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A Result on Order Statistics

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Let $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K$ be K i.i.d. $M \times 1$ complex Gaussian random vectors, i.e., $\boldsymbol{h}_i \sim \mathcal{CN}(0, \boldsymbol{I}_M)$. Let

$$j_{(1)} = \underset{j=1,\ldots,K}{\operatorname{argmax}} \|\boldsymbol{h}_j\|.$$

In effect, $\boldsymbol{h}_{j_{(1)}}$ is the vector that has the largest norm. Denote the remaining $K-1$ channel vectors other than $\boldsymbol{h}_{j_{(1)}}$ as $\tilde{\boldsymbol{h}}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1}$. These $\tilde{\boldsymbol{h}}_i$'s are indexed as follows.

$$\tilde{\boldsymbol{h}}_i = \begin{cases} \boldsymbol{h}_i, & i < j_{(1)} \\ \boldsymbol{h}_{i+1}, & i \geq j_{(1)}. \end{cases}$$

We have the following result.

Lemma 1: *Conditioned on $\boldsymbol{h}_{j_{(1)}}$, the vector that has the largest norm, the remaining vectors $\tilde{\boldsymbol{h}}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1}$ are i.i.d.. Furthermore, as K goes to infinity, conditioned on $\boldsymbol{h}_{j_{(1)}}$, each $\tilde{\boldsymbol{h}}_i$ converges to a complex Gaussian vector in distribution.*

Proof: For two $M \times 1$ complex vectors $\boldsymbol{z} = \boldsymbol{z}_r + i\boldsymbol{z}_i$ and $\boldsymbol{z}' = \boldsymbol{z}'_r + i\boldsymbol{z}'_i$, we write $\boldsymbol{z} \preccurlyeq \boldsymbol{z}'$ if every element of $\boldsymbol{z}_r$ and $\boldsymbol{z}_i$ is less than its counterpart in $\boldsymbol{z}'_r$ and $\boldsymbol{z}'_i$, respectively. We have

$$\begin{aligned} &\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1} \,\middle|\, \boldsymbol{h}_{j_{(1)}} = \boldsymbol{z}_{(1)}\right\} \\ = &\lim_{\Delta \boldsymbol{z} \to 0} \Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1} \,\middle|\, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta \boldsymbol{z}\right\} \\ = &\lim_{\Delta \boldsymbol{z} \to 0} \frac{\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta \boldsymbol{z}\right\}}{\Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta \boldsymbol{z}\right\}}, \end{aligned} \tag{1}$$

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where the real and imaginary parts of $\Delta\boldsymbol{z}$'s elements are all positive.

Note that

$$\begin{aligned}
&\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} \\
&= \sum_{i=1}^{K} \Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}, j_{(1)} = i\right\}. \qquad (2)
\end{aligned}$$

Because $\boldsymbol{h}_i$'s are i.i.d., $\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}, j_{(1)} = i\right\}$ are equal for all $1 \leq i \leq K$. Therefore, (2) becomes

$$\begin{aligned}
&\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} \\
&= K \Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}, j_{(1)} = K\right\} \\
&= K \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| \leq \|\boldsymbol{h}_K\|, \text{ for } 1 \leq i \leq K-1, \boldsymbol{z}_{(1)} < \boldsymbol{h}_K \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\}. \qquad (3)
\end{aligned}$$

Let $\pi(\Delta\boldsymbol{z})$ be the product of the real and imaginary parts of all the elements of $\Delta\boldsymbol{z}$, i.e., $\pi(\Delta\boldsymbol{z}) = \prod_{j=1}^{M} (\Delta\boldsymbol{z}_r)_j \prod_{j=1}^{M} (\Delta\boldsymbol{z}_i)_j$. Let $f_{\boldsymbol{h}}(\cdot)$ be the p.d.f. of the complex Gaussian vector $\boldsymbol{h} \in \mathbb{C}^M$. When $\Delta\boldsymbol{z}$ is very small, using Taylor expansion, we have [1]

$$\Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_K \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} = f_{\boldsymbol{h}}(\boldsymbol{z}_{(1)})\pi(\Delta\boldsymbol{z}) + O(|\pi(\Delta\boldsymbol{z})|^2). \qquad (4)$$

