NONLINEAR APPROXIMATION

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I. **ABSTRACT.** A summary is given of the research accomplishments for the 5-year period of my grant GP32940X for research on nonlinear approximation. Twenty one papers were written with the support of this grant. Three Ph.D. theses were written which led to three additional papers and one widely used computer program. The two principle accomplishments were (1) the first theoretical analysis and convergence proofs for adaptive quadrature and the exploitation of this idea for adaptive approximation and (2) the demonstration that standard finite differences is an inferior method for solving partial differential equations.

II. **RESEARCH PROPOSED FOR GP32940X AT THE BEGINNING OF THE PERIOD 1972-1977.**

There were seven areas of activity proposed:

1. Completion of some work in progress nearly complete.
2. Piecewise polynomial approximation with variable knots.
   
   This was a central topic in the research of the period and considerable progress was made.
3. Degree of convergence for spline approximation with variable knots.
   
   The main theoretical advances here were made by others, but several results were obtained on the degree of convergence of various constructive methods. New results on the degree of convergence for multivariate approximation are in progress.
4. New approach to fast, reliable, general purpose curve fitting.
   
   The discovery, theoretical analysis and perfection of adaptive approximation is one of the principle accomplishments of this research program. The theoretical development occurred first for adaptive quadrature, then for approximation and is now being extended to multivariate approximation.
5. Applications to differential equations.

The development, analysis and comparison of approximation theoretical methods for operator equations has been pursued extensively.


Carl de Boor (who was involved in this project for 1 year) pursued this topic with considerable success.


No progress was made here.

III. SPECIFIC PROJECTS AND PAPERS UNDER THIS RESEARCH PROGRAM.

The work of Carl de Boor in the early part of this period is not included.

The work is subdivided into three categories. Numbers in brackets refer to the publications of John Rice, other references are given in the form of [Author, date].

Piecewise Polynomials, Variable Knots and Adaptive Methods


   This paper [3] gave the first theoretical analysis of adaptive quadrature and a general convergence result. It also analyzes the structure of such algorithms in some detail.


The preceding four papers [2], [7], [8], [17] constitute an in-depth
study of the application of parallel computers to adaptive methods for numerical integration. All aspects are covered from theoretical convergence and speed-up to practical considerations of implementation. The fourth paper [8] presents the first (to my knowledge) formal proof of correctness for a non-trivial algorithm in numerical computation.

This paper [5] presents the first fast methods for computing smooth approximations. The rate of convergence is shown to be optimal for all functions of practical interest.


The preceding two papers [11], [14] present a Fortran algorithm for adaptive approximation and report on extensive experience with it. It behaves as the theory predicts.

This is an invited paper [12] for the East European congress approximation theory. It focuses on what is currently feasible in the computation of approximations.

This paper [4] announces the main results of items 1, 2, 3 and 4 above.
Computational Complexity


These two papers [1], [6] develop a theory of the complexity of computing various kinds of best approximations. There is the surprising result that there is little (theoretical) difference in the difficulty of computing $L_1$, $L_2$ and $L_\infty$ approximations. It is shown that piecewise polynomial approximations may be computed with optimal efficiency.

Application to Differential Equations


These lecture notes present a systematic framework for the analysis and comparison of approximation theoretic methods (e.g. collocation, least squares) for operator equations. It was never submitted for publication anywhere.


The principle result of these two papers [9], [15] is conclusive evidence that the standard finite difference method is inferior for
solving second order linear elliptic partial differential equations. Detailed computational characteristics of various numerical methods are presented.


These three papers [16], [18], [21] develop the theory and application of a new approach to high order finite difference equations discovered by R. E. Lynch (co-author of the above). This new method is good for ordinary differential equations, but its real potential lies in 2 and 3 dimensional problems (with or without rectangular geometry). See The HODIE method: A Brief Introduction with a Summary of Computational Properties, CSD-TR 170, Purdue University, November 18, 1975.

19. Software for Elliptic Partial Differential Equations. This is now a separate project that arose out of items 13, 14, 15, 17 above. It has been underway since early 1976 and is a collaborative effort with people at Harvard, the University of Texas and, perhaps, other institutions.

Research by Graduate Students. Three of my five Ph.D. Students were supported by this program.


Item 24. below is an outgrowth of this thesis and a computer program developed is now widely used through SPSS (Statistical


Results from this thesis have been developed into two separate papers [Houstis, 1976], [Houstis, 1977] now being published. Another set of Houstis' results appeared in the SIAM J. Numer. Anal. by other authors [Baker and Dougalis, 1976].


Item 5. above is an outgrowth of this thesis as well as a second paper now being published [Lemme, 1977].

Other Papers


This paper [10] gives a systematic treatment of the problem of selecting the best algorithm for a class of computations. The parallel with certain parts of approximation theory are developed in some detail.

24. Systematic Search in High-Dimensional Sets, SIAM J. Numer. Anal., This paper [13] shows that random or Monte Carlo search in high dimensional sets is inferior to systematic search. This contradicts widely held opinion. This search is applicable to starting computations in approximation and optimization.
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15. , Evaluation of Numerical Methods for Elliptic Partial
    Differential Equations (with E. Houstis, R. E. Lynch and T. S.

16. , The HODIE method for ordinary differential equations
    (with R. E. Lynch).

17. , Speedup in parallel algorithms for adaptive quadrature
    (with J. Lemme), J. Assoc. Comp. Mach.

18. , High accuracy finite difference approximation to
    solutions of elliptic partial differential equations (with R. E. Lynch),

19. , An adaptive algorithm for multivariate approximation
    with optimal convergence rate (with C. de Boor) to appear.


21. , The HODIE method for elliptic partial differential
    equations (with R. E. Lynch).