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Adaptive Multivariate Approximation Theory and Applications

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ADAPTIVE MULTIVARIATE APPROXIMATION THEORY AND APPLICATIONS

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ADAPTIVE MULTIVARIATE APPROXIMATION

THEORY AND APPLICATIONS

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I. CONVERGENCE THEORY REVIEW

ONE VARIABLE

LINEAR - POLYNOMIALS, SPLINES
PIECEWISE MOLYNOMIALS WITH VARIABLE KNUTS

MULTIVARIATE

LINEAR - POLYNOMIALS, TENSOR PRODUCT SPLINES
PIECEWISE POLYNOMIALS WITH VARIABLE CELLS

II. ADAPTIVE COMPUTATION

EXAMPLE
ONE VARIABLE ALGORITHMS AND CONVERENCE
MULTIVARIATE ALGORITHM AND CONVERGENCE

III. APPLICATIONS

MULTIVARIATE QUADRATURE
PARTIAL DIFFERENTIAL EQUATIONS

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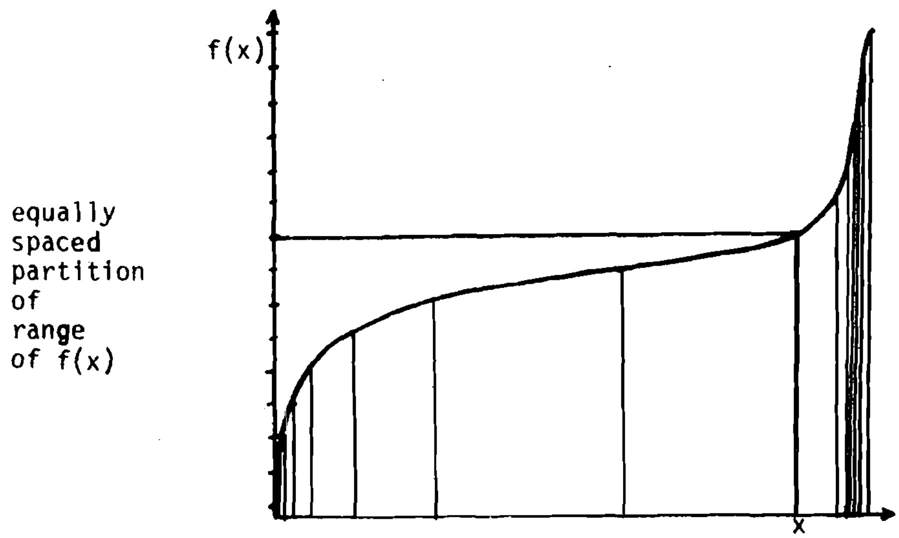
THEOREM 3 F(X) CONTINUOUS, MONOTONE WITH $F(0) = 0$, $F(1) = 1$ THEN

DIST_∞ (F, STEP FUNCTIONS OR BROKEN LINES) $\leq 1/K$



THEOREM 4 F(X) CONTINUOUS, OF BOUNDED VARIATION IFF

DIST_∞ (F, STEP FUNCTIONS) $= o(K^{-1})$

KAHANE (1961)

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THEOREM 5. $f(x)$ HAS SINGULARITIES $S = \{ s_i \}$, IS IN $Lip(\alpha)$, $\alpha > -1/p$ AND IS IN C^N EXCEPT ON S. IF $|f^{(N)}(x)| \leq$ Const. $|\Pi(x-s_i)|^{\alpha-N}$ THEN

$$\mathrm{dist}_p(f, \text{SPLINES } S_\pi^N) = O(K^{-N})$$

Rice (1969)

THEOREM 6. SET $\sigma = 1/(N+1/p)$, $||f||_\sigma = [\int |f^{(N)}|^\sigma]^{1/\sigma}$. IF $f \epsilon C^N$ THEN

$$\mathrm{dist}_p(f, \text{SPLINES } S_\pi^N) \leq \text{Const. } K^{-N} ||f||_\sigma$$

McClure (1970 for p=2), Burchard (1974)

