Equational Logic and Equational Theories of Algebras

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AND EQUATIONAL THEORIES OF ALGEBRAS

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CONTENTS

Introduction. .......................................................... 1

Chapter 1. The general theory. ..................................... 7

1.1 Syntax. .............................................................. 8
1.2 Semantics .......................................................... 12
1.3 Derivability and the completeness theorem of Birkhoff ........ 18
1.4 The characterization of varieties ............................... 24
1.5 Definitional equivalence of theories ......................... 26
1.6 Definitional equivalence of classes of algebras .......... 39
1.7 Theories of groups and other familiar algebraic structures .... 52
Problems ................................................................. 67

Chapter 2. The lattices of theories ................................. 68

2.1 The characterization problem. ................................ 70
2.2 Equationally complete theories ................................ 91
2.3 The structure of relativized lattices of theories
  and the covering relation. .................................... 118
2.4 Definability in the lattices $\mathfrak{Th}$ .................. 155
2.5 Identities of the lattices $\mathfrak{Th}[\theta]$ ................... ...
2.6 Notions of irreducibility in the lattices $\mathfrak{Th}[\theta]$ ... ...
2.7 Product of theories ...............................................

Chapter 3. Cardinality of bases of theories ...................... ...

Chapter 4. Decision problems ...................................... ...

Chapter 5. Model theory ............................................. ...
**Equational logic and equational theories of algebras**

**Introduction.**

Equational logic is a fragment of first-order logic. It constitutes that part which deals exclusively with sentences in the form of identities—the universal closure of equalities between terms—and the classes of structures defined by identities. Equational logic plays a special role in the metamathe-matics of algebra since the classes of algebras of most inter-est to algebraists are either axiomatically defined by identities or are closely related to such a class. As examples we have the class of semigroups defined by the associative law \( x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \) and the class of commutative semigroups defined by the associative law together with the commutative law \( x \cdot y \approx y \cdot x \).

Groups are also defined by equations but one must be careful here. Groups are usually presented as semigroups in which there exists a two-sided identity element and a two-sided inverse for every element. When these axioms are formalized we get, in addition to the associative law,

\[
\exists x \forall y [x \cdot y \approx y \land y \cdot x \approx x \land \exists z (z \cdot y \approx x \land y \cdot z \approx x)].
\]

This axiom is of course not an identity and it is easy to see
that there exists no set of identities logically equivalent to it together with the associative law. Indeed, considered as a special class of semigroups the class of groups is not even a universal class since it is not closed under the formation of subalgebras. (The semigroup of natural numbers is a subalgebra of the group of integers but it does not form a group.) However, when we conceive of groups as algebras with three fundamental operations—binary composition, $\cdot$, unary inverse, $^{-1}$, and a distinguished element (or $o$-ary operation), $e$—then the axioms become

$$e \cdot x = x, \quad x \cdot e = x, \quad x \cdot x^{-1} = e$$

together with the associative law; so groups in this second conception form an equational class. Algebraists do not generally bother to distinguish between these two conceptions of a group, but we see already that for metamathematical considerations the distinction is important.

Many other conceptions of a group are familiar to algebraists: as algebras with composition and inverse as the only fundamental operations, as algebras with the single operation of division (either right- or left-hand division). In all these conceptions the class of groups forms an equational class; for example, the following single equational found by Higman-Neuman [52] proves
to be sufficient to define the class of all groups with right-hand division as the only fundamental operation.

\[ x:(((x:x):y):z):(((x:x):x):z)) \approx y. \]

From the above discussion we see that the presumably well-defined notion of group, when looked at closely from a metamathematical point of view, ramifies into many different notions. This leads to the problem of defining exactly what a group is. This particular problem, which finds a solution in the general theory of definitional equivalence discussed in the last part of Chapter 1, can be thought of as fairly typical of the type of problems considered in an important part of equational logic.

