Adaptive Multivariate Approximation Theory and Applications

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ADAPTIVE MULTIVARIATE APPROXIMATION
THEORY AND APPLICATIONS

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1. CONVERGENCE THEORY REVIEW

ONE VARIABLE
LINEAR - POLYNOMIALS, SPLINES
PIECEWISE POLYNOMIALS WITH VARIABLE KNOTS

MULTIVARIATE
LINEAR - POLYNOMIALS, TENSOR PRODUCT SPLINES
PIECEWISE POLYNOMIALS WITH VARIABLE CELLS

II. ADAPTIVE COMPUTATION

EXAMPLE
ONE VARIABLE ALGORITHMS AND CONVERGENCE
MULTIVARIATE ALGORITHM AND CONVERGENCE

III. APPLICATIONS

MULTIVARIATE QUADRATURE
PARTIAL DIFFERENTIAL EQUATIONS
**Theorem 3**  
$F(x)$ continuous, monotone with $F(0) = 0, F(1) = 1$ then
$\text{DIST}_n(F, \text{step functions or broken lines}) \leq 1/k$

\begin{center}
\begin{tikzpicture}
\draw[->] (-1,0) -- (6,0) node[below] {$x$};
\draw[->] (0,-1) -- (0,6) node[right] {$f(x)$};
\draw (0,0) -- (6,0) -- (6,1) -- (2,1) -- (2,2) -- (0,2);
\draw (0,1) -- (6,1) -- (6,2) -- (2,2) -- (2,3) -- (0,3);
\draw (0,2) -- (6,2) -- (6,3) -- (2,3) -- (2,4) -- (0,4);
\draw (0,3) -- (6,3) -- (6,4) -- (2,4) -- (2,5) -- (0,5);
\draw (0,4) -- (6,4) -- (6,5) -- (2,5) -- (2,6) -- (0,6);
\node at (-1,0.5) {equally spaced partition of range of $f(x)$};
\end{tikzpicture}
\end{center}

**Theorem 4**  
$F(x)$ continuous, of bounded variation IFF
$\text{DIST}_n(F, \text{step functions}) = O(k^{-1})$

Kahane (1961)
THEOREM 5. \( f(x) \) HAS SINGULARITY: \( S = \{ a \_ 1 \} \), IS IN \( \text{Lip}(\alpha) \), \( \alpha > -1/p \) AND IS IN \( C^N \) EXCEPT ON \( S \). IF \( |f^{(N)}(x)| \leq \text{Const.} \ |x-a\_1|^\alpha-N \) THEN

\[
\text{dist}_p(f, \text{SPLINES } S^N_N) = O(K^{-N})
\]

Rice (1969)

THEOREM 6. SET \( \alpha = 1/(N+1/p) \), \( ||f||_\alpha = \left( \int |f^{(N)}|^\alpha \right)^{1/\alpha} \). IF \( f \in C^N \) THEN

\[
\text{dist}_p(f, \text{SPLINES } S^N_N) \leq \text{Const. } K^{-N} \ |f||_\sigma
\]


THEOREM 7. SET

\[
V_{1/N} = \{ f(x) \mid \sum_{i=1}^{\infty} \left( \text{dist}_\alpha(f, P_N) \right)^{1/N} \leq \text{Const. for all } N \}
\]

\[
V_{1/N}^0 = \text{CLOSURE IN } V_{1/N} \text{ OF FUNCTIONS WITH COMPACT SUPPORT}
\]

IF \( f(x) \) IS LOCALLY BOUNDED ON \((-\infty, \infty) \) THEN

(JACKSON-TYPE) \( f \in V_{1/N}^0 \) IMPLIES \( \text{dist}_\alpha(f, S^N_N) = O(K^{-N}) \)

(BRANCHING-TYPE) \( f \in C^0 \) AND \( \text{dist}_\alpha(f, S^N_N) = O(K^{-N}) \) IMPLIES \( \in V_{1/N}^0 \)


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\[ R^M = \text{M-space} \]
\[ D = \text{unit cube in } R^M \]
\[ W^N_p(D) = \text{Sobolev space of functions on } D \]

**THEOREM 8.** Let \( f \in W^N_p(D) \), \( \Omega \) uniform with side \( h \), then
\[ \text{Dist}_{p}(f, S^n) = O(h^N) = O(k^{-N/M}) \]

Morrey (1966), Birman and Solomyak (1967)

