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DIRECTION OF ARRIVAL ESTIMATION AND TRACKING OF NARROWBAND AND WIDEBAND SIGNALS

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DIRECTION OF ARRIVAL
ESTIMATION AND TRACKING OF
NARROWBAND AND WIDEBAND SIGNALS

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DIRECTION OF ARRIVAL ESTIMATION AND TRACKING OF
NARROWBAND AND WIDEBAND SIGNALS'

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF TABLES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xi</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>xix</td>
</tr>
<tr>
<td>1. INTRODUCTION AND OVERVIEW</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Direction of Amval (DOA) Estimation</td>
<td>2</td>
</tr>
<tr>
<td>1.1.1 Narrowband signals</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Wideband Signals - DOA Estimation and Tracking</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Multiple Signal Tracking and Data Association</td>
<td>6</td>
</tr>
<tr>
<td>1.4 Layout of the Thesis</td>
<td>10</td>
</tr>
<tr>
<td>2. MAXIMUM LIKELIHOOD ESTIMATION AND CRAMER-RAO BOUNDS OF DIRECTION OF ARRIVAL PARAMETERS OF A LARGE SENSOR ARRAY</td>
<td>11</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>11</td>
</tr>
<tr>
<td>2.1.1 Outline of the Problem</td>
<td>11</td>
</tr>
<tr>
<td>2.1.2 Contribution</td>
<td>12</td>
</tr>
<tr>
<td>2.1.2.1 ML estimation method</td>
<td>12</td>
</tr>
<tr>
<td>2.1.2.2 Cramer-Rao lower bounds</td>
<td>13</td>
</tr>
<tr>
<td>2.1.2.3 Resolution criterion</td>
<td>13</td>
</tr>
<tr>
<td>2.1.3 Chapter overview</td>
<td>13</td>
</tr>
<tr>
<td>2.2 Signal Model and Proposed Estimation Method</td>
<td>14</td>
</tr>
</tbody>
</table>
2.2.1 Signal model ................................................................. 14
2.2.2 Maximum likelihood estimation ........................................... 15
2.2.3 Proposed likelihood expression ............................................... 16
  2.2.3.1 Theorem 2.1 .......................................................... 17
2.2.4 Relation of proposed likelihood with other criterion functions ...... 18
2.2.5 Benefits of making approximation ......................................... 18
2.2.6 Necessary conditions for maximization .................................... 18
2.2.7 Estimation of signal power and noise variance ......................... 19
  2.2.7.1 First order expressions ............................................. 19
  2.2.7.2 Second order expressions .......................................... 19
  2.2.7.3 Closed form expressions for estimators ................................ 20
2.3 Explicit Expressions for Cramer-Rao Bounds on the DOA Estimates 21
  2.3.1 Motivation ................................................................. 21
  2.3.2 General form of the Fisher information matrix ....................... 21
    2.3.2.1 Notation and Symbols ........................................... 22
  2.3.3 Expressions for the CR bounds ........................................ 23
    2.3.3.1 Theorem 2.2 ....................................................... 23
    2.3.3.2 Observations from first order bounds ............................ 23
    2.3.3.3 Theorem 2.3 ....................................................... 24
    2.3.3.4 Observations from second order bounds .......................... 24
2.4 Properties of ML Estimates .................................................. 25
2.5 Performance of Proposed Method ............................................ 25
  2.5.1 Experimental results ................................................... 25
  2.5.2 Comparison with stochastic maximum likelihood .................... 26
  2.5.3 Comparison with signal subspace methods ............................ 28
2.6 Discussion on the Cramer - Rao Bounds ................................... 28
  2.6.1 Comparison of Fisher information matrices and computation of
       CRB ........................................................................... 30
  2.6.2 Effect of parameters on first and second order bounds .............. 32
  2.6.3 Summary of observations .............................................. 33
2.7 DOA Resolution Criterion .................................................... 34
  2.7.1 Description of the criterion ............................................. 34
  2.7.2 Illustration ................................................................ 34
  2.7.3 Usefulness of the proposed criterion .................................... 36
3. A MAXIMUM LIKELIHOOD APPROACH FOR ESTIMATING TRACK PARAMETERS OF NARROWBAND SIGNALS

3.1 Introduction

3.1.1 Motivation for the tracking problem

3.1.2 Existing approaches for angle parameter estimation

3.1.3 Proposed method for track parameter estimation

3.1.4 Crux of the tracking problem

3.1.5 Subspace methods and estimate association

3.1.6 Current approaches

3.1.7 Contribution

3.2 Signal Model and Estimation Method

3.2.1 Signal model

3.2.1.1 Fresnel Approximation

3.2.2 Likelihood Expression

3.2.2.1 Simplification of likelihood expression

3.2.2.2 Upshot of likelihood simplification

3.3. Development of the Proposed Tracking ALgorithm (TAL)

3.3.1 Parameter Estimation and Estimate Association

3.3.2 Maximum Likelihood estimation of DOA and range parameters

3.3.3 The proposed Tracking ALgorithm (TAL)

3.3.4 Features of proposed method

3.4 Performance of Proposed Method

3.4.1 Comparison with Swindlehurst and Kailath's method

3.4.2 Comparison with Sword's algorithm

3.4.3 Crossing target scenario

3.4.3.1 Experiment 1 (two targets, constant velocity, high SNR & small sampling interval)

3.4.3.2 Experiment 2 (two targets moving with constant velocity, low SNR and large sampling interval)

3.4.3.3 Experiment 3 (two targets and an interfering decoy)

3.4.3.4 Experiment 4 (three targets with parabolic trajectories)
3.4.3.5 Experiment 5 (statistical performance) ........................................ 62
3.4.4 Extensions .................................................................................................. 70
3.5 Asymptotic Cramer-Rao Lower Bound (CRB) Expressions for the Track Parameters ........................................................................................................ 70
  3.5.1 Theorem 3.1 .................................................................................................. 72
  3.5.1.1 Corollary 3.1 .......................................................................................... 72
3.6 Information on System Parameters from CR Bounds .................................... 73
  3.6.1 Theorem 3.2 .................................................................................................. 73
  3.6.2 Discussion .................................................................................................... 74
  3.6.3 Effect of angle on the CR bounds ................................................................. 75
  3.6.4 Effect of array spacing ................................................................................. 75
3.7 Conclusions ..................................................................................................... 78

4. A MAXIMUM LIKELIHOOD APPROACH FOR TRACKING MULTIPLE WIDEBAND SIGNALS ................................................................. 79
  4.1 Introduction ..................................................................................................... 79
  4.2 Signal Model ................................................................................................... 79
    4.2.1 Narrowband signal approximation ............................................................ 79
    4.2.2 Wideband Signal model ............................................................................ 80
  4.3 Maximum Likelihood Parameter Estimation ................................................ 81
    4.3.1 Direction of arrival estimation ................................................................ 81
    4.3.2 Estimation of Spectral density matrix ...................................................... 82
  4.4 Wideband Signal Model and Estimate Association ...................................... 83
  4.5 Bayes Classification to get Updated Target Estimates .................................. 84
  4.6 The Proposed Wideband Tracking Algorithm ............................................ 86
  4.7 Performance of Proposed Method .................................................................. 86
    4.7.1 A simulation result .................................................................................. 86
    4.7.2 Discussion .................................................................................................. 87
  4.8 Conclusion ..................................................................................................... 90

5. ESTIMATION OF SINGULARITIES FOR INTERCEPT POINT FORECASTING ........................................................................................................ 91
6.5.3.2 Algorithm CC ................................................................. 121
6.5.4 Evidence combination for linking (Algorithm DD) .................. 123
  6.5.4.1 Structure of the Link Matrix ...................................... 123
  6.5.4.2 Formation of the Ordering Matrix ............................. 123
  6.5.4.3 Forming the link matrix from the ordering matrix ......... 124
6.6 Proposed Tracking Algorithm ............................................. 124
6.7 Performance of Proposed Method ....................................... 125
  6.7.1 Experimental results .................................................... 125
    6.7.1.1 Experiment 1 ..................................................... 125
    6.7.1.2 Experiment 2 ..................................................... 127
    6.7.1.3 Experiment 3 ..................................................... 127
  6.7.2 Discussion ................................................................. 133
    6.7.2.1 Computational complexity of data association ............ 134
6.8 Conclusion .................................................................. 134

7. CONCLUDING REMARKS .................................................... 135

REFERENCES .................................................................... 139
APPENDICES .................................................................. 145
  APPENDIX I: PROOF OF THEOREM 2.1 .............................. 145
  APPENDIX II: PROOF OF COROLLARY 2.1 .......................... 146
  APPENDIX III ................................................................. 147
  APPENDIX IV: PROOF OF THEOREM 2.2 ...................... 149
    IV.1 Determination of Q_{pp} ........................................ 149
    IV.2 Determination of Q_{pv} ....................................... 150
  APPENDIX V: PROOF OF THEOREM 2.3 ......................... 152
    V.1 Determination of Q_{pp} ........................................ 152
    V.2 Determination of Q_{pv} ....................................... 153
  APPENDIX VI: TRANSFER VECTORS AND THEIR DERIVATIVES FOR SIMPLIFYING EQUATIONS (3.31) - (3.34) .................. 155
  APPENDIX VII: PROOF OF THEOREM 3.1 ...................... 156
    VII.1 Determination of Q_{66}^k ................................... 156
    VII.2 Determination of Q_{rr}^k ................................... 158
    VII.3 Determination of Q_{r6}^k ................................... 158
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Root Mean Square Error (RMSE) values of track parameters</td>
<td>68</td>
</tr>
<tr>
<td>4.1</td>
<td>True and estimated DOA values over tracking period</td>
<td>89</td>
</tr>
<tr>
<td>5.1</td>
<td>Mean and RMSE of intercept point forecasts for varying SNR</td>
<td>101</td>
</tr>
<tr>
<td>6.1</td>
<td>Statistics of intercept point between targets 2 and 3</td>
<td>131</td>
</tr>
</tbody>
</table>


## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.1</strong></td>
<td>The Direction of Arrival problem. A planar wave front from the source incident on a uniform linear array of M sensors. The spacing between the sensors in the array is denoted by 'd'. The wave front makes an angle ( \theta ) with the reference sensor. The wave is travels a constant distance ( dsin\theta ) from sensor to sensor as it passes across the array.</td>
</tr>
<tr>
<td><strong>2.1</strong></td>
<td>Variation of DOA estimates with SNR for five sources. DOAs are two degrees apart from each other. True DOAs are indicated in the figure by arrowheads.</td>
</tr>
<tr>
<td><strong>2.2</strong></td>
<td>Comparison of Root Mean Square Error of DOA estimates for the two source case from Proposed ML and Stochastic ML.</td>
</tr>
<tr>
<td><strong>2.3</strong></td>
<td>A comparison between the proposed ML and the Root-music methods for two closely spaced sources. The ML estimates are always close to true values irrespective of the correlation between the sources or SNR whereas Root-music estimates are far away from the true angles and converge to values in between the true DOAs.</td>
</tr>
<tr>
<td><strong>2.4</strong></td>
<td>Comparison of principal diagonal terms of CRB matrices ( CRBQ_f ) and ( CRBQ_e ) for varying correlation. Angular separation between the two sources is five degrees.</td>
</tr>
<tr>
<td><strong>2.5</strong></td>
<td>Comparison of principal diagonal terms of CRB matrices ( CRBQ_f ), ( CRBQ_s ), and ( CRBQ_e ) for varying separation. Correlation coefficient between sources = 0.3.</td>
</tr>
</tbody>
</table>
2.6 Comparison of principal diagonal terms of CRB matrices $\text{CRB}_{Qf}$, $\text{CRB}_{Qs}$, and $\text{CRB}_{Qe}$ for varying SNRs and varying Correlation coefficients. Separation between the two sources $= 40^\circ$..................31

2.7 Plot of eigen values of $Q_s$ matrix for reference condition. The resolution is given by the absolute difference between true DOA1 $= 45^\circ$ and DOA corresponding to one of the vertical lines. (~ 1.8 deg)................................. 35

2.8 Plot of eigen values of $Q_s$ mamx for $M = 32$. The resolution is given by the absolute difference between true DOA1 $= 45^\circ$ and DOA corresponding to one of the vertical lines. (~ 1 deg)................................. 35

2.9 Plot of eigen values of $Q_s$ matrix for $N = 128$. The resolution is given by the absolute difference between true DOA1 $= 45^\circ$ and DOA corresponding to one of the vertical lines. (~ 1.7 deg). Only a very little improvement for four fold increase in $N$................................. 37

2.10 Plot of eigen values of $Q_s$ mamx for correlation $= 0.8$. The resolution is given by the absolute difference between true DOA1 $= 45^\circ$ and DOA corresponding to one of the vertical lines. (~ 3.7 deg). Increase in correlation reduces resolving power ......................................................... 37

2.11 Plot of eigen values of $Q_s$ mamx for low SNR $= 0$ dB. The resolution is given by the absolute difference between true DOA1 $= 45^\circ$ and DOA corresponding to one of the vertical lines. (~ 3.9 deg). Decrease in SNR decreases resolving power ................................................................. 38

2.12 The plot shows the smallest spacing between the DOAs which can be resolved for increasing $M$ for the two source case. It is seen that this strict lower limit decreases with decrease in correlation or increase in $M$. The threshold separation is the minimum spacing value for which the $Q_s$ matrix just becomes positive definite [(2.46) & (2.47)]................................. 38
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Illustration of the two different time scales for data acquisition and track parameter estimation.</td>
</tr>
<tr>
<td>3.2</td>
<td>Angle estimates for two source case. Runs = 100. SNR = 20 dB.</td>
</tr>
<tr>
<td>3.3</td>
<td>Range estimates for two source case. Runs = 100. SNR = 20 dB.</td>
</tr>
<tr>
<td>3.4</td>
<td>Averaged DOA tracks of the two targets for TAL and Sword's Methods.</td>
</tr>
<tr>
<td>3.5</td>
<td>Log RMSE of DOA Estimates for both TAL and Sword's methods.</td>
</tr>
<tr>
<td>3.6</td>
<td>High SNR case (10 dB). TAL tracks the trajectories accurately.</td>
</tr>
<tr>
<td>3.7</td>
<td>SNR = 5 dB. TAL follows the wrong track and estimates are not accurate.</td>
</tr>
<tr>
<td>3.8</td>
<td>Target track plots for reduced SNR (5 dB) but velocity = 150 m/s. Now, TAL is able to provide accurate estimates of the tracks for reduced SNR.</td>
</tr>
<tr>
<td>3.9</td>
<td>Estimated and true angle tracks for TAL and Sword's algorithm. Sword's method fails to track the trajectories whereas accurate estimates are obtained for TAL for low SNR (5 dB) and reduced velocity (150 m/s).</td>
</tr>
<tr>
<td>3.10</td>
<td>Plot of estimated and true range tracks of the TWO targets. TAL is able to provide accurate estimates of the range tracks.</td>
</tr>
<tr>
<td>3.11</td>
<td>The tracks are simulated using equations of projectile motion and describe parabolic trajectories. Inspite of the decoy (dim target) crossing the path, TAL is able to track the trajectory of target 1. This experiment also demonstrates robustness of TAL inspite of lack of explicit information about the number of targets present. Both targets are moving towards the antenna array as indicated by the arrows. The dark dots show the estimated tracks and the light ones show the true tracks.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>3.12</td>
<td>Plot of estimated and true angle tracks of the TWO targets. The DOA estimates are accurate even when the decoy crosses target 1.</td>
</tr>
<tr>
<td>3.13</td>
<td>Plot of estimated and true range tracks of the TWO targets. The range estimates show significant error at the point where decoy crosses target 1, but TAL is able to correct the error and follow the true trajectory of target 2 correctly.</td>
</tr>
<tr>
<td>3.14</td>
<td>The tracks are simulated using equations of projectile motion and describe parabolic trajectories. Instead of the decoy (dim target) crossing the path, we have a third target (10 dB) crossing the track of target 1. Now, instead of two, we estimate three tracks. TAL is able to track the trajectory of all three targets. This experiment also demonstrates ability of TAL to track three crossing targets.</td>
</tr>
<tr>
<td>3.15</td>
<td>Plot of estimated and true angle tracks for the THREE target case.</td>
</tr>
<tr>
<td>3.16</td>
<td>Plot of estimated and true range tracks for the THREE target case. The TAL range estimates are accurate even when target 1 and 3 cross each other.</td>
</tr>
<tr>
<td>3.17</td>
<td>The averaged tracks over 10 trials of the two targets are plotted in X-Y coordinates. The two targets are moving towards each other as indicated by the arrows. Dark dots show estimated tracks and the light ones true tracks.</td>
</tr>
<tr>
<td>3.18</td>
<td>Plot of true and averaged angle tracks demonstrate very accurate DOA estimation by proposed method.</td>
</tr>
<tr>
<td>3.19</td>
<td>Plot of true and averaged range tracks demonstrate fairly accurate range estimation by proposed method even if tracks cross each other.</td>
</tr>
<tr>
<td>3.20</td>
<td>Theoretical Cramer-Rao bounds on DOA versus angle for varying SNR.</td>
</tr>
<tr>
<td>3.21</td>
<td>Theoretical Cramer-Rao bounds on range versus angle for varying SNR.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>3.22</td>
<td>Theoretical Cramer-Rao bounds on DOA versus range for varying d.</td>
</tr>
<tr>
<td>3.23</td>
<td>Theoretical Cramer-Rao bounds on range versus range for varying d.</td>
</tr>
<tr>
<td>4.1</td>
<td>True and estimated trajectories (DOAs) of the two targets plotted against tracking time. The motion of the targets is indicated by arrowheads. The proposed algorithm tracks the two targets correctly even after cross over.</td>
</tr>
<tr>
<td>4.2</td>
<td>True and estimated power spectrum of the two targets at the cross-over point (indicated in 4.1). The center frequency of both spectra is identically equal to 0.3 Hz. The center frequency of the estimated power spectrum given by the proposed method closely approximates the true value.</td>
</tr>
<tr>
<td>5.1</td>
<td>Illustration of the two different time scales for data acquisition and DOA estimation.</td>
</tr>
<tr>
<td>5.2</td>
<td>Figure illustrates idea behind the AES algorithm.</td>
</tr>
<tr>
<td>5.3</td>
<td>Estimated Target/Interceptor tracks along with true/forecasted intercept point for SNR = 0 dB. The target/interceptor tracks are formed from direction of arrival angles estimated at regular time instants. The forecasted interception point shown in figure is the estimated singularity of the nodal cubic fitted to the data comprising of DOA estimates. This forecast is obtained at target intercept ([\left(\hat{T}_r, \hat{\phi}_r, \hat{\alpha}_r\right)]), meaning, using all data up to the time of interception, (T^*). We observe that the forecasted intercept point lies in the neighborhood of the true value for all 10 different realizations of the same experiment. The closeness of the estimates to the true value (evident in figure) demonstrates satisfactory performance of the AES algorithm. However, the intercept angle estimates have a definite bias. We can possibly get better estimates by increasing SNR (Table 5.1) and also rate of data acquisition by the array.</td>
</tr>
</tbody>
</table>
Figure 5.4 Intercept angle forecasts \((\hat{\phi}, t, t = 1, 2, \ldots, T^*)\) over the tracking period. Rate of convergence of estimates is slow and exhibit a definite bias even close to the point of collision ................................................................. 102

5.5 Variation of intercept time \((\hat{T}, t, t = 1, 2, \ldots, T^*)\) forecasts over tracking period. Intercept time forecasts approach closer to the true value as the interceptor nears the target ................................................................. 102

5.6 Plot of RMSE of forecasted intercept angle versus tracking time for SNR = 0 dB. The RMSE of the forecasted angle is observed to be around 0.80 at intercept. Better estimates can be obtained by increasing sampling rate. 104

5.7 Plot of RMSE of forecasted intercept time versus tracking time for SNR = 0 dB. The RMSE of forecasted intercept time is observed to be around 0.56 s at intercept. RMSE is a small value close to the intercept point........... 104

6.1 Illustration of the idea behind Intercept Point Estimation................................. 112

6.2 Estimated tracks of two targets and an unlabeled measurement vector at time \(t_1\) ........................................................................................................ 118

6.3 Straight association between estimates at times \((t_1 - T)\) and \(t_1\) ................. 118

6.4 Cross association between estimates at times \((t_1, -T)\) and \(t_1\) ...................... 118

6.5 Illustration of two targets approaching each other............................................. 120

6.6 Illustration of two targets at cross-over point..................................................... 120

6.7 Illustration of two targets diverging after crossover.......................................... 120

6.8 Illustration of straight linking of two targets approaching each other ............. 122

6.9 Illustration of cross linking of two targets at cross-over.................................. 122
Figure | Description | Page
--- | --- | ---
6.10 | Illustration of straight linking of two targets diverging from each other | 122
6.11 | Plot of true and estimated trajectories for a single trial of the experiment | 126
6.12 | Plot of RLS intercept time estimates. The intercept time forecast converges to the neighborhood of the true value when the targets cross facilitating detection of the cross-over point | 126
6.13 | Plot of RLS intercept angle estimates. The intercept angle forecasts are accurate and also converge to the neighborhood of the true value facilitating detection of cross-over point | 128
6.14 | Four target case, true trajectories illustrating three targets crossing over at the same time | 128
6.15 | Estimated DOA tracks averaged over 20 runs at 0 dB SNR | 129
6.16 | Root mean square error of the estimated angle tracks over the entire tracking period. The error is high at the cross over points since true DOAs are closely spaced and hence estimates are noisy. The error reduces after cross-over with correct association and estimated tracks follow true tracks. The higher error in the initial portion of the track for targets 1 and 4 is because of the directionality property of the uniform linear array, i.e., its inability to give good estimates for targets close to end-fire | 129
6.17 | Plot of RMSE of three intercept time estimates over 20 trials. RMSE decreases rapidly as we approach the true value and has minimum value at and after cross-over. Small value of RMSE after cross-over indicates correct estimate association | 130
6.18 | Plot of RMSE of the corresponding three intercept angle estimates over 20 trials. We observe the same behavior in that RMSE decreases rapidly as we approach the true value and has minimum value at and after cross-over | 130
6.19 Dense target environment, 6 targets and 12 intercept points to be detected.  
6.20 Averaged estimated DOA tracks for the dense target environment case....
ABSTRACT

The research addresses estimation and tracking of direction of arrival (DOA) and associated parameters of narrowband and wideband signals impinging on a uniform linear array of sensors. The signals are modeled as sample functions of a Gaussian stochastic process. Computationally efficient, approximate maximum likelihood (ML) methods are developed for direction of arrival estimation of narrowband signals impinging on a large array of sensors. A new likelihood function is formulated based on a large M (# sensors) Taylor's series approximation of the original likelihood function. Asymptotic expressions for Cramer-Rao lower bounds on the DOA estimates are derived. From the positive definiteness property of the Fisher information matrix, a resolution criterion for closely spaced sources is proposed.

An algorithm for tracking multiple narrowband signal sources in near-field is proposed based on joint estimation of angle and range by the maximum likelihood principle. For sources modeled as wideband signals, a new scheme for tracking direction of arrival is proposed. The wideband signals are modeled as vector auto regressive models so that their spectral densities are characterized by a finite number of parameters. A Bayes classifier is employed for data association.

A new method is proposed for tracking and data association by estimation of singularity of higher order curves fitted to data (DOA estimates). At every tracking time instant, the intercept point forecast information of pairs of signal tracks obtained from existing track data is employed for data association. The forecasted intercept point is recognized as the estimated singularity of a single second order curve fitted to data from every pair. Data association is achieved by detecting cross-over from the knowledge of these forecasts, and by suitable evidence combination of cross-over detection.
1. INTRODUCTION AND OVERVIEW

Array signal processing deals with the processing of signals carried by the propagating wave phenomena. An array of sensors located at different points in space are used to receive signals that are reflected from the targets (objects) of interest--airplanes, helicopters, missiles etc. The intention is to estimate the unknown parameters of the received (noisy) signal field, e.g., its direction, range, speed of propagation etc. It is commonly assumed that objects of interest are point emitters radiating signals with certain energy. The signals may be either narrowband or wideband. Informally, a signal is considered narrowband if the signal bandwidth is small compared to the inverse of the transit time of the wave front across the array, else it is wideband. In this context, there are several problems of immediate interest.

(a) Assuming that sources radiating the signals are stationary, how to appropriately model the narrowband or wideband signals and then estimate the direction of arrival (DOA) and associated parameters like powers of the signal and noise fields. This is commonly referred to as 'direction of arrival estimation'.

(b) Multiple sources, radiating signals are in motion and the objective is to model the signals and track their trajectories as they move. The key aspect is to propose methods for establishing correspondence between measurements (e.g., DOA) of multiple targets over the tracking period so that the respective trajectories can be correctly tracked. This is termed 'data association' or 'estimate association' or the 'correspondence problem'. Information about the signals or their trajectories can be obtained as functions of estimated model parameters or trajectories and are employed for association.

The above two issues are addressed in the thesis and new algorithms are proposed for direction of arrival estimation and for tracking and data association of multiple signals. Since both areas have been well researched, motivation for the proposed approaches is best illustrated with a brief review of existing literature.
1.1 Direction of Arrival (DOA) Estimation

Multiple plane waves either narrowband or wideband are incident from different directions on a uniform linear array (ULA) of sensors distributed in space. The direction of arrival is specified by the radial directions (azimuth and elevation) of the incident plane waves using observations (snapshots) received by the array. The scenario is illustrated in Fig. 1.1 which shows the geometry of a ULA and an emitter radiating plane waves. The main aim is to estimate the DOAs from sensor data.

1.1.1 Narrowband signals

A vast number of techniques have been proposed for estimating the DOA of narrowband signals. The ones which stand out in terms of versatility and performance are the popular MUSIC algorithm [1] and the statistically based Maximum Likelihood (ML) [2] algorithm.

There are several versions of the ML method for DOA estimation [2, 4, 6, 7]. The 'Maximum Likelihood' estimate defined in several papers like [4] are not really ML estimates as defined by Fisher in the statistical literature (i.e., the ML estimate is the value of the DOA parameter $\phi$ which maximizes the joint probability density of all the observations given $Q$). These methods assume the input signals to be deterministic (also termed as 'Conditional ML'[4]) and hence called 'deterministic ML'. In view of the deterministic signal assumption, ND input signal values are unknown and have to be estimated in addition to N signal powers and the noise variance. Here, N is the number of observations or snapshots and D denotes the number of sources. Consequently, the total number of unknowns is of the same order as N, leading to inconsistent estimates with non zero bias even if N tends to infinity [4, 9]. Also, these deterministic methods do not meet the regularity conditions and thus do not achieve the Cramer-Rao Bound (CRB) on the estimate error variance. However, it is shown in [10] that the stochastic and deterministic methods are asymptotically equivalent when both M and N are large.

Alternatively, the input signals are assumed to be sample functions of a Gaussian random process (termed as 'Unconditional ML' [8]) and ML estimates for the DOAs and the covariance matrix are obtained in a separable form [2, 6, 7]. This is referred to as the 'stochastic ML' method. The stochastic likelihood function is regular, and the resultant ML estimate achieves CRB on the estimation error variance. Hence, this stochastic method is asymptotically efficient.
In many real-time applications, large sample results are of little use, due to a limited data collection time, a non-stationary scenario, and/or the effect of damped signal waveforms. To obtain accurate parameter estimates in these cases, it is generally necessary to employ a large number of sensors. Arrays up to 20,000 elements are not uncommon in radar systems [11].

If the non-separable stochastic ML method [2] is employed, then we would have to invert a $M \times M$ matrix where $M$ stands for the number of sensors. If the separable stochastic ML method [7] is used, the signal power estimates may become unstable because of ill-conditioning of the pseudo-inverse of the array manifold matrix, which does not ensure maximum likelihood DOA estimates. It is also a very complicated task to perform the requisite eigen decomposition of such a large sample covariance matrix for obtaining the signal subspace in the eigen vector based methods like MUSIC [1], root-MUSIC [12] etc. Another alternative is to perform an initial beamforming and process the data in beamspace, e.g. beamspace root-MUSIC [5].

However, the requirement of a statistical approach motivates development of a computationally efficient stochastic ML method based on a large sensor approximation of the original log likelihood function. This also forms the basis for development of tracking algorithms. Closed form expressions are derived for the Cramer-Rao bounds of the direction of arrival estimates [33] and a resolution criterion is also proposed. Contributions of Chapters 2 & 3 concentrate essentially on this idea.

Approximate ML methods (not included in thesis) based on signal-to-noise ratio (SNR) are proposed, both for narrowband and broadband signals in [32]. Two criterion functions--one for low SNR and another for high SNR are obtained from suitable approximations in the criterion function. For each of the two cases, a two-step estimation procedure is proposed. In the first step, closed form expressions are obtained for the estimates of the signal power (spectral density in wideband case) and noise power. In step two, these signal and noise power estimates are used to estimate the DOA angles. Comparison of experimental results with the ML method in [2] and MUSIC [1] for narrowband signals shows that this method does better for small SNR and snapshots.
The Direction of Arrival problem. A planar wave front from the source incident on a uniform linear array of $M$ sensors. The spacing between the sensors in the array is denoted by 'd'. The wave front makes an angle $\theta$ with the reference sensor. The wave is travels a constant distance $d \sin \theta$ from sensor to sensor as it passes across the array.
1.2 Wideband Signals - DOA Estimation and Tracking

Most of the research focused on narrowband applications are not applicable in problems such as passive tracking of spread-spectrum (broadband) signals. Bohme has given a tutorial discussion on the subject in [6]. The generalization of the stochastic ML method for narrowband signals to broad band signals is mentioned in his [6] paper but is not elaborated.

A similar method which adopts modeling of signals is proposed by Su and Morf [13] suggested a modal decomposition method based on all sources being modeled as ARMA processes. Each mode of one ARMA source, i.e., the residue vector corresponding to one pole of the ARMA process, spans a one-dimensional signal subspace. Hence, we can process each mode individually using known signal subspace approaches. Nevertheless, ARMA modeling is computationally very expensive and requires long data to estimate the residues accurately.

Kashyap has developed a ML approach for the estimation of DOA and associated parameters for wideband signals [15]. The proposed method evolves from Bohme's point about multi-frequency generalization of narrowband signals. The sources are modeled as sample functions of a Gaussian random process. The spectral density matrix is parameterized by a vector Auto-Regressive (AR) model so that it is characterized by a finite number of parameters. The spectral density is now a known function of these parameters. A maximum likelihood approach for estimating these AR parameters [40] has also been developed. The estimated AR parameters and variance of the driving noises are used in the log likelihood function which is maximized to obtain the DOA estimates. It is also shown that neither true knowledge of the parameters nor order of the model is necessary for estimation of the DOAs. The method outperforms in [15] most of the above methods like ARMA [13], CSS [16] and BASS-ALE [14]. However, the estimates are biased. It can also handle sources with identical spectra.

However, the thrust in the work on wideband signals is not to develop a new DOA estimation method, but to utilize [15] to track the angles of multiple moving targets [35]. The estimated power spectral densities of each of the signals are characterized by a finite set of parameters of a vector AR process. This forms a signature for each of the targets by which recognition and association become possible. The estimated AR parameters along with the DOA form a feature vector of a particular estimated class.
(target). A Bayes classifier [17] is designed to classify the feature into one of the D classes. Non-linear optimization is the biggest drawback of this method. This constitutes Chapter 4 of the thesis.

1.3 Multiple Signal Tracking and Data Association

One of the key functions of a surveillance system is to keep track of all targets of interest within the operating region of its sensors (e.g., phased array radar [11]). A key problem in multiple target tracking is the measurement/target data association problem; i.e., finding which estimate in the measurement set belongs to which target.

There are a number of reasons why the data association problem is hard. Usually predictions are made as to the expected locations of the current set of targets of interest. These predictions are then matched to the actual measurements. At this stage, ambiguities arise. Predictions may not be supported by measurements—and these targets have ceased to exist or are they simply not visible? There may also be unexpected measurements bringing up the question of whether they originate from newly visible targets or whether they are spurious readings from noisy sensors. It can also happen that multiple measurements can match a predicted feature. The question then is which is the correct measurement and what is the origin of the other measurements. On the other hand, if a single measurement matches multiple features, the question to be answered is, to which feature the measurement be assigned. Resolving these ambiguities is the essence of data association and tracking. Research on the tracking problem can be separated into two approaches.

The classical one, the target state model approach, uses a state model representation with the state vector consisting of the position, velocity and possibly the acceleration of the targets. Target tracking here consists of estimating the positions of the targets through the estimation of the state vector at each sampling instant [18]. The estimation process is thus based on a dynamic state model for the moving targets, i.e., to employ Kalman filtering techniques [19]. The salient features and tradeoffs of the state estimation approach to target tracking are discussed in detail in the excellent survey paper by Chang et. al [20].

In this approach, the following type of data association and tracking techniques can be distinguished:
The simplest sub optimal data-association algorithm is the nearest-neighbor algorithm [21] which assumes that each measurement originates from the closest corresponding feature, where closest is usually defined as the Mahalanobis distance. It uses the measurements only at the current time instant for making the decision. If a misassignment occurs, the Kalman filter may not even converge. Better results can be obtained by postponing the decision process in the hope that future measurements can clarify the ambiguities. The earliest is to attempt this was the track splitting filter.

In the track splitting filter [22], if more than one measurement is found significant while tracking a single target, rather than arbitrarily assigning the closest measurement to the track, a tree is formed. The two branches of the tree denote alternative assignments. Decisions are made after additional measurements are gathered. The implicit assumption is that ambiguities at time ‘t’ can be resolved by future measurements. Track trees can become very large very quickly due to simple combinatorial explosion. Hence, a likelihood measure of an assignment is needed so that unlikely hypotheses can be deleted from the track tree.

The joint likelihood method developed in [23] produces disjoint measurement partitions so that a measurement is assigned to a single unique target. The basic idea is to first group measurements into feasible tracks. This set of tracks is not necessarily disjoint, and a subset of disjoint tracks must therefore be selected. However, there are many such legal sets, and so a search is performed to find the best set of disjoint tracks. A joint-likelihood measure is used to quantify which is best. The main disadvantage is that it is a batch-process. The more important point is that track initiation and termination is not explicitly handled by the algorithm. These are addressed in the multiple-hypothesis testing algorithm.

The basic operation of the Multiple Hypothesis Tracking (MHT) algorithm [24] is described now. An iteration begins with the set of current hypotheses from iteration (t - T). Each hypothesis (leaf) consists of a set of active tracks and becomes the parent hypothesis for the current iteration. It also provides an interpretation of all past measurements consisting of a collection of disjoint tracks. Predictions are made to the expected locations of measurements and these predictions are matched to actual measurements using the Mahalanobis distance. Each measurement may belong to (a) a previously known track
(b) start of a new track
(c) a false alarm
(d) if tracks are not assigned to any measurements, delete the track

The resulting enumeration of associations produces a set of children (events) of the parent node, extending the depth of the tree by another level. Associated with each new leaf is a probability which can be computed. In the final step, the tree is pruned to remove unlikely correspondences.

The problem with the above described methods is, they have exponential complexity. Though heuristics can be used to constrain the search space, still a large memory and a lot of computation may be required. Sub-optimal algorithms are to be resorted to, one such algorithm being the joint-probabilistic data association filter.

The class of sub-optimal algorithms require a fixed number of computational resources per cycle. The joint-probabilistic data association algorithm weights all measurements with all tracks. The weights represent the probability that a measurement originated from a particular track. Hence, the term probabilistic data association. The original probabilistic data association filter (PDAF) assumed existence of only a single target which has already been initialized [25]. The JPDAF extended this to a fixed known number of targets [26].

There is a body of literature discussing the above approaches, the latest material is presented in a series of books edited by Bar-Shalom [27-28].

The second tracking approach, uses narrowband signals received from the targets to track their positions with respect to a fixed reference line [29-31]. The sensors in the array are passive in nature in contrast to say, a radar. This approach is commonly referred to as 'direction of arrival tracking'. In view of previous work on DOA estimation (Chapter 2), a natural motivation is to extend the ideas towards passive tracking using a ULA of sensors. Currently, only a few approaches are available for DOA tracking, e.g [29-31], and the work presented in Chapter 3 is similar to the approach adopted in [31].

Sword et. al [29] extend the results of direction of arrival (DOA) estimation by eigenvalue analysis (MUSIC [1]) to derive a recursive procedure based on a matrix quadratic equation. The solution of this matrix quadratic equation is then used to provide
updated target positions. Here, the data association problem is avoided since the sensor array output is comprised of a summation of signals from all targets but requires the signal powers of various targets are distinct.