The class of rings is equational defined if we choose the right fundamental operations, and, as in the case of groups, there are many different conceptions of rings. The class of modules over some fixed ring \( R \) also forms an equational class but in this case we have to consider an infinite number of fundamental operations (if the ring is infinite). Let \( + \) and \( - \) denote the Abelian group operations of addition and additive inverse in the module. Then for each element \( r \) of \( R \) let \( O_r \) denote the operation of scalar multiplication by \( r \). Then an algebra is a left \( R \)-module iff it satisfies the equational axioms for an Abelian group together with all of the following...
identities:

\[ O_{r}O_{s}(x) \approx O_{t}(x) \quad \text{for all } r, s, t \in R \quad \text{such that } t = r \cdot s \]

\[ O_{r}(x) + O_{s}(x) \approx O_{t}(x) \quad \text{for all } r, s, t \in R \quad \text{such that } t = r + s \]

\[ O_{r}(x+y) \approx O_{r}x + O_{r}y \quad \text{for all } r \in R. \]

Using this same idea many other important classes of algebras become equationally defined; e.g., linear algebras, Lie algebras, etc.

Boolean algebras form an important equational class and throughout the first third of the century much research was done on finding simple axioms for them; cf. Birkhoff [67], p.44 for references. The following system is due to Huntington and appears in loc. cit.

\[ x \lor y \approx y \lor x \]

\[ x \lor (y \lor z) \approx (x \lor y) \lor z \]

\[ (x' \lor y)' \lor (x' \lor y')' \approx x. \]

(The axiom system actually given in loc. cit. involves both join and meet but one of the axioms is in the form of a definition of join in terms of meet and complementation; thus meet can be eliminated on the basis of this definition to give the system of equations listed above.) There is an interesting open problem
in connection with this particular set of axioms. Can the third axiom be replaced by the axiom

\[ ((x \lor y)' \lor (x \lor y'))' = x? \]

This problem originates with Herbert Robbins. It is known only that every finite algebra which satisfies this axiom, together with the commutative and associative laws, is a Boolean algebra.

It seems to be generally agreed that the general theory of algebras, as a fully conceived mathematical discipline, began in 1935 with the paper Birkhoff [35]. Equational logic in its broadest conception as a part of the general theory of algebras was born in the same paper. Birkhoff proves two theorems: a completeness theorem for equational logic which is entirely analogous to the Gödel completeness theorem for first-order logic, and a characterization theorem which provides a purely algebraic characterization for equational classes. The first part of Chapter 1 of this paper is devoted to a discussion of these two theorems and related results.

Although it is impossible to make the separation complete, recent research in equational logic can be roughly divided along four lines: (I) the study of the structure of lattices of equational theories; (II) investigations relating to the cardinality of axiom systems for equational theories; (III) decision
problems; (IV) model theory. Furthermore, there is a more or less natural dichotomy of each of these areas corresponding to whether one is interested mainly in results of a general character or in results applying to various special kinds of algebras. The kinds of algebras whose equational metatheory has been studied in any detail are semigroups, lattices, and groups, and to a lesser extent rings, loops, quasigroups and some other algebras with an underlying group structure. In Chapters 2--5 we shall discuss each of the topics (I)--(IV), respectively. We will concentrate on the general theory but will discuss as many of the special results as possible while emphasizing open problems.

As the finally part of this introduction we mention some general references to equational logic and related subjects. For the general theory of algebras the following books are recommended: Cohn [65], Grätzer [68], Henkin-Monk-Tarski [71], Jonsson [72], Mal'cev [71], and Pierce [68]. All of these works include some discussion of general results of equational logic; Grätzer's and Cohn's books are probably most comprehensive in this regard. Mal'cev's is a collection of papers while the others are in the form of textbooks or monographs.

The only reference for equational logic in general now available is the survey paper Tarski [68] where results in the
areas (II) and (III) of a primarily general character are discussed. For results applying to the special algebras see the survey papers Evans [71a] for semigroups and B. H. Neumann [67] for groups; for more comprehensive treatments see the book H. Neumann [67] for groups and the paper Osborn [72] for other algebras with an underlying group structure.

Chapter 1. The general theory.

