] D. Stipanovic, G. Inalhan, R. Teo, and C. J. Tomlin. Decentralized overlapping control of a formation of unmanned aerial vehicles. *Automatica*, 2004. To appear.

[27] P. Tabuada, G. J. Pappas, and P. Lima. Feasible formations of multi-agent systems. In *Proc. American Control Conference*, volume 1, pages 56–61, Arlington, VA, Jun. 2001.

[28] H. Tanner, G. J. Pappas, and V. Kumar. Leader to formation stability. *IEEE Trans. Robotics and Automation*, 20(3):443–455, June 2004.

[29] M. Tillerson, L. Breger, and J. P. How. Distributed coordination and control of formation flying spacecraft. In *Proc. IEEE American Control Conference*, Denver, Co, Jun. 2003.

[30] Claire Tomlin, G.J. Pappas, and S. Sastry. Conflict resolution for air traffic management: a study in multi-agent hybrid systems. *IEEE Trans. on Automatic Control*, 43:509–521, 1998.

[31] J. D. Wolfe, D. F. Chichka, and J. L. Speyer. Decentralized controllers for unmanned aerial vehicle formation flight. In *Proc. AIAA Guidance, Navigation and Control Conference*, San Diego, CA, Jul. 1996.