In [30], Sastry et. al also use the DOA approach and obtain the estimate of the target angles by minimizing the norm of an error matrix function involving the covariance of the sensor outputs. A conjugate gradient search method is used for optimization. The DOAs are estimated at regular time intervals, during which the N snapshots are obtained from the sensors.

In the approach of [31], a dynamic model governing the motion of the targets is used. DOA estimates at each time interval are obtained using the 'stochastic ML' method [2]. These DOA estimates are refined using a Kalman filter wherein angular velocity and acceleration of the targets (components of the state vector in addition to the DOA) are also estimated. A necessary condition for correct data association in [30, 31] requires the targets to have different signal powers.

Based on joint estimation of DOA and range from the new ML method of chapter 2, an algorithm for tracking multiple targets called the Tracking ALgorithm (TAL) is proposed in Chapter 3 [34, 36]. The formulation of this likelihood function is such that a structure is imposed for ML estimation of range and angle parameters of each of D targets. With this, the estimates obtained at every instant of time are naturally ordered. Thus, the data association problem is automatically solved by associating the respective 2-tuples (range & angle estimates) of the targets at any two successive time instants.

Chapter 6 presents an entirely different strategy for the data association problem. Instead of using the signal power information for association as done in Chapter 4, a forecast of the point of interception obtained from estimated DOA trajectories is employed [37]. The forecasted intercept point is recognized as the estimated singularity of a single second order curve of the form fitted to a pair of estimated trajectories. Tracking of direction of arrival of targets is achieved by successive DOA estimation, intercept point forecasting and data association.

The developed algorithms have immediate applications in air-traffic control and in mobile communication systems for localization and tracking of mobiles.
In the algorithms, the number of signals/targets are assumed to be known and constant during the entire tracking period. However, this is not always the case and hence, the number of signals have to be estimated at every tracking instant from sensor data. Model order determination criteria [38, 39] from standard statistical theory could be used to estimate the number of signals.

A mention about the neural network approaches is certainly worthy in view of their recent popularity for solving the data association problems. In [41], a neural network has been designed for multiple target tracking on an optimal assignment basis. The optimal assignment hypothesis is that which maximizes the sum of likelihood functions of measurement-to-track file associations. A tracking system utilizing the neural network in conjunction with a Kalman filter can automatically delete and initiate track files. The solution to the data association problem and therefore to the design of a neural network is based on the minimization of a properly defined energy function. An approximation to the JPDAF using a neural network with a suitably chosen energy function is suggested in [42]. A computational method for solving the data association problem using parallel Boltzmann machines is proposed in [43].

1.4 Layout of the Thesis

In Chapter 2, the ML DOA estimation for narrowband signals is formulated and its performance is compared with other existing methods. It also contains derivation and discussion on Cramer-Rao lower bound (CRB) expressions on the estimation error variance of the angle estimates. In Chapter 3, the Tracking ALgorithm (TAL) for tracking multiple targets based on the maximum likelihood principle is presented. Its performance in comparison with other methods is also given. In addition, this chapter also has derivation and discussion of CR bound expressions for the variance of angle and range estimates. Chapter 4 constitutes a maximum likelihood approach for tracking wideband signals in noise. A new approach is presented in Chapter 5 to forecast the position of collision of the target and pursuing interceptor by tracking the singularity points of the fitted nodal cubic curve. In Chapter 6, a real-time recursive algorithm is developed to effectively track the angle of arrival of multiple moving targets based on intercept point estimation. Conclusions and suggestions for future research are included in the final chapter.
2. MAXIMUM LIKELIHOOD ESTIMATION AND CRAMER-RAO BOUNDS OF DIRECTION OF ARRIVAL PARAMETERS OF A LARGE SENSOR ARRAY

2.1 Introduction

Various high resolution methods have been proposed for estimating direction of arrival and associated parameters for multiple plane wave signals incident on a uniform linear array (ULA) of sensors. These include eigen structure based methods [1, 5, 12, 46] and maximum likelihood techniques [2, 4, 6, 7, 8]. With fractional beamwidth resolution, these methods provide enhanced performance for several important applications like interference rejection in radar and radio communication systems, underwater source localization etc. In practice, it is not very uncommon to see ULAs composed of thousands of elements. A maximum likelihood technique specifically incorporating this information will give the most accurate results. This key idea forms the underlying motivation of this chapter to develop an alternate maximum likelihood approach for estimating the DOA and signal and noise covariance parameters.

2.1.1 Outline of the problem

The problem of estimation of the direction of arrival and other parameters in a multi-source, multi-sensor array context can be reduced to estimating the parameters of the following model represented in a vector matrix format as

\[
y(\tau) = A(\phi)x(\tau) + u(\tau)
\]

where \(y(\tau)\) is the M-dimensional vector of observations received by the M sensors at time \(\tau\), \(\tau = 1, 2, ..., N\). \(N\) is referred to as the number of snapshots. \(x(\tau)\) is the D-dimensional vector of the input signal, \(u(\tau)\) is the complex additive noise vector which is assumed to be Gaussian zero-mean. \(A(\phi)\) is the \(M \times D\) matrix

\[
A(\phi) = [f(\phi_1) f(\phi_2) ... f(\phi_D)]
\]

where \(f(.)\) is a known vector function (called transfer vector between the kth signal and \(y(\tau)\)) and \(\phi_k, k = 1, 2, ..., D\) are the unknown direction of arrivals (DOA) to be estimated.
Most of the data models used, assume the noise to be a temporally and spatially uncorrelated Gaussian random process. However, based on models for source signals, two different methods for maximum likelihood estimation of the arrival angles have been proposed. The first approach treats the input signals, \( x(\cdot) \) as deterministic [4] and hence termed deterministic ML (DML). Thus, there will be \( DN \) input signal values \( x_k(z), k = 1, \ldots, D, \tau = 1, \ldots, N \) as unknowns in addition to the \( D \) unknown DOAs, \( \phi_1, \ldots, \phi_D \). If ML estimation is used, \((DN + D + 1)\) parameters are estimated with \( MN \) observations, i.e., both the number of parameters and the number of observations are of the same order \( N \) [4]. But, the interest really is in estimating \( \phi \) in the model of (2.1) and not in estimating the \( ND \) input variables \( \{x_k(z), k = 1, \ldots, D; t = 1, \ldots, N\} \). Hence, \( x_k(z) \) is treated as a narrowband stochastic process [2, 6, 7] \( x_k(z) = g(z)\exp(j\omega_k\tau) \) where \( \{g(z), \tau = 1, 2\ldots N\} \) are assumed to be independent, identically distributed zero mean, Gaussian random variables. \( g(z) \) and \( \omega_k \) may be correlated. Now, only a finite number of parameters are necessary to model these \( D \) stochastic sequences.

The source signals are thus modeled as sample functions of a Gaussian stochastic process and thus the method is called stochastic ML (SML). In addition, the original log likelihood function can be simplified and ML estimates of angles of arrival are obtained in separable form [7], i.e., angle estimates are obtained by maximizing a function of only the angle parameters. Estimates for \( \Gamma \) and \( \rho \) are given by an explicit formula [2, 6, 7].

### 2.1.2 Contribution
#### 2.1.2.1 ML estimation method

In this chapter, a new separable ML method is developed based on a large sensor (M) Taylor's series approximation of the inverse of the data covariance matrix \( R \) present in the original criterion function of the stochastic ML method. The key point of this manipulation is that the original criterion function can be simplified resulting in a computationally efficient algorithm for DOA estimation. By defining the 'B' matrix, a new likelihood function is formulated which does not require inversion of the \( M \times M \) data covariance matrix.

An interesting feature of this method is that, the relationship of the log likelihood function to other criterion functions for DOA estimation can easily be established. The criterion function reduces to the beamforming criterion [3] either when there is only one signal \( (D = 1) \) or when there is more than one signal, the signal DOAs are apart and \( M \) is
large. With this formulation of the ML DOA estimation problem, we can also handle wideband signals [15].

2.1.2.2 Cramer-Rao lower bounds

Analysis of ML estimates for large N and M can provide useful information. In particular, the ML estimate under certain regularity conditions is asymptotically efficient i.e., estimate error variance equals the Cramer-Rao lower bound (CRB), and an expression for CRB will be useful in evaluating accuracy of the DOA estimates. In [8], asymptotic CRB expressions under the stochastic signal model have been derived for large N and it is also shown that it is lower bounded by the CRB under the deterministic signal model for large N and M. In other words, the stochastic and deterministic ML methods are asymptotically equivalent for large M and N. This equivalence is verified for large M and arbitrary N in [10].

Simplified analytical expressions for Cramer-Rao bounds for the covariance matrix of all the unknown Direction Of Arrival (DOA) angles which behave like $K/NM^3$ for large M and finite N are derived [33]. In contrast to [8, 10], using these approximations, lower bound on variance of the estimate can be explicitly represented as a function of the number of snapshots (N), the number of sensors in the uniform linear array (M), signal-to-noise ratio (SNR), the correlation between the signal sources ($\gamma$), their separation ($\phi_i$) i.e., $\text{CRB} = f(M, N, \text{SNR}, \phi_i, \gamma)$. The approach adopted for deriving closed form expressions is different from that of [8, 10]. The Taylor's series approximations on $R^{-1}$ employed for deriving the ML algorithm are utilized whereas, a first order approximation of the likelihood function for large N is used in [8, 10].

2.1.2.3 Resolution criterion

Moreover, a consequence of the explicit CRB expressions is a resolution criterion for the DOAs. It is shown that, just by using the condition of positive definiteness of the Fisher Information Matrix, a tight lower limit on the minimum angle of resolution can be obtained for a given amount of correlation between the signal sources.

2.1.3 Chapter overview

In the next section, the ML method for narrowband signals is formulated. In Section 2.3, expressions for the CR bound on the variance of DOA estimates is derived.
In Section 2.4, statistical properties of the ML DOA estimates are discussed. In Section
2.5, performance of the estimation method in comparison with other high resolution
methods is given. Sections 2.6 and 2.7 contain discussion on the derived bounds and the
resolution criterion respectively. Conclusions are included in the final section.

2.2 Signal Model and Proposed Estimation Method

2.2.1 Signal model

Consider D narrowband signals impinging on a passive sensor array of M sensors
\((D < M)\). The input signals, \(x(.)\) represented as complex envelopes are described by
\[ x_{k}(\tau) = \hat{\theta}_{k}(\tau)e^{j\omega_{k}\tau} \quad k = 1, 2, ..., D \text{ and } \tau = 1, 2, ..., N. \] (2.3)

\(N\) is referred to as the number of snapshots.

(i) \(\hat{\theta}_{k}(\tau)\) is an independently identically distributed (i.i.d) Gaussian sequence \((0, \Gamma_{kk})\).

(ii) Two input signals \(x_{k}(\tau)\) and \(x_{\ell}(\tau)\) could be correlated,
\[ E[\hat{\theta}_{k}^{*}(\tau_{1})\hat{\theta}_{\ell}^{*}(\tau_{2})] = 0 \text{ if } \tau_{1} \neq \tau_{2} \]
since it is assumed that the signals are not correlated across time. The notation \(^*\) is used
to denote the complex conjugate of the quantity in question.
\[ E[\hat{\theta}_{k}(\tau_{1})\hat{\theta}_{\ell}^{*}(\tau_{2})] = \Gamma_{k\ell} \text{ if } \tau_{1} = \tau_{2} \]
since two signals observed at the same instant of time may be correlated.

The quantity \(E[x(\tau)x^{*}(\tau)] = \Gamma\) is defined as the \((D \times D)\) covariance matrix of the signals.

If \(\Gamma_{k\ell} = 0\), it is assumed that \(\omega_{k} = \omega_{\ell}\).

The additive noise sequence \(\{u(\tau) = 1, 2, ..., N\}\) is assumed to be i.i.d Gaussian \((0, \sigma^{2})\).
The variance of the additive noise is represented by \(\sigma^{2}\).

Estimating the track parameters, in a multi-target, multi-sensor array context can
be reduced to estimating the parameters of the following model
\[ y_{m}(\tau) = \sum_{k=1}^{D} f_{km}x_{k}(\tau) + u_{m}(\tau) \quad m = 1, 2, ..., M \text{ and } \tau = 1, 2, ..., N \] (2.4)

Let the vector function \(f(.)\) for the narrowband signals be
\[ f_{k} = \text{col}[1 e^{j\psi_{ka1}} e^{j\psi_{ka2}} \ldots e^{j\psi_{kam}}] \quad k = 1, 2, ..., D \] (2.5)

where \(\psi_{km}\) is the phase delay of the narrowband signal from the kth source at the mth
sensor. As the signal considered is narrowband, the direction of arrival is a function of
the phase delay of the signal wavefront as it passes across the array.

The phase delay \(\psi_{km}\) of the kth signal impinging on a uniform linear array can be
expressed as a function of the arrival angle in the form
\[ \psi_{km} = (m - 1)(\pi \sin \theta_{k}) \] (2.6)
where 'd' is the inter element array spacing. The choice $d = \lambda / 2$ prevents spatial aliasing where $\lambda$ is the carrier wavelength.

Let $\phi_k = \pi \sin \theta_k$, $k = 1, 2, ..., D$.  

(2.7)

Substituting (2.7) in (2.6), the phase delay is given by

$$\psi_{km} = (m-1)\phi_k, \quad k = 1, 2, ..., D$$  

(2.8)

Substituting (2.8) in (2.5), the vectors $f(.)$ are given by

$$f_k = \text{col.}[1, e^{j2\pi k}, ..., e^{j(D-1)k}], \quad k = 1, 2, ..., D$$  

(2.9)

$f_k$ being a function of $\phi_k$, is represented as $f(\phi_k)$ for $k = 1, 2, ..., D$. Substituting (2.9) in (2.4), the signal model of (2.1) is obtained and $A(\phi)$ is the $M \times D$ array manifold matrix (2.2) comprising of $D$ transfer vectors and $\phi_k$, $k = 1, 2, ..., D$ are unknown DOA parameters to be estimated.

The covariance matrix of the observation vector $y(t)$ is given by,

$$R = E[y(t)y^*(t)] = \rho I + A(\phi)\Gamma A^*(\phi)$$  

(2.10)

R is an $M \times M$ matrix where $M$ denotes the number of sensors.

2.2.2 Maximum likelihood estimation

The sequence $Y = \{y(\tau), \tau = 1, ..., N\}$ is independent identically distributed Gaussian with zero mean and covariance $R$, its joint probability density given $\phi, \Gamma, \rho$ can be written as follows:

$$p(Y|\phi, \Gamma, \rho) = \prod_{t=1}^{N} 2\pi^{-M/2}(\det R)^{-1/2} \exp \left\{ -\frac{1}{2} y^*(t)R^{-1}y(t) \right\}$$

$$= 2\pi^{-MN/2}(\det R)^{-N/2} \exp \left\{ -\frac{1}{2} \text{trace} \left[ R^{-1} \sum_{t=1}^{N} y(t)y^*(t) \right] \right\}$$  

(2.11)

Thus, the density possesses a sufficient statistic $\hat{R}$ given by

$$\hat{R} = \frac{1}{N} \sum_{t=1}^{N} y(t)y^*(t)$$  

(2.12)

Consequently, the log likelihood can be simplified as follows:

$$L(\phi, \Gamma, \rho) = \ln p(Y|\phi, \Gamma, \rho) = -\left[ \frac{MN \ln 2\pi}{2} + \frac{N}{2} J(\phi, \Gamma, \rho) \right]$$  

(2.13)

where $J(\phi, \Gamma, \rho) = \ln[\det R(\phi, \Gamma, \rho)] + \text{trace}[R^{-1}(\phi, \Gamma, \rho)\hat{R}]$  

(2.14)

Maximum likelihood estimates of parameter set $(\phi, \Gamma, \rho)$ in the new method are obtained by minimizing (2.14) subject to conditions that $\rho > 0$ and the signal power matrix, $\Gamma$ is positive definite.
2.2.3 Proposed likelihood expression

Direct minimization of (2.14) is complicated and involves a search over \([D(D+3)/2 + 1]\) unknowns. A simpler solution which is separable, for estimation of the D DOA parameters can be obtained by making certain assumptions. The proposed method is similar to [6, 7], that is, ML estimates of the D DOAs are obtained by maximizing a function over only D angles. Also, the ML estimates of the unknown signal covariance matrix, \(\Gamma\) and the additive noise variance, \(p\) are given by explicit closed form expressions. The separable solution in [6, 7] assumes that \(R\) and \(\hat{R}\) are positive definite, \(p > 0\) and no conditions are imposed on \(\Gamma\). \(\hat{R}\) is positive definite with probability one iff \(N \geq M\) (# of observations \(\geq\) # of sensors).

However, in this approach, a new likelihood function applicable to a large sensor array with sufficient number of sensors, subject to assumptions \(p > 0\) and \(\Gamma\) positive definite is derived from (2.13). Maximizing this log likelihood function, a separable solution for the DOA estimates can be obtained. The positive definiteness assumption on \(\Gamma\) permits the inverse of \(R\) (2.10) to be expressed in terms of a mamx of dimension \(DxD\). Using the mamx inversion lemma [44]

\[
R^{-1} = R^{-1}I - A(A^*A + \rho \Gamma^{-1})^{-1}A^*\rho^{-1}
\] (2.15)

It is to be noted that \((A^*A)\) = \(f_k^*f_k = M\) and \((A^*A)_{k\ell} = f_k^*f_{\ell} = \frac{(1 - e^{-j\Omega_{k\ell}})}{(1 - e^{-j\phi_k})}\) where \(\phi_d = \phi_\ell - \phi_k\) is the source separation. \(f_k^*f_{\ell}\) is finite for finite \(\phi_d\) even if \(M\) tends to infinity. The key to the new likelihood function is to express the term \((A^*A + \rho \Gamma^{-1})^{-1}\) in (2.15) as the summation of a Taylor's series expansion in terms of \(M\), the number of sensors. For simplification, the 'B' matrix is defined such that

\[
\left(A^*A + \rho \Gamma^{-1}\right) = (MI + B)
\] (2.16)

where the elements of the B mamx are given by

\[
B_{k\ell} = (\Gamma^{-1})_{k\ell} \rho + (1 - \delta_{k\ell})f_k^*f_{\ell}, \quad k, \ell = 1, 2, \ldots, D
\]
\[
\delta_{k\ell} = 1, \quad k = \ell
\]
\[
\delta_{k\ell} = 0, \quad \text{else}
\]

All elements of B remain finite as \(M \to \infty\). Taking inverse on both sides of (2.16),

\[
\left(A^*A + \rho \Gamma^{-1}\right)^{-1} = (MI + B)^{-1} = \left(\frac{1}{M}\right)\left[I - \frac{B}{M} + \left(\frac{B}{M}\right)^2\right] + O\left(\frac{1}{M^3}\right)
\] (2.18)
using a second order approximation of the Taylor’s series expansion. Substituting in (2.15), we have
\[ \mathbf{R}^{-1} = \left( \frac{1}{\rho} \right) \left[ I - \left( \frac{\mathbf{AA}^*}{\mathbf{M}} \right) + \left( \frac{\mathbf{ABA}^*}{\mathbf{M}^2} \right) \right] + \mathcal{O} \left( \frac{1}{\mathbf{M}^3} \right) \quad (2.19) \]

This is the approximate expression which will be employed in deriving the expression for the new log likelihood function. From (2.19), a first order expression for \( \mathbf{R}^{-1} \) can be written as
\[ \mathbf{R}^{-1} = \left( \frac{1}{\rho} \right) \left[ I - \left( \frac{\mathbf{AA}^*}{\mathbf{M}} \right) \right] + \mathcal{O} \left( \frac{1}{\mathbf{M}^2} \right) \quad (2.20) \]

The proposed criterion function for the estimation of DOA and associated signal power and noise variance parameters is given in Theorem 2.1.

2.2.3.1 Theorem 2.1

The log likelihood of the observations \( \{\mathbf{y}(t), t = 1, \ldots, N\} \) from \( D \) sources incident on the sensor array having \( M \) sensors is given by
\[
\ln p(Y \mid \phi, \Gamma, \rho) = \frac{N}{2\rho} \sum_{k=1}^{D} f_k^* \hat{\mathbf{R}} f_k - \frac{N}{2\rho M^2} \sum_{k=1}^{D} \sum_{l=1}^{D} f_k^* \hat{\mathbf{R}} f_l B_{kl} \\
- \frac{N}{2} \text{tr} \left( \frac{\mathbf{B}}{\mathbf{M}} \right) + \frac{N}{4} \text{tr} \left( \frac{\mathbf{B}}{\mathbf{M}} \right)^2 \\
+ \text{terms involving only } \Gamma, \rho \quad + \text{terms not involving } \phi, \Gamma, \rho \quad + \mathcal{O} \left( \frac{1}{\mathbf{M}^3} \right). \quad (2.21)
\]

The proof is in Appendix I.

The above form of the log likelihood emphasizes only terms which are functions of the angle parameters and thus can be maximized to obtain their ML estimates.

Corollary 2.1

For the single source case (\( D = 1 \)), the log likelihood stated in Theorem 2.1 simplifies exactly as follows:
\[
\ln p(Y \mid \phi, \Gamma, \rho) = \frac{N}{2\rho} \left( \frac{1}{\mathbf{M} + \rho / \Gamma} \right) f_k^* \hat{\mathbf{R}} f_k \\
- \frac{N}{2} \left[ M \ln 2\pi + (M - 1) \ln \rho + \ln(M\Gamma + \rho) + \frac{1}{\rho} \text{tr}(\hat{\mathbf{R}}) \right] \quad (2.22)
\]

The proof is in Appendix II.
2.2.4 Relation of proposed likelihood with other criterion functions

In presence a single source \( (D = 1) \), the B matrix in (2.17) is a scalar given by \( (p / T) \). Cross-terms like \( f_k^* \hat{f}_e \) which are present in the second term of (2.21) are absent. It is easy to see that, the corresponding criterion function for finding \( \phi \) with only one dominant term (first term) reduces to the Beamforming criterion. This establishes relationship of this criterion function to others employed for DOA estimation.

However, when \( D > 1 \), the second term in the likelihood function of (2.21) contains cross terms like \( f_k^* \hat{f}_e \) whose coefficient \( B_{k\ell} \) has \( f_k^* \hat{f}_e \) in it. But, \( f_k^* \hat{f}_e \) has \( \sin \left( \frac{\Phi_k}{2} \right) \) term in its denominator. Thus, if \( \left| \text{Msin}(\Phi_{k\ell}) \right| >> 1 \), (i.e., the DOA of any two sources are sufficiently far apart for the given value of M), then the effect of the second term in (2.21) is negligible and the ML estimates are close to that given by beamforming. But, if the above conditions are not satisfied, the ML estimates are different from the beamforming estimates. Thus, for closely spaced sources, the proposed ML method gives better results than the beamforming technique.

2.2.5 Benefits of making approximation

In contrast to (2.14), maximization of (2.21) does not involve inversion of the \( M \times M \) data covariance matrix, \( R \). Inverting \( R \) for large number of sensors is computationally expensive. Instead, computation of the B matrix (2.17) is required. This involves inversion of \( \Gamma \) which is only \( D \times D \) where \( D \) is the number of sources. If sources are uncorrelated, inversion of \( \Gamma \) can be trivially accomplished. Thus, the proposed criterion function is computationally more efficient when compared to (2.14).

2.2.6 Necessary conditions for maximization

For convenience of notation, let \( \beta = \text{col.}(\theta_1, \ldots, \theta_D) \) represent the true arrival angle parameter vector. Let \( \nabla_k \) denote the derivative \( (\partial / \partial \beta_k) \). From this definition, \[ \nabla_k \ln \det R = \text{tr} \left[ R^{-1} \nabla_k R \right] \text{ and } \nabla_k R^{-1} = -R^{-1} \nabla_k RR^{-1} \].

In order to determine necessary conditions for maximization, the original log likelihood given in (2.13) is considered. Its first derivative can be written as

\[
\nabla_k \ln p(Y|\beta) = - (N I 2) \text{tr} \left[ R^{-1} (\nabla_k R) \left( I - R^{-1} \hat{R} \right) \right] \tag{2.24}
\]

The second derivative is then given by,

\[
\nabla_k^2 \ln p(Y|\beta) = -(N I 2) \text{tr} \left[ \nabla_k \left( R^{-1} \nabla_k R \right) \left( I - R^{-1} \hat{R} \right) - R^{-1} \nabla_k RR \nabla_k R^{-1} \hat{R} \right] \tag{2.25}
\]

\( k, \ell = 1, 2, \ldots, D \)
It is easy to see that, $\hat{R} = R$ in (2.24) gives the necessary condition for existence of a maximizer. For sufficiency, we need $\nabla^2 \ln p(Y|\beta) < 0$. Substituting $\hat{R} = R$ in (2.25), the condition is given to be

$$\text{tr} \left[ \nabla^2 R \right] < 0 \quad (2.26)$$

Substituting for $R$ and $R^{-1}$ from (2.10) and (2.19) respectively and manipulating (2.26),

$$\text{tr} \left[ \nabla^2 (ABA^* - A(M\beta)A^*) \right] < 0 \quad (2.27)$$

A condition which satisfies the above inequality is $|\lambda_k^B| < M \quad k = 1, 2, ..., D \quad (2.28)$

where $\lambda_k^B$ is the $k$th eigen value of the $B$ matrix. This is also the condition required for the Taylor’s series expansion of (2.18) to be valid and hence is necessary for proving Theorem 2.1. In general, (2.28) is satisfied in presence of sufficient number of sensors.

**2.2.7 Estimation of signal power and noise variance**

Closed form expressions for $\hat{\gamma}$ and $\hat{\rho}$ as functions of data are desirable, thereby facilitating a separable solution wherein DOA estimates can be obtained by maximizing a function involving only the DOA parameters. This results in a substantial reduction in size of the parameter set for numerical optimization [7]. Estimation of $\Gamma$ and $\rho$ from log likelihood functions formed using both first and second order approximations on $R^{-1}$ is now taken up.

**2.2.7.1 First order expressions**

From $R^{-1}$ given by (2.20), the criterion function in (2.14) can be written as

$$J'(\phi, \Gamma, \rho) = (M - D) \ln p + D \ln M + \ln \det \Gamma + \text{tr} \left( \frac{B}{M} \right)$$

$$+ \left( \frac{1}{\rho} \right) \text{tr}(\hat{R}) - \left( \frac{1}{\rho \Lambda} \right) \text{tr}(A\Lambda^*\hat{R}) + O \left( \frac{1}{M^2} \right) \quad (2.29)$$

Solve for $\hat{\gamma}$ & $\hat{\rho}$ by setting $(\partial J'/\partial \Gamma) = 0$ and $(\partial J'/\partial \rho) = 0$ respectively in (2.29).

Omitting details, $\hat{\rho}_f = \left( \frac{1}{M - D} \right) \text{tr}(\hat{R}) - \left( \frac{1}{M(M - D)} \right) \text{tr}(A\Lambda^*\hat{R}) \quad (2.30)$

and $\hat{\gamma}_f = \left( \frac{\hat{\rho}_f}{M} \right) \quad (2.31)$

**2.2.7.2 Second order expressions**

The log likelihood in (2.21) written directly as the summation of (1.4) and (1.5) is

$$J''(\phi, \Gamma, \rho) = (M - D) \ln p + D \ln M + \ln \det \Gamma + \text{tr} \left( \frac{B}{M} \right) - \frac{1}{2} \text{tr} \left( \frac{B}{M} \right)^2$$
\[ \frac{1}{\rho} \text{tr}(\hat{\mathbf{R}}) - \frac{1}{\rho M} \text{tr}(\mathbf{A}^\top \hat{\mathbf{R}}) + \frac{1}{\rho M^2} \text{tr}(\mathbf{A} \mathbf{B}^\top \hat{\mathbf{R}}) + O\left(\frac{1}{M^3}\right) \]  

Setting \((\partial J'' / \partial \Gamma^{-1}) = 0\) in (2.32) suggests a simpler way of solving for \(\hat{\mathbf{r}}\). The following non-linear system of equations is obtained

\[ \hat{\mathbf{r}}_s^2 - C_1 \hat{\mathbf{r}}_s + (\hat{\mathbf{s}} / M)^2 \mathbf{I}_D = 0 \]  

where

\[ C_1 = \left( \hat{\mathbf{s}} - \frac{1}{M} \right) \left[ \begin{array}{cccc} f_1^r_1 I_{11} & \cdots & f_1^r_2 I_{21} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ f_2^r_1 I_{12} & \cdots & f_2^r_2 I_{22} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{array} \right] + \left( \frac{1}{M^2} \right) \sum_{k=1}^{D} \left( f_{1k}^r \hat{\mathbf{r}}_1 I_{1k} \right). \]

Here, \(\mathbf{I}_D\) is a \(D \times D\) identity matrix whereas \(\mathbf{I}_{k\ell}\) is a \(D \times D\) matrix in which only the \(k\ell\)th element is one and the rest zero. Setting \((\partial J'' / \partial \rho) = 0\), we obtain a cubic equation in \(\hat{\rho}\) which also involves \(\hat{\mathbf{r}}\) and is given by

\[ \mu_1 \hat{\rho}_s^3 + \mu_2 \hat{\rho}_s^2 + \mu_3 \hat{\rho}_s + \mu_4 = 0 \]  

where

\[ \mu_1 = \frac{\text{tr}(\hat{\mathbf{r}}_s^{2-})}{M^2}, \quad \mu_2 = -\left[ (\text{tr}\hat{\mathbf{r}}_s^{-1}) - \left( \frac{1}{2M^2} \right) \sum_{k=1}^{D} \sum_{\ell=1}^{D} \left( f_{1k}^r f_{1\ell} (\hat{\mathbf{r}}_s^{-1})_{k\ell} + f_{1k}^r f_{1\ell} (\hat{\mathbf{r}}_s^{-1})_{\ell k} \right) \right] \]

\[ \mu_3 = -\left[ (M - D) + \left( \frac{1}{M^2} \right) \text{tr}(\mathbf{A} \hat{\mathbf{r}}_s^{-1} \mathbf{A}^\top \hat{\mathbf{R}}) \right] \text{ and} \]

\[ \mu_4 = -\left[ \frac{1}{M} \text{tr}(\mathbf{A}^\top \hat{\mathbf{R}}) - \left( \frac{1}{M^2} \right) \text{tr}(\mathbf{A} \hat{\mathbf{r}}_s^{-1} \mathbf{A}^\top \hat{\mathbf{R}}) - \text{tr}\hat{\mathbf{R}} \right]. \]

### 2.2.7.3 Closed form expressions for estimators

Evidently, numerical optimization is required for obtaining estimates of \(\hat{\mathbf{r}}\) & \(\hat{\rho}\) since (2.33) and (2.34) form a system of non-linear equations to be solved simultaneously and hence not computationally feasible. However, we obtain closed form expressions ((2.30) & (2.31)) from first order approximation. The estimator for noise variance in (2.30) is similar to the stochastic ML estimator, \(\hat{\rho}_{\text{SML}}[6]\) given by

\[ \hat{\rho}_{\text{SML}} = \left( \frac{1}{M - D} \right) \text{tr}(\hat{\mathbf{R}}) - \left( \frac{1}{M - D} \right) \text{tr}[\mathbf{A}(\mathbf{A}^\top)^{-1}\mathbf{A}^\top \hat{\mathbf{R}}] \]  

A comparison of (2.35) and (2.30) shows that, in \(\hat{\rho}_{\text{F}}, (\mathbf{A}^\top \mathbf{A})\) is asymptotically diagonal in \(\mathbf{M}\) and is represented by \(\mathbf{M}\mathbf{I}_D\). This means that, transfer vectors corresponding to distinct track parameters are orthogonal for large \(M\). Also, from [6, 7],

\[ \hat{\mathbf{r}}_{\text{SML}} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top (\hat{\mathbf{R}} - \hat{\mathbf{r}}_{\text{ML}}) \mathbf{A}(\mathbf{A}^\top)^{-1} \]  

Since \(\mathbf{A}^\top \mathbf{A}\) is asymptotically diagonal in \(\mathbf{M}\), (2.36) can be rewritten as

\[ \hat{\mathbf{r}}_{\text{ML}} \equiv \left( \frac{(\mathbf{A}^\top \mathbf{R}_\mathbf{A})}{M^2} \right) - \left( \frac{\hat{\rho}_{\text{ML}}}{M} \right) \]
Equation (2.31) is similar to (2.37) with the $O(1/M^2)$ term in the latter being neglected. In summary, equations (2.30) and (2.37) give estimates of signal power and noise variance.

Hence, a \textit{computationally efficient separable} solution is obtained for DOA estimation from equations (2.21), (2.30) and (2.37). In practice, using these expressions for estimating $\Gamma$ and $\rho$ does not significantly affect the accuracy of DOA estimates.

2.3 Explicit Expressions for Cramer-Rao Bounds on the DOA estimates

2.3.1 Motivation

Asymptotic Cramer-Rao matrix Bounds (CRB) for variance of ML range and DOA estimates of the signal sources incident on a large uniform linear array of sensors are derived and implications discussed. The derived asymptotic expressions are based on the large $M$ Taylor's series expansion of the data covariance matrix $R^{-1}$ in the expression for the likelihood function (2.13). The approach followed in this paper is one of \textit{direct} evaluation of the Fisher information matrix under the approximation and is different from the approach followed in [8, 10]. In [8, 10] an expression for the asymptotic covariance matrix ($N >> 0$) is first derived. It is also known that the ML estimate is consistent under the stochastic signal model as $N \to \infty$. Thus, from standard statistical theory of ML estimators, it is concluded that the ML DOA estimate asymptotically achieves CRB. Hence, the CRB for large $N$ is given by the asymptotic covariance matrix.

In this section, simplified expressions are obtained from the series expansion and are expressed as explicit functions of the relevant parameters like $M$, $N$, signal-to-noise ratio, source separation and correlation. Though the derived expressions are only almost exact, they are easily computable. These equations also result in a resolution criterion for distinguishing between DOAs.

Here, the interest is only in the lower bound on the variance of the angle parameters and hence quantities $\Gamma$ and $\rho$ treated as nuisance parameters and assumed to be known. The ratio ($\Gamma / \rho$), $\Gamma$ and $\rho$ being true values, is defined as signal-to-noise ratio denoted by SNR.

2.3.2 General form of the Fisher information matrix

From the definitions and notations employed in Section 2.2.6,

\[
\nabla_i \ln \det R = \text{tr} \left[ R^{-1} \nabla_i R \right] \quad \text{and} \quad \nabla_i R^{-1} = - R^{-1} \nabla_i R R^{-1}.
\]  

(2.38)
\[ \nabla_k \ln p(Y|\beta) = - \left( N / 2 \right) \text{tr} \left[ R^{-1} (\nabla_k R) (1 - R^{-1} \hat{R}) \right] \] (2.39)

\[ \nabla_{k\ell}^2 \ln p(Y|\beta) = - \left( N / 2 \right) \text{tr} \left[ \nabla_k (R^{-1} \nabla_k R) (1 - R^{-1} \hat{R}) - R^{-1} \nabla_k R \nabla_k R^T \hat{R} \right] \quad k, \ell = 1, 2, \ldots, D \] (2.40)

where \( \beta = \text{col.}(\theta_1, \ldots, \theta_D) \) is the true parameter vector.

The only random variable in (2.40) is \( \hat{R} \) which occurs linearly and its expectation is \( R \). It is important to notice that \( E[\hat{R}] = R \) is an average over the sample functions of the random process with number of snapshots, \( N \) still a finite quantity. The expectation of the first term in (2.40) is zero. Defining the matrix \( Q \) whose elements comprise expectations of the terms in (2.40),

\[ Q_{k\ell} = E \left[ \frac{\partial^2 \ln p}{\partial \theta_k \partial \theta_\ell} \right] = -\text{Re} \left( N / 2 \right) \text{tr} \left[ \nabla_k R \nabla_k R^T \right] \quad k, \ell = 1, 2, \ldots, D \] (2.41)

Therefore, the CRB on the variance of \( \hat{\theta} \) is given by \( \text{CRB}_\theta = Q^{-1} \). (2.42)

Here \( Q \) is defined to be the Fisher Information Matrix (FIM).

Theorem 2.2 gives a simplified expression for the FIM, \( Q \) of the DOA angles for large \( M \) in the general \( D \) source case using the first order expression for \( R^{-1} \) given in (2.20). In Theorem 2.3, a closed form expression for \( Q \) is obtained for large \( M \) using the second order approximation on \( R^{-1} \) (2.19). Results of Appendix III are required for deriving results in Theorems 2.2 and 2.3.