As was mentioned in the introduction equational logic is a fragment of first-order logic and consequently many of the basic notions and fundamental results of the latter automatically apply to the former. For example, the notion of an equation $\mathcal{E}$ being formally derivable from a set $\Gamma$ of equations certainly has meaning as does the companion notion of $\mathcal{E}$ being true in every model of $\Gamma$, i.e., a logical consequence of $\Gamma$. Moreover, from the completeness theorem of first-order logic we immediately see that these two notions are the same in extension. In view of the special character of equations, however, one would expect that for the purpose of obtaining just the logical consequences of $\Gamma$ in the form of equations a much simpler deductive apparatus than that required for the full predicate logic would suffice, and this is indeed the case.

Actually, it proves most convenient to develop even the
basic parts of equational logic independently of first-order
theory. However it is helpful throughout this development to
always keep the first-order case in mind—to emphasize the
parallelism and to point out both similarities and differences.

Section 1.1. Syntax

As in the first-order case one builds the language of equational
logic by choosing an alphabet and describing formation rules by
means of which the well-formed formulas of the language, the
equations, are inductively defined.

The alphabet includes a fixed infinite sequence of distinct
variable symbols: \( v_0, v_1, v_2, \ldots \). There is only one logical
constant: the binary predicate symbol \( \approx \) denoting equality.
For non-logical constants the alphabet includes a doubly infinite
sequence of operation symbols for each rank: \( o^{(0)}_0, o^{(0)}_1, o^{(0)}_2, \ldots, \)
\( o^{(1)}_0, o^{(1)}_1, o^{(1)}_2, \ldots, o^{(2)}_0, o^{(2)}_1, o^{(2)}_2, \ldots, \). Thus \( o^{(\lambda)}_\kappa \) is the
\( \kappa \)-th symbol in the fixed infinite sequence of operation symbols
of rank \( \lambda \). Operation symbols of rank 0 are also called
constant symbols.

There are no non-logical relation symbols or symbols for
Boolean connectives. There are also no quantifier symbols; the
universal quantifier prefix of identities becomes superfluous when
these are the only sentences under consideration. Notice that
this formalizes a convention that is common in algebra.

We use \( x, y, z, x_1, y_1, z_1, \ldots \) as syntactic variables ranging over individual variable symbols; by convention they shall be assumed to denote distinct variable symbols in any given context—unless indicated otherwise. We also use \( P, Q, R, S, P_1, Q_1, \ldots \) to denote operation symbols and a convention similar to the one described above for variables shall apply. When the rank of the represented operation symbol is clear from context it will not be explicitly mentioned.

An expression is any finite sequence of letters of the alphabet. Expressions of length 1 are usually identified with the unique letter in the range; thus \( \langle P \rangle \) and \( P \) and \( \langle x \rangle \) and \( x \) are identified. The set of terms is defined to be the smallest set \( T \) of expressions such that (i) \( \forall \xi \in T \) for every \( \xi < \omega \); (ii) whenever \( \tau_0, \ldots, \tau_{\lambda-1} \in T \), then \( T \) also contains the concatenation of the \( \lambda+1 \) sequences \( \langle O^{(\xi)} \rangle, \tau_0, \ldots, \tau_{\lambda-1} \); this sequence is written

\[
O^{(\xi)} \tau_1 \cdots \tau_{\lambda-1}.
\]

Alphabet symbols and terms are often referred to respectively as letters and words; this is especially true when dealing with groups.

There is a simple algorithm for testing when an expression is a term. Let \( \tau \) be an arbitrary expression and let
be all the operation symbols occurring in $\tau$ where a symbol is counted each time it occurs in $\tau$. Let

$$\phi\tau = \sum_{\nu < \mu} (\lambda_{\nu})$$

Let $\psi\tau$ equal the number of occurrences in $\tau$ of variable symbols and operation symbols of rank 0.

**Theorem 1.1.1.** Let $\tau$ be any expression. Then a necessary and sufficient condition for $\tau$ to be a term is that $\phi\tau + 1 = \psi\tau$ and $\phi\sigma \geq \psi\sigma$ for every proper initial segment $\sigma$ of $\tau$.