THEOREM 7. SET

$$V_{1/N} = \{ f(x) | \sum_{I \epsilon \pi} (\mathrm{dist}_{\infty,I}(f, P_N))^{1/N} \leq \text{Const. for all } \pi \}$$

$$V_{1/N}^0 = \text{CLOSURE IN } V_{1/N} \text{ OF FUNCTIONS WITH COMPACT SUPPORT}$$

IF $f(x)$ IS LOCALLY BOUNDED ON $(-\infty, \infty)$ THEN

(JACKSON-TYPE) $f \epsilon V_{1/N}^0$ IMPLIES $\mathrm{dist}_\infty(f, S_\pi^N) = O(K^{-N})$

(BERNSTEIN-TYPE) $f \epsilon C^0$ AND $\mathrm{dist}_\infty(f, S_\pi^N) = O(K^{-N})$ IMPLIES $f \epsilon V_{1/N}^0$

Peetre and Bergh (1974), Brudnyi (1974), Burchard and Hale (1975)

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R^M = M-SPACE

D = UNIT CUBE IN R^M

$W_p^N(D)$ = SOBOLEV SPACE OF FUNCTIONS ON D

<u>THEOREM 8</u>. LET $f \in W_p^N(D)$, Π UNIFORM WITH SIDE h, THEN

$$\text{Dist}_p(f, S_\Pi^N) = O(h^N) = O(K^{-N/M})$$

Morrey (1966), Birman and Solomyak (1967)

<u>THEOREM 9</u>. THEOREM 7 CAN BE EXTENDED TO FUNCTIONS OF SEVERAL VARIABLES.

Brudnyi (1974, 1976)

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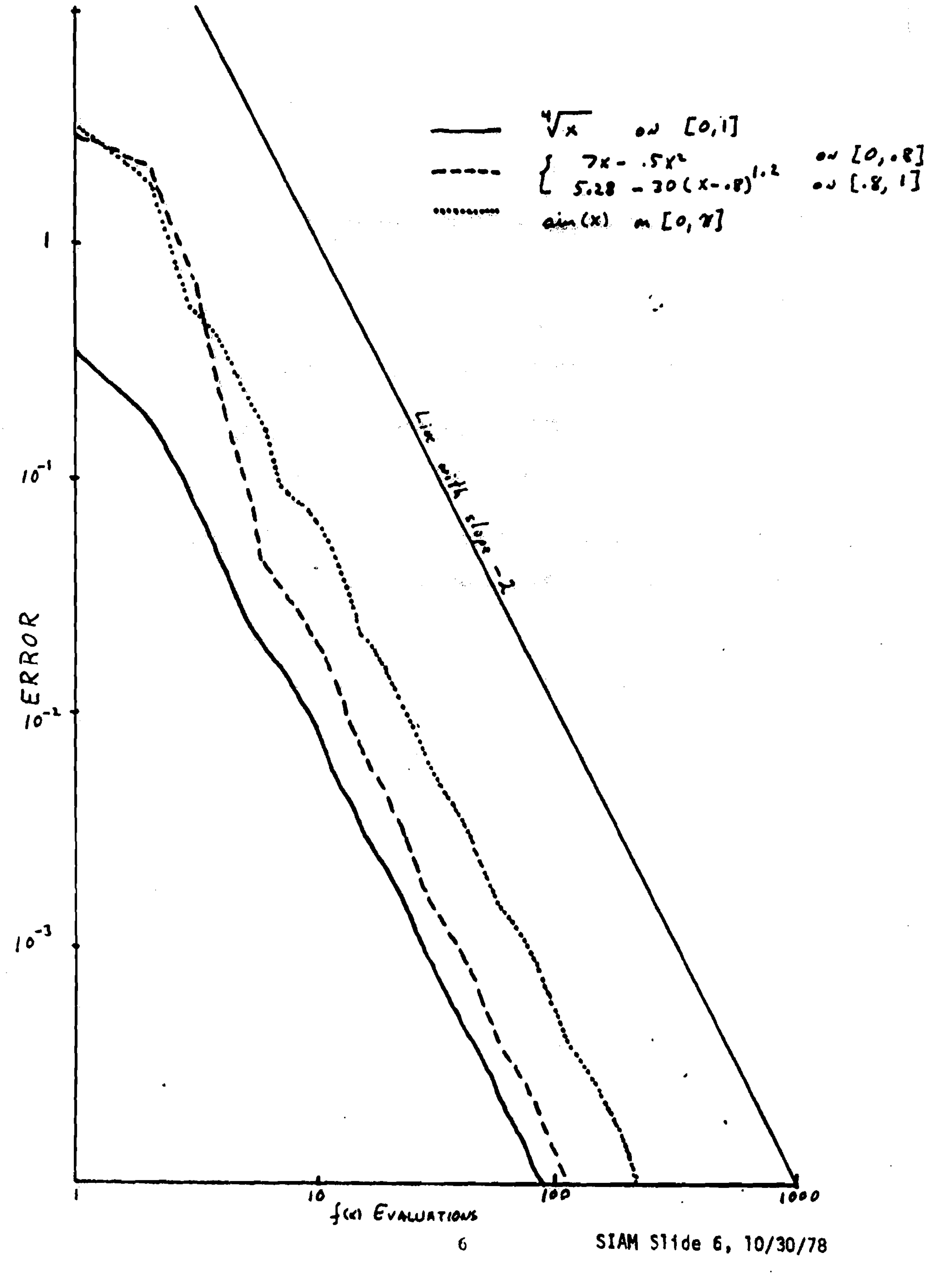