### 2.3.2.1 Notation and Symbols

Symbols for the various quantities are defined here for clarity.

- **FIM** = Fisher information matrix.
- \( Q_f \) = FIM for unbiased DOA estimate using \( I \) order approximation.
- \( Q_z \) = FIM for unbiased DOA estimate using \( II \) order approximation.
- \( Q_{ppf} \) = Principal diagonal terms of FIM \( Q_z \), \( p = 1, 2, \ldots, D \). (2.43)
- \( Q_{povf} \) = Off - diagonal terms of FIM \( Q_z \), \( p, \ell = 1, 2, \ldots, D \). (2.44)
- \( Q_{pps} \) = Principal diagonal terms of FIM \( Q_z \), \( p = 1, 2, \ldots, D \). (2.45)
- \( Q_{pws} \) = Off - diagonal terms of FIM \( Q_z \), \( p, \ell = 1, 2, \ldots, D \). (2.46)
- \( \text{CRB}_{\phi^t} \) = Derived CRB matrix on \( \phi \) for two source case, using \( I \) order approximation (2.47). Subscript ‘t’ stands for 2 sources.
- \( \text{CRBS}_{\phi^t} \) = Derived CRB matrix on \( \phi \) for two source case, using \( II \) order approximation (2.48).
- \( \text{CRBQ}_{\phi^t} \) = Computed CRB matrix for two source case using (2.42), (2.43) and (2.44).
\text{CRBQ}_{\phi_1} = \text{Computed CRB matrix for two source case using} \ (2.42), \ (2.46) \text{ and } (2.47).

\text{CRBQe} = \text{Computed Exact CR bound for two source case using} \ (2.41) \text{ and } (2.42).

2.3.3 Expressions for the CR bounds

2.3.3.1 Theorem 2.2

The Fisher Information Matrix \( \mathbf{Q}_f \) for the unbiased DOA estimate \( \hat{\phi} \) for large \( M \), using the first order approximation for \( \mathbf{R}^{-1} \) (2.20) is given by

\[
\mathbf{Q}_{pff} = \frac{N M^3 \Gamma_{pp}}{12\rho} - \frac{N M^2}{12} \sum_{\nu=1}^{D} (K_{\nu} - 2) \left( \frac{\Gamma_{\nu\nu}}{\rho} \right) + \mathcal{O}(M) \tag{2.43}
\]

and

\[
\mathbf{Q}_{pvr} = -N \left( \frac{\Gamma_{\nu\nu}}{\rho} \right) \left( \frac{M^2 K_{\nu}}{4} \right) + \mathcal{O}(M) \quad p, \nu = 1, 2, \ldots, D. \tag{2.44}
\]

The quantity \( K_{\nu} \) is a function of the separation between the sources and is given by

\[ K_{\nu} = \left\{ \sin \left( \frac{(2M - 1)\phi_d}{2} \right) / \sin \left( \frac{\phi_d}{2} \right) \right\}. \]

The proof of this theorem is given in Appendix IV.

Corollary 2.2

The CR matrix bound on the covariance matrix of the unbiased DOA estimate, \( \hat{\phi} \) for the two source case based on the first order approximation (using equations (2.43) and (2.44)) is

\[
\text{CRBF}_{\phi_1} \approx \frac{12}{N(SNR)M^3} \begin{bmatrix}
1 - \frac{\gamma(K_1 - 2)}{M} & -\left( \frac{3\gamma K_1}{M} \right) \\
-\left( \frac{3\gamma K_1}{M} \right) & 1 - \frac{\gamma(K_1 - 2)}{M}
\end{bmatrix} \tag{2.45}
\]

2.3.3.2 Observations from first order bounds

The expressions obtained for elements of the CRB on \( \hat{\phi} \) for the two source case facilitate comparison of the effect of individual parameters \( N, M, \text{SNR}, (\gamma) \) and \( (\phi_d) \) on the bound values. This point is dealt in Section 2.6. \( \phi_d \) is the angular separation between the two sources, \( \text{SNR} \) is the ratio of true signal and noise powers, \( \Gamma / \rho \) and \( \gamma \) is the correlation coefficient between the sources. In obtaining (2.45), it is assumed without loss of generality that \( \Gamma_{pp} = \Gamma, p = 1, \ldots, D \) and \( \Gamma_{p\nu} = \gamma \Gamma, \nu \neq p, \nu, p = 1, \ldots, D. \)

The expressions for \( Q \) in (2.43) and (2.44) are functions of \( \gamma \) and \( \phi_d \), but the effect of these parameters is only \( \mathcal{O}(M^2) \). This means that, effect of these two
parameters on the bound is of lesser magnitude than the other parameters N and SNR. This is further justified by the CRB expressions using the second order approximation. A detailed discussion of the bounds with experimental results is presented in Section 2.6.

2.3.3.3 Theorem 2.3

The $Q_s$ matrix for the unbiased DOA estimate $\hat{\phi}$ for large M, using the second order approximation for $R^{-1}$ (2.19) has the following expressions:

$$Q_{pp} = \frac{NM^3T_{pp} - NM^2}{12} - \frac{NM^2(D - 1)}{3} + O(M)$$

and

$$Q_{p_p} = \frac{NM^2}{4} + \frac{NM^2(1 + K)}{8} \left( \frac{\Gamma_{pp}}{\rho} \right) + O(M)$$

(2.46)

(2.47)

The proof is given in Appendix V.

Corollary 2.3

The CRB on the covariance matrix of the unbiased DOA estimate, $\hat{\phi}$ using second order approximation (2.19) for the two source case is given by

$$\text{CRB}_{s_{\phi}} = \frac{12}{N(\text{SNR})M^3} \begin{bmatrix}
\left( 1 - \frac{5}{M(\text{SNR})} \right) & -\left( \frac{3}{M(\text{SNR})} + \frac{3\gamma(1 + K)}{2M} \right) \\
-\left( \frac{3}{M(\text{SNR})} + \frac{3\gamma(1 + K)}{2M} \right) & \left( 1 - \frac{5}{M(\text{SNR})} \right)
\end{bmatrix}$$

(2.48)

2.3.3.4 Observations from second order bounds

Observations of Section 2.3.3.2 are also valid here. The 2x2 FIM for the two source case is

$$Q_{f(s)} = \begin{bmatrix}
Q_{pp(s)} & Q_{pp'(s)} \\
Q_{pp'(s)} & Q_{pp(s)}
\end{bmatrix}$$

(2.49)

and the CRB matrices are specified by (2.42) to be $\text{CRB}_{f(s)} = Q_{f(s)}^{-1}$. The elements of $Q_{f(s)}$ are obtained using results of Theorem 2.2 (2.3). Corollaries 2.2 and 2.3 are approximate CR bounds for the two source case which only serve to study the dependency of the various parameters. They are approximate since all terms other than the $O(M^6)$ term are neglected while computing the determinant (necessary for computing inverse) of the FIM.
2.4 Properties of ML Estimates

In [4], the question of the behavior of ML estimates as the number of sensors, \( M \) tends to infinity is considered. It can be shown that the ML estimates \( \hat{\phi} \) of the DOA are consistent as both \( N, M \to \infty \). The statement is consistent with the fact that the variance of the DOA estimate behaves like \( \frac{K}{N M^2} \) for large \( M \) and \( N \). This behavior is also observed from the approximate CRB expressions in (2.45) and (2.48). The asymptotic equivalence of the ML methods both under the stochastic and deterministic signal models for \( M \to \infty \) and arbitrary \( N \) is verified in [10].

However, the ML estimate of \( \Gamma \) is not consistent as \( M \to \infty \) with fixed \( N \). This follows from the expression of the variance of \( \hat{\Gamma} \) which tends to a constant as \( M \to \infty \). This is seen from the expressions for the CR Bounds on \( \hat{\Gamma} \) for the single and two source cases which are respectively given by (without proof)

\[
\text{CRB}_{\Gamma}^{\text{(single)}} = \frac{2(\Gamma + \rho/M)^2}{N} \quad \text{(2.50)}
\]

\[
\text{CRB}_{\Gamma}^{\text{(two)}} = \frac{M^2 \gamma}{3N M^2 + N(\Gamma + 1)(K_1 + 1)} \quad \text{(2.51)}
\]

where \( K_1 = \left\{ \sin \left( \frac{2M-1}{2} \phi_a \right) / \sin \left[ \frac{\phi_a}{2} \right] \right\} \). Evaluating the limits on (2.50) and (2.51) as \( M \to \infty \) gives a positive constant quantity.

2.5 Performance of Proposed Method

2.5.1 Experimental results

This section considers performance study of the proposed maximum likelihood method under different scenarios. Experiments are performed to illustrate performance of the proposed algorithm. In particular, the situation considers five narrowband plane signals impinging on a uniformly spaced linear array spaced \( \lambda/2 \) apart. Then, \( D = 5 \) and \( f(\phi_k) \) is

\[
f(\phi_k) = \text{col. } [1, e^{j\phi_1}, e^{j2\phi_2}, ... , e^{j(M-1)\phi_5}] \quad (k = 1, 2, ..., 5)
\]

where the \( \phi_k \)s are the DOAs to be estimated. The signals are assumed to be of equal powers and the \( \Gamma \) matrix (5x5) is given by

\[
\Gamma = \begin{bmatrix}
1 & \gamma & : & : & : \\
\gamma & 1 & : & : & : \\
: & : & \ddots & \ddots & : \\
: & : & \ddots & \ddots & \ddots \\
\gamma & : & : & : & 1
\end{bmatrix}
\]

where \( \gamma \) is the correlation coefficient and \( |\gamma| < 1 \).
For characterizing the separation between the signal sources, the half power beam width or beam width of a uniform linear broadside array is given by [45]

$$\text{HPBW} = \sin^{-1}\left( \pm \frac{M-1}{M} \lambda \right)$$

(2.52)

where $M$ = number of sensors in the array, $L$ = length of the array and $\lambda$, the spacing between the array elements.

In the first experiment, the number of sensors in the uniform linear sensor array are 20 with 128 snapshots. For this array configuration, the beam width is 5.74°. The DOAs are chosen to be closely spaced, 20° apart, the angles being 44, 46, 48, 50 and 52 degrees, i.e., the five sources are spaced within 1.4 beam widths. Angle estimates are obtained for two different values of $\gamma(0, 0.75)$ with SNR varying from -6 dB to 4 dB. All statistics are obtained for 50 independent runs of the experiment. A plot of estimated DOAs versus SNR (dB) is shown in Fig. 2.1. The method resolves the five sources. When sources are uncorrelated, the estimates are close to the true values for all given values of SNR. The sources are resolved even though they are correlated, but accuracy of the estimates goes down with decreasing SNR. From the plot, it is observed that the estimates are independent of $\gamma$ up to 0 dB SNR.

### 2.5.2 Comparison with stochastic maximum likelihood

Experimental results illustrating performance of proposed method in comparison with Stochastic ML[6, 7] are given in Fig. 2.2. Two closely spaced uncorrelated narrowband signals within a beamwidth are incident on a uniform linear array of 32 sensors subtending angles of 20 and 23.5 deg. The number of snapshots, $N = 40$. The two DOAs were estimated by both methods and the root mean square error (RMSE) of the DOA estimates averaged over 25 mals are plotted against SNR (dB) (Fig. 2.2).

Since the number of sensors is large, the new likelihood function is well suited and thus DOA estimation with this criterion function yields accurate results with smaller RMSE than the stochastic ML method. In the separable stochastic: ML[7], though inversion of R matrix is not needed, inversion of the $Dx\bar{D}$ matrix ($A^*A$) for estimating signal power can be unstable in the presence of large number of sensors, thus affecting DOA estimate accuracy. In general, the number of sensors in an array (M) is much greater than the number of DOAs (D) to be estimated. In the above example, $M = 32$ and
Fig. 2.1. Variation of DOA estimates with SNR for five sources. DOAs are two degrees apart from each other. True DOAs are indicated in the figure by arrowheads.

Fig. 2.2. Comparison of Root Mean Square Error of DOA estimates for the two source case from Proposed ML and Stochastic ML.
D = 2. The proposed ML method is also computationally efficient since, inversion of only a DxD matrix is required for DOA estimation.

2.5.3 Comparison with signal subspace methods

The proposed ML method is compared with the Root-Music [12] method in Fig. 2.3. For comparing resolving power of ML and Root-Music, the deviation of DOA estimates from their true values for varying values of correlation coefficient are compared. A 10 element array with half a wavelength spacing is considered and hence beam width as given by (2.52) is 11.53°. Two closely spaced sources with a separation of 0.16 beamwidths with true arrival angles of φ₁ = 7.30 and φ₂ = 9.150 are considered.

For a high value of SNR (10 dB), in Root-Music, the DOA estimate for the uncorrelated case is itself well away from the true value and the resolving capabilities reduce as γ is increased. When the sources become almost fully correlated, Root-Music is not able to resolve the two sources and the resultant estimate for the two sources is same, and lies in between the true values. But, in ML, the sources are clearly resolved with negligible deviation of the estimate from its true value for all values of γ between 0 and 1. For a low SNR value (0 dB), when Root-Music is employed, the deviation is almost uniform for all values of γ. Thus, the experiment demonstrates resolving capability of the proposed ML method.

2.6 Discussion on the Cramer - Rao Bounds

In Theorems 2.2 and 2.3, closed form expressions for the CR Bounds on the variance of the DOA estimates for the general D source case under the stochastic signal model are derived. The motivation arises since, CRB evaluated exactly from (2.42) does not give much insight in view of many nuisance parameters involved in the stochastic ML formulation.

To compute the value of the CR bound in practice, given N, M, SNR, (y) and (φ_d), the Q matrices are computed using equations in Theorems 2.2 and 2.3 and its inverse (2.42) gives either the first or second order CR bounds. These are discussed in Section 2.6.1 with some illustrations.
Fig. 2.3. A comparison between the proposed ML and the Root-music methods for two closely spaced sources. The ML estimates are always close to true values irrespective of the correlation between the sources or SNR whereas Root-music estimates are far away from the true angles and converge to values in between the true DOAs.

Fig. 2.4. Comparison of principal diagonal terms of CRB matrices CRBQf, CRBQs, and CRBQe for varying correlation. Angular separation between the two sources is five degrees.
On the other hand, the usefulness of expressions given in Corollaries 2.2 and 2.3 is to compare and contrast the effect of individual parameters $N$, $M$, SNR, $(y)$ and $(\phi_d)$ on the bound values. This point is dealt in Section 2.6.2.

### 2.6.1 Comparison of Fisher information matrices and computation of CRB

Both diagonal ($Q_{pp}$) and off-diagonal ($Q_{pd}$) terms of $Q_t$ (Theorem 2.2) are functions of $(y)$ and $(\phi_d)$. However, the diagonal terms of $Q_s$ ($Q_{pp}$) (Theorem 2.3) does not involve either $(y)$ or $(\phi_d)$. But, the non-diagonal terms ($Q_{d}$) are functions of $(y)$ and $(\phi_d)$. The key point is, although both $Q_t$ and $Q_s$ are functions of $(y)$ and $(\phi_d)$, there is only a weak dependency. The magnitude of terms which are functions of $(y)$ and $(\phi_d)$ is only $O(M^2)$ in both cases. These facts are evident in Corollaries 2.2 and 2.3 and in discussion (Section 2.6.2) where explicit expressions are given for CRB (two source case).

Hence, under the stochastic signal model, variance of the DOA estimates is almost independent of the amount of correlation between signals and also the separation between sources. A stronger conclusion is evident from the statistical properties of ML estimators. The DOA estimates obtained by maximizing (2.14) are indeed ML estimates since estimates of $\hat{\phi}$ and $\hat{\phi_d}$ are maximum likelihood estimates with probability one ($\Gamma$ is assumed to be positive definite in the formulation itself) and hence are asymptotically efficient. This implies that, under the signal model, accuracy of ML DOA estimates is almost independent of source separation and correlation. This fact is indeed evident when the proposed ML method and Root-Music are compared (Fig. 2.3).

Figures 2.4.2.5 and 2.6 illustrate comparison of first and second order CR bounds (computed using $Q_t$ and $Q_s$) and the exact CRB (eqns. (2.41) and (2.42)) for the two source case.

A comparison of CRB$_{Q_{f},}$, CRB$_{Q_{s},}$, and CRB$_{Q_e}$ (principal diagonal terms) versus varying correlation coefficient for the two source case is shown in Fig. 2.4. The exact bound (CRB$_{Q_e}$) as well as CRB$_{Q_{s},}$ are close to each other and almost independent of correlation, whereas CRB$_{Q_{f},}$ increases with increasing correlation. This shows that CRB$_{Q_{s},}$ is a better approximation of the true value (CRB$_{Q_e}$).

In Fig. 2.5, the three bounds are plotted versus varying separation. DOA1 is at $65^0$ and DOA2 is varied from $65^0$ to $75^0$. For increasing separation CRB$_{Q_{s},}$ and
Fig. 2.5. Comparison of principal diagonal terms of CRB matrices CRBQf, CRBQs, and CRBQe for varying separation. Correlation coefficient between sources = 0.3.

Fig. 2.6. Comparison of principal diagonal terms of CRB matrices CRBQf, CRBQs, and CRBQe for varying SNRs and varying Correlation coefficients. Separation between the two sources = 40°.
CRBQf, are close to each other and approach CRBQe. For small separations, CRBQf, matrix becomes unstable (indefinite / negative definite) and CRBQf, is negative. CRBQe mamx is positive definite but the principal diagonal term (variance of the DOA estimate) shows an oscillatory behavior. In contrast, CRBQs, matrix is positive definite and the principal diagonal term monotonically decreases with increasing separation. CRBQs, provides increased resolution when compared to CRBQf,., i.e., the range over which CRBQs, matrix is positive definite is more than CRBQf,.

The variation of the different CR bounds with SNR is illustrated in Fig. 2.6. For increasing SNR both CRBQs, and CRBQf, approach the exact value CRBQe. CRBQf, is lower than CRBQs, which is closer to the true bound. The family of curves plotted for four different correlations show little variation thus supporting the conclusions drawn from Fig. 2.4. The derived bounds are smaller when compared to the true value. The approximations made while deriving the bounds make them 'looser' than they really are. The bounds become 'tighter' with increase in order of approximation which is evident from the graph.

It is seen that CRBṣ, is closer to the exact CR bound than CRBf, in all scenarios except when sources are located near endfire. In this situation, the bounds are close to each other. CRBṣ, also has a better range and hence can provide a better resolution. The range for which the bounds are valid is dictated by the positive definiteness of the Q matrix. This fact is utilized to derive a resolution criterion for the DOAs in the Section 2.7.

2.6.2 Effect of parameters on first and second order bounds

Equations (2.45), and (2.48) give simplified explicit expressions for the CRB mamx of the DOA estimates exclusively in terms of the number of snapshots (N), the number of sensors (M), signal-to-noise ratio (SNR), the correlation between the signal sources (y) and their separation (d) for the two source case. Here, we discuss the properties of the CRB matrices derived using both first and second order approximations on R⁻¹ (Corollaries 2.2 and 2.3).

Firstly, the principal diagonal terms of both the CRB matrices (CRBf, CRBṣ,) which represent lower bound on the variance of the DOA are O(K / NM³), but the non-diagonal terms are no bigger than O(K / NM⁴). This shows a reduced effect of non
diagonal terms for increasing $M$, $N$ and SNR. The principal diagonal terms of the CRB matrices in (2.45) and (2.48) are given by

$$\text{CRB}_{\phi_i}(p, p) = \frac{12}{N(SNR)M^3} - \frac{12}{N(SNR)M^3} \left( \frac{\gamma(K_i - 2)}{M} \right), \quad p = 1, 2 \quad (2.53)$$

$$\text{CRB}_{\phi_i}(p, p) = \frac{12}{N(SNR)M^3} - \frac{12}{N(SNR)M^3} \left( \frac{5}{M(SNR)} \right), \quad p = 1, 2 \quad (2.54)$$

Also, the expression for CRB on $\hat{\phi}$ for the single source case, is given by (proof is trivial)

$$\text{CR Bound on } \hat{\phi} = \frac{12 (M + \rho/\Gamma)}{N(M^3)(M^2 - 1)} \left( \rho/\Gamma \right)$$

For $\frac{\Gamma}{\rho} \gg 1$, CR Bound on $\hat{\phi} = \frac{12}{N(SNR)M^3}$ \quad (2.55)

It is interesting to note that (2.55) is the same as the first terms in (2.53) and (2.54). In other words, the second terms of (2.53) and (2.54) can be defined as 'augmentation' factors on the single source bound which are functions of the parameters. The augmentation factor in the first order bound is a function of $M$, $(\gamma)$ and $(\phi_d)$ whereas the augmentation factor in (2.54) is a function of $M$ and SNR. A key point is that, the augmentation factor for the second order bound is $\textit{independent}$ of the correlation coefficient $(\gamma)$ and the separation between the sources $(\phi_d)$, further reinforcing the 'weak dependency' factor discussed in the previous section. Another point is, given an array with sufficient $M$ (which is the underlying assumption), the difference in CRB from CRB$_{\phi_1}$ is only dependent on the SNR.

2.6.3 Summary of observations

(1) The first order bound with $O(M^2)$ terms suggested a diminishing dependency but was a function of $(\gamma)$ and $(\phi_d)$.

(2) Second order bound suggests a weak dependence on $(\gamma)$ and $(\phi_d)$, (these terms only appear in the off-diagonal terms of CRB$_{\phi_1}$) though diagonal terms are independent of $(\gamma)$ and $(\phi_d)$.

This shows that the second order approximation results in a bound which best models the true functional relationship between variance of DOA estimate and the parameters involved.
The CRBF bound approaches CRBS for increasing values of M, N, and SNR. However, the difference between these approximations and CRBe (exact numerical value of CRB from (2.41)) becomes large for very small angular separations. This is because, the expressions are valid for only small but finite separation angles in accordance with the definition of the B matrix (2.17). Finally, both CRBs and CRBF reduce to the uncorrelated sources case (y = 0) in the absence of augmentation factors, i.e., for large SNRs, M and uncorrelated sources, the effect of augmentation factors damps out and the CRB expressions in both cases are identical.

2.7 DOA Resolution Criterion

2.7.1 Description of the criterion

A criterion for the resolution of any two DOAs is stated, based on the eigenvalues of the Fisher Information Matrix (Q). Q is explicitly expressed as a function of M, N, SNR, φ, and y in (2.46) and (2.48). The FIM as defined in (2.41) is a function of R and R⁻¹. R⁻¹ is a function of the B matrix (2.17), which is directly dependent on source separation (Section 2.2.3). Thus, behavior of the FIM, Q, is a function of source separation. The eigen values of Q, thus follow variation of the eigen values of the B matrix. This motivates the resolution criterion based on the positive definiteness of the FIM. The criterion is as follows:

For a given array, N, SNR and correlation y between the sources, there is a minimum value of φd for which the Qₜ matrix (Eqns. 2.46 and 2.47) just becomes positive definite. This φd value can be regarded as an approximate measure of the smallest spacing for which the DOAs will be resolved. It is true that the second order approximation of the true R⁻¹ becomes worse as DOAs come closer which can make the FIM indefinite. Hence, this resolution criterion is tighter than the true resolution.

2.7.2 Illustration

The usefulness of the resolution criterion can be illustrated by this example:

Consider a narrowband source located at 45° with reference to a 16 element uniform linear array. A second source is considered such that the correlation between the two DOAs, y = 0.4. The SNR of both sources is fixed at 10 dB with 32 snapshots. This specifies the operating point. The eigen values of the FIM, Q, are plotted against varying values of DOA₂ (40° - 500) keeping DOA₁ fixed. In Fig. 2.7, it is observed that the two eigen values have their maximum and minimum values at the true value of DOA₁.
(condition when the two DOAs are identical). Around the true values of DOA1 (45°), there is a symmetric region in which one of the eigen value is less than zero, i.e., $Q_i$ is

![Fig. 2.7](image1.png)

Fig. 2.7. Plot of eigen values of $Q_{max}$ for reference condition. The resolution is given by the absolute difference between true DOA1 = 45° and DOA corresponding to one of the vertical lines. (≈ 1.8 deg)

![Fig. 2.8](image2.png)

Fig. 2.8. Plot of eigen values of $Q_{max}$ for $M = 32$. The resolution is given by the absolute difference between true DOA1 = 45° and DOA corresponding to one of the vertical lines. (≈ 1 deg)
indefinite. This range defines the minimum separation for which the two DOAs can be resolved. In this example the minimum separation is 1.8\(^0\) which means that, for the given operating point, two DOAs can be resolved up to 1.8\(^0\).

Figures 2.8 to 2.11 illustrate the effect of other parameters on the resolution criterion. In Fig. 2.8, M is increased to 32. As expected, resolution capability increases and the minimum separation is now 10\(^0\). In Fig. 2.9, N is increased to 128. Now the minimum separation is 1.70\(^0\) which is almost the same as the original value. Thus, increasing the observations fourfold shows negligible improvement which implies that the DOA estimates are efficient for increasing M rather than for increasing N. Increasing the correlation between the DOAs decreases resolution capability (3.70\(^0\)) as shown in Fig. 2.10. Lastly, decreasing SNR to 0 dB increases minimum separation to 3.90\(^0\) (Fig. 2.11). These experiments illustrate the usefulness of the resolution criterion. With decreasing values of \(\gamma\), the threshold value also decreases which implies that the resolution capability becomes better when sources are more uncorrelated.

2.7.3 Usefulness of the proposed criterion

The resolution capability increases with increase in number of sensors in the array (Fig. 2.12). The plot shows the smallest spacing between the DOAs which can be resolved for increasing M for the two source case. It is seen that this strict lower limit decreases with decrease in correlation or increase in M. The threshold separation is the minimum spacing value for which the \(Q_s\) matrix just becomes positive definite. Eqns (2.46) & (2.47) are used (II order approx.) to determine the threshold separation. \(Q_s\) is selected to derive the resolution criterion since, it has already been shown that the second order expansion models the effects better.

The derived resolution criterion, though strict can be very useful in practical situations such as an interceptor tracking a target. For tracking, DOAs of both target and interceptor are continuously estimated and associated. The trajectory of the interceptor, the number of sensors in the tracking radar and the sampling rate is under ground control. Based on the minimum angle of resolution criterion, it is possible to dynamically change these parameters thus varying trajectory of the interceptor so that the target can eventually be intercepted. Computation of eigen values of the FIM is straightforward from the derived closed form expressions.
Fig. 2.9. Plot of eigen values of $Q_s$ matrix for $N = 128$. The resolution is given by the absolute difference between true $\text{DOA}_1 = 45^0$ and DOA corresponding to one of the vertical lines. ($\approx 1.7$ deg). Only a very little improvement for four fold increase in $N$.

Fig. 2.10. Plot of eigen values of $Q_s$ matrix for correlation = 0.8. The resolution is given by the absolute difference between true $\text{DOA}_1 = 45^0$ and DOA corresponding to one of the vertical lines. ($\approx 3.7$ deg). Increase in correlation reduces resolving power.
Fig. 2.11. Plot of eigen values of $Q_x$ matrix for low SNR = 0 dB. The resolution is given by the absolute difference between true DOA1 = 45° and DOA corresponding to one of the vertical lines. (~3.9 deg). Decrease in SNR decreases resolving power.

Fig. 2.12. The plot shows the smallest spacing between the DOAs which can be resolved for increasing M for the two source case. It is seen that this strict lower limit decreases with decrease in correlation or increase in M. The threshold separation is the minimum spacing value for which the $Q_x$ matrix just becomes positive definite [(2.46) & (2.47)].
2.8 Summary

Using stochastic modeling of signals, a new separable ML method for estimation of DOA and associated parameters for estimating the direction of arrival and associated parameters of narrowband signals have been developed for large arrays. A new likelihood expression has been derived based on the Taylor's series expansion of the covariance matrix of observations. The analytical study of the ML method shows potential to outperform other high resolution methods for DOA estimation.

The more significant point is the derivation of explicit expressions for the CR bound on the arrival angles for the general D source case. The derivation is based on using the same Taylor series approximation for $R^{-1}$ in terms of the order of $M$, the number of sensors in the uniform linear array. The expressions obtained for the CRB on the variance of the DOA estimate explicitly in terms of $N$, $M$, SNR, $\phi_d$, the separation between the signal sources and $\gamma$, the correlation coefficient provide a means to evaluate the CRB directly as a function of the parameters of the model. The test for positive definiteness of the Fisher Information matrix is useful to determine the separation between two sources to be just resolved. However, the resolution criterion is strict because of the large $M$ approximation used.

In this chapter, the number of signals, $D$ was assumed to be known. However, it is not always known and has to be estimated. Model order determination criteria [38, 39] from standard statistical theory can be used for estimating number of source signals.
3. A MAXIMUM LIKELIHOOD APPROACH FOR ESTIMATING TRACK PARAMETERS OF NARROWBAND SIGNALS

The previous chapter was devoted towards the estimation of direction of arrival of stationary (fixed) narrowband sources from an observation sequence obtained from a large uniform linear array of sensors. However, recent interest in this topic has involved designing methods for tracking angle of arrival of multiple signals [29-31] wherein the sources are moving, perhaps with a constant velocity or acceleration which are also unknown parameters in the estimation problem. In this chapter, development of efficient algorithms for tracking multiple moving sources in near-field is considered.

3.1 Introduction
3.1.1 Motivation for the tracking problem

This suggests short range tracking applications like, air-traffic control in the vicinity of crowded airports wherein an efficient tracking algorithm would be invaluable for tracking incoming, outgoing and circling aircraft. In more recent developments, phased array technology is being incorporated at the base-stations of a cellular communications network. This enhances capability of locating and tracking of mobiles and also provides the opportunity to efficiently utilize the available radio spectrum, thus increasing the overall capacity of the network [47]. Since many base stations are located within a certain designated area for receiving and transmitting messages from the mobiles, the problem of locating and tracking of these mobiles is inherently a near-field problem which requires continuous estimation of range and bearing angles. Moreover, the operating frequencies for cellular communication systems is around 900 MHz (0.9 GHz) for which an efficient phased array antenna with a uniform linear array of sensors can be designed. Yen and Reudink [48] discuss applications which include 24 element base station antennas.

With this underlying motivation, the objective is to develop a computationally efficient method for joint bearing and range estimation of the narrowband sources and utilize it for tracking the moving sources (mobiles/aircraft).
3.1.2 Existing approaches for angle parameter estimation

In view of the interest in the passive sensor array based approach for tracking, a good first step is to select the appropriate high resolution angle estimation method. The presently available high resolution subspace based direction of arrival estimation algorithms [1, 12, 46] and their variants provide accurate estimates only in the presence of the true covariance matrix or asymptotically in N, the number of observations. However, if the data collection time is limited (finite data), these methods will not be very accurate, especially in typical multipath environments. The Maximum Likelihood (ML) based source localization methods [4, 6, 71] have a lower signal-to-noise ratio threshold and provide asymptotically efficient estimates for large N, but are computationally intensive. The approach adopted is to include more elements in the sensor array (large M) to enhance estimation accuracy and develop a maximum likelihood approach for bearing and range estimation.

3.1.3 Proposed method for track parameter estimation

The problem of estimating the 'track parameters' (a collective name for DOA and range parameters) in near-field is formulated in a manner similar to the ML DOA estimation method in the previous chapter wherein the signal sources (targets) are modeled as sample functions of a Gaussian stochastic process. The power of the proposed method comes from the formulation of a new likelihood function (in contrast to [6, 30]) which is maximized to get both DOA and range estimates of the D targets from the same sequence of N snapshots. Extraction of range information is made possible by expressing the phase delay as a function of both range and angle of the respective targets.

The new likelihood function is based on the inverse of the data covariance matrix being approximated by a second order Taylor's series expansion in terms of M, the number of sensors in the array. Thus, maximization of the new likelihood function does not involve inversion of the MxM data covariance matrix, R where value of M is significant for a large array. Instead, we need to invert a DxD matrix, where D stands for number of targets. In practice, D<<M and hence the algorithm is computationally efficient. Addition of more number of sensors does not markedly increase computational complexity, but, guarantees accurate estimates of DOA and range. However, there is an
upper bound on number of sensors in the array for getting good estimates and is discussed in Section 3.6.

3.1.4 Crux of the tracking problem

Multiple moving targets are tracked by repeated implementation of the proposed estimation scheme. However, the key problem in multiple target tracking is the measurement/target 'data association' problem: that is, determining which measurement in the measurement set corresponds to which target. For the case of D targets, standard state model based methods like track splitting [22], Probabilistic Data Association [21] etc., involve searching over the D! possible combinations. In sensor array based methods for angle target tracking, data association implies association of DOA estimates of different targets at two successive time instants and is termed 'estimate association'.

3.1.5 Subspace methods and estimate association

The problem with using eigen methods like MUSIC [1] or Root-MUSIC [12] for DOA target tracking is that of estimate association. The dominant eigen values of the data covariance matrix give the DOAs in MUSIC whereas DOAs are obtained by polynomial rooting in Root-MUSIC. Both eigen values as well as the zeros of the characteristic polynomial do not suggest any ordering of the DOA estimates and hence, there is no way to associate the DOA estimates obtained at two contiguous instants.

3.1.6 Current approaches

The DOA based method of Sword et. al [29] avoids the estimate association problem by deriving a recursive procedure for obtaining updated angle estimates at regular intervals of time whereas in the method of Sastry et. al [30] estimates of target angles are obtained by minimizing the norm of an error matrix function involving the covariance of the sensor outputs. Both these methods use MUSIC [1] algorithm to obtain estimates of initial DOA and number of targets at regular time intervals. But, the above DOA based approaches do not use target dynamics for obtaining updated target positions in the sense that range and velocity of targets are not estimated.

1 Although no assumptions are made on array structure for developing the ML estimation scheme, bounds are derived considering a uniform linear array (ULA) of sensors and hence all references to the array imply an ULA. In view of the derived bound, reference to a large array implies an array with sufficiently large number of sensors.
In the approach of [31], a dynamic model governing the motion of the targets is used. DOA estimates at each time interval are obtained using the 'stochastic ML' method [6]. These DOA estimates are refined using a Kalman filter wherein angular velocity and acceleration of the targets (components of the state vector in addition to the DOA) are also estimated. A necessary condition for correct estimate association in [30, 31] requires the targets to have different signal powers.

3.1.7 Contribution

Based on joint estimation of DOA and range from the new ML method, an algorithm for tracking multiple targets called the Tracking Algorithm (TAL) is proposed [34, 36]. The formulation of the likelihood function is such that a structure is imposed for ML estimation of range and angle parameters of each of D targets. With this, the estimates obtained at every instant of time are naturally ordered. Thus, the estimate association problem is automatically solved by associating the respective 2-tuples (range & angle estimates) of the targets at any two successive time instants. By joint estimation of DOA and range, the position of the target in a plane can be uniquely defined.

In Section 3.2, the problem is formulated using stochastic Gaussian model for signals (targets) and the new ML estimation procedure for the track parameters is derived. Section 3.3 discusses the TAL algorithm. Implementation of TAL and comparison with two other methods of [29] and [49] are discussed in Section 3.4. In Section 3.5, asymptotic Cramer-Rao Bound (CRB) expressions on the variance of DOA and range estimates are derived. Section 3.6 discusses the CRB expressions and conclusions are included in the final section.

3.2 Signal model and Estimation Method

3.2.1 Signal model

Consider D moving targets to be tracked by a passive array of M sensors (D < M). The sensors are uniformly spaced and separated by a distance 'd'. The targets emitting narrowband signals describe arbitrary trajectories in the r-θ plane and at time t = t, impinge on the array from distinct directions specified by \( \{ \theta_k(t), k = 1, \ldots, D \} \) with corresponding ranges \( \{ r_k(t), k = 1, \ldots, D \} \).
Fig. 3.1. Illustration of the two different time scales for data acquisition and track parameter estimation.
Remark 3.1

There are two different time scales (Fig. 3.1), one over which sensor data is collected called the 'snapshot time scale' and another over which the track parameters are estimated. The latter is referred to as the 'tracking time'. Both time scales can be represented over the entire tracking period by a single expression given by

\[ t = t_1 + nT + \left( \frac{T}{N} \right)(1 - \delta)\tau, \quad n = -(t_1 - 1)/T, \ldots, -1, 0, 1, \ldots, \hat{T} \text{ and } \tau = 1, 2, \ldots, N \]

for each 'n'.

where \( \delta = 0 \) snapshot time scale

and \( \delta = 1 \) tracking time scale.

\( n \equiv \) tracking time index.

\( \tau \equiv \) snapshot index.

\( N \equiv \) number of snapshots collected over a time interval \( T \).