**THEOREM 9.** THEOREM 7 CAN BE EXTENDED TO FUNCTIONS OF SEVERAL VARIABLES.

Brudnyi (1974, 1976)
\[ \sqrt{x} \quad \text{on } [0, 1] \]
\[ 7x - 5x^2 \quad \text{on } [0, 0.8] \]
\[ 5.28 - 20(x - 0.8)^{1.2} \quad \text{on } [0.8, 1] \]
\[ \sin(x) \quad \text{on } [0, 2] \]
ADAPTIVE APPROXIMATION ALGORITHM

LOCAL APPROXIMATION OPERATOR $T_I^p$: $f(x) \rightarrow A_I^p(f, x)$

I = SUBINTERVAL OF $[0,1]$ IS ACTIVE IF $\|f - A_I^p(f)\|_I \geq \varepsilon$

U = COLLECTION OF ACTIVE INTERVALS

SET $U = [0,1]$

UNTIL $U$ IS EMPTY DO

CHOOSE $I$, HALVE IT TO OBTAIN $I_{LEFT}$, $I_{RIGHT}$

FIND $T_{I_{LEFT}}^p$, $T_{I_{RIGHT}}^p$

DISCARD $I_{LEFT}$ OR $I_{RIGHT}$ IF $\|f - A_{I_{LEFT}}^p(f)\| < \varepsilon$

OR IF $\|f - A_{I_{RIGHT}}^p(f)\| < \varepsilon$, OTHERWISE RETURN THEM TO $U$.

ASSUME: 1. $f(x) \in C^n$ EXCEPT AT SINGULARITIES $S$, $\|f^{(n)}\| \leq \text{Const.} \ |x-s|^{\alpha-n}$

2. $\|f - A_I^p(f)\| \leq \text{Const.} \ |f^{(n)}|_I^n$ IF $S \cap I$ IS EMPTY.

3. $\|f - A_I^p(f)\| \leq \text{Const.} \ |I|^{\alpha}$ IF $S \cap I$ IS NOT EMPTY.

THEOREM 10. IF $\alpha > \varepsilon$ THEN THE ALGORITHM TERMINATES WITH A GLOBAL APPROXIMATION $A(x)$ SO THAT

$\|f - A\|_\infty = O(K^{-n})$

WHERE $K$ IS THE NUMBER OF PIECES OF $A(x)$.

RICE (1976)
EXISTING ONE DIMENSIONAL ALGORITHMS

1. RICE AND DEBOOR (1968). LEAST SQUARES WITH CUBIC SPLINES. NOT ADAPTIVE, USES NONLINEAR MINIMIZATION SCHEME.

2. ICHIDA, KIYONZ AND YOSHIMOTO (1977). LEAST SQUARES WITH HERMITE CUBICS FOR DISCRETE DATA.

3. RICE (1978). $L_p$-APPROXIMATION BY PIECEWISE POLYNOMIALS OF ORDER $n \leq 13$ AND SMOOTHNESS $< (n + 1)/2$ FOR AN INTERVAL.

4. HULL AND TAYLOR (1979). $L_2$ OR $L_\infty$ - APPROXIMATION BY PIECEWISE POLYNOMIALS OF ORDER $n$ AND SMOOTHNESS $< n-1$ FOR DISCRETE DATA.
MULTIVARIATE ADAPTIVE APPROXIMATION ALGORITHM

U = COLLECTION OF ALLOWABLE CELLS; CLOSED, CONVEX AND NOT THIN.
U CONTAINS ALL TRANSLATIONS AND SCALINGS OF ITS CELLS
E = ERROR BOUND FOR APPROXIMATION ON A CELL.
C \subseteq C_1 \text{ THEN } E(C) \leq E(C_1)

SPLITTING ALGORITHM SO THAT IF C_1 COMES FROM C THEN \( |C_1|/|C| > \beta > 0 \)
D = DOMAIN OF APPROXIMATION IN R^M
S = SMOOTH MANIFOLD OF SINGULARITIES OF DIMENSION L
ASSUME \( f(x) \) SATISFIES

1. \( |f^{(N)}(C)| \leq \text{Const. dist} (S,C)^{a-N} \)
2. IF \( S \cap C \) NOT EMPTY THEN \( \text{dist } p,C(f,F_N) \leq \text{Const. } |C|^{1/p} (\text{diam } C)^a \)
3. \( F(C) = \min F(C), G(C) \)
   \( F(C) = \text{dist} (S,C)^{a-N} (\text{diam } C)^N |C|^{1/p} \)
   \( G(C) = (\text{dist} (S,C) + \text{diam } C)^a |C|^{1/p} \)

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**THEOREM 12.** Suppose \( a > \ln M - \frac{(M-L)}{p} \) then the algorithm produces a partition \( H \) so that
\[
\text{dist}_{\rho_D}(f, \mathcal{H}_n) = O(K^{-N/M})
\]

The condition on \( a \) is necessary for an optimal convergence rate.