⁴√x on [0,1]
{ 7x - .5x² on [0,.8]
5.28 - 30(x-.8)^1.2 on [.8,1]
sin(x) on [0,π]
Line with slope -2
1
10⁻¹
10⁻²
10⁻³
ERROR
1
10
100
1000
f(x) Evaluations

ADAPTIVE APPROXIMATION ALGORITHM

LOCAL APPROXIMATION OPERATOR T_I: $f(x) \rightarrow A_I(f,x)$

I = SUBINTERVAL OF [0,1] IS ACTIVE IF $||f - A_I(f)||_I \geq \varepsilon$

U = COLLECTION OF ACTIVE INTERVALS

SET U = [0,1]

UNTIL U IS EMPTY DO

CHOOSE I, HALVE IT TO OBTAIN ILEFT, IRIGHT

FIND $T_{ILEFT}f$, $T_{IRIGHT}f$

DISCARD ILEFT OR IRIGHT IF $||f - A_{ILEFT}(f)|| < \varepsilon$

OR IF $||f - A_{IRIGHT}(f)|| < \varepsilon$, OTHERWISE RETURN THEM TO U.

ASSUME:

1. $f(x) \in C^n$ EXCEPT AT SINGULARITIES S, $||f^{(n)}|| \leq \text{Const.} \, |\Pi(x-s_i)|^{\alpha-n}$
2. $||f - A_I(f)|| \leq \text{Const.} \, ||f^{(n)}||_I \, |I|^n$ IF $S \cap I$ IS EMPTY.
3. $||f - A_I(f)|| \leq \text{Const.} \, |I|^\alpha$ IF $S \cap I$ IS NOT EMPTY.

THEOREM 10. IF $\alpha > 0$ THEN THE ALGORITHM TERMINATES WITH A GLOBAL APPROXIMATION A(x) SO THAT

$$||f - A||_\infty = O(K^{-n})$$

WHERE K IS THE NUMBER OF PIECES OF A(x).

RICE (1976)

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EXISTING ONE DIMENSIONAL ALGORITHMS

1. RICE AND DEBOOR (1968). LEAST SQUARES WITH CUBIC SPLINES. NOT ADAPTIVE, USES NONLINEAR MINIMIZATION SCHEME.

2. ICHIDA, KIYONZ AND YOSHIMOTO (1977). LEAST SQUARES WITH HERMITE CUBICS FOR DISCRETE DATA.

3. RICE (1978). L_p -APPROXIMATION BY PIECEWISE POLYNOMIALS OF ORDER $N \leq 13$ AND SMOOTHNESS $< (N + 1)/2$ FOR AN INTERVAL.

4. HULL AND TAYLOR (1979). L_2 OR L_∞ - APPROXIMATION BY PIECEWISE POLYNOMIALS OF ORDER N AND SMOOTHNESS $<$ N-1 FOR DISCRETE DATA.

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MULTIVARIATE ADAPTIVE APPROXIMATION ALGORITHM

U = COLLECTION OF ALLOWABLE CELLS; CLOSED, CONVEX AND NOT THIN.