\( t_1 \equiv \) reference time.

\( T \equiv \) final tracking time index.

For convenience, the tracking time instants will be referred to by variable 't' and the snapshots by 't' throughout the thesis. The following description of the signal model is confined to one tracking time interval, \( T \) and \( 1 \leq \tau \leq N \), over which data is collected. From this data, track parameters of the D targets for one tracking time instant, say, \( t = t_1 \), are estimated. The statistical properties of the signal and noise are assumed to remain constant over the entire tracking period and hence the model is applicable to all tracking time instants \( t, t - T, \ldots, t + T, \ldots \).

Since the targets are assumed to emit narrowband signals, the input signals, \( x(.) \) represented as complex envelopes are described by

\[ x_k(\tau) = 6, (\tau) e^{j\omega_\tau} \quad k = 1, 2, \ldots, D \text{ and } \tau = 1, 2, \ldots, N. \]

(3.1)

\( N \) is referred to as the number of snapshots (sensor data). The symbol 't' denotes discrete time instants on the array data collection time scale. The signal model is identical to the one adopted in the previous chapter.

(i) \( 6, (\tau) \) is an independently identically distributed (i.i.d) Gaussian sequence \((0, \Gamma_{kk})\).

(ii) Two input signals \( x_k(\tau) \) and \( x_l(\tau) \) could be correlated,

\[ E[\delta_k(\tau_1)\delta_l^*(\tau_2)] = 0 \text{ if } \tau_1 \neq \tau_2 \]

\[ \text{In view of Remark 3.1, the dependence of the signal model on the tracking time } t_1 \text{ is neglected.} \]
since it is assumed that the signals are not correlated across time. The notation * is used to denote the complex conjugate of the quantity in question.

\[ E[\theta_k(\tau_1)\theta^*_k(\tau_2)] = \Gamma_{kk} \text{ if } \tau_1 = \tau_2 \]

since two signals observed at the same instant of time may be correlated.

The quantity \( E[x(\tau)x^*(\tau)] = \Gamma \) is defined as the \((DxD)\) covariance matrix of the signals.

If \( \Gamma_{kk} = 0 \), it is assumed that \( \omega_k = \omega_k \).

The additive noise sequence \( \{u(\tau) = 1, 2, ..., N\} \) is assumed to be i.i.d Gaussian \((0,\sigma^2)\).

The variance of the additive noise is represented by \( \sigma^2 \).

Estimating the track parameters, in a multi-target, multi-sensor array context can be reduced to estimating the parameters of the following model

\[ y_m(\tau) = \sum_{k=1}^{K=M} f_{km} x_k(\tau) + u_m(\tau) \text{ m = 1, 2, .., M and } \tau = 1, 2, .., N \]  

(3.2)

Let the vector function \( f(.) \) for the narrowband signals be

\[ f_k = \text{col.}[1 e^{j\omega_{1k}} e^{j\omega_{2k}} ... e^{j\omega_{Mk}}] k = 1, 2, .., D \]  

(3.3)

where \( \psi_{km} \) is the phase delay of the narrowband signal from kth target at the mth sensor.

### 3.2.1.1 Fresnel Approximation

It can be shown [50] that the phase delay \( \psi_{km} \) of the kth signal impinging on a uniform linear array can be expressed as a function of bearing and range parameters as

\[ \psi_{km} = (m-1)(\pi \cos \theta_k) + (m-1)^2 \left(\frac{\pi d \sin^2 \theta_k}{r_k}\right) \]  

(3.4)

where 'd' is the inter element array spacing. The choice \( d = \lambda / 2 \) prevents spatial aliasing where \( \lambda \) is the carrier wavelength.

Equation (3.4) is referred to as the Fresnel approximation [50]. In near field, the planar approximation is invalid, instead the wavefronts are modeled as quadratic surfaces.

Let \( \phi_k = \pi \cos \theta_k \) and \( \xi_k = \left(\frac{\pi d \sin^2 \theta_k}{r_k}\right) \) \( k = 1, 2, ..., D \).

(3.5)

Substituting (3.5) in (3.4), the phase delay is given by

\[ w_{km} = [(m-1)\phi_k + (m-1)^2 \xi_k] \text{ k = 1, 2, ..., D} \]  

(3.6)

Substituting (3.6) in (3.3), the array manifold vectors, \( f(\cdot) \) are

\[ f_k = \text{col.}[1 e^{j(\phi_k+\xi_k)} e^{j(2\phi_k+2^2\xi_k)} ... e^{j(M-1)\phi_k+(M-1)^2\xi_k}] \text{ k = 1, 2, ..., D} \]  

(3.7)

Since \( f_k \) is a function of both \( \phi_k \) and \( \xi_k \), \( k = 1, 2, ..., D \), it is represented in the form \( f(\phi_k, \xi_k) \) for \( k = 1, 2, ..., D \). Substituting (3.7) in (3.2), the signal model can be rewritten in vector - matrix format as

\[ y(\tau) = A(\phi, \xi)x(\tau) + u(\tau) \text{ } \tau = 1, 2, ..., N \]  

(3.8)
where $y(\tau)$ is the M-vector of observations received by the M sensors. $x(\tau)$ is the D-vector of the input signal, $u(\tau)$ is the complex Gaussian additive noise vector. $A(\theta, \xi)$ is the $M \times D$ 'track' matrix comprising of D transfer vectors and is given by

$$A(\phi, \xi) \equiv [f(\phi_1, \xi_1) f(\phi_2, \xi_2) \ldots f(\phi_D, \xi_D)]$$

(3.9)

$\phi_k, \xi_k, k = 1, 2, \ldots, D$ are the unknown track parameters to be estimated. These are the primary parameters of interest. $\phi_k$ is a function of the arrival angle and $\xi_k$ is the function of both the angle and the range of the kth target.

The covariance matrix of the observation vector $y(\tau)$ is given by,

$$R = E[y(\tau)y^*(\tau)] = \rho I + A(\phi, \xi)\Gamma A^*(\phi, \xi)$$

(3.10)

R is an $M \times M$ matrix where M denotes the number of sensors.

### 3.2.2 Likelihood Expression

Since $Y = \{y(\tau), \tau = 1, \ldots, N\}$ is an independent Gaussian sequence with zero mean and covariance R, its joint probability density given $\phi, \xi, \Gamma, \rho$ can be written as

$$p(Y|\phi, \xi, \Gamma, \rho) = \prod_{\tau=1}^{N} 2\pi^{-M/2}(\det R)^{-1/2} \exp\left[-\frac{1}{2} y^*(\tau)R^{-1}y(\tau)\right]$$

$$= 2\pi^{-MN/2}(\det R)^{-N/2} \exp\left[-\frac{1}{2} \text{trace}\left(R^{-1} \sum_{\tau=1}^{N} y(\tau)y^*(\tau)\right)\right]$$

(3.11)

Thus, the density possesses a sufficient statistic $\hat{R}$ given by

$$\hat{R} = \frac{1}{N} \sum_{\tau=1}^{N} y(\tau)y^*(\tau)$$

(3.12)

Consequently, the log likelihood can be simplified as follows:

$$L(\phi, \xi, \Gamma, \rho) = \ln p(Y|\phi, \xi, \Gamma, \rho) = -\left[\frac{MN \ln 2\pi}{2} + \frac{N}{2} J(\phi, \xi, \Gamma, \rho)\right]$$

(3.13)

where $J(\phi, \xi, \Gamma, \rho) = \ln(\det R(\phi, \xi, \Gamma, \rho)) + \text{trace}(R^{-1}(\phi, \xi, \Gamma, \rho)\hat{R})$

(3.14)

Maximum likelihood estimates of parameter set $(\phi, \xi, \Gamma, \rho)$ are obtained by minimizing (3.14) subject to the condition that $\rho > 0$ and $\Gamma$ is positive definite. Equation (3.14) is identical to the 'stochastic ML' [6] criterion function for DOA estimation except that we are estimating D additional range parameters.

#### 3.2.2.1 Simplification of likelihood expression

Direct minimization of (3.14) is complicated and involves a search over $[2D(2D+3)/2 + 1]$ unknowns. A simpler solution which is separable, for estimation of the
2D track parameters can be obtained by proceeding in a manner similar to Section 2.2.3. The Taylor's series approximation on the inverse of the R matrix is used to simplify (3.14). Hence, the new simplified log likelihood of observations \( \{y(\tau), \tau = 1, ..., N\} \) from D targets incident on an array of M sensors is given by

\[
\ln p(Y|\phi, 5, \Gamma, p) = \frac{N}{2} \rho \left( \sum_{k=1}^{D} f_k^* \hat{R} f_k - \frac{N}{2} \rho M^2 \right) - \frac{N}{2} \text{tr} \left( \frac{B}{M} \right) - \frac{N}{4} \text{tr} \left( \frac{B}{M} \right)^2
+ \text{(terms involving only } \Gamma, p) + \text{(terms not involving } \phi, 5, \Gamma, p) + O \left( \frac{1}{M^L} \right). \tag{3.15}
\]

The form of the new likelihood function is identical to that derived in Theorem 2.1 except that it is also a function of the unknown range parameters through parameter \( \xi_k, k = 1, 2, .., D \). Hence, the necessary and sufficient conditions on the likelihood function for existence of a maximum are also identical to that obtained in Section 2.2.6. Additionally, closed form expressions for signal power and noise variance are given by

\[
\hat{\rho}_{\text{ML}} = \left( \frac{1}{M - D} \right) \text{tr} \left( \hat{R} \right) - \left( \frac{1}{M(M - D)} \right) \text{tr} [A(\phi, \xi) A(\phi, \xi) \hat{R}]
\tag{3.16}
\]

\[
\hat{\Gamma}_{\text{ML}} = \left( \frac{A^*(\phi, \xi) \hat{R} A(\phi, \xi)}{M^2} \right) - \left( \frac{\hat{\rho}_{\text{ML}} I_D}{M} \right) \tag{3.17}
\]

where \( \hat{R} \) is given by (3.12).

### 3.2.2.2 Upshot of likelihood simplification

A **computationally efficient separable** solution is thus obtained for track parameter estimation from equations (3.15)-(3.17). The important point is, approximating the log likelihood function subject to the assumption that \( \Gamma \) is positive definite gives a simple straightforward solution to both parameter estimation and estimate association. This is discussed in the ensuing section.

### 3.3. Development of the Proposed Tracking Algorithm (TAL)

#### 3.3.1 Parameter Estimation and Estimate Association

The proposed tracking algorithm involves joint estimation of DOA and range along with estimates of signal power and noise variance of the D targets at every instant of time wherein a sequence of N snapshots are obtained. Successive DOA estimation at different time instants from estimation methods like MUSIC [1] **cannot** be used for tracking, since peak picking from MUSIC spectrum **does not** suggest any ordering of estimates at contiguous time instants. In contrast, in ML methods [4, 6, 7], parameters
are estimated by minimizing a cost function involving the array response matrix or the 'track' matrix in which a structure is imposed on the parameters. This means that the kth column of the track matrix in (3.9) is a function of the parameters $\phi_k, \xi_k, k = 1, 2, \ldots, D$ which are to be estimated. This is the first step towards solving the estimate association problem. However, certain important conditions are to be satisfied in the estimation procedure before we can achieve ordering of estimates at contiguous time instants by mere one-to-one association.

In [31], two forms of the criterion function\(^3\) (3.14) are considered:

(a) $\Gamma$ and $p$ are unknown.

(b) $\Gamma$ and $p$ are known quantities.

For case (a), closed form expressions for $\hat{\Gamma}(\hat{\Gamma}_{\text{SML}})$ & $\hat{\phi}(\hat{\phi}_{\text{SML}})$ [6, 7] are given by

$$\hat{\Gamma}_{\text{SML}} = (A^*A)^{-1}A^*(\hat{R} - \hat{\rho}_{\text{ML}})A(A^*A)^{-1}$$ (3.18)

and

$$\hat{\phi}_{\text{SML}} = \left(\frac{1}{M - D}\right)\text{tr}(\hat{R}) - \left(\frac{1}{(M - D)}\right)\text{tr}[A(A^*A)^{-1}A^*\hat{R}]$$ (3.19)

But, the expression for $\hat{\Gamma}(3.18)$ is symmetric in the components of the DOA parameters, and the resulting criterion function formed by substituting $\hat{\Gamma}$ & $\hat{\phi}$ in (3.14) is also symmetric in $\phi_k, k = 1, 2, \ldots, D$. This means that, if $\hat{\phi}_k$ is an ML estimate, then, any vector obtained by permutating the components of $\hat{\phi}_k$ is also an ML estimate. Thus, if all parameters $\phi_k, k = 1, 2, \ldots, D$ are estimated using (3.14), correct association between estimates of the same target at two successive time instants is not possible.

However, if $\Gamma$ and $p$ are known, (case(b)), and $\hat{\Gamma}$ is positive definite ($\hat{\Gamma}_{\text{SML}}$ is positive definite asymptotically in N), the data covariance matrix, $R$ and its inverse are known quantities and the log likelihood function of (3.13) can be directly used to estimate the DOA parameters. Since the expression for $\hat{\Gamma}_{\text{SML}}$ is not used, the criterion function is no longer symmetric and hence a unique ordering of estimates for different time instants is obtained. A sufficient condition for identifiability requires targets to have different signal powers.

In the proposed method, the expression for $\hat{\Gamma}_{\text{ML}}(3.17)$ is not symmetric in view of the Taylor's series expansion on $R^{-1}$. When $\hat{\Gamma}_{\text{ML}}$ is substituted in the criterion function

\(^3\)Though range is not estimated, the form of the log likelihood function in [31] is the same as in (3.13). By including ranges, only the dimension of the parameter set is extended by D.
in (3.15), it is easy to see that, the resultant log likelihood is not symmetric in the components of \( \xi_k \), \( k = 1, 2, \ldots, D \). Hence, for different permutations of the component vector, the value taken by the criterion function in (3.15) will be different.

Thus, even if \( \Gamma \) and \( \rho \) are not known, using the proposed estimation scheme, estimates of two targets at two successive time instants can be uniquely associated by mere one-to-one association. Moreover, the modified form of the criterion function (3.14) assumes that \( \Gamma \) is positive definite and hence maximum likelihood estimates for \( \Gamma \) are obtained.

### 3.3.2 Maximum Likelihood estimation of DOA and range parameters

The procedure for maximum likelihood estimation of angle, range of the \( D \) targets using \( N \) snapshots \( Y = \{ y(\tau), \tau = 1, \ldots, N \} \) is outlined.

**Step 1:** From initial guess values for angle and range, estimate noise variance and signal powers by

\[
\hat{\rho}_{\text{ML}} = \left( \frac{1}{M-D} \right) \text{tr}(\hat{R}) - \left( \frac{1}{M(M-D)} \right) \text{tr}[A(\phi, \xi)A(\phi, \xi)\hat{R}] \\
\hat{\Gamma}_{\text{ML}} = \left( \frac{(A^*(\phi, \xi)\hat{R}A(\phi, \xi))}{M^2} \right) - \left( \frac{\hat{\rho}_{\text{ML}} I_M}{M} \right) \tag{3.20}
\]

**Step 2:** Using \( \hat{\rho}_{\text{ML}} \) and \( \hat{\Gamma}_{\text{ML}} \), obtain \( (\hat{\phi}_k)_{\text{ML}} \) and \( (\hat{\xi}_k)_{\text{ML}} \), \( k = 1, 2, \ldots, D \) by minimizing (3.15) with respect to \( \hat{\phi}_k \) and \( \hat{\xi}_k \), \( k = 1, \ldots, D \) respectively (separable solution).

**Step 3:** From (3.5), ML estimates\(^4\) of the DOA and range parameters at time \( t = t \), are given as

\[
(\hat{\theta}_k(t_1))_{\text{ML}} = \cos^{-1}\left( \frac{(\hat{\phi}_k(t_1))_{\text{ML}}}{\pi} \right), \ k = 1, 2, \ldots, D \tag{3.22}
\]

and

\[
(\hat{r}_k(t_1))_{\text{ML}} = \left( \frac{\pi d \sin^2(\hat{\theta}_k(t_1))_{\text{ML}}}{(\hat{\xi}_k(t_1))_{\text{ML}}} \right), \ k = 1, 2, \ldots, D \tag{3.23}
\]

which completes the estimation procedure. The estimates in (3.22) and (3.23) are indeed maximum likelihood estimates since estimates of \( (\hat{\phi}_k(t), \hat{\xi}_k(t)), k = 1, 2, \ldots, D \) obtained from Step 2 are necessarily ML estimates. This is the invariance property of any maximum likelihood estimator.

---

\(^4\)The time index '1' represents the track parameter estimates at tracking time \( t_1 \).
3.3.3 The proposed Tracking ALgorithm (TAL)

TAL comprises of the following steps:

(i) The inputs to the TAL algorithm at time $t = t_1$ are estimates $(\hat{\theta}_k(t_1 - T))$, and $(\hat{\phi}_k(t_1 - T))$, $k = 1, 2, \ldots, D$ of the track parameters obtained at time $t = t_1 - T$ and $N$ snapshots acquired at time $t = t_1$.

(ii) (Steps 1-3 above). Use estimates of track parameters at the $t = t_1 - T$ as initial guess values and estimate $(\hat{\phi}_k(t_1))$, and $(\hat{\phi}_k(t_1))$, $k = 1, 2, \ldots, D$ by minimizing (3.15). Then compute $(\hat{\phi}_k(t_1))_{\text{ML}}$ and $(\hat{\phi}_k(t_1))_{\text{ML}}$, $k = 1, 2, \ldots, D$ using (3.22) & (3.23). This gives updated positions of the targets.

(iii) Associate respective D 2-tuples (range and angle estimates) obtained at times $t = t_1$ and $t = t_1 - T$.

(iv) Make $t = t_1 + T$ and repeat steps (i)-(iii) till end of tracking period.

Note: The number of targets is always assumed to be known and is not estimated. Later on, it is experimentally verified that reasonable performance can be obtained even without knowledge of the targets.

3.3.4 Features of proposed method

(1). The criterion function maximized to obtain the track parameter estimates is different from that of 'stochastic ML' [6, 31]. In our method, the inverse of data covariance matrix $R$ is approximated by a second order Taylor series expansion for large $M$, the number of sensors in the linear array. A separable solution for estimation of the track parameters is obtained, i.e., the log likelihood function is maximized only over the track parameter set. Also, the estimation does not involve inversion of the $M \times M$ $R$ matrix. Instead, we need to invert a $D \times D$ matrix, where $D$ stands for number of targets, when computing the $B$ matrix. In practice, $D << M$ and this makes the ML estimation algorithm computationally efficient. In contrast to [6, 31], more number of sensors can be added into the array without significant increase in computation. This is because size of the matrix to be inverted in (3.15) does not depend on $M$ but only on $D$. We can thus get more accurate estimates of DOA and range. However, there is an upper bound on the number of sensors given by (3.40).

(2). The proposed criterion function in TAL is not symmetric in the components of the track parameters. This is in view of the large $M$ approximation on the data covariance
matrix and hence the estimates we obtain are naturally ordered. This fact is true even if the signal and noise powers are unknown.

(3). Another important point is computational efficiency of the tracking algorithm. In [6], the tracking algorithm needs the equivalent of (3.14) to be maximized at every time instant in order to maintain association between estimates and this requires inversion of the $M \times M$ $R$ matrix. The computationally efficient separable solution for DOA estimation cannot be used because it does not guarantee association. This implies, computation of $R^{-1}$ is essential for all time instants except the first where the separable solution is needed to estimate $\Gamma$ and $p$. However, for tracking with TAL, inversion of $R$ is not needed at any point in time.

(4). Both range and angle of targets are estimated taking into account the curvature of the impinging wavefront. Hence, unique localization of targets on a plane is possible in near field.

(5). TAL is iterative and information is processed in 'blocks' i.e., we get a set of snapshots in a small time interval, estimate the track parameters and these estimates (information) are utilized as initial guesses along with the next block of data for estimating the new track parameters. However, convergence to the global minimum is guaranteed only if initial guess values are in vicinity of the true values.

(6). With regard to asymptotic properties of estimators, it is shown in [10] that, accuracy of both deterministic [4] and stochastic maximum likelihood [6, 7] methods for DOA estimation with finite $N$ and a large enough $M$ are the same and that, the estimates are asymptotically consistent in $M$. Hence, the deterministic maximum likelihood method (DML) could also be used for tracking. However, if DML is employed, all parameters of the $D$ waveforms (corresponding to $D$ targets) should be estimated at every time instant to ensure proper estimate association and this is computationally not feasible.

3.4 Performance of Proposed Method

A situation with two moving targets is considered. Data for experiments are generated according to the model equations (3.7) and (3.8). Statistics of the estimated parameters like root mean-square error are computed to evaluate performance.
To start with, the TAL algorithm will be compared with Swindlehurst and Kailath's method [49] of estimating the DOA and range parameters for near-field sources. Though the method in [49] involves joint estimation of DOA and range and is not used for tracking purposes, a comparison is useful because the proposed method relies on simultaneous estimation of angle and range for target tracking.

3.4.1 Comparison with Swindlehurst and Kailath's method

The method in [49] uses a spatial analog of the Wigner-Ville distribution for estimating the DOA and range of a near-field sources. A signal subspace method, ESPRIT [46] is employed for estimating the Wigner-Ville kernel frequencies. The advantages in [49] is that, the source locations are estimated more accurately, closely-spaced sources are easily resolved and no search of a spectral surface is required. Using TAL, DOAs can be estimated equally well or better and superior range estimates are obtained. TAL has been simulated for all their test cases but their experimental results are directly taken from the paper for comparison. In addition, TAL can also handle correlated targets (multi-path reflections).

For all simulations shown, we assume a uniform linear array of 16 sensors with inter element spacing $d = \frac{\lambda}{4}$ (adequate for preventing spatial aliasing). Estimates are obtained after averaging over 100 runs of the algorithm. For the two target case, the range and angle estimates are plotted versus the number of snapshots. The signal-to-noise ratio (SNR) chosen is 20 dB. Good DOA estimates for both methods in all cases are obtained as shown in Fig. 3.2. However, the range estimates in [49] deteriorate for small number of snapshots. At close range (Fig. 3.3), it is seen that TAL gives good estimates whereas the range estimates get mixed up in the case of [49]. If this result were to be used for target tracking, it would result in wrong association of estimates. In comparison, even for small range difference of the order of $5\lambda$ (close range), TAL is able to distinguish the targets correctly and gives good estimates.

3.4.2 Comparison with Sword's algorithm

Trajectories of two targets moving in a straight line with constant velocity are tracked using the proposed TAL algorithm. In reality, the scenario considered could represent two aircraft on a sortie or surveillance mission. The proposed method is compared with Sword's algorithm [29] by considering two targets approaching the sensor array. Approaching targets are considered because rate of change of angle of arrival in
Comparison of Angle Estimates for Two Source Case, M = 16.

Number of snapshots, N

Fig. 3.2. Angle estimates for two source case. Runs = 100. SNR = 20 dB.

Comparison of Range Estimates for the Two Source Case, M = 16.

Number of Snapshots, N

Fig. 3.3. Range estimates for two source case. Runs = 100. SNR = 20 dB
Fig. 3.4. Averaged DOA tracks of the two targets for TAL and Sword's Methods.

Fig. 3.5. Log RMSE of DOA Estimates for both TAL and Sword's methods.
near-field is appreciable when compared to that of a target moving parallel to the sensor array. The number of snapshots in a sampling interval is 100 for all experiments.

As already mentioned, only angle tracks are obtained from [29] since range is not estimated. A comparison is made between DOA estimates obtained from the two methods. The estimates of [29] used for comparison were obtained by actually implementing Sword's algorithm for the specified scenarios.

The results of comparison are shown in Figs. 3.4 and 3.5. From the angle tracks in Fig. 3.4, the TAL DOA estimates are closer to the true values than the DOA estimates in [29]. More importantly, as time increases, the RMSE in TAL shows a decreasing trend whereas it continuously increases in Sword's method. The log root mean-square error (RMSE) of the DOA estimates averaged over 20 runs is plotted versus the track period in Fig. 3.5. This shows that, in Sword's method, error in estimate gets propagated along the track. DOA estimates are not stable probably because of the Taylor's series expansion and the subsequent first order approximation of the transfer vector function. Moreover, the algorithm uses information of only the first column of the data covariance matrix. The differentiation operation introduces noise into the system which gets propagated.

Estimation of track parameters in TAL involves multi-dimensional non-linear optimization. Two methods of optimization from the MATLAB optimization toolbox are implemented - the Nelder-Mead simplex search method and the Quasi-Newton (BFGS) method. The former method was found to be more robust to variations in number of targets and associated parameters like SNR, number of sensors, number of snapshots etc. Though TAL is iterative and is dependent on the accuracy of ML estimates, it is more stable than Sword's method. Having a sufficiently large number of sensors and a high sampling rate results in the initial estimates to be in vicinity of the true values and thus the corresponding ML estimates will converge to the global minimum. However, errors can propagate in TAL if ML estimates at time $t = t_0 - T$, which are fed as initial guesses to the estimation procedure at the following time instant, $t = t_1$, are not in vicinity of the true values.

3.4.3 Crossing target scenario
A more realistic situation wherein the targets cross each other is considered. The estimate association problem becomes crucial for this case. With various experimental
results, it is shown that TAL is able to track the targets by correctly associating track parameter estimates over the duration of the track. To describe different types of trajectories, the following definitions are stated:

**Target Track**: The trajectory of the targets in Cartesian co-ordinates (spatial domain).

**Range track**: Path described by the range estimates of a target over the tracking period.

**DOA track**: Path described by the DOA estimates of a target over the tracking period.

### 3.4.3.1 Experiment 1 (two targets, constant velocity, high SNR & small sampling interval)

Two fast moving targets are considered with \( v = 200 \text{ m/s} \). The targets are assumed to be moving in a straight line with constant velocities. The sampling interval is small (0.11s) and the SNR is 10 dB implying favorable operating conditions. The true and the estimated tracks of the TAL algorithm are plotted as shown in Fig. 3.6. The plotted track is the average of two test runs. The TAL algorithm is able to follow the targets' true trajectory over the entire tracking period accurately. It is also interesting to note that the errors increase slightly in track (2) near the end of the track period. We see from Fig. 3.6 that the target is close to endfire and going away from the broadside array. This explains the drop in performance of the method at low-angles. This can also be seen from the CRB expressions (Figs. 3.20 and 3.21) where the bounds on the DOA and range increase rapidly close to 0 or 180 deg.

### 3.4.3.2 Experiment 2 (two targets moving with constant velocity, low SNR and large sampling interval)

Experiment 2 illustrates that SNR and sampling interval are important factors in correct tracking and association. The trajectories are the same as in experiment 1, but SNR is decreased to 5 dB and sampling interval is doubled. The resulting target tracks plotted in the spatial domain in Fig. 3.7 show that the TAL method is not able to follow the correct track after cross-over. Reducing SNR (5 dB) and increasing sampling interval causes the following problem: the track parameter estimates are not close to their true values at cross over. This means, for estimating the track parameters at the following time instant, these erroneous estimates which are used as initial guess values are not in the neighborhood of the true values. This causes further deterioration in TAL estimates and thus the algorithm follows the wrong target track. A large increase in error after cross-over occurs since the targets tracks are flipped over. This experiment illustrates problems in convergence of the optimization procedure for lower sampling rates.
Fig. 3.6. High SNR case (10 dB). TAL tracks the trajectories accurately.

Fig. 3.7. SNR = 5 dB. TAL follows the wrong track and estimates are not accurate.
Fig. 3.8. Target track plots for reduced SNR (5 dB) but velocity = 150 m/s. Now, TAL is able to provide accurate estimates of the tracks for reduced SNR.
Fig. 3.9. Estimated and true angle tracks for TAL and Sword's algorithm. Sword's method fails to track the trajectories whereas accurate estimates are obtained for TAL for low SNR (5 dB) and reduced velocity (150 m/s). 

Fig. 3.10. Plot of estimated and true range tracks of the TWO targets. TAL is able to provide accurate estimates of the range tracks.
Effect of reducing SNR can be countered by reducing velocity of the targets. Targets are tracked correctly at a reduced SNR (5 dB) if the velocity is decreased from 200 m/s to 150 m/s as seen in Figs. 3.8, 3.9 and 3.10. Results of Sword's algorithm (Fig. 3.9) for the same conditions demonstrate the inability of the method to perform reasonably. Although, a trade-off in performance with SNR, sampling interval and velocity is expected, the threshold values appear to be significantly better in TAL indicating robustness of the method.

3.4.3.3 Experiment 3 (two targets and an interfering decoy)

Two targets describing non-linear (parabolic) trajectories with 10 dB SNRs are moving towards the sensor array. A 'dim' target with SNR 4 dB crosses the path of target 1 (Fig. 3.11). In practice, this can be a decoy. The snapshots (data) at the array have information about all three targets. The objective is to estimate the track parameters of only the two targets. TAL is able to follow the trajectory of target 1 inspite of the interfering decoy (Fig. 3.11) and estimates of angle track are accurate (Fig. 3.12). The range track estimates of the target close to the decoy (Fig. 3.13) are significantly noisy when decoy crosses target 1, but TAL is able to follow the true trajectory of target 1 correctly. This experiment demonstrates robustness of TAL and also shows that information about number of targets is not crucial for obtaining good estimates.

3.4.3.4 Experiment 4 (three targets with parabolic trajectories)

SNR of the third target in the previous experiment is also 10 dB and the sampling interval is smaller. Now, the three trajectories are correctly tracked (Fig. 3.14). Perfect angle tracks are obtained (Fig. 3.15), though range tracks are noisy, moreso for target 2 (Fig. 3.16).

It is thus seen SNR, velocity and the sampling interval are the important factors affecting the performance of the TAL method. The method performs well in different situations as indicated by the simulations and can be made feasible for practical implementation.

3.4.3.5 Experiment 5 (statistical performance)

This experiment illustrates statistical performance of TAL for the case in Fig. 3.17. The estimated angle and range tracks are shown in Figs. 3.18 and 3.19 respectively. The root mean square error (RMSE) of track parameter estimates obtained for 10 runs of
Fig. 3.11. The tracks are simulated using equations of projectile motion and describe parabolic trajectories. Inspite of the decoy (dim target) crossing the path, TAL is able to track the trajectory of target 1. This experiment also demonstrates robustness of TAL Inspite of lack of explicit information about the number of targets present. Both targets are moving towards the antenna array as indicated by the arrows. The dark dots show the estimated tracks and the light ones show the true tracks.
Fig. 3.12. Plot of estimated and true *angle* tracks of the TWO targets. The DOA estimates are accurate even when the decoy crosses target 1.

Fig. 3.13. Plot of estimated and true *range* tracks of the TWO targets. The range estimates show significant error at the point where decoy crosses target 1, but TAL is able to correct the error and follow the true trajectory of target 2 correctly.
The tracks are simulated using equations of projectile motion and describe parabolic trajectories. Instead of the decoy (dim target) crossing the path, we have a third target (10 dB) crossing the track of target 1. Now, instead of two, we estimate three tracks. TAL is able to track the trajectory of all three targets. This experiment also demonstrates ability of TAL to track three crossing targets.
Fig. 3.15. Plot of estimated and true angle tracks for the THREE target case.

Fig. 3.16. Plot of estimated and true range tracks for the THREE target case. The TAL range estimates are accurate even when target 1 and 3 cross each other.
Fig. 3.17. The averaged tracks over 10 trials of the two targets are plotted in X-Y coordinates. The two targets are moving towards each other as indicated by the arrows. Dark dots show estimated tracks and the light ones true tracks.
Table 3.1
Root Mean Square Error (RMSE) values of track parameters

<table>
<thead>
<tr>
<th>Time Instant (Sec)</th>
<th>RMSE DOA1 (Deg)</th>
<th>RMSE DOA2 (Deg)</th>
<th>RMSE Range1 (m)</th>
<th>RMSE Range2 (m)</th>
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Fig. 3.18. Plot of true and averaged angle tracks demonstrate very accurate DOA estimation by proposed method.

Fig. 3.19. Plot of true and averaged range tracks demonstrate fairly accurate range estimation by proposed method even if tracks cross each other.
the experiment are given in Table 3.1. The RMSE values for angle estimates are very accurate with an average RMSE of 0.04° over the tracking period. Range estimates are also accurate and average RMSE over the tracking duration is about 12 m. TAL is thus seen to provide consistently accurate estimates over the tracking period.

3.4.4 Extensions

(1) Simultaneous angle and range estimation facilitates estimation of velocity, acceleration of targets using a Kalman filter.

(2) Using estimated angle, range and velocity and the nature of trajectory (e.g., linear), one-step ahead prediction of position of targets can be achieved. The predicted angle and range estimates can be used as initial guesses for the ML estimation procedure to get better estimates of trajectories. But the above experiments, instead of the predicted estimates, the estimates themselves are used as initial guess values to attain rapid convergence.

3.5 Asymptotic Cramer-Rao Lower Bound (CRB) Expressions for the Track Parameters

Asymptotic Cramer-Rao matrix Bounds (CRB) are derived for the variance of ML range and DOA estimates of the signal sources incident on a large uniform linear array of sensors and their implications discussed. The derived asymptotic expressions are based on the large M Taylor's series expansion of the data covariance matrix $R^{-1}$ in the expression for the likelihood function (3.13). The interest is only in the lower bound on the variance of the track parameters and hence quantities $\Gamma$ and $p$ treated as nuisance parameters and assumed to be known. The ratio $(\Gamma / p)$, $\Gamma$ and $p$ being true values, is defined as signal-to-noise ratio denoted by SNR.

For convenience of notation, let $\beta = \text{col.}(\theta_1, \ldots, \theta_D, r, \ldots, r)$ represent the true track parameter vector. For $k, \ell = 1, 2, \ldots, D$, let $\nabla_k$ denote the derivative $(\partial / \partial \beta_k)$. From this definition,

$$\nabla_k \ln \det R = \text{tr} \left[ R^{-1} \nabla_k R \right]$$

and

$$\nabla_k R^{-1} = -R^{-1} \nabla_k RR^{-1}.$$  \hspace{1cm} (3.24)

The first derivative of the original log likelihood given in (3.13) can be written as

$$\nabla_k \ln p(Y|\beta) = - \left( N / 2 \right) \text{tr} \left[ R^{-1} (\nabla_k R) (I - R^{-1} \hat{R}) \right]$$  \hspace{1cm} (3.25)

The second derivative is then given by,

$$\nabla_k^2 \ln p(Y|\beta) = - \left( N / 2 \right) \text{tr} \left[ \nabla_k (R^{-1} (\nabla_k R) (I - R^{-1} \hat{R}) - R^{-1} \nabla_k R \nabla_k R^{-1} \hat{R}) \right]$$  \hspace{1cm} (3.26)
The only random variable in (3.26) is \( \hat{R} \) which occurs linearly and its expectation is \( R \). It is well known that \( E[\hat{R}] = R \) as the number of observations, \( N \) becomes large. The expectation of the first term in (3.26) is zero. Defining the matrix \( Q \) whose elements comprise the expectations of the terms in (3.26),

\[
Q_{k\ell} = E \left[ \frac{\partial^2 \ln p}{\partial \beta_k \partial \beta_\ell} \right] = -\text{Re} \left( \frac{N}{2} \text{tr} \left[ \nabla_k R \nabla_\ell R^{-1} \right] \right) \quad k, \ell = 1, 2, \ldots, D \quad (3.27)
\]

Therefore, the CRB on the variance of \( \hat{\beta} \) is given by \( \text{CRB}_\beta = Q^{-1} \). \( (3.28) \)

Here \( Q \) is defined to be the Fisher Information Matrix (FIM). The bound given by \( (3.28) \) is asymptotic in the number of sensors (\( M \)) and for arbitrary number of snapshots (\( N \)).