A proof of this theorem can be found in Rosenbloom [50].

It has many interesting consequences. For example, no proper initial or terminal segment of a term can be a term. Also, for any expression $\sigma$ there exist (possibly empty) expressions $\rho, \pi$ such that $\rho\sigma\pi$ is a term. As another consequence of the theorem we have that, for each term $\tau$ that is not a variable or constant symbol, there exists a unique operation symbol $O_{\lambda}$ and a unique sequence $\sigma_0, \ldots, \sigma_{\lambda-1}$ of terms such that

$$\tau = O_{\lambda} \sigma_0 \cdots \sigma_{\lambda-1};$$

thus an arbitrary term can be parsed in one and only one way.
Moreover, if \( \tau = Q \sigma_0 \cdots \sigma_{\lambda-1} \) is any term and \( \rho \) any proper subterm of \( \tau \) (i.e., \( \tau = \pi \rho \xi \) for some pair of expressions \( \pi, \xi \) not both of which are empty), then \( \rho \) must be a subterm of one of the \( \sigma_i \).

We are using the simpler Polish notation for constructing terms (i.e., all operation symbols are prefixed and parentheses are eliminated). However, in certain cases, especially when we are dealing with some special kind of algebras such as groups for which there is a standard notation, we will often write binary operation symbols as infixed. Also, in certain situations when we are dealing with a complicated term and want to clarify its structure we shall represent it by its parsing tree. For example, if \( Q \) is a binary operation symbol, then the same term can be represented in any one of the following three ways:

\[
Q Q x y z, \quad (x Q y) Q z, \quad \begin{array}{c}
\mapsto \\
Q \\
x \\
y \\
z 
\end{array}
\]

Also, within a term strings of consecutive occurrences of the same symbol are abbreviated using exponents; for example \( Q Q Q \times x x x y \) can be written \( Q^3 \times Q x y \).

An equation is any expression of the form \( \tau \approx \sigma \) where \( \tau \) and \( \sigma \) are terms; we shall always write equations in the form \( \tau \approx \sigma \). The set of all variables, terms, and equations are denoted by \( \text{Va} \), \( \text{Te} \), and \( \text{Eq} \), respectively. Usually we will want to
consider not the entire language but some fragment of it
determined by an arbitrary subset \( I \) of operation symbols; it
is obvious how this sublanguage is defined. The set \( I \) is
called the **type** of the language and the corresponding sets of
terms and equations are denoted by \( \text{Te}_I \) and \( \text{Eq}_I \). Any equation
of the form \( \tau \equiv \tau \) is called a **tautology** and \( \text{Ta}_I \) denotes the
set of all tautologies of type \( I \). In a context in which the
particular type \( I \) is either understood or of no consequence
we simply write \( \text{Te}, \text{Eq}, \text{and} \text{Ta} \) for \( \text{Te}_I, \text{Eq}_I, \text{and} \text{Ta}_I \),
respectively; a similar remark applies to other notation introduced
below.

**Section 1.2. Semantics**

We will express the fact \( f \) is a function from \( A \) into \( B \)
by writing \( f: A \to B \); we write \( f: A \torightarrow B \) and \( f: A \to\to B \) when
\( f \) is respectively one-one and onto. If \( \langle A_m : m \in M \rangle \) is any
system of sets we let

\[
P_m A_m = \{ f : \operatorname{M} \to \bigcup_{m \in M} A_m, \text{fm} \in A_m \text{ for each } m \in M \}
\]

be the **Cartesian product**. The set of all functions from \( A \)
into \( B \) is written \( A^B \); this is also called the **\( A \)'th Cartesian
power** of \( B \). For any ordinal \( \xi \), \( \xi A \) is then the set of all
\( \xi \)-termed sequences of elements of \( A \). By an **operation** on \( A \) we
mean any function \( f: \mathcal{A} \rightarrow A \) where \( \xi \) is an ordinal called the rank of \( f \). Notice that \( \mathcal{A} \) is the set of all operations on \( A \) of rank \( \xi \).