U CONTAINS ALL TRANSLATES AND SCALINGS OF ITS CELLS

E = ERROR BOUND FOR APPROXIMATION ON A CELL

$C \subseteq C_1$ THEN $E(C) \leq E(C_1)$

SUBDIVISION ALGORITHM SO THAT IF C_1 COMES FROM C THEN $|C_1|/|C| \geq \beta > 0$

D = DOMAIN OF APPROXIMATION IN R^M

S = SMOOTH MANIFOLD OF SINGULARITIES OF DIMENSION L

ASSUME f(x) SATISFIES

1. $|f^{(N)}(C)| \leq$ Const. dist $(S,C)^{\alpha-N}$
2. IF $S \cap C$ NOT EMPTY THEN dist $_{p,C}(f,P_N) \leq$ Const. $|C|^{1/p}$ (diam C)$^{\alpha}$
3. $E(C) = \min F(C), G(C)$

   $F(C) = \text{dist } (S,C)^{\alpha-N} (\text{diam } C)^N \; |C|^{1/p}$

   $G(C) = (\text{dist } (S,C) + \text{diam } C)^{\alpha} \; |C|^{1/p}$

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THEOREM 12. SUPPOSE $\alpha > LN/M - (M-L)/p$ THEN THE ALGORITHM PRODUCES A PARTITION Π SO THAT

$$\text{dist}_{p;D}(f, S^N_\Pi) = O(K^{-N/M})$$

THE CONDITION ON α IS NECESSARY FOR AN OPTIMAL CONVERGENCE RATE.

deBoor and Rice (1979)

INTERPRETATION. MANIFOLDS OF SINGULARITIES CAN RUIN THE OPTIMAL RATE OF CONVERGENCE. NOTE THAT $f \in L_p(D)$ ONLY REQUIRES $\alpha > -(M-L)/p$. THEOREM 12 IS ALSO TRUE FOR ADAPTIVE BLENDING FUNCTION APPROXIMATION.

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ELEMENTS FOR MULTIVARIATE APPROXIMATION

ELEMENTS = (CELLS + FUNCTIONS)

L. L. SCHUMAKER: Fitting Surfaces to Scattered Data , 1976

R. E. BARNHILL : Representation and Approximation of Surfaces , 1977

TENSOR PRODUCTS

LOCAL APPROXIMATION SCHEMES (FINITE ELEMENTS)

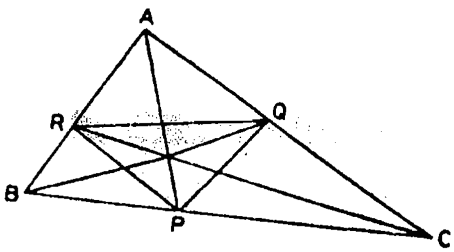
C^0 - Continuity : (a) Triangles and Rectangles with linear, quadratic, cubic elements

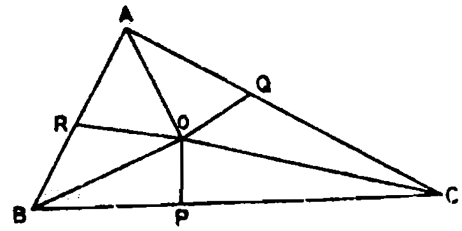
C^1 - Continuity : Triangles with quartics, quintics, Clough-Tocher

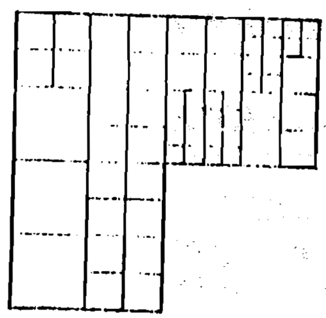
Blending functions (Coon's Patches)

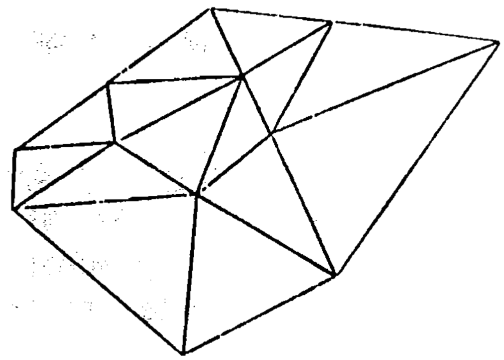
FACT: It is difficult to devise schemes which give

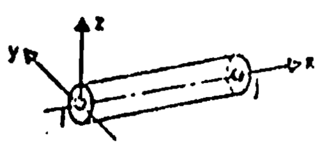
1. Smooth approximation
2. Accurate approximation
3. Local determination
4. Good "Shape" Representation