Almost-exact closed form CRB expressions are derived for angle and range parameters in terms of the system parameters \( N, M, \text{SNR}, f, d, \Theta \) and their interaction is discussed. Expressions are derived first for the single source case (\( D = 1 \)). The CRB matrix for the case of \( D \) uncorrelated targets is \( 2D \times 2D \) block diagonal and is given by

\[
\text{CRB} \equiv \begin{bmatrix}
\text{CRB}_{\theta\theta}^1 & \text{CRB}_{\theta r}^1 & 0 & 0 & 0 & 0 \\
\text{CRB}_{r\theta}^1 & \text{CRB}_{rr}^1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \text{CRB}_{\theta\theta}^D & \text{CRB}_{\theta r}^D \\
0 & 0 & 0 & 0 & \text{CRB}_{r\theta}^D & \text{CRB}_{rr}^D \\
\end{bmatrix}^{-1}
\]

From \( (3.29) \), it is observed that, the CRB matrix for the general \( D \) target case is a simple extension of the single target case. For the single source case (\( D = k \) implies the \( k \)th source),

\[
\text{CRB}_k \equiv Q_k^{-1} = \begin{bmatrix}
Q_{\theta\theta}^k & Q_{\theta r}^k \\
Q_{r\theta}^k & Q_{rr}^k \\
\end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix}
Q_{\theta\theta}^k & -Q_{\theta r}^k \\
-Q_{r\theta}^k & Q_{rr}^k \\
\end{bmatrix} \quad (3.30)
\]

where \( Q_k \) is the Fisher information matrix (FIM). The parameter set \( \beta = \text{col.} (8, r) \). The elements of the FIM are given by

\[
Q_k = \begin{bmatrix}
Q_{\theta\theta}^k & Q_{\theta r}^k \\
Q_{r\theta}^k & Q_{rr}^k \\
\end{bmatrix}
\]

From \( (3.27) \), the elements of \( Q_k \) are given by
The CRB matrix and the FIM are symmetric and positive definite. Closed form expressions for (3.31) - (3.34) are derived in terms of the quantities $r$, $\theta$, $N$, $M$, SNR and $d$. These are then used to derive the CRB matrix bound on the range and angle estimates.

A theorem on the CRB matrix is now stated.

### 3.5.1 Theorem 3.1

The Fisher information matrix for the unbiased track parameter estimates, $\hat{r}$ and $\hat{\theta}$ for large $M$ for the single source case is given by

\[
Q_{\theta \theta}^k = -\text{Re} \frac{N}{2} \text{tr} \begin{bmatrix} \frac{\partial R}{\partial \theta_k} \frac{\partial R}{\partial \theta_k} \end{bmatrix} \tag{3.31}
\]

\[
Q_{\theta r}^k = -\text{Re} \frac{N}{2} \text{tr} \begin{bmatrix} \frac{\partial R}{\partial \theta_k} \frac{\partial R}{\partial r_k} \end{bmatrix} \tag{3.32}
\]

\[
Q_{r \theta}^k = -\text{Re} \frac{N}{2} \text{tr} \begin{bmatrix} \frac{\partial R}{\partial r_k} \frac{\partial R}{\partial \theta_k} \end{bmatrix} \tag{3.33}
\]

\[
Q_{rr}^k = -\text{Re} \frac{N}{2} \text{tr} \begin{bmatrix} \frac{\partial R}{\partial r_k} \frac{\partial R}{\partial r_k} \end{bmatrix} \tag{3.34}
\]

The values of all parameters appearing in these expressions are their true values. SNR is given by $(\Gamma / \rho)$.

**Proof:**

The proof is in Appendix VII. A list of identities involving the transfer vectors and their derivatives is given in Appendix VI which are useful for simplifying equations (3.31) - (3.34).

### 3.5.1.1 Corollary 3.1

The CRB matrix given by (3.30) has the following closed form expression:

\[
\text{CRB} = \begin{bmatrix} \text{CRB}_{\theta \theta}^k & \text{CRB}_{\theta r}^k \\ \text{CRB}_{r \theta}^k & \text{CRB}_{rr}^k \end{bmatrix}
\]
\[
\frac{12r_k}{NM^3(SNR)(\pi \sin \theta_k)^2} \begin{bmatrix}
\frac{16}{r_k - 64M \cos \theta_k} & -15 \\
-15 & \frac{16C - 15}{(M \sin \theta_k)^2} - 15C \cos \theta_k
\end{bmatrix}
\]

(3.38)

where \( C = 1 - \frac{(4M \cos \theta_k)}{r_k} \).

The determinant in (3.30) is computed by \( A = Q_{\theta\theta}^k Q_{\theta r}^k - Q_{\theta r}^k Q_{\theta \theta}^k \) using (3.35) - (3.37).

Also, results (3.35) - (3.37) substituted in (3.29) give FIM and CRB matrices for the general D target case.

3.6 Information on System Parameters From CR bounds

Closed form expressions of FIM and CRB on DOA and range are derived ((3.35) - (3.38)) as explicit functions of the various parameters in tracking system \(-M, N, SNR,\) sensor spacing, range and DOA. The interest is not in the actual values of the CRB persay, but in the study of how these system parameters affect the accuracy of DOA and range estimates. The CRB expressions derived in the previous section are used as a tool towards this purpose. The family of curves obtained by varying these parameters suggest possible regions of operation of the tracking system for obtaining good estimates.

Also, based on the expressions for \( \text{CRB}_{\theta\theta}^k \) & \( \text{CRB}_{\theta r}^k \), an inequality which suggests both an upper bound on the number of sensors and a lower bound on range of the targets can be obtained from (3.38).

3.6.1 Theorem 3.2

The minimum range of a target with a DOA \( \theta_k \), which can be measured for a given array aperture is given by the inequality

\[
r_k > 64 \text{Md} \sqrt{(1 - \sin^2 \theta_k)}
\]

(3.39)

where \( d \) = spacing between the sensors in the ULA, \( M \) = number of sensors in the ULA, \((M-1)d)\) is defined to be the array aperture and \( \cos \theta_k = \sqrt{(1 - \sin^2 \theta_k)} \) in order to make it symmetric in the interval between 0 and 180 degrees. This is because the phase/time delay is an even function of the angle over this interval.

The inequality of (6.1) can also be expressed in the form

\[
M < \frac{r_k}{64d \sqrt{(1 - \sin^2 \theta_k)}}
\]

(3.40)
which gives the upper limit on the number of sensors in the array.

The proof follows trivially, since, CRB mamx given by (3.38) is positive definite only if (3.39) is satisfied.

3.6.2 Discussion

This chapter has three key expressions - the phase delay relation (3.4), the log likelihood function for the proposed ML method (3.15) and the CR bounds (3.35-3.38). The inequality of (3.39) provides a link between each of the three expressions.

The ML method facilitates an increase in M, the number of sensors for obtaining better estimates with reduced complexity. A second order Taylor's series approximation of the inverse of the data covariance matrix allows us write the modified likelihood function in a form wherein only a DxD mamx needs to be inverted, D representing the number of targets. A second order Taylor's approximation is made, in order to express the phase delay as a function of both DOA and range of the D targets. The derived CR bound expressions inherently comprise both of these approximations.

Though the ML method allows for increase in M, (3.40) gives an upper bound on M. This is because, the approximation used for the phase delay becomes inaccurate for increasing number of sensors. There is a trade off between the number of sensors for parameter estimation and the deterioration of the approximation of phase delay which is reflected in the inequality of the CR bound. Thus, it is reasonable to expect that CRB expressions have to satisfy certain conditions to be positive definite. In other words, if the CRB mamx is not positive definite, it implies that the phase delay expression is not valid in which case there is no point in joint range and DOA estimation. So, the inequality provides information on what range each of the parameters should be set up so that good estimates of DOA and range can be obtained.

Considering (3.39), if array aperture is pre-specified, i.e., for a constant sensor spacing and array size, the lower limit on range (3.39) is a large value for \( \theta \) close to 0 and \( \pi \). This means, for end-fire conditions, targets situated very close to the array cannot be resolved. This is also intuitively obvious since, for a broadside array, the beam is strong near \( \theta = \pi/2 \). As \( \theta \) approaches \( \pi/2 \) (array broadside), the lower limit according to (3.39) decreases which means even very small ranges can be resolved. Also, as 'd' is decreased, i.e., sensors are close to each other, targets closer to the array can be tracked.
In (3.40), for a given carrier frequency and angle, as range increases, the upper limit on number of sensors increases. There should be more number of sensors in the array as range increases. From (3.38), the ratio is \( \left( \frac{r_t^4}{M^4} \right) \). This conveys the important idea that, M is the key parameter which has to be increased in order to reduce the variance of the range estimate. In other words, (3.38) suggests that, for good estimates of range, a larger array is required. It is also true from the array perspective i.e., for a broadside ULA, the beamwidth is \( \frac{2}{(M-1)} \) and more number of sensors reduces beamwidth thus increasing resolution. Though there is an upper bound on M, good results are obtained when the number of sensors chosen is close to the upper bound. For lower operating frequencies, d is large which limits the number of sensors in the array and hence the accuracy of the estimates are affected.

A similar argument holds for varying \( \theta \). At array endfire, \( \sin \theta \) is close to its minimum value, so that \( \sqrt{1 - \sin^2 \theta} \) is maximum. By (3.40), we can only have a small number of sensors which limits the accuracy.

### 3.6.3 Effect of angle on the CR bounds

The behavior of the CRB for varying angles is illustrated in Figs. 3.20 and 3.21. The principal diagonal terms of the CRB matrix in (3.38) represent lower bound on the variance of DOA and range estimates of the target. The magnitude of non-diagonal terms ensure the matrix is positive definite. As the expression of (3.38) indicates, the combination of sine, and \( \cos \theta \) terms in the denominator makes CRB large at the edges and minimum at \( \theta = \pi/2 \). Intuitively, it is obvious that CRB on angle and range will be minimum for \( \theta = \pi/2 \) since directivity of a broadside array is maximum in this direction. The effect of SNR is straightforward; the bound is small for higher SNRs and has a large value for lower SNRs.

### 3.6.4 Effect of array spacing

The spacing between sensor elements is directly proportional to the wavelength of the incident signal and thus inversely proportional to operating frequency. Figs. 3.22 and 3.23 give plots of CR bounds for angle and range estimates respectively versus range. Three curves are plotted for different spacing values. While CRB for angle increases for increasing d, CRB for the range decreases with increase in d.
Fig. 3.20. Theoretical Cramer-Rao bounds on DOA versus angle for varying SNR.

Fig. 3.21. Theoretical Cramer-Rao bounds on range versus angle for varying SNR.
Fig. 3.22. Theoretical Cramer-Rao bounds on DOA versus range for varying d.

Fig. 3.23. Theoretical Cramer-Rao bounds on range versus range for varying d.
Physically, as array spacing increases, the antenna aperture increases which means the beam is not concentrated in one particular direction and so the angular resolution decreases. Error in angle thus increases which means that the bound on the angle should increase. Also, a decrease in array spacing implies the sensors are clustered closely which increases directivity and thus better angular resolution and better angle estimates are obtained. This implies the bound on the error should be smaller.

Considering range estimates, the physics of the problem can be explained as follows. A decrease in array spacing for a given number of sensors implies that the beam strength or power of the antenna is reduced. Since measurement accuracy of 'range' or 'distance' of a target from the sensor array is directly proportional to the strength of the signal, it is clear that reduction of antenna power affects its ability to measure range. Analytically, the \((1/d^2)\) term in the \(\text{CRB}_\theta\) expression ((3.38)) coupled with the \(r^4\) term blows up the value very quickly which can be observed in Fig. 3.23. The only way to increase antenna power is to have more number of sensors in the array but it is bounded by (3.40). Equation (3.40) suggests a trade-off for angle and range accuracy. Analytically, \(M\) should be increased in \((r^4 / M^4)\) ((3.38)) so that \(\text{CRB}_\theta^\lambda\) reduces. In practice, for maximizing performance of the tracking system, the need is to minimize beamwidth and maximize power simultaneously which means, having smaller and larger spacing simultaneously. A possible solution would be to design an algorithm which uses information from two phased arrays operating at different frequency bands to obtain the track parameter estimates.

3.7 Conclusions

The key idea of TAL is the Taylor's series approximation which gives a non-symmetric form for the signal covariance estimator (3.21). This forces the corresponding likelihood function to be non-symmetric in the components of the track parameters and ensures automatic ordering of estimates at successive time instants. The new criterion function is computationally efficient because, it needs inversion of the \(D \times D\) B matrix whereas in [6, 31], inversion of the \(M \times M\) data covariance matrix is needed. Asymptotic expressions are derived for CR bounds on DOA and range estimates as explicit functions of the system parameters namely DOA, range, snapshots, sensors and SNR facilitating choice of tracking system with accurate track parameter estimates. The bounds lead to a tradeoff in estimation accuracy and indicate use of two linear arrays with different sensor spacing which could minimize error simultaneously in DOA and range estimation.
4. A MAXIMUM LIKELIHOOD APPROACH FOR TRACKING MULTIPLE WIDEBAND SIGNALS

4.1 Introduction

In this chapter, a new scheme to track direction of arrival (DOA) of multiple wideband signals involving autoregressive (AR) parameter estimation and pattern classification is proposed. The radiating sources are modeled as wideband signals implying that the signal bandwidth (W) is very much greater than the reciprocal of transit time (T_{trans}) of the wavefront across the array, i.e., WT_{trans} >> 1. Inherently, wideband signals have higher noise rejection capability. Thus accurate estimates of azimuth/bearing angle parameters can be obtained in a low signal to noise ratio environment. The key idea is that, the wideband signals are modeled as vector autoregressive (AR) processes so that their spectral densities are characterized by a finite number of parameters. The AR parameters and hence the DOAs are estimated by a maximum likelihood approach. The estimated AR parameters of the sources are utilized for estimate association. A brief review of the different source localization techniques for wideband signals was given in Chapter 1.

4.2 Signal Model

4.2.1 Narrowband signal approximation

A discussion about the nature of the narrowband signal approximation is now provided, motivating the choice of the wideband signal model. With the complex envelope representation, the kth source signal over N observation times is given by

\[ x_k(\tau) = w_k(\tau)e^{i\nu_k(\tau)}e^{i\omega_0 \tau}, \tau = 1, 2, \ldots, N \text{ and } k = 1, 2, \ldots, D \]

where \( \omega_0 \) is the center frequency. D defines the number of sources/targets. For a narrowband signal, the bandwidth of the amplitude (\( w_k(\cdot) \)) and phase (\( \nu_k(\cdot) \)) is small when compared to \( \omega_0 \), i.e., the amplitude and phase are slowly varying functions. The sequence \( \{x_k(\tau), \tau = 1, 2, \ldots, N\} \) represents the kth signal waveform as observed from the reference point in the array. The combined signal received from the D sources at the mth sensor at time \( \tau \) can be written as
\[ y_m(\tau) = \sum_{k=1}^{D} x_k(\tau - \psi_{km}) + u_m(\tau) \quad \tau = 1, 2, ..., N, \quad m = 1, 2, ..., M. \] (4.2)

where \( \psi_{km} \) is the propagation delay of the kth signal impinging on the array at the mth sensor from the reference point. The propagation delay at the mth sensor is given by
\[ \psi_{km} = (m - 1)\phi_k \] (4.3)
and
\[ \phi_k = \pi \sin \theta_k \] (4.4)
where \( \theta_k \) is the arrival angle of the kth signal source.

The quantity \( u_m(\tau) \) is the additive Gaussian noise at the mth sensor. From the narrowband assumption, variations in the amplitude and phase modulation are insignificant during the transit time of the wavefront across the array. Now,
\[ x_k(\tau - \psi_{km}) = w_k(\tau - \psi_{km}) e^{j\nu_k(\tau - \psi_{km})} e^{j\phi_{km}} e^{-\alpha_{km}} \] (4.5)

By the narrowband approximation,
\[ w_k(\tau - \psi_{km}) \equiv w_k(\tau) \quad \text{and} \quad \nu_k(\tau - \psi_{km}) \equiv \nu_k(\tau) \] (4.6)
Substituting (4.6) into (4.5),
\[ x_k(\tau - \psi_{km}) \equiv \left( w_k(\tau)e^{j\nu_k(\tau)}e^{j\phi_{km}}e^{-\alpha_{km}} \right) \] (4.7)

From the definition in (4.1), (4.7) becomes
\[ x_k(\tau - \psi_{km}) \equiv \left( x_k(\tau)e^{-\alpha_{km}} \right) \] (4.8)

This implies that the time delay is transformed into only a pure phase delay of the reference signal. This phase delay depends only on the center frequency, the separation between sensors and direction of arrival. However, it is independent of the time variable. But, in practice this need not be satisfied. If the entire information is concentrated in a narrowband of frequencies, it is more susceptible to broadband jamming signals as in a typical hostile environment. Also, when a signal is sent out, instead of the signal being reflected off the target, there may be some absorption or the medium through which the signal propagates can be dispersive. This results in attenuation and compression of the signal. In such situations, it is more appropriate to consider the unapproximated version (4.5) which specifies a wideband signal model.

### 4.2.2 Wideband signal model

Let \( \{ Y_m(L), L = 0, 1, ..., N-1 \} \) be the Discrete Fourier Transform (DFT) of the array output \( \{ y_m(\tau), \tau = 1, 2, ..., N \} \), i.e.,
\[ Y_m(L) = \sum_{\tau=1}^{N} \exp \left\{ -\frac{j2\pi L \tau}{N} \right\} y_m(\tau) \quad L = 0, 1, ..., N-1. \] (4.9)
Similarly, the DFT of \( \{u_m(z), \tau = 1, 2, \ldots, N\}, u_m(.) \) being i.i.d Gauss \((0, \sigma)\) is given by \( U_m(\ell) \). Also, DFT of \( \{x_k(\tau), \tau = 1, 2, \ldots, N\} \) is \( X_k(\ell), \ell = 0, 1, \ldots, N - 1 \).

In contrast to the narrowband assumption, the phase delay, is a function of frequency with \( \omega_\tau \Delta (2\pi \ell / N) \) from (4.9). From (4.2) and (4.9),

\[
Y_m(\ell) = \sum_{k=1}^{D} \exp\{-j(m-1)\omega_k\phi_k\} X_k(\ell) + U_m(\ell) \tag{4.10}
\]

Equation (4.10) can be written in the form

\[
Y(\ell) = AX(\ell) + U(\ell) \tag{4.11}
\]

where \( A = [f_{1\ell}(\phi_1) f_{2\ell}(\phi_2) \ldots f_{D\ell}(\phi_D)] \) is the vandermonde wideband DOA matrix.

For a uniform linear array, the array manifold vector is given by

\[
f_{k\ell}(\phi_k) = \text{col} \{1, e^{-j\omega_k\phi_k}, \ldots, e^{-j\omega_k(M-1)\phi_k}\} \tag{4.12}
\]

From (4.11), the spectral density of \( y(\cdot) \) is defined as

\[
S_f = E[Y(\ell)Y^*(\ell)] = N(A_\ell^\tau \Gamma(\omega_\tau)A_\ell + \rho I) \tag{4.13}
\]

where \( \Gamma(\omega_\tau) \) is the spectral density matrix of \( x(\cdot) \) at frequency \( \omega_\tau = (2\pi \ell / N) \Delta \).

### 4.3 Maximum Likelihood Parameter Estimation

#### 4.3.1 Direction of arrival estimation

The problem of simultaneously estimating the direction of arrival and the spectral densities can be treated as a two-dimensional (2D) spectral estimation problem. A method based on the maximum likelihood (ML) principle is given for estimation of the DOAs.

It is well known that sequence \( \{Y(\ell), \ell = 0, 1, \ldots, N - 1\} \) is i.i.d with Gaussian density \((0, S_f)\) [65]. The joint probability density of \( \{Y, = Y(0) Y(1) \ldots Y(N - 1)\} \) is

\[
p(Y, \phi, \Gamma(\omega_\tau), \rho) = \prod_{\ell=0}^{N-1} (2\pi)^{-M/2} |\text{det}(S_f)|^{-1/2} \exp\left[-\frac{Y^*(\ell)S_f^{-1}Y(\ell)}{2}\right] \tag{4.14}
\]

Simplifying (4.14), maximum likelihood estimates of the parameter set \( \{\phi, \Gamma(\omega_\tau), \rho\} \) are obtained by minimizing the criterion function

\[
J(\phi, \Gamma(\omega_\tau), \rho) = \frac{1}{2} \sum_{\ell=0}^{N-1} \left\{ \ln|\text{det}S_f^\tau| + \text{trace}[S_f^{-1}Y(\ell)Y^*(\ell)] \right\} \tag{4.15}
\]

\( S_f^\tau \) is a function of both unknown DOA and spectral density. However, simultaneous optimization of (4.15) is too complicated. Instead, it is possible to estimate the spectral density, \( \Gamma(\omega_\tau) \) and noise variance, \( \rho \) separately as a function of only the DFT of sensor data and then substitute their estimates in (4.13) to get \( \hat{S}_f(\phi) \). \( \hat{S}_f(\phi) \) substituted in (4.15) provides the separable form of likelihood expression. The resulting function of only the arrival angles and data to be minimized to obtain the D DOA estimates is given by
\[ J(\phi) = \frac{1}{2} \sum_{\ell=0}^{N-1} \left\{ \ln \det \hat{S}_\ell(\phi) + \text{trace}[\hat{S}_\ell^{-1}(\phi)Y(\ell)Y^*(\ell)] \right\} \] (4.16)

By parameterization of the \textit{wideband} signal \( x(.) \) by a vector \( AR \) model, the spectral density matrix and noise variance can be estimated in terms of the \( AR \) parameters and variance of the driving process noise.

### 4.3.2 Estimation of Spectral density matrix

Now, the \textit{wideband} signals are modeled as vector \( AR \) models so that their spectral densities are characterized by a finite number of \( AR \) parameters [15]. The spectral density becomes a function of these parameters which can be estimated. This problem is one of \textit{ML} estimation of \( AR \) parameters from a noisy \( AR \) process [40].

The \textit{wideband} signals \( x(.) \) modeled as stationary autoregressive processes are defined by

\[ x_k(\tau) = \sum_{p=1}^{P} \alpha_{kp} x_k(\tau - p) + \tilde{n}_k(\tau) \quad k = 1, 2, \ldots, D \] (4.17)

where the driving noise, \( \tilde{n}_k(\tau) \) is assume to be i.i.d Gauss \((0, \rho_k)\). \( P \) is the order of the model. \( \{\alpha_{kp}\}_{p=1}^{P} \) are the \( AR \) coefficients of the \( k \)th signal. For convenience, \( P \) is chosen to be 2, i.e., each signal is modeled as a second order stationary \( AR \) process. The principal diagonal elements of the spectral density matrix of the signals is given by

\[ \Gamma_{kk}(\omega) = \rho_k / \left[ (1 + \alpha_{k1}^2 + \alpha_{k2}^2 + 2\alpha_{k1}\alpha_{k2} - 1) \cos(\omega) - 2\alpha_{k2} \cos(2\omega) \right] \] (4.18)

\( k = 1, 2, \ldots, D \) and \( \ell = 0, 1, \ldots, (N - 1) \).

The spectral density matrix is a function of the \( AR \) parameters \( \{\alpha_{k1}, \alpha_{k2}, \rho_k\} \), \( k = 1, 2, \ldots, D \).

To estimate \( AR \) parameters, consider output of the first sensor.

\[ y(\tau) = \sum_{k=1}^{D} x_k(\tau) + u(\tau), \quad \tau = 1, 2, \ldots, N \] (4.19)

Taking DFT on both sides,

\[ Y(\ell) = \sum_{k=1}^{D} X_k(\ell) + U(\ell) \quad \ell = 0, 1, \ldots, (N - 1) \] (4.20)

Taking discrete Fourier transform of (4.17), for \( P = 2 \),

\[ X_k(\ell) = \alpha_{k1} \lambda^{-\ell} X_k(\ell) - \alpha_{k2} \lambda^{-2\ell} X_k(\ell) + \hat{N}_k(\ell) \] (4.21)

where \( \lambda = \exp(2\pi j / N) \).
Rearranging (4.21), taking expectation and simplifying,
\[ \Gamma_{kk}(\omega) = \mathbb{E}[X_k(\ell)X_k^*(\ell)] = \frac{\mathbb{E}[\hat{N}_k(\ell)\hat{N}_k^*(\ell)]}{\|h_k(\ell)\|^2} \]  
(4.22)

where \( h_k(\ell) = 1 - \alpha_k \lambda^{-\ell} - \alpha_{k+1} \lambda^{-2\ell} \)

Also, \( \mathbb{E}[\hat{N}_k(\ell)\hat{N}_k^*(\ell)] = Np_k \)  
(4.23)

(4.24)

From (4.22) and (4.24), we get the spectral density matrix to be
\[ \Gamma(\omega) = \mathbb{E}[X(\ell)X^*(\ell)] = N \left\{ \sum_{k=1}^{p} \frac{\rho_k}{\|h_k(\ell)\|^2} \right\} \]  
(4.25)

In (4.20), since \( X(.) \) and \( U(.) \) are zero mean, \( Y(.) \) is also zero mean.
Also, \( \mathbb{E}[U(\ell)U^*(\ell)] = Np \)

Hence, from (4.20) and (4.26),
\[ \mathbb{E}[Y(\ell)Y^*(1)] = B(\ell) = N \left\{ \sum_{k=1}^{p} \frac{\rho_k}{\|h_k(\ell)\|^2} + \rho \right\} \]  
(4.27)

Hence the likelihood function can be written as,
\[ p(Y(0), Y(2), \ldots Y(N-1)|\Theta) = \prod_{\ell=0}^{N-1} \left[ \frac{\exp\left\{ -\frac{Y(\ell)Y^*(\ell)}{2B(\ell)} \right\}}{\sqrt{2\pi}B(\ell)} \right] \]  
(4.28)

where the parameter vector is \( \Theta = \{ a, a_r, \rho_k, \rho \} \), \( k = 1, \ldots, D \).

From (4.28), for obtaining the ML estimates of the AR parameters, the criterion function to be minimized becomes
\[ J(\Theta) = \sum_{\ell=0}^{N-1} \left[ \ln(B(\ell)) + \frac{Y(\ell)Y^*(\ell)}{B(\ell)} \right] \]  
(4.29)

Once \( \hat{\Theta}_{ML} \) is obtained from (4.29), \( \hat{S}_\ell = S_\ell(\hat{\Theta}_{ML}) \) and so \( S \) becomes a known function. Now, (4.16) can be used to estimate the DOAs.

**Remark 4.1**

The order of the AR model for each signal is assumed apriori for estimation. If signals are closely spaced, a higher order may be required to characterize the information, increasing dimensionality of optimization. The model order can also be estimated [39].

### 4.4 Wideband Signal Model and Estimate Association

The wideband source signal model and the resulting parametric form of its spectral density is crucial for tracking and estimate association. The spectral parameters
provide a distinct feature for each signal source which allows for unique identification. Hence, accurate estimate association can be achieved even for crossing targets (signals). As usual, there are two different time scales, one over which snapshots are collected (indexed by $\tau$) and another, representing tracking time (indexed by $t$).

At time instant $t = t_0$, in the narrowband case, an unlabeled DOA estimate is assigned to a particular track whose signal power estimate (at $t = t_0 - T$) is closest to the signal power estimate of the unlabeled DOA at time $t = t_0$. However, in wideband case, a feature vector is formed from estimated AR parameters (characterizing the spectral density (signal power)) and DOA of the source. The Bayes classifier [17] is then used to assign the unlabeled feature vector to a particular track. Estimate association is performed in higher dimensional space ($\mathfrak{R}^{p+2}$), $p$ specifying order of the AR model. While deriving the discriminant functions, the parameters in the feature vector are assumed to obey a multi-variate Gaussian density with arbitrary covariance structure. Association in higher dimension ensures better quality of association in low SNR situations and also when the spectra of different signals are almost identical. Nevertheless, there exists a trade-off in complexity of spectral estimation.

In the narrowband case, a closed form solution can be obtained for the ML signal power and noise variance estimates (3.16 and 3.17). However, in spectral estimation, an extended range of parameters have to be estimated simultaneously (4.29) since no closed form solution exists for the estimated spectral density. This can result in a loss of information, thus affecting estimate association accuracy. In the presence of $D$ targets, each modeled by an AR ($p$) process, thus, $[(p+1)D + 1]$ parameters are to be estimated in addition to the $D$ arrival angles. Thus, the ML method of AR parameter estimation becomes unmanageable in the presence of large number of targets.

4.5 Bayes Classification to get Updated Target Estimates

The estimate association problem is solved by using the Bayes classifier. A feature vector is formed for a particular signal (target) from estimates of the AR parameters, driving noise variance and the DOA at that time instant. Each target is defined as a class into which each DOA estimate is classified. Each class comprises of estimated target trajectory and estimates of the AR parameters.
An unassociated feature vector of a particular class at tracking time instant $t = t_1$ is defined as $\hat{\psi}_k^i \equiv \text{col}[\hat{\theta}_k^i, \hat{\omega}_k^i, \hat{\sigma}_{2k}^i, \hat{\rho}_{k}^{i}] \ k \in \{1, 2, \ldots, D\}$ where $\hat{\theta}_k^i$ is the unlabeled ML DOA estimate at time $t = t_1$. The other components of $\hat{\psi}_k^i$ are the estimated spectral parameters of the signal at time $t = t_1$. The feature vector, $\hat{\psi}_k^i$ is assumed to be multivariate Gauss (in 4 variables as noticed from above definition). Without venturing into details, the discriminant function, $g_k(\psi)$ of the Bayes classifier for the $k$th signal can be written as

$$g_k(\psi) = -\frac{1}{2}(\psi - \mu_k)^T \Sigma_k^{-1}(\psi - \mu_k) + \log P(x_k), \ k = 1, 2, \ldots, D.$$  

(4.30)

where $\mu_k$ is the 4-dimensional mean vector for the $k$th signal, $\Sigma_k$ is the covariance matrix and $P(x_k)$ is the apriori probability.

At time $t = t_1$, the mean and covariance of each class is given by the estimated (sample) mean and covariance

$$\hat{\mu}_k^i = \left(\frac{1}{N_{\text{track}}}\right) \sum_{t=1}^{t_1-1} \hat{\psi}_k^i, \ k = 1, 2, \ldots, D$$

(4.31)

$$\hat{\Sigma}_k^i = \left(\frac{1}{N_{\text{track}}-1}\right) \sum_{t=1}^{t_1-T} (\hat{\psi}_k^i - \hat{\mu}_k^i)(\hat{\psi}_k^i - \hat{\mu}_k^i)^T \ k = 1, 2, \ldots, D$$

(4.32)

where $N_{\text{track}}$ is the number of tracking instants at time $t = t_1$, including $t = t_1 - T$. $\hat{\psi}_k^i$, $t = 1, 2, \ldots, t_1 - T$ are labeled feature vectors of the $k$th signal.

If the apriori probabilities are the same (each target is equally likely), they can be ignored. To classify an unknown feature vector, $(\hat{\psi}_k^i, k \in \{1, 2, \ldots, D\})$ into one of $k$ classes, the discriminant functions $g_k(\hat{\psi}_k^i), k = 1, 2, \ldots, D$ are computed with $\hat{\mu}_k^i$ and $\hat{\Sigma}_k^i$ given by (4.31) and (4.32) respectively. This is the Mahanalobis distance of the given $\hat{\psi}_k^i$ with each of the $D$ signal mean vectors. $\hat{\psi}_k^i$ is classified to the signal to which the computed discriminant function value (4.30) is the largest, i.e., to which cluster $\hat{\psi}_k^i$ is closest. Thus, each feature vector is assigned to a different target and the estimated trajectories are constructed by linking the corresponding DOA estimate. This process is repeated at every tracking time instant.

Thus, the overall tracking algorithm contains three parts:

(i) Maximum likelihood estimation of AR parameters (spectral density of signals).
(ii) ML Estimation of angles of arrival.
(iii) Bayes classification for generating correct updated target estimates.
4.6 The Proposed Wideband Tracking Algorithm

The ML wideband signal tracking algorithm comprises of the following steps:

(i). The inputs to the algorithm at time $t = t$, are direction of arrival estimates ($\hat{\theta}_k^{\text{ML}}$), spectral parameter estimates $\left\{\hat{\alpha}_k^{\text{ML}}, \hat{\beta}_k^{\text{ML}}\right\}$, $k = 1, \ldots, D$ obtained at time $t = t - T$ and N snapshots acquired at time $t = t$.

(ii). **(Spectral density estimation)** Using $\left\{\hat{\alpha}_k^{\text{ML}}, \hat{\beta}_k^{\text{ML}}\right\}$ as initial guess values and the DFT of the N snapshots obtained, estimate AR parameters and noise variance of the D signals by minimizing (4.29). The updated estimates are given by $\left\{\hat{\alpha}_k^{\text{ML}}, \hat{\beta}_k^{\text{ML}}\right\}$, $k = 1, \ldots, D$.

(iii). **(DOA estimation)** From updated estimates and DFT of data, estimate the corresponding D DOAs by minimizing (4.16).

(iv). Form feature vectors $(\hat{\psi}_k, k \in \{1, 2, \ldots, D\})$ with updated estimates from steps (ii) and (iii).

(v). **(Estimate association)** Use Bayes classifier ((4.30) - (4.32)) to assign the DOAs to respective signal sources/targets.

(iv). Make $t = t + T$ and repeat steps (i)-(v) till end of tracking period $(T)$.

**Note:** 'T' defines the tracking time interval, i.e., the trajectories are only defined at those discrete instants starting at $t = 1$ and separated by $T$. The number of targets is always assumed to be known and is not estimated. The algorithm is started off with initial guess values of DOA and AR parameters in the neighborhood of the true values.

4.7 Performance of Proposed Method

4.7.1 A simulation result

The experiment addresses an extreme situation and serves to illustrate the performance of the proposed method. In contrast to the usual constant velocity assumption, non-linear trajectories with two fast moving targets crossing each other are considered. For the simulation, an 8 element uniform linear sensor array with sensors spaced half a wavelength apart is considered. For simplicity, the sensors are assumed to be omnidirectional with a flat frequency response. The true target tracks are shown in Fig. 4.1. The two targets have identical SNR of 7 dB so that this information cannot be used for distinguishing them. The spectra are assumed to have the same center frequency of 0.3 Hz [13], or else identification is fairly easy with an FFT. The targets are
parameterized by a wide sense stationary AR(2) process with spectral factors given by [13]

\[
\text{Target 1 : } \frac{1}{1 + 0.556z^{-1} + 0.81z^{-2}} \\
\text{Target 2 : } \frac{1}{1 + 0.494z^{-1} + 0.64z^{-2}}
\]

The true target spectra are shown in Fig. 4.2. For generating data, the driving noise variance for both targets is assumed to be unity. The data from the two AR(2) processes are correctly normalized to get the same SNR. The number of observations at every tracking instant is 256. The targets are tracked every 0.5 sec for a period of 5 sec.

Figure 4.1 shows the result of applying the proposed algorithm for a single realization. The estimated trajectories closely follow the true tracks. The estimates after cross over are correctly associated. The true and estimated DOA values are tabulated in Table 4.1.

Figure 4.2 shows estimated AR spectra of the two targets at the cross over point. Fairly accurate spectral estimates are obtained though the targets are separated only by a degree at cross over (Fig. 4.1). This ensures correct estimate association at cross-over.

4.7.2 Discussion

The preliminary result specifically demonstrates the workability of the algorithm in a worst case scenario. This can be further quantified by more experiments. The higher dimensionality of the feature vector ensures high probability of correct association at cross-over. To qualify these results in the two target case, a bound on the probability of classification error can be derived for the Bayes classifier. This can be expressed as a function of target SNR, center frequency, separation and order of chosen AR model.

For the two target case, 6 AR parameters are to be estimated at every tracking instant (4.29) which is done accurately as indicated in Fig. 4.2. But, for a dense target environment, the number of parameters to be estimated increases depending on number of targets and order of the chosen AR model. ML estimation in this case is computationally involved and this is a drawback of the method. But, in general, the ML
Fig. 4.1 True and estimated trajectories (DOAs) of the two targets plotted against tracking time. The motion of the targets is indicated by arrowheads. The proposed algorithm tracks the two targets correctly even after cross over.