DeBoor and Rice (1979)

**INTERPRETATION.** Manifold of singularities can ruin the optimal rate of convergence. Note that if \( f \in L_p(D) \) only requires \( a > -\frac{(M-L)}{p} \). **Theorem 12** is also true for adaptive blending function approximation.
ELEMENTS FOR MULTIVARIATE APPROXIMATION
ELEMENTS = (CELLS + FUNCTIONS)

L. L. SCHUMAKER: Fitting Surfaces to Scattered Data , 1976
R. E. BARNHILL : Representation and Approximation of Surfaces , 1977

TENSOR PRODUCTS

LOCAL APPROXIMATION SCHEMES (FINITE ELEMENTS)

$C^0$ - Continuity: (a) Triangles and Rectangles with
linear, quadratic, cubic elements

$C^1$ - Continuity: Triangles with quartics, quintics,
Clough-Tocher

Blending functions (Coon's Patches)

FACT: It is difficult to devise schemes which give

1. Smooth approximation
2. Accurate approximation
3. Local determination
4. Good "Shape" Representation
TANGENT PIPE ELEMENT

THICK SHELL ELEMENT
WORKING ADAPTIVE ALGORITHMS

MULTIVARIATE APPROXIMATION:
NONE EXISTS AT PRESENT

MULTIVARIATE QUADRATURE

A. THERE ARE VARIOUS WAYS TO EMPLOY 1-VARIABLE METHODS FOR MULTIVARIATE PROBLEMS

B. KAHANER AND WELLS (1979) ANALYZE IN DETAIL THE IMPLEMENTATION OF SUCH ALGORITHMS. THEY EMPHASIZE METHODOLOGIES OF DATA STRUCTURES, MODULAR PROGRAMMING, SYMBOLIC MATHEMATICAL PROCESSING, ETC. TO OVERCOME THE INHERENT COMPLEXITIES OF THESE ALGORITHMS.
PARTIAL DIFFERENTIAL EQUATIONS

REMINDER: OPTIMAL PARTITIONS ARE UNIFORM IN SOME MEASURE RELATED TO THE ERROR.

FACT: PDE PROBLEMS ARE GLOBAL, SO PARTITIONING (ADAPTATION) MUST NOT BE DONE ONE CELL AT A TIME.

FOUR ADAPTIVE APPROACHES

1. INTUITIVE, HUMAN DIRECTED, FINITE ELEMENTS
   Engineering Application

2. WEAK FORMULATION OF PDE, FINITE ELEMENTS
   Babuska and Rheinholdt (1970 - Present)

3. APPROXIMATION THEORY BASED, FINITE ELEMENTS
   deBoor and Dodson (1972), and Pereyra and Sewell (1975), Sewell (1976)

4. MULTIGRID, FINITE DIFFERENCES
   Brandt (1972 - 77)

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OPTIMALITY DERIVED FROM APPROXIMATION THEORY

$$f \, || \, u^{(k)} \, || \, v = \text{CONSTANT FOR OPTIMALITY}$$

$$\alpha = \text{CONSTANT} < 1 \text{ DEPENDS ON NORM AND K}$$

$$\lambda = 1 + \text{POLYNOMIAL DEGREE OF ELEMENTS}$$

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WEAK FORM \[ B(u, v) = f \cdot v \] ALL \( v \in \text{TEST SPACE} \)

\[ B = \text{BILINEAR FORM} \]

ERROR BOUND = \[ \Sigma \int_{\Omega_i} |e_i|^2 \] \( \Omega_i = \text{CELL} \)

\( E_i = \text{ERROR INDICATOR DERIVED LOCALLY FROM } B \)

OPTIMAL PARTITION HAS \[ \int_{\Omega_i} |e_i|^2 = \text{CONSTANT} \]

SHADED \[ \int_{\Omega_i} |e_i|^2 < e_{\text{LOW}} \] ACCEPT SOLUTION

SHADED \[ \int_{\Omega_i} |e_i|^2 > e_{\text{HIGH}} \] REFINES CELLS

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REFERENCES FOR SIAM TALK - 10/30/78

ADAPTIVE MULTIVARIATE APPROXIMATION: THEORY AND APPLICATIONS

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Slide 3:


Slide 4:


Slide 5:


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denBoor, C. W. and Rice, J. R., Least squares cubic spline approximation II - Variable knots, CSD-TR 21 Computer Science, Purdue University (1968). Also IMSL subroutine ICUVKU.


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