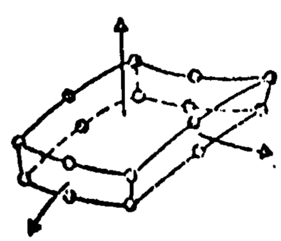






TANGENT

PIPE ELEMENT



THICK SHELL ELEMENT

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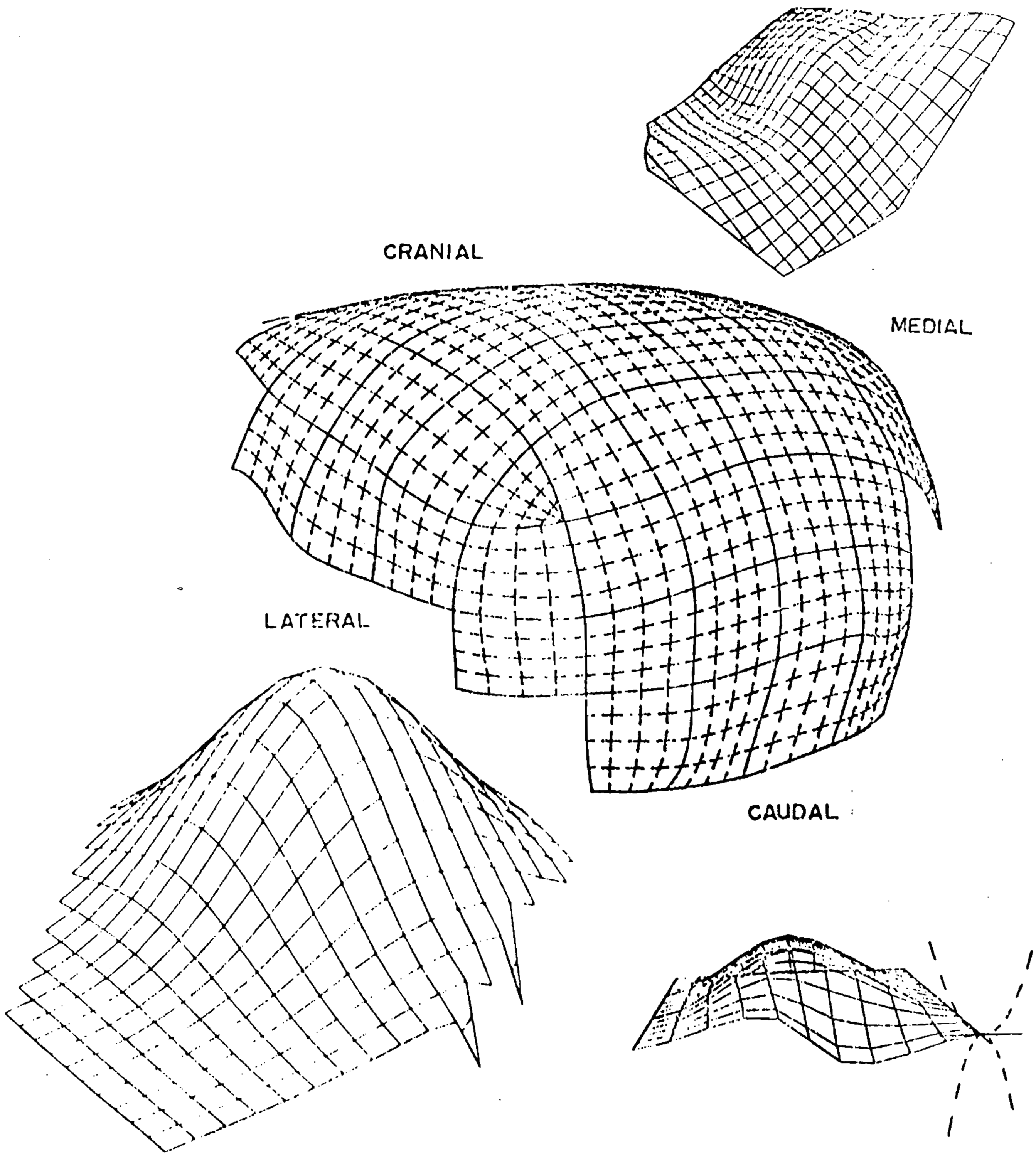