Fig. 4.2 True and estimated power spectrum of the two targets at the cross-over point (indicated in Fig. 4.1). The center frequency of both spectra is identically equal to 0.3 Hz (normalized). The center frequency of the estimated power spectrum given by the proposed method closely approximates the true value.
TABLE 4.1
True and estimated DOA values over tracking period

<table>
<thead>
<tr>
<th>TIME (Sec)</th>
<th>TRUE DOA1 (Deg)</th>
<th>TRUE DOA2 (Deg)</th>
<th>ESTIMATE D DOA1 (Deg)</th>
<th>ESTIMATE D DOA2 (Deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>59.036</td>
<td>80.538</td>
<td>58.975</td>
<td>80.509</td>
</tr>
<tr>
<td>0.5</td>
<td>60.299</td>
<td>78.671</td>
<td>60.263</td>
<td>78.644</td>
</tr>
<tr>
<td>1.0</td>
<td>61.735</td>
<td>76.587</td>
<td>61.481</td>
<td>76.496</td>
</tr>
<tr>
<td>1.5</td>
<td>63.381</td>
<td>74.250</td>
<td>63.145</td>
<td>74.117</td>
</tr>
<tr>
<td>2.0</td>
<td>65.283</td>
<td>71.622</td>
<td>64.463</td>
<td>71.081</td>
</tr>
<tr>
<td>2.5</td>
<td>67.500</td>
<td>68.657</td>
<td>65.486</td>
<td>68.780</td>
</tr>
<tr>
<td>3.0</td>
<td>70.109</td>
<td>65.307</td>
<td>73.329</td>
<td>66.262</td>
</tr>
<tr>
<td>3.5</td>
<td>73.209</td>
<td>61.517</td>
<td>73.525</td>
<td>61.552</td>
</tr>
<tr>
<td>4.0</td>
<td>76.931</td>
<td>57.237</td>
<td>77.221</td>
<td>57.234</td>
</tr>
<tr>
<td>4.5</td>
<td>81.437</td>
<td>52.423</td>
<td>81.470</td>
<td>52.418</td>
</tr>
<tr>
<td>5.0</td>
<td>86.927</td>
<td>47.049</td>
<td>87.036</td>
<td>47.047</td>
</tr>
</tbody>
</table>
method has the potential for yielding accurate estimates with small bias under stringent conditions like low M or N or correlated signals. This indicates a trade off between better accuracy and more computation.

In [51], methods are proposed to choose outgoing signals and to process echoes of those signals from a dense target environment so as to determine the density function of the targets. The signals chosen are wideband in nature. It is shown in [51] that the narrowband approximation on the Doppler shift is not at all necessary for tracking. From the results of this chapter, it is felt that modeling of signals as wideband for tracking holds a lot of promise for obtaining accurate results at the cost of more computation.

4.8 Conclusion

A new Maximum Likelihood approach for multiple target tracking is proposed involving a combination of AR parameter estimation and pattern classification in association with DOA estimation. The targets are treated as wideband sources. AR parameter estimation and pattern classification are used for generating the updated estimates for continuous tracking of the targets. Experiments demonstrate the workability of the algorithm in a stringent tracking environment.
5. ESTIMATION OF SINGULARITIES FOR INTERCEPT POINT FORECASTING

5.1 Introduction

Perhaps the most challenging task of guidance and control of an interceptor (missile) in pursuit of a highly maneuverable target is that of midcourse guidance [52, 53]. This consists of estimation of target motion, the generation of guidance commands to optimally steer interceptor towards target intercept and the control of the coupled, nonlinear, multi-variable and uncertain dynamics of the interceptor. Midcourse guidance can be implemented with the target track information being uplinked to the interceptor which uses this information in addition to its self knowledge obtained from an on-board inertial navigator. Both target and interceptor can be tracked by the ground system. For optimal steering, it is desirable to have an algorithm to forecast the Intercept Point (IP) which points the interceptor to a direct collision course to meet the target, thereby reducing interceptor maneuverability. This reduces the amount of propellant required for maneuvering, which is of consequence especially in Space Based Interceptors (SBI) wherein lofting fuel into orbit is very expensive [54]. In [54, 55], approaches are proposed to predict the intercept point by estimating target state via Kalman Filtering.

The emergence of passive sensors like IRST and FLIR (forward-looking infrared) has had a significant impact on the design of weapons systems traditionally dependent on radars. The fusion of information from IR and EM (electromagnetic) sensors render the system less susceptible to target counter-measures and to the destruction of one type of sensors by a preemptive strike. With this motivation, a scheme is proposed for forecasting the intercept point using the Direction of Arrival (DOA) angle information of both the target and interceptor obtained from a passive sensor array. A sequence of observations or snapshots is obtained at regular intervals of time from a uniform linear array (ULA) of passive sensors by which the DOAs of target and interceptor are estimated. The target and interceptor are assumed to be sources of narrowband signals moving in far-field so that information of their trajectories can be characterized only by their DOA.
5.1.1 Focus of the problem

The two commonly used midcourse guidance schemes [52] are (a) Explicit guidance and (b) Kappa guidance scheme. In the former, the target state data computed by a Kalman filter is uplinked to the interceptor where it is combined with the interceptor state data obtained from the on-board IRU to form the guidance commands. The forecasted intercept point is the information which is uplinked and is employed by the interceptor along with target/interceptor state information to compute its eventual trajectory. In the latter, both target and interceptor are tracked using a Kalman tracker from ground, guidance commands are computed and uplinked to interceptor. Here, intercept point information is used to compute the eventual guidance commands.

It is the estimation of this intercept point that will be explicitly addressed in this paper. The objective is to minimize the 'miss distance' and provide a reasonably good estimate of the eventual collision point so that the target can eventually be intercepted during terminal homing phase.

![Diagram of two different time scales for data acquisition and DOA estimation](image)

**Fig. 5.1.** Illustration of the two different time scales for data acquisition and DOA estimation.
5.2 Motivation

The underlying motivation is to use the available angle estimates over a period of time as data to forecast the point of intercept. Fig. 5.1 illustrates the two different time scales, one for data acquisition and another for DOA estimation and is similar to Fig. 3.1 except that range is not estimated. Henceforth, all variables and notations are similar to that in Chapter 3. At the outset, the following definitions are stated.

**Definition**: Data Acquisition Time, $\tau$

The discrete time instants over which snapshots are collected from sensor array.

**Definition**: Tracking Time, $t$

The discrete time instants over which direction of arrival is estimated using array data (snapshots).

**Definition**: DOA Estimate, $\hat{\theta}_k(t)$

The direction of arrival (DOA) angles of target or interceptor at the discrete tracking time instant, '$t$', $t \in \{1, 2, \ldots, t_1-T, t_1, \ldots, T^*, \ldots, t, t+T, \ldots\}$, based on a sequence of $N$ observations or snapshots, $\{y_m(z), \tau = 1, \ldots, N, m = 1, \ldots, M\}$ obtained from the sensor array. $T^*$ is the time instant at which the target is intercepted. The superscript 'n' (Fig. 5.1) is omitted since 't' represents the tracking time index.

The sensor array is assumed to be uniform linear with $M$ sensors and hence the sensor output vector $y_m(\tau)$ is of dimension $M \times 1$. The DOA estimates over a period of time form data for forecasting the intercept point.

**Definition**: Data

Data available at $t = t_i$ comprises of DOA estimates of both target and interceptor given by the vector sequence $\{\hat{\theta}(t), t_0 \leq t \leq t_i\}$, $\hat{\Theta}(t) = \text{col.}[\hat{\theta}_{k1}(t), \hat{\theta}_{k2}(t)]$, $k1 \neq k2$, $k1, k2 \in \{K, I\}$. For convenience, $t_0 = 1$. 'K' represents the target and 'I', the interceptor indices.

**Remark 5.1**

An implicit assumption is that, only two unlabeled estimates are obtained at every time instant, one belonging to the target and another to the interceptor.

**Definition**: Estimated angle tracks

The tracks formed from data by associating the nearest angle estimates at every instant of time.
Definition: Intercept Point, \((T^*, \theta^*)\)

The true point of intersection of the angle tracks of target and interceptor in the \(t - \theta\) plane. The intercept point or collision point is completely specified by the DOA angle at interception called Intercept Angle, \(\theta^*\) and time of interception called Intercept Time, \(T^*\) (refer to Fig. 5.2).

Definition: Forecasted Intercept Time, \(\hat{T}(t_i)\)

\[ \hat{T}(t_i) \equiv \text{estimate of } T^* \text{ based on } \{\hat{\theta}(t), t_0 < t \leq t_i\}. \]

Definition: Forecasted Intercept Angle, \(\hat{\theta}(t_i)\)

\[ \hat{\theta}(t_i) \equiv \text{estimate of } \theta^* \text{ based on } \{\hat{\theta}(t), t_0 < t \leq t_i\}. \]

Definition: Forecasted Intercept Point, \((\hat{T}(t_i), \hat{\theta}(t_i))\)

This is the estimate of the true intercept point \((T^*, \theta^*)\) based on data up to time \(t_i\).
5.2.1 Choice of curve fit for data modeling

The choice of the curve fitted to data (comprising of DOA estimates) is dictated by the following requirements:

1. In order to achieve target interception, the interceptor has to be continuously accelerated towards the intercept point. This reduces maneuverability of the interceptor, thus minimizing attendant propellant expenditure. The objective then, is to fit a curve to data which maximizes the intercept velocity.

2. A regression line fit to data of target/interceptor implies an underlying constant velocity dynamics assumption, which in this case is erroneous since the requirement is for a varying acceleration. Thus, a better model for target/interceptor dynamics requires a higher order curve fit.

3. The prediction of the intercept point by fitting two regression lines to the data, one for the target and another for the interceptor assumes that origin of each of the data is known, i.e., whether it belongs to the target or interceptor. However, there can exist situations with uncertainty in the origin of the DOA estimates, for e.g.: the target can generate jamming signals which makes it indistinguishable from the interceptor.

The key idea to simultaneously overcome these problems is, to fit a single higher order curve to all available data. The nodal cubic in $\theta'$ and $t'$ provides a linearly varying acceleration fit to the DOA data. The important idea is to recognize that the intercept point can be represented as the singularity of a nodal cubic curve [56] fitted to all available data. The significant point is, since only one curve is fitted to the entire data, $\{\hat{\theta}(t), t_0 \leq t \leq t_1\}$, instead of two regression lines (Fig. 5.2), it is not required to recognize whether measurements belong to the target or interceptor. The forecast of the intercept point becomes more accurate with time as more information (data) is obtained. Fig. 5.2 illustrates the essential idea behind the proposed algorithm.

The prediction of the intercept point is thus a two-step process. A recursive method is proposed and performs these two steps at every time instant using DOA estimates of the current time and a function of the previous estimates. In other words, the singularity (intercept point), $(T^*\theta^*)$ which is assumed to be an unknown deterministic
quantity is continuously tracked. However, the intercept point can also vary with tracking time, albeit slowly.

5.3 Direction of Arrival Estimation

For estimating DOA, signal magnitudes (voltage) are the observations and phase delay information of the array outputs is used to estimate the bearing angles of targets / interceptors. The targets / interceptors are assumed to be narrowband signals in far-field impinging on a linear array of passive sensors. The narrowband signals are modeled as sample functions of a Gaussian stochastic process. The phase delay of these signals is expressed as a function of bearing angle or DOA of the respective targets. The angle parameters of all targets / interceptors, referred to as 'track parameters' are then simultaneously estimated. Any one of the available high resolution methods like MUSIC [1], root - MUSIC [12], stochastic maximum likelihood (ML) [2] etc. can be employed for DOA estimation. The choice of DOA estimation method is dictated by the achievable accuracy of the estimation method close to target intercept when DOAs of both target and interceptor are closely spaced. ML methods provide accurate estimates when target and interceptor are close to each other and under low SNR and hence stochastic ML method of [2] is selected. The problem formulation for maximum likelihood DOA estimation using stochastic Gaussian model for signals is given in Section 2.2.

5.4 Chapter Overview

The outline of this chapter is as follows. In the following section, the Algorithm for Estimation of Singularity (AES), is developed. A recursive weighted least squares procedure is formulated for estimating the intercept point co-ordinates. In Section 5.6, various steps for tracking the intercept point from radar returns upto target intercept, i.e., the overall tracking algorithm is given. Section 5.7 discusses the experimental results and comparison of the proposed approach with other existing methods. Conclusions are summarized in the final section.

5.5 Algorithm for Estimation of Singularity (AES)

5.5.1 Singularity estimation

In this section, an algorithm (AES) for dynamically estimating the intercept point is proposed. Fig. 5.2 motivates the strategy to fit a single curve, a nodal cubic, to data consisting of DOA estimates of both target and interceptor. The use of a recursive
method allows the interceptor to adaptively track the target. For angle tracks, the ordinate of the singularity is the intercept angle and abscissa the intercept time.

It is known that the interceptor is to have a continuously increasing (variable) acceleration. However, target dynamics are unknown. In absence of this knowledge, the form of the chosen nodal cubic is such that, both target and interceptor trajectories are modeled to have a linearly varying acceleration. The form of the nodal cubic satisfying this requirement is

\[ \Lambda: F(t, \hat{\theta}) = \hat{\theta}^2 - t^2 - t^3 = 0 \]  

(5.1)

where 't' represents tracking time, '\hat{\theta}' is the DOA estimate, \( \hat{\theta}, t \in \mathbb{R} \). Though DOA estimates are obtained only at discrete time instants, \( \hat{\theta} \) and \( t \) are continuous, and take values on the real line. All operations are restricted to 1 quadrant of the \( t - \hat{\theta} \) plane, since for a linear array \( \hat{\theta} \in (0, \pi) \) and \( t \geq 0 \). The singularity lies on the origin of the \( t - \hat{\theta} \) plane in (5.1). However, within the I quadrant, position of the intercept point is arbitrary. Hence, the form in (5.1) needs to be translated.

The equation of the translated nodal cubic parameterizing the data is given by

\[ F(t, \hat{\theta}) = (\hat{\theta} - \theta^*)^2 - \alpha_0 (t - T^*)^2 - \alpha_1 (t - T^*)^3 = 0 \]  

(5.2)

where the parameters to be estimated are \( \{T^*, \theta^*, \alpha_0, \alpha_1\} \), \( \alpha_0, \alpha_1 \in \mathbb{R} \).

From (5.2), \( (\hat{\theta} - \theta^*)^2 = \alpha_0 (t - T^*)^2 + \alpha_1 (t - T^*)^3 \)  

(5.3)

Equation (5.3) is to be solved for \( \{T^*, \theta^*, \alpha_0, \alpha_1\} \). Equation (5.3) can be written as

\[ \hat{\theta}^2 + A\hat{\theta} + Bt^3 + Ct^2 + Dt + E = 0 \]  

(5.4)

where \( A = -2\theta^* \)

\( B = -\alpha_i \)

\( C = 3\alpha_i T^* - \alpha_0 \)

\( D = 2\alpha_0 T^* - 3\alpha_i T^{*2} \)

and \( E = \alpha_i T^{*3} - \alpha_0 T^{*2} + \theta^{*2} \).  

(5.5)

From (5.5), it is evident that five parameters are required to fit a nodal cubic. The parameter vector at time 't' is defined as \( \psi_p(t) \equiv \text{col.} \{E(t), D(t), C(t), B(t), A(t)\} \). A nodal cubic is to be fitted at each instant of time 't' to the data consisting of a sequence of pairs of DOA estimates from target and interceptor up to time 't'. Hence, for \( 2t > 5 \), the over determined system of 2t equations with 5 unknowns that is to be solved to obtain \( \hat{\psi}_p(t) \) is given by

\[ z(t) = \Xi(t)\psi_p(t) \]  

(5.6)
At time $t = t_1$, the data available is $\{\hat{\theta}(t), t_0 \leq t \leq t_1\}$. The initial value is assumed to be $t_0 = 1$ and thus there are $t_1$ pairs of DOA estimates (one for target and another for interceptor). For $t = t_1$, this system is given by

$$
\begin{bmatrix}
(\hat{\theta}_{k1}(t_0))^2 \\
(\hat{\theta}_{k2}(t_0))^2 \\
\vdots \\
(\hat{\theta}_{k1}(t_1))^2 \\
(\hat{\theta}_{k2}(t_1))^2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & t_0^2 & t_0^3 & \hat{\theta}_{k1}(t_0) \\
1 & t_0^2 & t_0^3 & \hat{\theta}_{k2}(t_0) \\
\vdots \\
1 & t_1^2 & t_1^3 & \hat{\theta}_{k1}(t_1) \\
1 & t_1^2 & t_1^3 & \hat{\theta}_{k2}(t_1) \\
\end{bmatrix}
\begin{bmatrix}
E(t_1) \\
D(t_1) \\
C(t_1) \\
B(t_1) \\
A(t_1) \\
\end{bmatrix}

$$

(5.7)

The vector $\hat{\psi}(t_1)$ is the least squares solution for the linear system in (5.7). Knowing $\hat{\psi}_p(t_1)$, the system in (5.5) is solved to obtain $\{\hat{T}(t_1), \hat{\phi}(t_1), \hat{\alpha}_0(t_1), \hat{\alpha}_1(t_1)\}$ which are estimates of $\{T^*, \theta^*, \alpha_0, \alpha_1\}$ at time $t_1$ as follows:

$$
\hat{\alpha}_1(t_1) = -\hat{B}(t_1) \\
\hat{\phi}(t_1) = -\hat{A}(t_1) / 2 \\
\hat{T}(t_1) = -\left(2\hat{D}(t_1) / 5\hat{C}(t_1)\right) \\
\hat{\alpha}_0(t_1) = \left(\hat{T}(t_1)\hat{C}(t_1) + \hat{D}(t_1)\right) / \hat{T}(t_1)
$$

(5.8) \hspace{1cm} (5.9) \hspace{1cm} (5.10)

The intercept point estimate $\hat{T}(t_1)$ is chosen to be

$$
\hat{T}(t_1) = -\left(2\hat{D}(t_1) / 5\hat{C}(t_1)\right)
$$

$$
+ \left[(4/25)(\hat{D}(t_1) / \hat{C}(t_1))^2 + \left(3(\hat{\phi}^2(t_1) - \hat{E}(t_1)) / 5\hat{C}(t_1)\right)\right]^{1/2}
$$

(5.11) \hspace{1cm} (5.12)

Equations (5.9) and (5.12) give closed form solutions to the forecasted intercept point $(\hat{T}(t_1), \hat{\phi}(t_1))$ at time $t = t_1$. Also, (5.8) and (5.11) give estimates of the coefficients of the fitted nodal cubic at time $t = t_1$ denoted by $(\hat{\alpha}_0(t_1), \hat{\alpha}_1(t_1))$. In the sequel, a recursive approach (Recursive Least Squares (RLS)) is proposed and implemented for solving the system in (5.6).

5.5.2 Recursive least squares estimation

The Recursive Least Squares (RLS) approach [57] is used to estimate the $\psi_p(t)$ vector and the corresponding estimate at time $t_1$ is denoted by $\hat{\psi}_p(t_1)$. For a vector
observation at every time instant, i.e., for a pair of DOA estimates, one from the target and another from the interceptor, the RLS recursions are given by

\[ \hat{\psi}_p(t_i) = \hat{\psi}_p(t_i - T) + L(t_i)[z(t_i) - \xi^T(t_i)\hat{\psi}_p(t_i - T)] \]  \hspace{1cm} (5.13)

\[ L(t_i) = Q^{-\lambda}(t_i)\xi(t_i) = P(t_i - 1)\xi(t_i)[\lambda(t_i)I + \xi^T(t_i)P(t_i - 1)\xi(t_i)]^{-1} \]  \hspace{1cm} (5.14)

\[ P(t_i) = \frac{1}{\lambda(t_i)}[P(t_i - 1) - L(t_i)\xi^T(t_i)P(t_i - 1)] \]  \hspace{1cm} (5.15)

\[ P(t_i) = Q^{-\lambda}(t_i) \]  \hspace{1cm} (5.16)

\[ z(t_i) \] is the vector observation of dimension 2 x 1. \[ L(t_i) \] is the gain matrix of dimension 5 x 2, the two columns representing the two targets. \( \lambda(t_i) \) is called the 'forgetting factor' which discounts the effect of the older measurements exponentially. The value of \( \lambda(t_i) \) is chosen to be \( \lambda < 1 \), that ensures the gain does not reduce to zero. Therefore, the matrix

\[ Q(t_i) = \sum_{k=1}^{N} \lambda^{i-k} \xi(k)\xi^T(k) \]

where, \( \xi(k) \) is the 5 x 2 regressor matrix.

Equations (5.9, (5.12) and (5.13) through (5.16) summarize the AES algorithm.

### 5.6 The Overall Algorithm for Intercept Point Tracking

For every tracking time instant, \( t = 1, 2, \ldots, T, t_1, t_1 + T, \ldots, T^* \) with unit tracking interval, where \( T^* = \) target intercept, repeat the following steps (1) to (4).

1. From a sequence of \( N \) snapshots, \( \{y_m(\tau), \tau = 1, \ldots, N, m = 1, \ldots, M\} \), estimate the direction of arrival and associated parameters of both target and interceptor, \( \hat{\theta}(t) = \text{col.}[\hat{\theta}_{k_1}(t), \hat{\theta}_{k_2}(t)] \), \( k_1 \neq k_2 \quad k_1, k_2 \in \{K, I\} \), \( \hat{\Gamma}_{ML} \) and \( \hat{\rho}_{ML} \) by the stochastic ML method of Section 2.2.

2. Using these DOA estimates, obtain \( \hat{\psi}_p(t) \) using the recursions in AES (Eqns (5.13) to (5.16)).

3. Substitute elements of \( \hat{\psi}_p(t) \) in (5.9) & (5.12) to compute forecasted intercept point \( \left( \hat{T}(t), \hat{\phi}(t) \right) \).

4. Assuming that the engagement between that target and interceptor goes up to target intercept, repeat Steps (1) to (3) up to time \( t = T^* \).

**Initialization:**

For initializing the recursion in AES, wait until time say, \( t_k \geq 5 \) when \( P(t_k) \) becomes invertible. \( \hat{\psi}_p(t_k) \) is then computed using the least squares estimate

\[ \hat{\psi}_p(t_k) = [\Xi^T(t_k)\Xi(t_k)]^{-1}\Xi^T(t_k)z(t_k) \]  \hspace{1cm} (5.17)

This approach uses entire data \( \{\hat{\theta}(t), t_0 \leq t \leq t_k\} \) up to time \( t_k \). Also, \( t_0 \equiv 1 \) is assumed.
5.7 Performance of Proposed Method

5.7.1 Experimental results

Some results from experiments to demonstrate working of the AES algorithm are presented. For simulation purposes, the true intercept point \((T^*, \theta^*)\) is assumed to be constant over the tracking time. Here, angle tracks of target and interceptor are generated from DOA estimation as shown in Fig. 5.3. The true co-ordinates of the intercept point are \((T^*, \theta^*) = (43.24\, s, 42.29\degree)\). The simulations analyze how well it can be forecasted.

The uniform linear sensor array has 64 sensors spaced at half a wavelength and 100 snapshots are used for every DOA estimation of target and interceptor. For simulation purposes, the total tracking period, \((t_0, T^*)\) is divided into an integer number of time instants, \(L\). DOAs are estimated at these time instants and the obtained data used to fit the nodal cubic and determine the resulting singularity. The intercept point is forecasted continuously up to point of interception and performance of the algorithm is analyzed at target intercept. The forgetting factor \(\lambda = 0.906\) in all experiments.

Figure 5.3 demonstrates closeness of the forecasted intercept point at collision, \((\hat{T}^*, \hat{\theta}^*)\) to the actual value, \((T^*, \theta^*)\). The plot shows forecasts for 10 different realizations of the same experiment. Each realization is obtained by choosing different seeds for generating data for DOA estimation at every time instant. The forecasts of the intercept point are for all realizations are clustered around the true value. The signal-to-noise ratio for DOA estimation was chosen to be -4 dB. These plots illustrate satisfactory performance of AES for low SNRs. Table 5.1 tabulates the results of intercept point forecasts for varying SNR values.

Figures 5.4 and 5.5 show variation of intercept angle \((\hat{\phi}(t)|t, t = 1, 2, ..., T^*)\) and intercept time \((\hat{T}(t)|t, t = 1, 2, ..., T^*)\) forecasts at every time instant over the entire tracking period, i.e., the 'average nearness' of the forecasted intercept point at intercept, \((\hat{T}^*, \hat{\theta}^*)\) is compared to the true value \((T^*, \theta^*)\).

Mean of forecasted Intercept Angle at intercept = \(\frac{1}{N_R} \sum_{r=1}^{N_R} \hat{\phi}(T^*_r)\)

Mean of forecasted Intercept Time at intercept = \(\frac{1}{N_R} \sum_{r=1}^{N_R} \hat{T}(T^*_r)\)

where, \(N_R\) is the number of realizations (= 10 in the above experiment).

The forecasts are obtained ten trials of the experiment. It is seen that intercept time forecasts are close to true value as the interceptor nears the target. However, the rate of
Table 5.1
Mean and RMSE of intercept point forecasts for varying SNR.

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>Mean Intercept Angle (Deg)</th>
<th>Mean Intercept Time (s)</th>
<th>RMSE of Intercept Angle (Deg)</th>
<th>RMSE of Intercept Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>23.5747</td>
<td>28.5303</td>
<td>1.3343</td>
<td>0.5725</td>
</tr>
<tr>
<td>-4</td>
<td>23.5746</td>
<td>28.5377</td>
<td>1.3344</td>
<td>0.5716</td>
</tr>
<tr>
<td>-6</td>
<td>23.5744</td>
<td>28.5521</td>
<td>1.3345</td>
<td>0.5683</td>
</tr>
</tbody>
</table>

N = 100, M = 64, tracking period = 29.001 s, sampling interval = 0.5 s (2 data DOA estimate/s - 200 snapshots/s). True IP, (T^{*}, \theta^{*}) = (29.09s, 24.91^\circ), \lambda = 0.906, 10 trials.

Fig. 5.3. Estimated Target /Interceptor tracks along with true/forecasted intercept point for SNR = -4 dB. The target/interceptor tracks are formed from direction of arrival angles estimated at regular time instants. The forecasted intercept point is the estimated singularity of the nodal cubic fitted to data (DOA estimates). Forecast is at target intercept \(\{\hat{T}_T, \hat{\theta}_T\}\), i.e., using all data upto time of interception, T^{*}. The forecasted intercept point is close to the true value for all 10 different realizations of the experiment which shows satisfactory performance of the AES algorithm.
Fig. 5.4. Intercept angle forecasts ($\hat{\theta}_t, t = 1, 2, \ldots, T^*$) over the tracking period. Rate of convergence of estimates is slow and exhibit a definite bias even close to the point of collision.

Fig. 5.5. Variation of intercept time ($\hat{T}_t, t = 1, 2, \ldots, T^*$) forecasts over tracking period. Intercept time forecasts approach closer to the true value as the interceptor nears the target.
convergence of intercept angle estimates is slow and exhibit a definite bias even close to the point of collision. The plots show that the proposed AES algorithm is able to give fairly accurate forecasts of intercept point even at low SNR. Better estimates can possibly obtained by increasing SNR and also rate of data acquisition by the array.

Figures 5.6 and 5.7 illustrate the root mean square error (RMSE) of the forecasted intercept angle and time over the entire tracking period \((t_0, T^*)\), i.e., up to target intercept for a DOA estimation SNR = -4 dB.

\[
\text{RMSE at time } t \text{ of forecasted Intercept Angle} = \left( \frac{1}{N_R} \sum_{r=1}^{N_R} (\hat{\theta}(t_r) - \theta^*)^2 \right)^{1/2} \quad (N_R = 10)
\]

\[
\text{RMSE at time } t \text{ of forecasted Intercept Time} = \left( \frac{1}{N_R} \sum_{r=1}^{N_R} (\hat{T}(t_r) - T^*)^2 \right)^{1/2} \quad (N_R = 10)
\]

The RMSE of both \(\theta^*\) and \(T^*\) decrease monotonically with tracking period. However, a slow rate of decrease of error in the intercept angle is observed. This is due to the choice of an identical forgetting factor, \(\lambda\) for both parameters. For \(\lambda\) close to 1, intercept angle estimates are accurate and for \(\lambda\) away from unity, intercept time estimates become accurate. An adaptive forgetting factor can alleviate the effect of this trade-off considerably and investigations will be carried out in this direction.

### 5.7.1.1 Effect of DOA spacing on intercept point estimates

The proposed algorithm is a mid course guidance scheme with control being transferred to the interceptor during terminal phase. However, it is important to analyze the effect of nearness of DOA estimates of both target and interceptor on the accuracy of the intercept point estimate itself, especially in proximity of target intercept.

It is well known that the high resolution DOA estimation methods [1, 2, 6, 12, Chapter 2] have a threshold in angle spacing beyond which the DOAs are not resolved. This can be alleviated by increasing the number of sensors in the ULA. Hence, accuracy of intercept point estimates for closely spaced sources is best analyzed by varying number of sensors in the array. Incidentally, the ML method in [6] gives best DOA estimates for low SNR and for closely spaced sources. In view of the finite spacing assumption which comes into play in the proposed method of Chapter 2, more number of sensors are required than the method in [6] for resolving angles of target and interceptor with the same accuracy at around target intercept.
Fig. 5.6. Plot of RMSE of forecasted intercept angle versus tracking time for SNR = -4 dB. The RMSE of the forecasted angle is around 1.33° at intercept. Better estimates can be obtained by increasing sampling rate.

Fig. 5.7. Plot of RMSE of forecasted intercept time versus tracking time for SNR = -4 dB. The RMSE of forecasted intercept time is observed to be around 0.57 s at intercept. RMSE is a small value close to the intercept point.
It is shown in [59] that, for closely spaced DOAs, estimation accuracy of the ML method of [6] progressively degrades as the number of sensors is reduced from 32 to 12. However, the RMSE of intercept angle and time estimates show no significant change especially for \( M \geq 20 \). This implies that intercept point estimation is robust to DOA estimation error variances of the stochastic ML method [6] even when target and interceptor are closely situated. This is a consequence of the idea of fitting a single curve to all data points.

5.7.2 Discussion

This chapter offers a real-time solution to the midcourse guidance problem of intercept point prediction. A good prediction of the intercept point reduces the need for interceptor maneuvers and reduces the attendant propellant expenditure. This method is based on recognizing that the intersection point can be represented as a singularity of a single nodal cubic curve which is fitted to the entire data.

The objects to be tracked are assumed to be moving in far-field so that only direction of arrival angles are sufficient to determine their trajectories (angle tracks). Consequently, the data used are DOA estimates of the target and interceptor which are estimated from a sequence of observations obtained from a passive uniform linear array. An important feature of the method is, it does not require apriori knowledge about target or interceptor identity to perform intercept point prediction. In contrast to state estimation methods [55], operation of an active radar to provide state estimates like range, range rate etc. is not required.

The Kalman filter based methods assume that target/interceptor trajectories can be tracked independently. This is reasonable since almost exact information about dynamics of the interceptor is known. But it may so happen that the target itself may send jamming signals so that independent estimates of target and interceptor state may be erroneous. This being the basis for intercept point prediction, erroneous prediction of the intercept point may lead to unnecessary maneuvers of the interceptor. The Kalman filter may not even converge. This problem does not occur in the proposed method, since the intercept point prediction does not depend on identifying the measurement correctly. Moreover, even if either target/interceptor bearing is entirely missed due to the jamming signals, prediction of the intercept point will not deteriorate rapidly since a single cubic is fitted to both angle tracks.
Another advantage of uplinking information to the interceptor is that it provides a means for estimating errors in the states of the Inertial Navigation System (INS) which are not directly measurable. In-flight calibration of on-board sensors is also possible which could allow a reduction in their quality and a consequent saving in cost [58]. In the proposed method, additional information provided to the interceptor which is the intercept point forecast is derived using only a passive uniform linear array. This reduces cost of the ground antenna system and of the sensors on-board interceptor.

Consider the usual case wherein, instead of fitting a single cubic, two regression lines are fitted, one for the target and another for the interceptor track and the singularity is predicted, reasonable estimates of the intercept point can be obtained. But, the prediction is very much dependent on correctly knowing the origin of each measurement. It is possible for a regression line to completely flip in the presence of a few outlier points. It is likely for this problem to occur if DOA estimates are incorrectly assigned.

The relative magnitudes of the parameter estimates $\hat{\alpha}_0(t,)$ and $\hat{\alpha}_1(t,)$ decide the contributions of the curvature of the fitted curve. For example, the data in Fig. 5.3 follows a linear trend and hence, the estimate $\hat{\alpha}_1(t,)$ at intercept, has a small value given by $-0.8645 \times 10^{-3}$ (for the case $M = 64$) when compared to $\hat{\alpha}_0(t,)$ which is $1.3636$. This implies the small contribution of the cubic term in (5.2) and that a pair of straight lines (second order curve) are sufficient to fit the data at/around target intercept. However, outliers in DOA estimation when target and interceptor are far apart can affect the accuracy of the intercept point estimates and the tracker requires a finite amount of time to come back to its previous accuracy. A Kalman filter at the front end can render the intercept point forecast robust to errors in tracking.

A recursive algorithm (AES) is developed to forecast the intercept point of a target and pursuing interceptor. The implemented method is able to forecast the intercept point in real time fairly accurately and does not impose high sampling rates on the data acquisition system. Accuracy of forecast can be enhanced by increasing sampling rate. The singularity parameter estimates, $(\hat{T}(t), \hat{\phi}(t))$ have a non-linear one to one mapping with the parameter vector $\hat{\psi}_p(t)$ and possess the same asymptotic convergence properties [57] as that of $\hat{\psi}_p(t)$. A quantitative confirmation is provided by above results.
Another important consequence of intercept point forecasting is tracking of multiple moving targets. A method is proposed [37] and discussed in the next chapter for estimate association wherein the key idea is to utilize the intercept point forecast of any two intersecting targets to assign the DOA estimate to the correct target.

The forgetting factor, his used as an adaptation parameter for learning from the data. Employing adaptive estimation techniques like the LMS algorithm [57] can yield better estimates as well as model the time varying nature of the singularity. However, for effective operation, the interceptor trajectory is chosen so that large variations in location of the intercept point are minimized. To minimize the quantity of fuel expended, only small perturbations about its value is feasible. This fact also justifies the assumption that the intercept point is unknown deterministic rather than a random variable.

5.7.2.1 Drawbacks of the method

It is also possible to get better estimates of intercept point forecast by introducing a rotation parameter in addition to the translation parameters which are currently used. However, this formulation of applying a rotational transformation for a nodal cubic becomes complicated and in addition requires the solving a non-linear system of equations for obtaining estimates of the intercept point. This disadvantage is offset in the next chapter by considering a second order curve which provides an elegant solution.

Argument against the motivation of the method stems from assuming no prior information about the interceptor which is entirely within our control. Also, if the target is identified and tracked before launching the interceptor by using a Kalman tracker to obtain adequate prior information about its dynamics, chances of confusion would be minimized when target and interceptor are tracked simultaneously. Nevertheless, it is possible not to have adequate time to determine target dynamics accurately (rapid strike).

5.8 Summary

A new approach is presented to forecast the position of collision of the target and pursuing interceptor by tracking the singularity points of the fitted nodal cubic curve. The approach involves a combination of maximum likelihood DOA estimation and singularity estimation and provides a simple solution for intercept point forecasting without assuming any probability density on the data. Simulation results demonstrate viability of the proposed method.
6. TRACKING DIRECTION OF ARRIVAL BY SINGULARITY ESTIMATION

In this chapter, the objective is to once again address the primary issue of data association in developing an efficient method for tracking angle of arrival of multiple moving targets. A new method (Algorithm for Data Association (ADA)) is proposed to maintain association between direction of arrival (DOA) estimates of different targets. At each time instant, \( t \), the intercept point forecast information of the intersecting targets obtained from existing data is employed for estimate association. The data comprises of direction of arrival estimates of targets obtained from the stochastic Maximum Likelihood (ML) DOA estimation method. The forecasted intercept point is recognized as the estimated singularity of a single second order curve of the form \( \hat{\theta}^2 = \alpha_0 t^2 \) fitted to the data where the parameter \( \alpha_0 \) characterizes the curve. The estimated singularity and coefficient of the second order polynomial are given by a recursive least squares solution. Tracking of direction of arrival of targets is achieved by successive DOA estimation, intercept point forecasting and data association.

6.1 Current Approaches for Data / Estimate Association

As far as estimate association strategies are concerned, the existing DOA based approaches in [30, 31] and in Chapters 3 and 4 use the signal power or power spectral density information in some fashion to perform estimate association. A simple assumption that signal powers of different targets are different is almost always satisfied ensuring identifiability of targets. In addition, signal powers are assumed to be unknown deterministic quantities that are constant throughout the tracking period in [30, 31] and Chapter 3.

In Chapter 4, the source signals are wideband and hence are specified by their spectral densities. The AR model assumed for the signals parameterizes the spectral density. The estimated AR parameters characterizing the power spectral density are assumed to be Gaussian random variables and a Bayes classifier is constructed. The measurements are then classified to one of D target classes.
However, in this chapter, instead of this information, predictive information about the estimated target tracks, in particular, the intercept point forecast is employed for association. This is similar in principle to the classical approaches (Chapter 1) wherein, a prediction of the track measurement is made and the measurement nearest to this prediction (in deterministic or probabilistic sense) is chosen. The decisions made are aposteriori in that the resolution of ambiguities for tracking instant $t_1$ are also based upon data collected from the future, at instants $(t_1 + T), (t_1 + 2T), \ldots, (t_1 + nT), n \in Z^+$.