Let \( I \) be any set of operation symbols. By an algebra of type \( I \) we mean any structure

\[
\mathfrak{U} = \langle A, Q^{(\mathfrak{U})} \rangle_{Q \in I}
\]

where \( A \) is a non-empty set and \( Q^{(\mathfrak{U})} \) is an operation on \( A \) with same rank as \( Q \). The superscript \( (\mathfrak{U}) \) on \( Q^{(\mathfrak{U})} \) is usually suppressed when the algebra \( \mathfrak{U} \) is clear from context. Capital German letters are used to represent algebras and the corresponding capital latin letter the universe, or carrier, of the algebra.

For algebras \( \mathfrak{U} \) and \( \mathfrak{B} \) of the same type we write \( \mathfrak{U} \subseteq \mathfrak{B} \) if \( \mathfrak{U} \) is a subalgebra of \( \mathfrak{B} \). The universe of a subalgebra of \( \mathfrak{B} \) is called a subuniverse; thus a subuniverse of \( \mathfrak{B} \) is any set \( A \subseteq B \) that is closed under all the fundamental operations of \( \mathfrak{B} \). To express the fact that \( h \) is a homomorphism from \( \mathfrak{U} \) into \( \mathfrak{B} \) we write \( h: \mathfrak{U} \rightarrow \mathfrak{B} \); if \( h \) is onto then \( \mathfrak{B} \) is a homomorphic image of \( \mathfrak{U} \). If \( \langle \mathfrak{U}_m : m \in M \rangle \) is any system of algebras of the same type \( I \), then by the Cartesian product, in symbols \( \prod_{m \in M} \mathfrak{U}_m \), we mean the algebra \( \mathfrak{B} \) of type \( I \) with universe \( \prod_{m \in M} A \) such that

\[
Q^{(\mathfrak{B})}(f_0, \ldots, f_{\xi-1})(m) = Q^{(\mathfrak{U}_m)}(f_0^m, \ldots, f_{\xi-1}^m)
\]
for all \( Q \in I, f_0, \ldots, f_{\kappa - 1} \in P_{m \in M} A_m \), and \( m \in M \). If \( \mathcal{U}_m = \mathcal{U} \) for each \( m \in M \), then \( P_{m \in M} \mathcal{U}_m \) is called the \( M \)th Cartesian power of \( \mathcal{U} \) and its universe is \( M^A \); it is also written \( \mathcal{U}^M \).

Let \( \xi \) be an equation of type \( I \) and \( \mathcal{U} \) any algebra of type \( I \). \( \mathcal{U} \) is said to be a model of \( \xi \), or \( \xi \) is said to be identically satisfied or to hold in \( \mathcal{U} \), or \( \xi \) is said to be an identity or a law of \( \mathcal{U} \), if \( \mathcal{U} \) is a model of the universal closure of \( \xi \) in the sense of predicate logic. In symbols,

\[ \mathcal{U} \models \xi. \]

This definition can be re-formulated in a way that emphasizes its algebraic character. By the term algebra of type \( I \) we mean the algebra

\[ \mathcal{T}_I = \langle T_{E_1}, Q^I \rangle_{Q \in I} \]

whose universe is the set of terms of type \( I \) and whose operations are the obvious ones: for any \( Q \in I \) of rank \( \kappa \) and any \( \kappa \) terms \( \tau_0, \ldots, \tau_{\kappa - 1}, Q^{(\mathcal{T}_I)} (\tau_0, \ldots, \tau_{\kappa - 1}) = Q^\tau_0 \cdots \tau_{\kappa - 1} \).