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WORKING ADAPTIVE ALGORITHMS

MULTIVARIATE APPROXIMATION:

NONE EXISTS AT PRESENT

MULTIVARIATE QUADRATURE

A. THERE ARE VARIOUS WAYS TO EMPLOY 1-VARIABLE METHODS FOR MULTIVARIATE PROBLEMS

B. KAHANER AND WELLS (1979) ANALYZE IN DETAIL THE IMPLEMENTATION OF SUCH ALGORITHMS. THEY EMPHASIZE METHODOLOGIES OF DATA STRUCTURES, MODULAR PROGRAMMING, SYMBOLIC MATHEMATICAL PROCESSING, ETC. TO OVERCOME THE INHERENT COMPLEXITIES OF THESE ALGORITHMS.

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PARTIAL DIFFERENTIAL EQUATIONS

REMINDER: OPTIMAL PARTITIONS ARE UNIFORM IN SOME MEASURE RELATED TO THE ERROR.

FACT: PDE PROBLEMS ARE GLOBAL, SO PARTITIONING (ADAPTATION) MUST NOT BE DONE ONE CELL AT A TIME.

FOUR ADAPTIVE APPROACHES

1. INTUITIVE, HUMAN DIRECTED, FINITE ELEMENTS
   Engineering Application
2. WEAK FORMULATION OF PDE, FINITE ELEMENTS
   Babuska and Rheinholdt (1970 - Present)
3. APPROXIMATION THEORY BASED, FINITE ELEMENTS
   deBoor and Dodson (1972), and Pereyra and Sewell (1975), Sewell (1976)
4. MULTIGRID, FINITE DIFFERENCES
   Brandt (1972 - 77)

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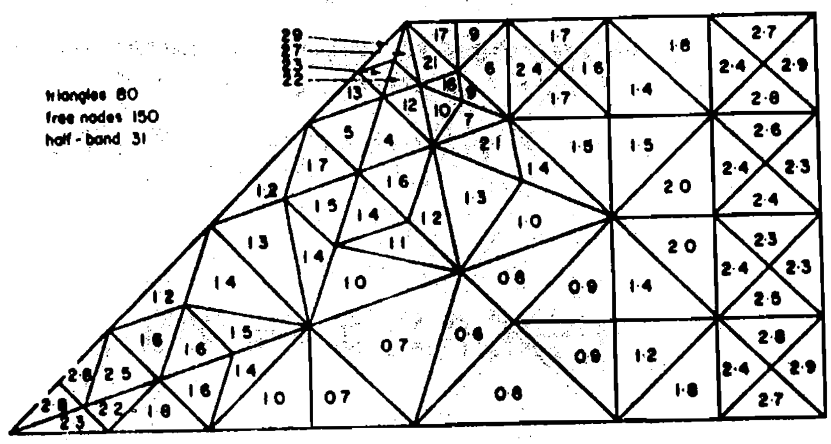
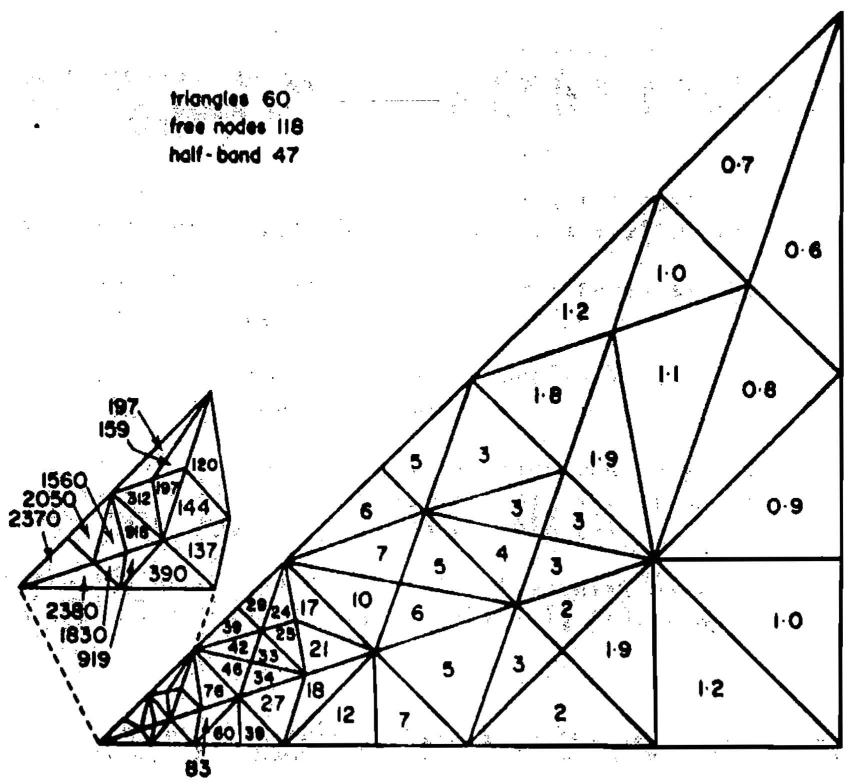
OPTIMALITY DERIVED FROM APPROXIMATION THEORY