6.2 Proposed Approach

In this chapter, a new method for estimate association is proposed wherein the idea is utilize the forecast of the point of intersection (intercept point (IP)) of any two intersecting targets to assign the correct target label to an unlabeled angle estimate. The key observation is, the intercept point can be represented by the singularity point of a second order curve fitted to data comprising of angle estimates of the two targets. If D targets are present, $D(D-1)/2$ intercept point estimates are obtained. Since $D(D-1)/2 > D$ for $D > 3$, a method is proposed to suitably combine these intercept point estimates to achieve correct association.

Remark 6.1

In the usual situation, the target track is extrapolated by fitting a least squares line (constant velocity assumption) and the predictive estimate along with new data is used to compute the filtered estimate. The important point is, by including the intercept point information, it is possible to show that the likelihood of correct estimate association is always greater than just using predictive estimates, especially when the targets are maneuvering or the tracking interval is large. This is because, when targets change direction, the predictive estimate being local will have a large error. It is possible for misassignment to occur in such a situation and, the resulting Kalman filter will not even converge. On the other hand, the intercept forecast gives global information about the trajectory of the targets. This ensures detection of maneuvers (if sufficiently smooth) and appropriate measures can be taken.

Thus, the proposed overall angle of arrival tracking algorithm consists of three parts -- estimating DOAs from array data, using estimated DOAs as data for estimating
the \( \frac{D(D-1)}{2} \) intercept points and then combining evidence of these intercept point forecasts to perform estimate association.

6.3 Overview of Proposed Method
6.3.1 Direction of arrival estimation

The targets are assumed to be narrowband emitters moving in far-field so that their trajectories can be characterized only by their DOA. A sequence of snapshots at regular time instants from a uniform linear array of passive sensors is obtained by which the DOAs of targets are estimated. The targets and the sensor array are assumed to be in the same plane. For DOA estimation, the stochastic ML method of Chapter 2 is used. This is step one of the proposed algorithm. There are two different time scales (Fig. 5.1), one over which sensor data is collected and the other over which the angles of arrival are estimated. The second time scale, i.e., time instants over which angles are estimated is defined as the tracking time. This convention is similar to the one proposed in Chapter 3. The DOA estimates form data for forecasting intercept points of the target tracks.

6.3.2 Motivation for intercept point forecasting

To begin with, the targets are assumed not to have crossed each other. For targets moving far apart, associating estimates is straightforward, nearest neighbors at successive time instants are associated (initial part of the track in Fig. 6.1). However, at cross-over, estimates are closely spaced and additional information is crucial for correct association. The primary goal is to detect whether two target tracks have crossed each other and follow their trajectories correctly after cross-over.

The additional information employed to detect cross-over is the forecast of the point of intersection at time \( t = t_1 \) of any two targets from available DOA data at time \( t = t_1 \). The important idea is to recognize that intercept point forecast of the intersecting targets at each tracking time instant, can indeed be used for estimate association, i.e., cross-over of target tracks is detected from the nature of intercept point forecast. The DOA estimates over several tracking time instants form data for forecasting the intercept point. The intercept point is completely specified by the DOA angle at interception called Intercept Angle and time of interception called Intercept Time (Fig. 6.1).
Fig. 6.1. Illustration of the idea behind Intercept Point Estimation.
6.3.3 Choice of curve fit

If origin of each of the data is known, i.e., to which target it belongs, then prediction of intercept point can be accomplished by fitting two regression lines to the data, one for each of the targets and finding their point of intersection. However, there is uncertainty in the origin of DOA estimates. The key idea to overcome this problem is to fit a single curve to all available data. It is recognized that the intercept point can be represented as the singularity of a second order curve of the form $\hat{\theta}^2 = \alpha_0 t^2$ which is fitted to available data. '8' is the DOA estimate and 't' is the tracking time. $\alpha_0$ characterizes the second order curve. This forms the second step of the developed algorithm. A recursive weighted least squares procedure called Singularity Estimation Method (SEM) is formulated for estimating the intercept point co-ordinates. Figure 6.1 illustrates the essential idea.

The idea of fitting a single curve to data follows from the work in Chapter 5. Chapter 5 just dealt with forecasting the intercept point by fitting a single nodal cubic to all DOA data. In this chapter, the key idea is to associate and track direction of arrival of multiple signals (targets) by incorporating a 'single curve fit' and estimating the point of interception. The data for fitting the curve can be some other variable which physically has a linear/quadratic functional behavior with the independent variable on the x-axis (usually time).

However, an important difference is in the nature of the curve chosen the data. Since targets can assume arbitrary trajectories in space, it should be possible to apply a translation and rotational transformation to the chosen curve. The resulting transformation for a nodal cubic results in a complex non-linear system of equations which do not have closed form solutions. Also, choice of nodal cubic implies a linearly varying acceleration assumption on the target dynamics. This is a realistic physical choice on a target trajectory of a seeker applicable only in situations when a target is to be intercepted.

Drawing upon the nodal cubic fit and intercept point estimation strategies of Chapter 5, a more powerful formulation for intercept point estimation, association and tracking is developed wherein the second order curve model gives a realistic model to the physical scenario.
This results in the Algorithm for Data Association (ADA) for associating the tracks. Depending on the position of the forecasted intercept point, the decision of whether the targets have crossed each other is made and DOA estimates are assigned to respective target tracks. ADA is the **final** step of the tracking algorithm.

### 6.3.4 Tracking algorithm summary

Summarizing the idea, the major steps of the proposed tracking algorithm are:

**Step 1** DOA estimation of D targets at every time instant using the stochastic ML method in Chapter 2.

**Step 2** A forecast of the intercept point between every pair of targets is obtained by fitting a single curve of the form $\hat{\Theta}^2 = \alpha_\Theta t^2$ to $\theta'(\eta \in Z^*)$ angle estimates of each of the two targets available at that time instant and estimating the singularity. The method is termed 'Singularity Estimation Method' (SEM) and is described in Section 6.4. In this manner, $D(D - 1)/2$ intercept point forecasts are obtained.

**Step 3** Using $D(D - 1)/2$ intercept point forecasts obtained at time $t = t_i$, the D unlabeled DOA estimates at current time instant $t = t$, are associated to the labeled estimates at previous time instant $t = t_i - T$, where $T$ is the interval over which N snapshots are collected and DOAs estimated. The proposed algorithm is called 'Algorithm for Data Association (ADA)' and is given in Section 6.5.

The various steps of the tracking algorithm are given in Section 6.6. Section 6.7 has some experimental results illustrating performance of the proposed method for various scenarios followed by a discussion. Section 6.8 concludes the chapter.

**Remark 6.2**

In the proposed tracking algorithm, number of targets are assumed to be known and constant during the entire tracking period. However, this is not always the case and hence, the number of targets have to be estimated at every tracking instant from sensor data. Model order determination criteria [38, 39] from standard statistical theory could be used to estimate the number of targets.
6.4 Singularity Estimation Method (SEM)

6.4.1 Estimation of the intercept point

The objective is to fit a single curve to every two estimated tracks (data) and estimate the intercept point. At every time instant, \( t = t_i \), \( D \) unlabeled DOA estimates are obtained. The strategy is to associate each of the \( D \) unlabeled samples to their nearest neighbor from the previous estimated tracks to form tentative tracks. These tracks selected two at a time, form data for intercept point estimation. The second order curve chosen to fit data from any pair of tracks indexed by \( k = k_1, k_2, k_1, k_2 \in \{1, 2, \ldots, D\} \) is

\[
\Lambda: F(t, \hat{\theta}) = \hat{\theta}^2 - t^2 = 0
\]

(6.1)

where 't' represents tracking time, '6' is the DOA estimate, \( \hat{\theta}, t \in \mathbb{R} \). Though DOA estimates are obtained only at discrete time instants, \( \hat{\theta} \) and \( t \) are continuous, and take values on the real line. All operations are restricted to I quadrant of the \( t - \hat{\theta} \) plane, since for a linear array \( \hat{\theta} \in (0, \pi) \) and \( t \geq 0 \). The general form of the chosen curve suggests a constant acceleration assumption on the target dynamics. The singularity lies on the origin of the \( t - \hat{\theta} \) plane in (6.1). However, within the I quadrant, the position of the intercept point is arbitrary. Hence, the general form in (6.1) needs to be translated. Equation (6.1) is also subjected to a rotational transformation to accommodate for arbitrary orientations of the trajectories. The translated and rotated curve parameterizing the data is given by

\[
\left[ (\hat{\theta} - \theta^*) \cos \beta - (t - T^*) \sin \beta \right]^2 = \alpha_0 \left[ (t - T^*) \cos \beta + (\hat{\theta} - \theta^*) \sin \beta \right]^2
\]

(6.2)

where \( \theta^* \) = true intercept angle, \( T^* \) = true intercept time, \( \alpha_0 \) = parameter describing the curve, \( \beta \) = angle of rotation of the curve.

Equation (6.2) needs to be solved for the four parameters \([T^*, \theta^*, \alpha_0, \beta]\). Equation (6.2) is rewritten as

\[
\hat{\theta}^2 = A\hat{\theta} + Bt^2 + Ct + Dt\hat{\theta} + E
\]

where

\[
A = 2\theta^* - \frac{T^*(\sin 2\beta)(1 + \alpha_0)}{\cos^2 \beta - \alpha_0 \sin^2 \beta}, \quad B = -\frac{\sin^2 \beta - \alpha_0 \cos^2 \beta}{\cos^2 \beta - \alpha_0 \sin^2 \beta}, \quad D = \frac{(\sin 2\beta)(1 + \alpha_0)}{\cos^2 \beta - \alpha_0 \sin^2 \beta}
\]

\[
C = 2T^*(\sin^2 \beta - \alpha_0 \cos^2 \beta) - \theta^*(\sin 2\beta)(1 + \alpha_0)
\]

and

\[
E = \frac{\theta^*T^*(\sin 2\beta)(1 + \alpha_0) - (T^2 - \theta^2\alpha_0)\sin^2 \beta - (\theta^* - T^*\alpha_0)\cos^2 \beta}{\cos^2 \beta - \alpha_0 \sin^2 \beta}
\]

(6.3)
From (6.3), five parameters are needed to fit the curve in (6.2). We define the parameter vector to be \( \psi_p(t) \equiv \text{col.}\{E(t), D(t), C(t), B(t), A(t)\} \). Hence, for \( 2t > 5 \), the over determined system of \( 2t \) equations with 5 unknowns that is to be solved to obtain \( \hat{\psi}_p(t) \) is given by

\[
\mathbf{z}(t) = \Xi(t)\psi_p(t) = \begin{bmatrix} -\hat{\theta}_{k_1}(t_0) \\
-\hat{\theta}_{k_2}(t_0) \\
\vdots \\
-\hat{\theta}_{k_1}(t_t) \\
-\hat{\theta}_{k_2}(t_t) \end{bmatrix}
\begin{bmatrix} t_0 \quad t_0^2 \quad t_0 \hat{\theta}_{k_1}(t_0) \quad t_0 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_0^2 \quad t_0 \hat{\theta}_{k_1}(t_0) \quad t_0 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \end{bmatrix}
\begin{bmatrix} E(t_0) \\
D(t_0) \\
C(t_0) \\
B(t_0) \\
A(t_0) \end{bmatrix}
\]

(6.4)

At time \( t = t_0 \), data \( \{\hat{\theta}_{k_1}(t), \hat{\theta}_{k_2}(t), t_0 \leq t \leq t_1 \}, k_1, k_2 \in \{1, 2, \ldots, D\} \) available is from two tentative tracks formed from angle estimates obtained at discrete tracking time instants in the interval \( t_0 \leq t \leq t_1 \). We assume that the initial value \( t_0 = 1 \) and the tracking time interval to be of length \( T \). Thus, there are \( t \), 2-vectors forming the data. The resulting system is given by

\[
\mathbf{z}(t) = \Xi(t)\psi_p(t) = \begin{bmatrix} -\hat{\theta}_{k_1}(t_0) \\
-\hat{\theta}_{k_2}(t_0) \\
\vdots \\
-\hat{\theta}_{k_1}(t_t) \\
-\hat{\theta}_{k_2}(t_t) \end{bmatrix}
\begin{bmatrix} t_0 \quad t_0^2 \quad t_0 \hat{\theta}_{k_1}(t_0) \quad t_0 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_0^2 \quad t_0 \hat{\theta}_{k_1}(t_0) \quad t_0 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \\
1 \quad t_1 \quad t_1^2 \quad t_1 \hat{\theta}_{k_1}(t_0) \quad t_1 \hat{\theta}_{k_2}(t_0) \end{bmatrix}
\begin{bmatrix} E(t_0) \\
D(t_0) \\
C(t_0) \\
B(t_0) \\
A(t_0) \end{bmatrix}
\]

(6.5)

Solving the linear system in (6.5), \( \hat{\psi}_p(t_1) \) is obtained which is the estimated parameter vector \( \psi_p(t) \) at time \( t_1 \). Knowing \( \hat{\psi}_p(t_1) \), the system in (3.3) is solved to get \( (\hat{f}(t_1), \hat{\phi}(t_1)) \), the intercept point estimates of the true intercept point \( \{T^*, \theta^*\} \) at time \( t_1 \).

Thus, from (6.3),

\[
\begin{align*}
C / B &= -2T^* - \theta^* (D / B) \\
C / D &= -2T^* (B / D) - \theta^*
\end{align*}
\]

(6.6)

Solving the simultaneous equations in (6.6) & (6.7), \( \text{(linear in } \theta^*, T^*) \),

\[
\hat{T}(t_1) = \frac{-2\hat{C}(t_1) + \hat{A}(t_1)\hat{D}(t_1)}{4\hat{B}(t_1) + \hat{D}^2(t_1)}
\]

(6.8)

and

\[
\hat{\phi}(t_1) = \left(\frac{4\hat{B}(t_1) + \hat{D}^2(t_1)}{2\hat{A}(t_1)\hat{B}(t_1) - \hat{C}(t_1)\hat{D}(t_1)}\right)
\]

(6.9)

\( (\hat{f}(t_1), \hat{\phi}(t_1)) \) represents forecasted Intercept Point or singularity of the second order curve at time \( t_1 \).
Remark 6.3
Solutions for $\alpha_0(t_1)$ and $\beta_0(t_1)$ are not obtained since the interest is only in intercept point estimation. The estimation of $\alpha_1(t_1)$ and $\beta_0(t_1)$ involves solving a non-linear system of equations.

The Recursive Least Squares (RLS) approach [57] is used to estimate the $\hat{\psi}_p(t)$ vector and the corresponding estimate at time $t_1$ is denoted by $\hat{\psi}_p(t_1)$. For a vector observation at every time instant, i.e., a pair of DOA estimates, one from each target, the RLS recursions are given by:

$$
\hat{\psi}_p(t_1) = \hat{\psi}_p(t_1 - T) + L(t_1) [z(t_1) - \xi^T(t_1) \hat{\psi}_p(t_1 - T)] \\
L(t_1) = Q^{-1}(t_1) \xi(t_1) = P(t_1 - 1) \xi(t_1) [\lambda(t_1) I + \xi^T(t_1) P(t_1 - 1) \xi(t_1)]^{-1} \\
P(t_1) = \frac{1}{\lambda(t_1)} [P(t_1 - 1) - L(t_1) \xi^T(t_1) P(t_1 - 1)] \\
P(t_1) = Q^{-1}(t_1)
$$

$z(t_1)$ is the vector observation of dimension 2 x 1. $L(t_1)$ is the gain matrix of dimension 5 x 2, the two columns representing the two targets. $\lambda(t_1)$ is called the 'forgetting factor' which discounts the effect of the older measurements exponentially. The value of $\lambda(t_1)$ is chosen to be $\lambda<1$, that ensures the gain does not reduce to zero. Therefore, the matrix $Q(t_1) = \sum_{k=2}^{5} \lambda^{t_1-k} \xi(k) \xi^T(k)$, where, $\xi(k)$ is the 5 x 2 regressor matrix. Equations (6.8-6.13) summarize the SEM algorithm.

6.5 Algorithm for Data Association (ADA)
At time $t_1$, we get an un-associated D-vector of DOAs (also called measurement vector). The task is to associate this vector to an already associated D-vector of DOAs (at time $(t_1 - T)$) as shown in Fig. 6.2. The following four issues are of concern:
(a) Forming initial estimated tracks for starting recursion (Algorithm AA).
(b) Selecting appropriate data for intercept point estimation (Algorithm BB).
(c) Cross-over detection using intercept point forecasts (Algorithm CC).
(d) Evidence combination of $D(D-1)/2$ intercept point forecasts to establish the $D$ links between successive time instants (Algorithm DD).

6.5.1 Forming initial estimated tracks (Algorithm AA)
To start with, the targets are assumed to be away from each other. At every time instant $\text{upto}$ a certain time $t_n$, the new un-associated D-vector of angle estimates is
**Fig. 6.2.** Estimated tracks of two targets and an unlabeled measurement vector at time $t_1$.

**Fig. 6.3.** Straight association between estimates at times $(t_1 - T)$ and $t_1$.

**Fig. 6.4.** Cross association between estimates at times $(t_1 - T)$ and $t_1$. 
linked to the previous associated D-vector based on their order statistics, i.e., the highest is associated with the highest, the second highest with the second highest and so on, thus forming the initial portion of the tracks. This is defined as **straight linking**.

### 6.5.2 Selection of appropriate measurement vectors (Algorithm BB)

For recursive intercept point estimation at each time instant \( t = t_1, t_1 > t \), \( 2\eta \) angle estimates (\( \eta \) for each of two targets chosen) are used for fitting the curve and estimating the intercept point, out of which \( 2(\eta - 1) \) have been previously associated and two which still have to be associated (Fig. 6.2). The number (\( \eta \)) of DOA estimates (data points) to get a good fit is greater than \((5 	imes 2 = 10)\) because at least five points are needed to estimate \( \psi_p(t) \) (6.5). The factor '2' accounts for selection of two targets at a time. For \( D \) targets, two targets can be chosen in \( \binom{D}{2} \) different ways. Thus, \( D(D-1)/2 \) intercept point estimates are obtained.

The **key hypothesis** is, for intercept point estimation at time \( t_1 \), the unlabeled DOA estimates at time \( t_1 \) are **not** associated, but just **straight linked** to the associated ones at time \((t_1 - T)\). Two cases are to be justified:

- **Targets far away**: If the targets are away from each other, this is the most natural thing to ensure correct association to get accurate forecasts of the intercept point.
- **Targets in proximity**: If the targets are close, i.e., the DOA estimates are very close to each other, **straight linking** will still ensure that the intercept point estimates are accurate, because, the DOAs are so close that it does not matter which one is going to be picked for determining the recursive intercept point forecast.

After selecting the appropriate \( \eta \) estimates of every pair of estimated tracks, \( D(D-1)/2 \) intercept point forecasts are obtained using SEM (Section 6.4).

### 6.5.3 Cross-over detection from intercept point forecast (Algorithm CC)

#### 6.5.3.1 Idea behind Algorithm CC

Consider angle estimates from estimated tracks of any two targets moving towards each other as in Fig. 6.2. A new vector of unlabeled measurements are obtained at time \( t_1 \). There exist two possible associations as seen from Figs. 6.3 and 6.4. We need to choose one of two hypotheses to determine the appropriate linking using the intercept
Fig. 6.5. Illustration of two targets approaching each other.

Fig. 6.6. Illustration of two targets at cross-over point.

Fig. 6.7. Illustration of two targets diverging after crossover.
point forecast. For the two targets under consideration, the following three situations can occur (Figs. 6.5 - 6.7).

(i) If two targets are approaching (Fig. 6.5), the intercept point forecast is \textit{in front} of the latest measurement vector (at time $t_i$).

(ii) If the two targets are at crossover (Fig. 6.6), the intercept point forecast is \textit{in between} the 2 measurement vectors (at times $(t_i - T)$ and $t_i$).

(iii) If the two targets are diverging (moving away) from each other (Fig. 6.7), the intercept point forecast is \textit{behind} both measurements.

This observation forms the basis for cross-over detection.

6.5.3.2 Algorithm CC

Check for the intercept point forecast in the region enclosed by measurements at times $(t_i - T)$ and $t_i$. If intercept point is present, cross-over is detected, hence cross-link DOAs (Fig. 6.9). If intercept point is not present, cross-over is not detected, hence straight link DOAs (Fig. 6.8 & 6.10).

Remark 6.4

The intercept point forecast from the RLS fit should accurately depict each situation. Experimental results confirm accuracy of RLS estimates of point of intersection. More importantly, at time $t_i$, there are $D$ estimates and $D(D-1)/2$ intercept points, i.e., the number of intercept points is greater than number of targets. Thus, information of all intercept points should be combined to decide upon the $D$ links between $(t_i - T)$ and $t_i$.

Remark 6.5

The hypothesis of algorithm BB seems to imply that the algorithm cannot handle cases where targets are come close to each other but \textit{do not} cross-over because the intercept point would still come in between and would suggest the estimates be cross-linked. This can happen if targets move towards each other till they are very close and then maneuver away. However, for sufficiently slow rate of change of direction, the intercept point estimate will still be well in front and hence the algorithm is able to conclude that tracks do not intercept.
Fig. 6.8. Illustration of straight linking of two targets approaching each other.

Fig. 6.9. Illustration of cross linking of two targets at cross-over.

Fig. 6.10. Illustration of straight linking of two targets diverging from each other.
6.5.4 Evidence combination for linking (Algorithm DD)

In view of remark 6.4, the current objective becomes one of linking D estimates of targets at successive time instants using \( D(D-1)/2 \) intercept point forecasts. An algorithm is proposed to handle situations wherein more than one intercept point occurs within one tracking interval or multiple intercept points exist in successive tracking instants.

6.5.4.1 Structure of the Link Matrix

Every element of the D-vector of DOA estimates at time \((t_1 - T)\) is associated and labeled because the measurement vectors at times \((t_1 - T)\) and \((t_1 - 2T)\) are associated. This labeling information is stored in a link matrix, \( LM(t_1 - T) \) of dimension \( D \times 2 \).

The first column of the link matrix consists of DOA estimates arranged in descending order of magnitude and the second column identifies the measurements, i.e., it contains the target label. The second column of the link matrix is called index vector. One possible link matrix configuration at time \((t_1 - T)\) is given by

\[
LM(t_1 - T) = \begin{bmatrix}
\hat{\theta}_1(t_1 - T) & \cdots & \hat{\theta}_k(t_1 - T) & \cdots & \hat{\theta}_D(t_1 - T)
\end{bmatrix}^T
\]

where \( \hat{\theta}_1(t_1 - T) \geq \hat{\theta}_2(t_1 - T) \geq \hat{\theta}_k(t_1 - T) \geq \cdots \geq \hat{\theta}_D(t_1 - T) \) \hspace{1cm} (6.14)

Without loss of generality, the link matrix at the first time instant, \( LM(t = 1) \) consists of the DOA estimates arranged in descending order with the first estimate getting the label of target '1' and the last one getting label 'D'.

6.5.4.2 Formation of the Ordering Matrix

Based on intercept point forecasts at time \( t \), and the link matrix, \( LM(t_1 - T) \) at time \((t, - T)\), a \( D \times D \) symmetric matrix called Ordering Matrix, \( OM(t_1) \) is formed. The elements of this matrix contain integer valued target labels. The upper and lower triangular elements give ordering information for association of measurements at \( t_1 \) with the already associated ones at \((t, - T)\). The ith row and column contain information about the ith target interacting with the other \((D-1)\) targets. Labels to \( D^2 \) entries of \( OM(t_1) \) are assigned as follows:

- \( OM_{(i,j)}(i,j) = 1, i = 1, 2, ..., D. \) \hspace{1cm} (6.15)
- For assigning labels to off-diagonal elements, the intercept point forecasts and the index vector of the link matrix (6.14) are used. The order in which these \( D(D-1)/2 \)
elements are filled is determined by choosing indices \((i,j)\) of elements \(\text{OM}_{(t)}(i,j)\), \((i \neq j)\) as follows:

Two elements \(i\) and \(j\) of the index vector are chosen at a time, beginning from the first row and proceeding downwards till the \((D-1)\)th row. The entries of \(\text{OM}(t)\):

\[
\text{OM}_{(t)}(i,j) = \text{OM}_{(t)}(j,i) = i, \text{ if cross - over is detected (6.16a)}
\]

\[
O \quad M(i,j) = O \quad M(j,i) = j, \text{ if no cross - over is detected (6.16b)}
\]

where \((i, j) \in \{1, 2, ..., D\}\).

**Remark 6.6**

\[\hat{\theta}_1(t_1 - T) \geq \hat{\theta}_j(t_1 - T), \ (i, j) \in \{1, 2, ..., D\} \] since the DOA column vector in the link mamx is always sorted and arranged in descending order.

**Remark 6.7**

The detection of cross-over which is used as a basis for ordering is assumed to be accurate which is indeed a crucial assumption (see Remark 6.4).

The \(D\) required links are established from the link mamx as illustrated below.

### 6.5.4.3 Forming the link matrix from the ordering matrix

- The first column of the link mamx, \(\text{LM}(t)\) consists of unlabeled DOA estimates of time \(t_1\) arranged in descending order.

- The \(i\)th row of the reordering matrix represents the \(i\)th target. The number of non-zero entries obtained by taking absolute difference between elements of the \(i\)th row and a row vector consisting of '1's (index \(i\)) gives the order statistic of the \(i\)th target. This procedure is repeated for all rows and the labels of targets are obtained. Thus, the index vector of the link matrix at time \(t\) is formed. The above insertion sort procedure completes estimate association between angle estimates between times \((t, - T)\) and \(t_\). 

### 6.6 Proposed Tracking Algorithm

The complete algorithm for tracking multiple moving targets is as follows:

1. At time instant \(t = t_\), the inputs to the tracking algorithm are the direction of arrival estimates \(\{\hat{\theta}_k(t), (t, -(\eta - 1)T) \leq t \leq (t_1 - T), k = 1, 2, ..., D\}\) and snapshots \(\{y_m^{\alpha=0}(\tau), \tau = 1, ..., N, m = 1, ..., M\}\).
From the sequence of snapshots, \( \{ y_{m}^{\text{seq}}(\tau), \tau = 1, \ldots, N, m = 1, \ldots, M \} \), estimate direction of arrival parameters \( \{ \hat{\theta}_{k}(t_{i}) \}_{\text{ML}}, k = 1, 2, \ldots, D \), \( \hat{\rho}_{\text{ML}} \), \( \hat{\Gamma}_{\text{ML}} \) (2.21, 2.30 and 2.37) of the D targets by ML method of Chapter 2.

For every pair of targets \([D(D-1)/2 \text{ pairs}]\),

(a) Using respective DOA estimates, obtain \( \hat{\Psi}_{p}(t_{i}) \) from recursions in SEM (Eqns (6.10) to (6.13)).

(b) Substitute elements of \( \hat{\Psi}_{p}(t_{i}) \) in (6.8) and (6.9) to compute forecasted intercept point \( (\hat{T}(t_{i}), \hat{\phi}(t_{i})) \).

Associate the estimates using forecasted intercept points by ADA procedure (Section 6.5).

Use associated estimates as initial guesses for DOA estimation at succeeding time instant.

Make \( t = t + T \) and repeat the following steps (1) to (5) till \( t = \hat{T} \), where \( \hat{T} \) = last instant of tracking period.

Initialization :

For initializing the recursion in SEM, wait until time say, \( t_k \geq 5 \) when \( P_i \) becomes invertible. \( \hat{\Psi}_{p}(t_k) \) is then computed using the least squares estimate

\[
\hat{\Psi}_{p}(t_k) = [\Xi^T(t_k)\Xi(t_k)]^{-1}\Xi^T(t_k)\hat{\xi}(t_k)
\]  (6.17)

The entire data \( \{ \hat{\theta}_{k_1}(t), \hat{\theta}_{k_2}(t), t_0 \leq t \leq t_k, k_1, k_2 \in \{1, 2, \ldots, D\} \} \) used.

6.7 Performance of Proposed Method

6.7.1 Experimental results

Presented here are different experiments to illustrate performance of proposed method.

6.7.1.1 Experiment 1

The simplest two target case with only one intercept point is considered. Targets move rapidly with rate of change of DOA being 30°/s. The chosen SNR is -3 dB with number of sensors = 8. 40 snapshots are used to estimate the 2 DOAs at every sampling instant. The proposed algorithm is able to track the two targets even at and after cross-over (Fig. 6.11). Figures 6.12 and 6.13 show forecasts of the intercept angle and time at every tracking instant respectively. It is observed that, the intercept time estimate (21.2 s) converges close to the true value (21.1613 s), so that the tracks flip exactly one time.
Fig. 6.11.  Plot of true and estimated trajectories for a single trial of the experiment.

Fig. 6.12.  Plot of RLS intercept time estimates. The intercept time forecast converges to the neighborhood of the true value when the targets cross facilitating detection of the cross-over point.
instant after true cross-over. The key point is, the values converge to the neighborhood of their true values *rapidly enough* thus facilitating correct detection of target intercept and subsequent tracking.

The **RLS** intercept point estimates are fairly accurate even at low SNR and with small number of sensors. Accuracy of the intercept point forecasts depends on the accuracy of the DOA estimates (data). Since the **ML** method gives very accurate angle estimates, the data used for curve fitting is essentially outlier free. The effective SNR of data used for obtaining intercept point forecasts is very high, though the array data itself can be very noisy. The chosen SNR and number of sensors are thresholds for the 2 target case. More number of sensors are required to get accurate intercept point forecasts as number of targets increase.

**6.7.1.2 Experiment 2**

There are 4 targets describing linear trajectories. The number of sensors is 16 and SNR is still at 0 dB. Although three targets cross each other at *almost* the same time (Fig. 6.14), the algorithm is still able to track the trajectories correctly (Fig. 6.15). The intercept point estimates are accurate and are within the area enclosed by the estimates at cross-over and hence ADA ensures that appropriate angle estimates are linked. Even if the DOA estimates are not accurate at cross-over, the accuracy of the intercept point estimates is not affected. This fact is one of the important features of the proposed method. The incorporated rotation transformation is useful for accurate intercept point estimation for arbitrarily oriented tracks, for *e.g.*, target pairs 1&2 and 3&4.

Figure 6.16 gives the RMSE (root mean square error) in the angle estimates for the entire tracking period for 20 runs of the experiment which illustrates the average performance of the tracking algorithm. RMSE of the intercept time and angle estimates are given in Figs. 6.17 and 6.18. A table of numerical results of illustrations 6.17 and 6.18 is also given in Table 6.1.

**6.7.1.3 Experiment 3**

A dense target environment is considered, with SIX fast moving targets with linear trajectories. The array has 16 sensors with data SNR being 0 dB. There are 12 *intercept points* with a span of 30 s in this complex scenario, with targets intersecting
Fig. 6.13. Plot of RLS intercept angle estimates. The intercept angle forecasts are accurate and also converge to the neighborhood of the true value facilitating detection of cross-over point.

Fig. 6.14. Four target case, true trajectories illustrating three targets crossing over at the same time.
Fig. 6.15. Estimated DOA tracks averaged over 20 runs at 0 dB SNR.

Fig. 6.16. Root mean square error of the estimated angle tracks over the entire tracking period. The error is high at the cross over points since true DOA's are closely spaced and hence estimates are noisy. The error reduces after cross-over with correct association and estimated tracks follow true tracks. The higher error in the initial portion of the track for targets 1 and 4 is because of the directionality property of the uniform linear array, i.e., its inability to give good estimates for targets close to end-fire.
Fig. 6.17. Plot of RMSE of three intercept time estimates over 20 trials. RMSE decreases rapidly as we approach the true value and has minimum value at and after cross-over. Small value of RMSE after cross-over indicates correct estimate association.

Fig. 6.18. Plot of RMSE of the corresponding three intercept angle estimates over 20 trials. We observe the same behavior in that RMSE decreases rapidly as we approach the true value and has minimum value at anti after cross-over.
TABLE 6.1  
Statistics of intercept point between targets 2 and 3.

\[ M = 16, N = 40, \text{SNR} = 0 \text{dB}, T^* = 41.00 \text{sec.}, \theta^* = 0.00^\circ \]

<table>
<thead>
<tr>
<th>Tracking Time (Sec)</th>
<th>Mean Estimated Intercept Time ( \hat{T}(t) ) (Sec)</th>
<th>Mean Estimated Intercept Angle ( \hat{\phi}(t) ) (Deg)</th>
<th>RMSE Intercept Time (Sec)</th>
<th>RMSE Intercept Angle (Deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>41.1397</td>
<td>0.0901</td>
<td>0.3592</td>
<td>0.1110</td>
</tr>
<tr>
<td>36</td>
<td>40.9417</td>
<td>0.1039</td>
<td>0.2373</td>
<td>0.1176</td>
</tr>
<tr>
<td>37</td>
<td>40.7244</td>
<td>0.1228</td>
<td>0.1832</td>
<td>0.1288</td>
</tr>
<tr>
<td>38</td>
<td>40.6928</td>
<td>0.1298</td>
<td>0.3217</td>
<td>0.1457</td>
</tr>
<tr>
<td>39</td>
<td>40.6887</td>
<td>0.1311</td>
<td>0.3456</td>
<td>0.1532</td>
</tr>
<tr>
<td>40</td>
<td>40.7027</td>
<td>0.1325</td>
<td>0.3472</td>
<td>0.1559</td>
</tr>
<tr>
<td>41</td>
<td>40.7280</td>
<td>0.1367</td>
<td>0.3328</td>
<td>0.1561</td>
</tr>
<tr>
<td>42</td>
<td>40.7501</td>
<td>0.1424</td>
<td>0.3078</td>
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<tr>
<td>43</td>
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<td>0.2852</td>
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</tr>
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<td>40.8453</td>
<td>0.1620</td>
<td>0.2317</td>
<td>0.1739</td>
</tr>
</tbody>
</table>
Fig. 6.19. Dense target environment, 6 targets and 12 intercept points to be detected.

Fig. 6.20. Averaged estimated DOA tracks for the dense target environment case.
with one another arbitrarily (Fig. 6.19). The output of the tracking algorithm (Fig. 6.20) shows that all intercept points were detected and targets tracks were followed correctly.

6.7.2 Discussion

A computationally simple and efficient solution to the data association problem is proposed. The number of sensors (M) in the uniform linear array is the key factor for good performance of the algorithm. Increasing the number of sensors causes intercept point forecasts to converge rapidly to their true values. This is because, increasing M results increases SNR and hence highly accurate DOA estimates are obtained which in turn are data for intercept point forecasting. If M is sufficient, the algorithm works well even at low SNR (0, -3 dB). The required number of sensors increases with increase in number of targets. Good results are obtained with 16-20 sensors for up to six targets.

Signal-to-noise ratio is not a significant factor for accurate estimate association. The number of snapshots required for DOA estimation at every time instant is around 40 (even with 4-6 targets) indicating good performance of the ML method used. The fitted curve accounts rotation and translation transformations and hence method works well for arbitrarily oriented tracks. This allows a simple recursive least squares solution for the intercept point which is assumed to be deterministic and unknown. Since the transformation in (6.4) is linear, the least squares estimate for \( \psi_p(t_i) \) is also the maximum likelihood estimate for Gaussian distributed errors in the parameters characterizing the second order curve. By the invariance property of the ML estimator, intercept point estimates in (6.8) and (6.9) which are functions of \( \hat{\psi}_p(t_i) \) are also ML estimates. In addition, the random variable assumption on the intercept point may offer a more robust estimator in situations where the intercept point varies, with time. e.g., tracking of slowly varying sinusoids in additive noise.

The second order fit of the angle track data implies the assumption of constant acceleration on target motion which is reasonable. The method does not require apriori knowledge about targets (e.g. SNR) identity to perform intercept point prediction. Even if either target bearing is entirely missed due to jamming signals, the prediction of the intercept point will not deteriorate rapidly since a single curve is fitted to both angle tracks, i.e., the prediction is robust to errors in tracking. Also, instead of fitting a single curve, if two regression lines are fitted, the line can completely flip in presence of a few
outlier points. It is likely for this problem to occur if DOA estimates are incorrectly assigned.

6.7.2.1 Computational complexity of data association

The usual data association algorithms require every value to be compared with every other value for data association in addition to using prior information. Hence, the complexity is of order $D^2$. Certain algorithms provide improvements under certain circumstances, but still retain their $D^2$ worst-case behavior. By determining the intercept points, we reduce the number to $D(D-1)/2$ instead of $D^2$ to associate $D$ targets. This reduction in the number of comparisons can make a significant difference when $D$ is large. For e.g., for $D = 10$, $D^2 = 100$ whereas $D(D-1)/2 = 45$. Since we estimate $D(D-1)/2$ intercept points at every time instant (assuming the number of targets to be a constant), the worst case complexity of the proposed method is also $D(D-1)/2$. Additional overheads are minimal, involving formation of the $DxD$ reordering mamx, the $Dx2$ link mamx and a $DxD$ mamx subtraction for determining the order statistic.