The term algebra has the property that given any algebra \( \mathcal{U} \) of type \( I \) and any function \( f: Va \rightarrow A \) from the set of variables into the universe of \( \mathcal{U} \), there exists a unique homomorphism \( h: \mathcal{T}_I \rightarrow \mathcal{U} \) such that \( h \) extends \( f \). (To see that such an \( h \)
exists, think of $f$ as a subset of the Cartesian product $T_e \times A$ and let $h$ be the subuniverse of the Cartesian product $T_m \times \mathcal{U}$ that is generated by $f$. Then it is easy to see that $h$ has all the properties of a homomorphism from $T_m$ into $\mathcal{U}$ except possibly the property of being a function. However the fact that a term can be parsed in only one way is exactly the property needed guarantee that $h$ is a function.)

Given any equation $\epsilon$ and algebra $\mathcal{U}$ of type $\mathbf{I}$, we can alternatively define $\mathcal{U} \vDash \epsilon$ to mean that, for every homomorphism $h: T_m \rightarrow \mathcal{U}$, $h\epsilon_L = h\epsilon_R$ where $\epsilon_L$ and $\epsilon_R$ are the left- and right-hand terms of $\epsilon$, respectively. Thus we have two definitions of the notion of model and in view of the basic property of the term algebra described in the previous paragraph it is clear that these definitions are equivalent; even more, they really are the same definition looked at in two different ways, one from the more traditional viewpoint of logic, and the other from a purely algebraic point of view. This division of viewpoints (if not of content) exists throughout the entire subsequent discussion, although even this distinction begins to fade as the trend toward algebraization in logic gains momentum.

The term algebra can also be used to give a very satisfactory algebraic characterization of the fundamental syntactical operation of substitution. Let $\varphi: V_a \rightarrow T_e$ be any assignment of terms
to variables. Then by the basic property of the term algebra
\( \varphi \) can be uniquely extended to an endomorphism of \( \mathcal{M}_I \) denoted
by \( \varphi \); conversely, an endomorphism of \( \mathcal{M}_I \) is of the form
\( \psi \) for some assignment of terms to variables. For any term
\( \tau \), \( \varphi \tau \) is just the term obtained from \( \tau \) by substituting for
each occurrence of a variable \( x \) the term \( \varphi x \) assigned to it.
Thus the familiar and important syntactical operations associated
with substituting terms for variables can be characterized simply
as the endomorphisms of the term algebra. A term \( \sigma \) is a
substitution instance of \( \tau \) if \( \sigma = \varphi \tau \) for some \( \varphi : \mathcal{V} \to \mathcal{T}_{\mathcal{T}} \).

For any \( \Gamma \subseteq \mathcal{E} \), define

\( \text{Mor} = \{ \mathcal{M} : \mathcal{M} \text{ algebra of type } I, \mathcal{M} \models \epsilon \text{ for every } \epsilon \in \Gamma \} \).

\( \text{Mor} \) is called the model class of \( \Gamma \). For any class \( K \) of
algebras of type \( I \)

\( \text{Th} K = \{ \epsilon : \epsilon \in \mathcal{E}, \mathcal{M} \models \epsilon \text{ for each } \mathcal{M} \in K \} \).

\( \text{Th} K \) is called the (equational) theory of \( K \); by an equational
class, or a variety or a primitive class, we mean the model class
of some set \( \Gamma \) of equations. By an (equational) theory we mean
the theory of some class of algebras. For any \( \epsilon \in \mathcal{E} \) and
\( \Gamma \subseteq \mathcal{E} \), \( \epsilon \) is a consequence of \( \Gamma \), in symbols \( \Gamma \models \epsilon \), if
\( \epsilon \in \text{ThMor} \). Notice that \( \epsilon \) is a consequence of \( \Gamma \) just in
case it is one in the sense of first-order logic. Thus an equational class is a special kind of elementary class. Observe also that an equational theory can be alternatively characterized in any of the following ways: (i) as the theory of some variety; (ii) as a set of equations closed under consequence; (iii) as a set of all consequences of some set \( \Gamma \) of equations; (iv) as the equational part of some first-order theory axiomatized by identities exclusively.

The sets \( \text{Tar}_I \) and \( \text{Eq}_I \) are the smallest and largest theories, respectively; they are also referred to as the trivial and inconsistent theories. An algebra with a single binary operation is called a groupoid; any theory of a type of a single binary operation symbol is called a theory of groupoids.