$$\int_{\Omega_i} \| u^{(K)} \|^{\sigma} = \text{CONSTANT FOR OPTIMALITY}$$

σ = CONSTANT < 1 DEPENDS ON NORM AND K

K = 1 + POLYNOMIAL DEGREE OF ELEMENTS





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BABUSKA - RHEINBOLDT

WEAK FORM $B(u,v) = f \cdot v$ ALL $v \epsilon$ TEST SPACE

B = BILINEAR FORM

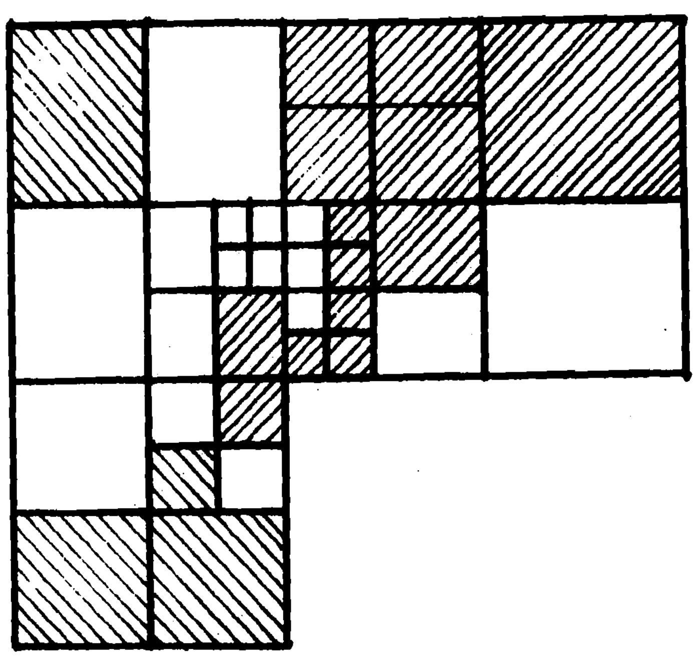
ERROR BOUND = $\sum_i \int_{\Omega_i} (EI)^2$ Ω_i = CELL

EI = ERROR INDICATOR DERIVED LOCALLY FROM B

OPTIMAL PARTITION HAS $\int_{\Omega_i} (EI)^2$ = CONSTANT

 SHADED $\int_{\Omega_i} (EI)^2 \leq \epsilon_{LOW}$ ACCEPT SOLUTION

 SHADED $\int_{\Omega_i} (EI)^2 > \epsilon_{HIGH}$ REFINE CELLS



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REFERENCES FOR SIAM TALK - 10/30/78

ADAPTIVE MULTIVARIATE APPROXIMATION: THEORY AND APPLICATIONS

Slide 2:

Birkhoff, G. and deBoor, C. W., Piecewise polynomial interpolation and approximation, in Approximation of Functions (H. Garabedian, ed.) Elsevier, Amsterdam (1965) pp. 164-190.

Meir, A. and Sharma, A., Degree of approximation by spline interpolation, J. Math. Mech. 15 (1966) pp. 759-767.

deBoor, C. W., On uniform approximation by splines, J. Approx. Thy., 1 (1968) pp. 219-235.

Slide 3:

Kahane, J. P., Teoria Constructiva de Funciones. Cursos y Seminarios de Matematica, No. 5, Univ. de Buenos Aires, (1969).