Hence the probability for any two independent vectors $\boldsymbol{h}_i$ and $\boldsymbol{h}_j$ to lie within the interval $[\boldsymbol{z}_{(1)}, \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}]$ is

$$\Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_i \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}, \boldsymbol{z}_{(1)} < \boldsymbol{h}_j \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} = |f_{\boldsymbol{h}}(\boldsymbol{z}_{(1)})\pi(\Delta\boldsymbol{z})|^2 + O(|\pi(\Delta\boldsymbol{z})|^3).$$

Consequently, similar to [1, page 11], the probability that at least one vector from $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{K-1}$ falls into the interval $[\boldsymbol{z}_{(1)}, \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}]$ is of $O(|\pi(\Delta\boldsymbol{z})|^2)$. Therefore,

$$\begin{aligned}
&\Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| \leq \|\boldsymbol{h}_K\|, \text{ for } 1 \leq i \leq K-1, \boldsymbol{z}_{(1)} < \boldsymbol{h}_K \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} \qquad (5) \\
&= \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| \leq \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \leq i \leq K-1\right\} \Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_K \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} + O(|\pi(\Delta\boldsymbol{z})|^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_{j_{(1)}} \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} \\
&= K \Pr\left\{\|\boldsymbol{h}_i\| \leq \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \leq i \leq K-1\right\} \Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_K \leq \boldsymbol{z}_{(1)} + \Delta\boldsymbol{z}\right\} + O(|\pi(\Delta\boldsymbol{z})|^2). \qquad (6)
\end{aligned}$$

Using (1) through (6), we have

$$
\begin{aligned}
&\Pr\left\{\tilde{\boldsymbol{h}}_1 \preccurlyeq \boldsymbol{z}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1} \preccurlyeq \boldsymbol{z}_{K-1} \left| \boldsymbol{h}_{j_{(1)}} = \boldsymbol{z}_{(1)} \right.\right\} \\
&= \lim_{\Delta \boldsymbol{z} \to 0} \frac{\Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| \le \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \le i \le K-1\right\} \Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_K \le \boldsymbol{z}_{(1)} + \Delta \boldsymbol{z}\right\} + O(|\pi(\Delta \boldsymbol{z})|^2)}{\Pr\left\{\|\boldsymbol{h}_i\| \le \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \le i \le K-1\right\} \Pr\left\{\boldsymbol{z}_{(1)} < \boldsymbol{h}_K \le \boldsymbol{z}_{(1)} + \Delta \boldsymbol{z}\right\} + O(|\pi(\Delta \boldsymbol{z})|^2)} \\
&= \frac{\Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| \le \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \le i \le K-1\right\}}{\Pr\left\{\|\boldsymbol{h}_i\| \le \|\boldsymbol{z}_{(1)}\|, \text{ for } 1 \le i \le K-1\right\}} \\
&= \frac{\prod_{i=1}^{K-1} \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\|\right\}}{\prod_{i=1}^{K-1} \Pr\left\{\|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\|\right\}} \\
&= \prod_{i=1}^{K-1} \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i \left| \|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\| \right.\right\}. \qquad (7)
\end{aligned}
$$

Hence, $\tilde{\boldsymbol{h}}_i, \ldots, \tilde{\boldsymbol{h}}_{K-1}$ are i.i.d. with CDF $\Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i \left| \|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\| \right.\right\}$.

But,

$$\lim_{K \to +\infty} \|\boldsymbol{z}_{(1)}\| = +\infty.$$

Therefore,

$$
\begin{aligned}
\lim_{K \to +\infty} \Pr\left\{\|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\|\right\} &= 1, \\
\lim_{K \to +\infty} \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i, \|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\|\right\} &= \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i\right\}. \qquad (8)
\end{aligned}
$$

We have

$$\lim_{K \to +\infty} \Pr\left\{\boldsymbol{h}_i \preccurlyeq \boldsymbol{z}_i \left| \|\boldsymbol{h}_i\| < \|\boldsymbol{z}_{(1)}\| \right.\right\} = F_{\boldsymbol{h}}(\boldsymbol{z}_i),$$

where $F_{\boldsymbol{h}}(\cdot)$ is the CDF of the complex Gaussian vector $\boldsymbol{h} \in \mathbb{C}^M$. Hence $\tilde{\boldsymbol{h}}_1, \ldots, \tilde{\boldsymbol{h}}_{K-1}$ are i.i.d. and converge to complex Gaussians in distribution. ∎

REFERENCES

[1] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed. John Wiley & Sons, Inc., 2003.