A base for a theory \( \theta \) is any set \( \Gamma \) of equations such that \( \theta = \text{ThMor} \); in this case we also say that \( \Gamma \) generates \( \theta \). For any cardinal \( \alpha \) \( \theta \) is \( \alpha \)-based if \( \theta \) has at least one base of cardinality \( \alpha \); \( \theta \) is finitely-based if it is \( \kappa \)-based for some \( \kappa < \omega \). As opposed to the situation in first-order logic an equational theory can be finitely based without being \( 1 \)-based; in fact for each \( \kappa < \omega \) theories of groupoids can be found that are \((\kappa+1)\)-based but not \( \kappa \)-based. Questions concerning the cardinality of bases for equational theories are considered in Chapter 3.
Section 1.3: Derivability and the completeness theorem of Birkhoff.

We now describe a deductive system for equational logic.

There are two inference rules:

**Substitution:** $\xi$ is directly derivable from $\delta$ by substitution if $\xi = \text{su} \delta$ for some $\varphi: \forall a \rightarrow \text{Te}_1$.

For any equation $\delta = (\delta \simeq \delta')$, by $\text{su} \delta$ we mean $\text{su} \delta \simeq \text{su} \delta'$; other syntactical transformations of terms which we shall consider are automatically extended to equations in the same way.

**Replacement:** $\xi$ is directly derivable from $\gamma$ and $\delta$ by replacement if one of the two sides of $\delta$, say $\delta'_L$, occurs as a subterm of a side of $\gamma$, and $\xi$ is the result of replacing this occurrence of $\delta'_L$ by $\delta'_R$.

**Logical axioms:** The single tautology $\forall_0 \equiv \forall_0$. Given any set of non-logical axioms $\Gamma \subseteq \text{Eq}_1$, a derivation from $\Gamma$ is a finite sequence $\xi_0, \xi_2, \ldots, \xi_{\kappa-1}$ such that each $\xi_\lambda$ is either the logical axiom, a member of $\Gamma$, or is directly derivable from earlier equations in the sequence by one of the two rules of inference. Finally an equation $\xi$ is derivable from $\Gamma$, in symbols $\Gamma \vdash \xi$, if there exists a derivation from $\Gamma$ whose last member is $\xi$. As in the first-order case this system can be modified by eliminating the substitution rule and replacing each
axiom (logical and non-logical) by its corresponding axiom schema, that is, the set of all its substitution instances.

For any set $\Gamma \subseteq \operatorname{Eq}_I$ we let

$$\Theta_I[\Gamma] = \{ \varepsilon : \varepsilon \in \operatorname{Eq}_I, \Gamma \vdash \varepsilon \}.$$  

If the type is clear from context we write $\Theta[\Gamma]$ for $\Theta_I[\Gamma]$; if $\Gamma = \{ \varepsilon_0, \ldots, \varepsilon_{k-1} \}$ we write $\Theta[\varepsilon_0, \ldots, \varepsilon_{k-1}]$ for $\Theta[\Gamma]$. A pair of terms $\tau, \sigma$ are called $\Gamma$-equivalent, in symbols $\tau \sim_{\Gamma} \sigma$, if $\tau \approx_{\Gamma} \sigma \in \Theta[\Gamma]$; $\varepsilon$ is $\Gamma$-derivable from $\Delta \subseteq \operatorname{Eq}_I$ if $\varepsilon \in \Theta[\Gamma \cup \Delta]$; finally two sets $\Delta, \Delta'$ of equations are $\Gamma$-inter-derivable if $\Theta[\Gamma \cup \Delta] = \Theta[\Gamma \cup \Delta']$.

The following theorem is the completeness theorem of equational logic that was first proved by Birkhoff [35].

**Theorem 1.3.1.** For any type $I$ and any $\Gamma \subseteq \operatorname{Eq}_I$ and $\varepsilon \in \operatorname{Eq}_I$,

$$\Gamma \vdash \varepsilon \text{ iff } \Gamma \vdash \varepsilon.$$  

We shall outline the proof. We begin by describing a second deductive system which turns out however to generate the same relation of derivability. There are four rules of inference.