Slide 4:

Rice, J. R., On the degree of convergence of nonlinear spline approximation, in Approximation Theory with Special Emphasis on Spline Functions (I. J. Schoenberg, ed.), Academic Press (1968) pp. 349-365.

McClure, C. B., Feature selection for the analysis of line patterns, Tech. Rpt. Div. Appl. Math., Brown University (1970).

Burchard, H. G., Splines with optimal knots are better, Applicable Anal., 3 (1974) pp. 309-319.

Peetre, J. and Berg., J., On the spaces $V_p (o < p \leq \infty)$, Bull. Un. Ital., 10 (1974) pp. 632-648.

Brudnyi, Ju. A., Spline approximation and functions of bounded variation, Dokl. Akad. Nauk SSSR, 215 (1974) pp. 511-513. Also Soviet Math. Dokl., 15 (1974) pp. 518-521.

Burchard, H. G. and Hale, D. F., Piecewise polynomial approximation on optimal meshes, J. Approx. Thy., 14 (1975) pp. 128-147.

Slide 5:

Morrey, D. B., Multiple Integrals in the Calculus of Variations, Springer (1966), New York.

Birman, M. S. and Solomyak, M. F., Piecewise polynomial approximation of functions of classes W_p^α, Mat. Sbornik, 73 (1967) pp. 295-317. Also Math. USSR - Sbornik, 2 (1967) pp. 295-317.

Brudnyi, Ju. A., Piecewise polynomial approximation, embedding theorem and rational approximation, in Approximation Theory, Bonn 1976, (R. Schaback and K. Scherer, eds.), Lecture Notes Math. 556 (1976) Springer, Heidelberg, pp. 73-98.

Slide 6

Rice, J. R., A metalgorithm for adaptive quadrature, J. Assoc. Comp. Mach., 22 (1975) pp. 61-82.

Slide 7

Rice, J. R., Adaptive approximation, J. Approx. Thy., 16 (1976) pp. 329-337.

Slide 8

deBoor, C. W. and Rice, J. R., Least squares cubic spline approximation II - Variable knots, CSD-TR 21 Computer Science, Purdue University (1968). Also IMSL subroutine ICUVKU.

Ichida, K., Kiyono, J. and Yoshimoto, F., Curve fitting by a one-pass method with a piecewise cubic polynomial, ACM Trans. Math. Software, 3 (1977) pp. 164-174.

Rice, J. R., Algorithm 525 - ADAPT: Adaptive smooth curve fitting, ACM Trans. Math. Software, 4 (1978) pp. 82-94.

Hull, J. A. and Taylor, G. D., Restricted range adaptive curve fitting, Dept. of Math., Colorado State University, (1978).

Slide 10

deBoor, C. W. and Rice, J. R., An adaptive algorithm for multivariate approximation giving optimal convergence rates. MRC Tech. Rpt. #1773, University of Wisconsin, (1977). 27 pages. Also J. Approx. Thy. to appear.

Slide 11

Schumaker L. L., Fitting surfaces to scattered data, in Approximation Theory II (Lorentz, Chui and Schumaker, eds.) Academic Press, New York, (1976) pp. 203-268.

Barnhill, R. E., Representation and approximation of surfaces, in Mathematical Software III (J. Rice, ed.), Academic Press, New York (1977) pp. 69-120.

Slide 13

Kahaner, D. K. and Wells, M. B., An algorithm for N-dimensional adaptive quadrature using advanced programming techniques, Los Alamos Rpt. LA-UR-16-2310 (1976) 107 pages. Also, shorter version ACM Trans. Math. Software, 5 (1979).

Slide 14

Bab ska, I. and Rheinholdt, W., Error estimates for adaptive finite element computations, SIAM J. Numer. Anal., 15 (1978)pp.736-754.

Zave, Pamela and Rheinholdt, W., Design of an adaptive parallel finite element system. ACM Trans. Math. Software, 5 (1979).

Sewell, G., An adaptive computer program for the solution of Div (p(x,y)Grad u)= F(x,y,n) on a polygonal region, in The Mathematics of Finite Elements and Applications II (J. Whiteman, ed.) Academic Press, New York (1976) pp. 543-553.

Brandt, A., Multi-level adaptive solutions to boundary value problems, Math. Comp., 31 (1977) pp. 333-390.