The proposed algorithm can be implemented partly in parallel, the intercept point forecasting for the $D(D-1)/2$ pairs can be done simultaneously.

6.8 Conclusion

A real-time recursive algorithm was developed to effectively track the angle of arrival of multiple moving targets based on intercept point estimation. The intercept point estimates are accurate and converge rapidly to the true values. The method works reasonably well for a variety of situations - fast moving targets, very closely located intersection points and for slowly varying non-linear trajectories. For sufficient number of sensors in the uniform linear array, the algorithm works well even at low SNR. However, performance of the algorithm is significantly dependent on the accuracy of the intercept point forecast (Remark 6.7). In situations where the targets are very fast and not linear, the RLS forecast takes a finite amount of time to converge and if targets cross before convergence, the cross-over is not detected. The proposed algorithm could also be used for tracking slowly varying sinusoids which cross each other in additive noise, for example, the fundamental frequencies of different speakers, the 60 Hz power supply frequency, bio-medical signals etc.
7. CONCLUDING REMARKS

In this thesis, the problem of estimation and tracking of direction of arrival of multiple signals was addressed. A maximum likelihood method for DOA estimation and three different approaches for tracking and estimate association of multiple source signals were presented.

In Chapter 2, a computationally efficient separable maximum likelihood solution was obtained for the direction of arrival and associated parameters of multiple narrowband sources incident on a large uniform linear array of sensors. Cramer-Rao bounds on the variance of the DOA estimates were derived. A new resolution criterion for resolving closely spaced sources was proposed based upon the positive definiteness of the Fisher information matrix.

The maximum likelihood approach for tracking and estimate association (Chapter 3) utilized the non-symmetric, separable signal covariance estimator to ensure that the resultant likelihood function is non-symmetric, thus ensuring automatic ordering of estimates at successive tracking time instants. The second approach (Chapter 4) used a wideband signal model instead of the usual narrowband signal approximation and parameterized the resultant spectral density of the signals by AR parameters. The estimated AR parameters (spectral signatures) were used for estimate association. In the third and final approach (Chapter 6), data from estimated trajectories of the sources/targets were used to obtain forecasts of the intercept point between pairs of targets. Association was achieved from the knowledge of these forecasts by detecting cross-over and suitably combining the evidence of cross-over detection.

An immediate extension to the work would be the analysis of the wideband estimate association scheme (Chapter 4) for the case of two signals (the problem is reduced to a two-class Bayes classifier [17]) by deriving expressions for the probability of misclassification (Bayes error) as a function of parameters of the feature vector and the center frequency. Also, a closed form solution for the spectral estimates can reduce
co~nputational complexity which ensures real time implementation. With wideband signal models, increasingly accurate results can be obtained even under multi-path reflections and the narrowband approximation on Doppler shift will not be necessary.

Chapter 6 consists of a qualitative formulation of the Algorithm for Data Association (ADA). It is possible to formulate this rigorously as a hypothesis testing problem. It can also be shown that error in association due to improper decisions or estimates of the intercept point can be detected and corrected. This formulation will be pursued in the immediate future. The proposed method is likely to give better performance in conjunction with a Kalman filter. In the proposed formulation, the intercept point is assumed to be a deterministic quantity, a Bayesian formulation will be attempted with the intercept point assumed to be a random variable. Additionally, efforts will be directed towards establishing statistical properties of intercept point estimator.

At any tracking instant, the singularity can be in one of three (different positions (Figs. 6.5 - 6.7). The location can be regarded as output of a observable Markov model. In other words, the singularity is modeled by a three state discrete vector-Markov chain. The Markov chain is a vector chain since two co-ordinates (angle, time) are needed to uniquely define the position of the intercept point in two space. By this formulation, it is possible to characterize different type of trajectories by state transitions. It overcomes the drawback of the proposed approach by being able to track trajectories of targets which come very close to each other and then diverge. Work on this topic is presently being pursued.

Due to the nature of the second order curve (a pair of lines) fitted to data in Chapter 6, the proposed algorithm can be used to detect and track line segments and smooth curves in noisy images. The motivation is to provide for a robust and computationally efficient recursive algorithm with minimal memory requirements. Presently, work is progressing in this direction. Preliminary results demonstrate computational speed up over the conventional Hough Transform [60] technique and higher robustness than the TLS-ESPRIT [46] based SLIDE [61] method. In SLIDE, lines are estimated based on the angle and the offset with respect to the vertical (Y) axis of the image. SLIDE is a very elegant, computationally simple technique with high resolution angle estimating capability. However, a practical implementation of the SLIDE shows that the offset estimates are quite sensitive to outliers. For parallel lines, though the lines
themselves are far away from each other, DOAs subtended are identical and hence TLS-ESPRIT has a high probability of failure.

In mobile communication applications, accurate localization and tracking is required in a multipath environment and array signal processing techniques are becoming increasingly popular (Section 3.1), arrays serving as base station antennas. However, a good array signal model for modeling the numerous multipath reflections from vicinity of the mobile is still evading researchers. The standard model is forced upon this scenario in [47], which is not very accurate. The objective is to develop a good signal model considering wideband signals (because of its inherent noise rejection capability) and use the approach of Chapter 6 for localization and tracking of the mobiles. Correspondence and message association are crucial when hand-offs occur.

In speech signals [62], often the first two (lower) formant frequencies are sufficient to characterize the syllable. Tracking of these formant signals will also be considered. A typical application would be to track the voice of a moving speaker in a conference room amidst interference from other people and the medium. An important application is in co-channel talker interference suppression [66] wherein accurate tracking of closely located and crossing frequencies forms the crux of the problem.

Lastly, the association schemes developed by the target tracking community [21, 27, 28] are being applied for motion correspondence by computer vision researchers [63]. An excellent review of statistical data association techniques for motion correspondence is provided in [64]. It is anticipated that the proposed method can be applied with good measure for this problem.
LIST OF REFERENCES


APPENDICES
APPENDIX I
PROOF OF THEOREM 2.1

The objective is to simplify (2.14). Consider evaluation of the first term in (2.14).
\[
\det R = \det \left[ \rho \left( I + A \Gamma A^* / \rho \right) \right] = \rho^M (\det(\Gamma / \rho)) \det(\rho \Gamma^{-1} + A^* A).
\]

Using (2.14), \( \det R = \rho^{M-B} (\det \Gamma) \det(B + M) = \rho^{M-B} \mu \det(\Gamma) \det(I + B / M) \) (1.1)

Taking \( \log \) on both sides of (1.1) gives
\[
\ln \det R = (M - D) \ln \rho + D \ln M + \ln \det \Gamma + \ln \det(I + B / M) + O(1 / M^3)
\]

Expanding \( \ln \det(I + B / M) \) in Taylor series about \( B \), and after some manipulation,
\[
\ln \det(I + B / M) = \left( \frac{1}{M} \right) \ln \det(I + B / M) + O\left( \frac{1}{M^3} \right)
\]

Substituting (1.3) in (1.2),
\[
\ln \det R = (M - D) \ln \rho + D \ln M + \ln \det \Gamma + \left( \frac{1}{M} \right) \ln \det(I + B / M) + O\left( \frac{1}{M^3} \right)
\]

For the second term in (2.14), using the second order expression for \( R^{-1} \) in (2.17),
\[
\text{tr} \left( R^{-1} \hat{R} \right) = \left( \frac{1}{\rho} \right) \text{tr} \left( \hat{R} \right) - \left( \frac{1}{\rho M} \right) \text{tr} \left( A A^* \hat{R} \right) + \left( \frac{1}{\rho M^2} \right) \text{tr} \left( A B A^* \hat{R} \right) + O\left( \frac{1}{M^3} \right)
\]

Using properties of trace [44],
\[
\text{tr} \left( A A^* \hat{R} \right) = \sum_{k=1}^{D} f_k^* \hat{R} f_k \text{ and tr} \left( A B A^* \hat{R} \right) = \sum_{k=1}^{D} \sum_{\ell=1}^{D} f_k^* \hat{R} f_{\ell} B_{k \ell}
\]

where elements of the \( B \) matrix are given by (2.17). Substituting the above results in (1.5),
\[
\text{tr} \left( R^{-1} \hat{R} \right) = \left( \frac{1}{\rho} \right) \text{tr} \left( \hat{R} \right) - \left( \frac{1}{\rho M} \right) \sum_{k=1}^{D} f_k^* \hat{R} f_k + \left( \frac{1}{\rho M^2} \right) \sum_{k=1}^{D} \sum_{\ell=1}^{D} f_k^* \hat{R} f_{\ell} B_{k \ell} + O\left( \frac{1}{M^3} \right)
\]

(1.4) and (1.6) together make up (2.14). Hence, (2.13) is given by
\[
\ln p(Y | \phi, \Gamma, \rho) = -\frac{NM \ln 2\pi}{2} - \frac{N}{2} \left[ \ln \det R + \text{tr} \left( R^{-1} \hat{R} \right) \right]
\]
\[
= \frac{N}{2\rho} \left( \frac{1}{M} \right) \sum_{k=1}^{D} f_k^* \hat{R} f_k - \frac{N}{2\rho M} \sum_{k=1}^{D} \sum_{\ell=1}^{D} f_k^* \hat{R} f_{\ell} B_{k \ell} - \frac{N}{2} \text{tr} \left( \frac{B}{M} \right) + \frac{N}{4} \text{tr} \left( \frac{B}{M} \right)^2
\]

+ (terms involving only \( \Gamma, \rho \)) + (terms not involving \( \phi, \Gamma, \rho \)) + O\left( \frac{1}{M^3} \right)

and the proof is finished.
APPENDIX II
PROOF OF COROLLARY 2.1

When \( D = 1 \), \( R^{-1} (2.15) \) has the following exact expression:

\[
R^{-1} = \rho^{-1} - \frac{1}{\rho} (f'f^*) \frac{1}{M + \rho / \Gamma}
\]

Substituting in the expression in (2.13), leads to the desired log likelihood expression

\[
\ln p(Y | \phi, \Gamma, \rho) = \frac{N}{2\rho} \left( \frac{1}{M + \rho / \Gamma} \right) f^* \hat{R} f
\]

\[
- \frac{N}{2} \left[ M \ln 2\pi + (M - 1) \ln \rho + \ln(M\Gamma + \rho) + \left( \frac{1}{\rho} \right) \text{tr}(\hat{R}) \right]
\]

and thus (2.22) is proved.
APPENDIX III

In this appendix, some simple formulae are given for manipulation of the transfer vectors of a uniform linear array which are useful in deriving the CRB relationship for the general D source case.

\[ f_p = [1, e^{j\phi_p}, \ldots, e^{j(M - 1)\phi_p}]^T \quad p = 1, 2, \ldots, D \]

\[ f'_p = [0, j e^{j\phi_p}, 2 j e^{j2\phi_p}, \ldots, j(M - 1) e^{j(M - 1)\phi_p}]^T \]

\[ f_p^* = [1, e^{-j\phi_p}, \ldots, e^{-j(M - 1)\phi_p}] \]

\[ f_p f_p^* = M \]

\[ f_p f'_p = jM(M - 1)/2 \]

\[ f_p f^* = -jM(M - 1)/2 \]

\[ f_p f'_p = M(M - 1)(2M - 1)/6 \]

\[ f_p^* f_v = \sum_{m=0}^{M-1} e^{jm\phi_d} \quad \text{where} \quad \phi_d = \phi_p - \phi_v \quad p, k = 1, 2, \ldots, D, \quad \phi_d \text{ is the angular separation between any two signal sources.} \]

\[ f_p^* f_v^* = \sum_{m=0}^{M-1} e^{jm\phi_d} \]

\[ f_p f'_v = \frac{M}{2} \left[ \frac{\sin(M - 1)}{2} \right] \quad f_p^* f'_v = \frac{M}{2} \left[ \frac{\sin(M - 1)}{2} \right] \]

\[ \sum_{m=0}^{M-1} \frac{m\cos m\phi_d}{2} = \frac{M}{2} \left[ \frac{\sin(M - 1)}{2} \right] \quad - \frac{1}{4} \left[ \frac{1 - \cos(M - 1)}{\sin^2(M - 1)} \right] \]

\[ \sum_{m=0}^{M-1} \frac{m\sin m\phi_d}{2} = \frac{1}{4} \left[ \frac{\sin(M - 1)}{2} \right] \quad - \frac{M}{2} \left[ \frac{\cos(M - 1)}{\sin^2(M - 1)} \right] \]

\[ \sum_{m=0}^{M-1} \frac{\cos m\phi_d}{2} = \frac{1}{2} \left[ \frac{1 + \sin(1 - 1)}{2} \right] \]

\[ \frac{1}{2} \left[ \frac{1 + \sin(1 - 1)}{2} \right] \]
\[ M^{-1} \sum_{m=0}^{\infty} \sin m \phi_d - \sum_{m=0}^{\infty} \frac{\sin M \phi_d \sin \left( \frac{M - 1}{2} \phi_d \right)}{\sin \left( \frac{\phi_d}{2} \right)} = -K_5 \]

\[ \sum_{m=0}^{M-1} m^2 \cos \phi_d = \frac{M^2 K_1}{2} \]

\[ \sum_{m=0}^{M-1} m^2 \sin \phi_d = K_6 \]

The definitions of \( K_1 \) upto \( K_6 \) follow from the above expressions. e.g.

\[ K_1 = \left\{ \sin \left[ \left( \frac{2M - 1}{2} \right) \phi_d \right] / \sin \left[ \left( \frac{\phi_d}{2} \right) \right] \right\}. \]

\[ C_1 = \left( \sum_{m=0}^{M-1} \cos m \phi_d \right) \left( \sum_{m=0}^{M-1} \sin m \phi_d \right) = \left( \frac{MK_1}{2} - \frac{K_2}{4} \right) \]

\[ C_2 = \left( \sum_{m=0}^{M-1} m \sin m \phi_d \right) \left( \sum_{m=0}^{M-1} m \sin m \phi_d \right) = \frac{K_3}{4} - \frac{MK_4}{2} \]

The above two expressions \( C_1 \) and \( C_2 \) are used in deriving Theorem 2.3.
APPENDIX IV
PROOF OF THEOREM 2.2

The proofs of parts (a) and (b) are given in a combined manner since (b) follows from a minor modification of the proof of (a).

Equation (2.10) can be written as,

\[ \mathbf{R} = \mathbf{pI} + \sum_{k=1}^{D} \sum_{l=1}^{D} \mathbf{f}_k \mathbf{f}_l' \Gamma_{kl} \]  

For any \( p \)th source,

\[ \frac{\partial \mathbf{R}}{\partial \phi_p} = (\mathbf{f}_p^* \mathbf{f}_p' + \mathbf{f}_p \mathbf{f}_p^*) \Gamma_{pp} + \sum_{v=l}^{D} (\mathbf{f}_v^* \mathbf{f}_v' \Gamma_{pv} + \mathbf{f}_v \mathbf{f}_v') \Gamma_{pp} \]  

Using the first order approximation for \( R^{-1} \) given by (2.20),

\[ \frac{\partial \mathbf{R}^{-1}}{\partial \phi_p} = -\frac{1}{\rho M} \sum_{k=1}^{n} \sum_{l=1}^{D} \mathbf{f}_k \mathbf{f}_l' + O(1/M^2) \]  

Thus,

\[ \frac{\partial \mathbf{R}^{-1}}{\partial \phi_p} = -\frac{1}{\rho M} \mathbf{f}_p \mathbf{f}_p' + O(1/M^2) \]  

(IV.3) and (IV.4) are key equations for obtaining closed form expressions for the CRB.

IV.1 Determination of \( Q_{pp} \)

The principal diagonal terms of the Fisher Information matrix are defined by (2.41) to be

\[ Q_{pp} = -\text{Re} \frac{N}{2} \text{tr} \left( \mathbf{J} \right) \quad \text{p} = 1, 2, ..., D. \]  

Using (IV.2) and (IV.4),

\[ \frac{N}{2 \rho M} \text{tr} \left\{ \mathbf{f}_p^* \mathbf{f}_p' \right\} \Gamma_{pp} + \sum_{v=l}^{D} \mathbf{f}_v \mathbf{f}_v' \Gamma_{pv} \]  

Substituting the results from Appendix III into (IV.6), taking only real parts and simplifying,
\[ Q_{pp} = \frac{N}{2pM} \left[ -\frac{M^2(M-1)^2}{2} + \frac{M^2(M-1)(2M-1)}{3} \right] \Gamma_{pp} + \frac{N}{2pM} \sum_{\nu \neq \rho} \frac{M(M-1)(2M-1)(1+K_i)}{12} - \frac{M^2(M-1)}{4} K_i \left( \Gamma_{\nu \nu} + \Gamma_{\nu \rho} \right) \]  

(IV.7)

where \( K_i = \{ \sin(\frac{2M-1}{2}) \phi_i \} / \sin(\frac{\phi_i}{2}) \).

In (IV.7), it can be assumed without loss of generality that \( \Gamma_{\nu \nu} = \Gamma_{\nu \rho} \). Including the \( O(M^2) \) terms also by neglecting only \( O(M) \) terms for large \( M \) in (IV.7) and simplifying,

\[ Q_{PP} = \frac{NM^3\Gamma_{pp}}{12} - \frac{NM^2}{12} \sum_{\nu \neq \rho} (K_i - 2) \left( \frac{\Gamma_{\nu \nu}}{\rho} \right) + O(M) \]  

(IV.8)

The effect of \( O(M^2) \) terms is considered in order to represent \( Q_{pp} \) as a function of \( \gamma \) and \( (\phi_\rho) \).

The second term on the RHS of (IV.8) can be further simplified and thus, (IV.8)

\[ Q_{pp} = \frac{NM^3\Gamma_{pp}}{12} \]  

which proves equation (2.43) in Theorem 2.2.  

(QED)

**IV. 2 Determination of \( Q_{pv} \)**

The non-diagonal terms of the Fisher Information matrix \( Q \) are defined by (2.41) to be

\[ Q_{pv} = -\text{Re} \left( \frac{N}{2} \right) \text{tr} \left[ \frac{\partial R}{\partial \phi_p} \frac{\partial R^{-1}}{\partial \phi_v} \right] \]  

(IV.9)

The \( \frac{\partial R}{\partial \phi_p} \) term is given by (IV.2) and \( \frac{\partial R^{-1}}{\partial \phi_v} = -\frac{1}{\rho M} \left( f_\nu f_\nu^* + f_\nu f_\nu^* \right) + O(1/M^2) \) (IV.10)

which is similar to (IV.4).

Therefore, using (IV.2) and (IV.10),

\[ Q_{pv} = \text{Re} \left( \frac{N}{2pM} \right) \text{tr} \left[ \left( f_\rho f_\rho^* + f_\rho f_\rho^* \right) \left( f_\nu f_\nu^* + f_\nu f_\nu^* \right) \Gamma_{pp} \right] \]  

\[ + \text{Re} \left( \frac{N}{2pM} \right) \text{tr} \left[ \left( f_\rho f_\rho^* \Gamma_{pv} + f_\rho f_\rho^* \Gamma_{vp} \right) \left( f_\nu f_\nu^* + f_\nu f_\nu^* \right) \right] + O(M^2) \]  

(IV.11)

Expanding and taking trace gives
Using results in Appendix III and simplifying,

term \( A = O(M) \) and term \( B = -\left( \frac{N \Gamma_{pv} K_1}{4 \rho} \right) M^2 + O(M) \) where it has been assumed without loss of generality that \( T_{pv} = T_{vp} \). Including \( O(M^2) \) terms to be significant terms and neglecting only \( O(M) \) terms gives us

\[
Q_{pv} = -N \left( \frac{\Gamma_{pv}}{\rho} \right) \left( \frac{M^2 K_1}{4} \right) + O(M) \]

proving (2.44) of Theorem 2.2. This finishes the proof for Theorem 2.2. (QED)
APPENDIX V
PROOF OF THEOREM 2.3

Definitions of $R$ and $\frac{\partial R}{\partial \phi_p}$ are taken from (IV.1) and (IV.2). The key point is that $R^{-1}$ is defined using the second order approximation from (2.19).

$$R^{-1} = \frac{1}{\rho} \left[ I - \frac{\frac{AA^*}{M} + \frac{ABA^*}{M^2}}{M} \right] + O(1/M^3) \quad (V.1)$$

Now, $R^{-1}$ can be written in terms of the elements of the B matrix as

$$R^{-1} = \frac{1}{\rho} \left[ I - \frac{1}{M} \sum_{k=1}^{B} f_k^* f_k^{**} + \frac{1}{M^2} \left( \sum_{k=1}^{B} f_k^* B_{kk} + \sum_{k=p}^{B} \sum_{k=p}^{B} f_k^* B_{kp} \right) \right] + O(1/M^3)$$

Hence, $\frac{\partial R^{-1}}{\partial \phi_p}$ for any $p$th source, is given by

$$\frac{\partial R^{-1}}{\partial \phi_p} = \left( \frac{1}{\Gamma_p M^2} - \frac{1}{\rho M} \right) (f_p^* f_p^* + f_p f_p^*) + \frac{1}{\rho M^2} \sum_{v=1}^{B} \left( f_v^* B_{vp} \right) + \frac{1}{\rho M^2} \sum_{v=1}^{B} \left( f_v^* B_{vp} \right)$$

$$+ \frac{C_1}{\rho M^2} \sum_{v=1}^{B} \left( f_v^* f_v^* - f_v f_v^* \right) - \frac{C_2}{\rho M^2} \sum_{v=1}^{B} \left( f_v^* f_v^* \right) + O(1/M^3) \quad (V.2)$$

where $C_1 = \left( \sum_{m=0}^{M-1} m \cos m \phi_d \right)$ and $C_2 = \left( \sum_{m=0}^{M-1} m \sin m \phi_d \right)$.

V.II Determination of $Q_{pp}$
The principal diagonal elements of the FIM given by (2.41) are

$$Q_{pp} = -\text{Re} \frac{N}{2} \text{tr} \left[ \frac{\partial R}{\partial \phi_p} \frac{\partial R^{-1}}{\partial \phi_p} \right] \quad (V.3)$$

It is required to use (IV.2) and (V.2) in (V.3) to get $Q$.

The strategy is given briefly. There are five terms in (V.2), each of which will be multiplied with the three terms in (IV.2) to generate five main terms. Each of these five terms will have three parts due to (IV.2). These five terms will be then simplified in a manner similar to Appendix IV. Since a second order approximation is being used, terms of order $O(M^2)$ and higher will be considered significant. Only the real parts are taken according to the definition in (V.3) and results given in Appendix III are used for obtaining simplified expressions. Details are omitted since the algebra is tedious. The final expressions for each of five terms are as follows:
- $Q_{pp} =$

\[
- \frac{NM^3}{12} \left( \frac{\Gamma_{pp}}{\rho} \right) + \frac{NM^2}{12} + \frac{NM^2}{12} \sum_{v \neq p} \left( \frac{\Gamma_{pv}}{\rho} \right)K_i - \frac{NM^2}{6} \sum_{v \neq p} \left( \frac{\Gamma_{pv}}{\rho} \right) \\
\]

\[
\frac{NM^2(D - 1)}{6} + \frac{NM^2}{12} \sum_{v \neq p} \left( \frac{\Gamma_{pv}}{\rho} \right)(1 + K_i) + \frac{NM^2(D - 1)}{6} + \frac{NM^2}{12} \sum_{v \neq p} \left( \frac{\Gamma_{pv}}{\rho} \right)(1 + K_i) \\
\]

\[
- \frac{NM^2}{4} \sum_{v \neq p} \left( \frac{\Gamma_{pv}}{\rho} \right)K_i + O(M)
\]

(V.4)

where $K_i$ as before is given by $K_i = \frac{\sin \left( \frac{2M - 1}{2} \phi_d \right)}{\sin \left( \phi_d \right)}$

It is to be noted that, term 5 being of $O(M)$, is included within the $O(M)$ term in (V.4). It is assumed without loss of generality that $\Gamma_{pv} = \Gamma_{vp}$. A straightforward manipulation of (V.4) yields the expression for the principal diagonal elements of the FIM using second order approximation and is denoted by $Q_{pps}$. Thus, we get

\[
Q_{pps} = \frac{NM^3}{12} \left( \frac{\Gamma_{pp}}{\rho} \right) - \frac{NM^2}{12} - \frac{NM^2(D - 1)}{3} + O(M)
\]

which proves equation (2.46) in Theorem 2.3. (QED)

**V. 2 Determination of $Q_{pv}$**

The non-diagonal terms of the Fisher Information matrix $Q$ are defined by (2.41) to be

\[
Q_{pv} = -\text{Re} \frac{N}{2} \text{tr} \left[ \frac{\partial R}{\partial \phi_p} \frac{\partial R^{-1}}{\partial \phi_v} \right]
\]

(V.5)

The $\frac{\partial R}{\partial \phi_p}$ term is given by (IV.2) and $\frac{\partial R^{-1}}{\partial \phi_v}$ is similar to (V.2) with all subscripts 'p' replaced by 'v'. A similar strategy is followed as above. Substituting (IV.2) and (V.2) into (V.5), five main terms each of which have three parts are obtained. These are simplified using results of Appendix III. Here, in all the summations, only one significant term exists, i.e., when $k = p$ and $k = v$. Omitting simplification details, the final expressions for each of the five terms are:
where \( K_1 \) is defined as before. Since term 5 is only of \( O(M) \), it is included within the \( O(M) \) term in (V.6). Now, (V.6) can be easily simplified to give the expression for the non-diagonal terms of the FIM for the second order approximation and is denoted by \( Q_{pvs} \). Thus,

\[
Q_{pvs} = \left[ \frac{NM^2}{4} + \frac{NM^2(K_1 + 1)}{8} \left( \frac{\Gamma_{pv}}{\rho} \right) \right] + O(M)
\]

which is the same as equation (2.47) of Theorem 2.3. (QED)
APPENDIX VI
TRANSFER VECTORS AND THEIR DERIVATIVES FOR SIMPLIFYING EQUATIONS (3.31) - (3.34)

The derivatives of the transfer vectors are denoted by the 'prime' symbol as shown.

The kth column vector of the tracking matrix can be written in expanded form as

\[ f_k = \begin{bmatrix} 1 e^{j \pi \cos \theta_k + \left( \frac{m \sin^2 \theta_k}{l_2} \right)} e^{2j \pi \cos \theta_k + 2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)} \cdots e^{(M-1)j \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)} \end{bmatrix} \]

for \( k = 1, 2, \ldots, D \).

Also,

\[ f_k' = \begin{bmatrix} 1 e^{j \pi \cos \theta_k + \left( \frac{m \sin^2 \theta_k}{l_2} \right)} e^{2j \pi \cos \theta_k + 2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)} \cdots e^{(M-1)j \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)} \end{bmatrix} \]

\[ f'_{k0} = \begin{bmatrix} 0 \cdots (j \pi (M-1) \sin \theta_k) \left( \frac{2d(M-1) \cos \theta_k}{r_k} - 1 \right) \left( \frac{M-1) \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)}{r_k} \right) \end{bmatrix} \]

\[ f'_{kr} = \begin{bmatrix} 0 \cdots j \pi \left( \frac{(M-1) \sin \theta_k}{r^2} \right) \left( \frac{M-1) \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)}{r_k} \right) \end{bmatrix} \]

\[ f'_{k\theta} = \begin{bmatrix} 0 \cdots (j \pi (M-1) \sin \theta_k) \left( 1 - \frac{2d(M-1) \cos \theta_k}{r_k} \right) \left( \frac{(M-1) \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)}{r_k} \right) \end{bmatrix} \]

\[ f'_{kr} = \begin{bmatrix} 0 \cdots j \pi \left( \frac{(M-1) \sin \theta_k}{r^2} \right) \left( \frac{(M-1) \pi \cos \theta_k + (M-1)^2 \left( \frac{m \sin^2 \theta_k}{l_2} \right)}{r_k} \right) \end{bmatrix} \]

\[ \sum_{m=1}^{M-1} m = \frac{M(M-1)}{2} \]

\[ \sum_{m=1}^{M-1} m^2 = \frac{(M-1)^2 M^2}{4} \]

\[ \sum_{m=1}^{M-1} m^3 = \frac{M(M-1)(2M-1)(3M-1)^2 + 3M + 4}{30} \]

\[ \sum_{m=1}^{M-1} m^4 = \frac{M(M-1)(2M-1)(3M-1)^2 + 3M + 4}{30} \]
APPENDIX VII

PROOF OF THEOREM 3.1

VII.1 Determination of \( Q_{60}^k \)

From (3.10), the data covariance matrix for the single source case is given by

\[
R = \rho I + f_k f_k^\Gamma
\]  

(VII.1)

Using matrix inversion lemma,

\[
R^{-1} = \rho^{-1} (I - K f_k f_k^\Gamma)
\]  

(VII.2)

where \( K = \frac{\Gamma/\rho}{1 + M \Gamma/\rho} \)  

(VII.3)

Differentiating (VII.1) and (VII.2) w.r.t. \( \theta \),

\[
\frac{\partial R}{\partial \theta_k} = \nabla_k R = \left( f_k f_k^\star + f_{k_0} f_{k_0}^\star \right) \Gamma
\]  

(VII.4)

\[
\frac{\partial R^{-1}}{\partial \theta_k} = \nabla_k R^{-1} = \frac{-K}{\rho} \left( f_k f_k^\star + f_{k_0} f_{k_0}^\star \right)
\]  

(VII.5)

Substituting (VII.4) and (VII.5) in (3.31) and multiplying terms,

\[
Q_{60}^k = \frac{N}{2} \left[ \frac{K \Gamma}{\rho} \right] \operatorname{trace} \left( f_k f_k^\star + f_{k_0} f_{k_0}^\star \right)^2 \]

Using properties of the trace,

\[
Q_{60}^k = \frac{N}{2} \left[ \frac{K \Gamma}{\rho} \right] \left( \left( f_k f_k^\star + f_{k_0} f_{k_0}^\star \right)^2 \right) + 2 \left( f_k f_k^\star + f_{k_0} f_{k_0}^\star \right) \left( f_k f_{k_0}^\star + f_{k_0} f_k^\star \right)
\]

(VII.6)

The various terms in (VII.6) have to be evaluated. It is seen that,

\[
\left( f_k f_{k_0}^\star \right) = -\left( f_{k_0} f_k^\star \right)
\]

(VII.7)

Using results from Appendix VI and after some manipulations,

\[
\left( f_k f_{k_0}^\star \right) = \left( j \pi \sin \theta_k \right) \frac{M^2}{2} - \left( j \pi \sin 2\theta_k \frac{M^3}{3} \right)
\]

Thus,

\[
\left( f_{k_0} f_k^\star \right) = -\left( \pi \sin \theta_k \right)^2 \frac{M^4}{4} + 2 \left( \pi \sin \theta_k \right) \left( \pi \sin 2\theta_k \right) \left[ \frac{d}{r_k} \right] \frac{M^5}{6}
\]

(VII.8)

From (VII.7), \( \left( f_k f_{k_0}^\star \right)^2 \) is the same as (VII.8). Also,

\[
\left( f_k f_{k_0}^\star \right) = M
\]

(VII.9)

\[
\left( f_{k_0} f_k^\star \right) = \left( \pi \sin \theta_k \right)^2 \left[ \frac{M^3}{3} - M^4 \cos \theta_k \left( \frac{d}{r_k} \right) \right]
\]

(VII.10)

Only the terms corresponding to the highest order of \( M \) are selected in view of a large \( M \) approximation.
\[
\frac{M(M-1)}{2} \approx \frac{M^2}{2} \quad \text{and} \quad \frac{M(M-1)(2M-1)}{6} \approx \frac{M^3}{3}
\]

Also, in deriving (VII.8), (VII.9), (VII.10), \(O\left(\frac{d}{r_k}\right)^2\) and lesser order terms are neglected since \(d \ll r_k\).

**Combining** results (VII.8) through (VII.10), substituting in (VII.6) and simplifying,

\[
Q_{06}^k = \frac{N}{2} \left( \frac{K\Gamma}{\rho} \right) \left( \frac{\pi^2 M^4 \sin^2 \theta_k}{6} \left( 1 - \frac{4Md\cos\theta_k}{r_k} \right) \right)
\]  
(VII.11)

For \(M, \text{SNR} \gg 1\),

\[
\left( \frac{K\Gamma}{\rho} \right) = \frac{\text{SNR}}{\rho}
\]  
(VII.12)

Hence, the final result is

\[
Q_{06}^k = \frac{\pi^2 N \text{SNR} M^3 \sin^2 \theta_k}{12} \left( 1 - \frac{4Md\cos\theta_k}{r_k} \right)
\]

which is the same as (3.35).  
(QED)

**VII.2 Determination of** \(Q_n^k\)

In a manner similar to the determination of \(Q_{06}^k\),

\[
\frac{\partial R}{\partial r_k} = \nabla_k R = \left( f_k f_k' + f_k' f_k \right) \Gamma
\]  
(VII.13)

\[
\frac{\partial R}{\partial r_k} = \nabla_k R = -\frac{K}{\rho} \left( f_k f_k' + f_k' f_k \right)
\]  
(VII.14)

Thus, substituting (VII.13) and (VII.14) in (3.34),

\[
Q_n^k = \frac{N}{2} \left( \frac{K\Gamma}{\rho} \right) \text{trace} \left[ \left( f_k f_k' + f_k' f_k \right)^2 \right]
\]

Using properties of the trace of a matrix,

\[
Q_n^k = \frac{N}{2} \left( \frac{K\Gamma}{\rho} \right) \left[ \left( f_k f_k' \right)^2 + \left( f_k f_k' \right)^2 + 2(f_k f_k') (f_k f_k') \right]
\]  
(VII.15)

Now, (VII.15) is simplified using expansion from Appendix VI for large M. It is seen that, (algebra omitted)

\[
Q_n^k = \frac{N}{2} \left( \frac{K\Gamma}{\rho} \right) \left[ \left( \frac{\pi d \sin^2 \theta_k}{r_k} \right)^2 - \frac{M^6}{9} - \frac{M^6}{9} + \frac{2M^6}{5} \right].
\]

Using (VII.12) and simplifying,
\[ Q^k = \left( \frac{4}{45} \right) N M^5 \text{SNR} \left( \frac{\pi d \sin^2 \theta_k}{r_k^2} \right)^2 \] which is the same as \((3.37)\) \hspace{1cm} (QED)

**VII.3 Determination of \( Q^k \)**

Once again,
\[
\frac{\partial R}{\partial r_k} = \nabla_k R = \left( f_k f_k' + f_k f_k' \right) \Gamma \tag{VII.16}
\]
\[
\frac{\partial R^{-1}}{\partial \theta_k} = \nabla_k R^{-1} = -\frac{K}{\rho} \left( f_k f_k' + f_k f_k' \right) \tag{VII.17}
\]

Substitution of (VII.16) and (VII.17) in (3.33) gives
\[
Q^k_{r_0} = \frac{N}{2} \left( \frac{K \Gamma}{\rho} \right) \text{trace} \left[ \left( f_k f_k' + f_k f_k' \right) \left( f_k f_k' + f_k f_k' \right) \right] \hspace{1cm} \text{which when simplified results in}
\]
\[
Q^k_{r_0} = \frac{N}{2} \left( \frac{K \Gamma}{\rho} \right) \left[ \begin{array}{c}
\text{term I} \\
\text{term II} \\
\text{term III} \\
\text{term IV}
\end{array} \right]
\]
\[
\text{which when simplified results in}
\]
\[
Q^k_{r_0} = \frac{N}{2} \left( \frac{K \Gamma}{\rho} \right) \left[ \begin{array}{c}
\text{term I} \\
\text{term II} \\
\text{term III} \\
\text{term IV}
\end{array} \right]
\]
\[
\text{(VII.18)}
\]

After large M approximation and neglecting \( \mathcal{O} \left( \frac{d^2}{r^3} \right) \) terms in (VII.18), it is seen that, (algebra omitted),
\[
Q^k_{r_0} = Q^k_{r_0} = \frac{N M^4 \text{SNR}}{12} \left( \frac{\pi^2 d \sin^3 \theta_k}{r_k^2} \right) \hspace{1cm} (QED)
\]

which is the same as \((3.36)\). Thus, the proof of Theorem 3.1 is finished.