**Transitivity:** the equation $\sigma \approx \tau$ is directly derivable from the two equations $\sigma \approx \rho$ and $\rho \approx \tau$. 
Symmetry: the equation $\sigma \approx \tau$ is directly derivable from
from the equation $\tau \approx \sigma$.

Equality: $\epsilon$ is directly derivable from $\delta^{(0)},\delta^{(1)},\ldots,\delta^{(x-1)}$
if there is a operation symbol $Q$ of rank $x$ such that

$$
\epsilon_\ell = Q\delta^{(0)}_\ell \delta^{(1)}_\ell \cdots \delta^{(x-1)}_\ell \quad \text{and} \quad \epsilon_r = Q\delta^{(0)}_r \delta^{(1)}_r \cdots \delta^{(x-1)}_r.
$$

Substitution: same as in the first system.

Also, as in the first system, $v_0 \approx v_0$ is the only logical
axiom, and the notions of a derivation from $\Gamma$ and $\epsilon$ being
derivable from $\Gamma$ are defined in the same way. It is now an
easy matter to show that $\epsilon$ is derivable for $\Gamma$ in the second
system just in case it is derivable in the first system.

We now turn to the proof of the completeness theorem.
Consider any $\Gamma \subseteq \text{Eq}_I$ and $\epsilon \in \text{Eq}_I$. It is easy to prove by
induction on the length of the derivation that $\Gamma \vdash \epsilon$ implies
$\Gamma \models \epsilon$. For the implication in the opposite direction assume that

$\Gamma \not\vdash \epsilon$, i.e., $\epsilon \not\in \mathcal{G}[\Gamma]$. We have just seen that $\mathcal{G}[\Gamma]$ can be
characterized as the smallest set of equations that includes $\Gamma$
and the tautology $v_0 \approx v_0$ and is closed under the transitivity,
symmetry, equality, and substitution rules. Let us identify

$\mathcal{G}[\Gamma]$ with the set of all ordered pairs of terms $(\sigma, \tau)$ such
that the equation $\sigma \approx \tau \in \mathcal{G}[\Gamma]$; thus $\mathcal{G}[\Gamma]$ can be thought of
as a binary relation on the universe of the term algebra. Since

$\mathcal{G}[\Gamma]$ contains $v_0 \approx v_0$ and is closed under substitution it is
(as a relation) reflexive, and since it is closed under the transitivity and symmetry rules, it is an equivalence relation. Moreover, closure under the equality rule implies that \( \approx[\Gamma] \) is an congruence relation on the term algebra. Finally, closure under the substitution rule guarantees that \( \approx[\Gamma] \) is a congruence relation invariant under all endomorphisms of \( \mathbb{M}_I \); such a congruence relation is called completely invariant in analogy with the notion of a completely invariant normal subgroup of a group. Now consider the quotient algebra \( \mathbb{M}_I/\approx[\Gamma] \); since \( \approx[\Gamma] \) contains \( \Gamma \) and is completely invariant it is not hard to see that \( \mathbb{M}_I/\approx[\Gamma] \) is a model of \( \Gamma \); on the other hand, since \( \notin \approx[\Gamma] \) by assumption, \( \mathbb{M}_I/\approx[\Gamma] \) is not a model of \( \in \). Thus \( \Gamma \not\models \in \) and the proof of the completeness theorem is finished.

Some interesting facts come out of this proof. First of all, we obtain a new, purely algebraic characterization of equational theories as completely invariant congruence relations on the term algebra; this is especially useful in the study of the structure of the lattice of equational theories which is done in Chapter 2. Secondly, the model \( \mathbb{M}_I/\approx[\Gamma] \) of \( \Gamma \) constructed in the proof was seen to fail to be a model of every \( \in \) which is not a consequence of \( \Gamma \). Therefore, the theory generated by \( \Gamma \), and hence every theory, is the theory of a single model. Here we see the first point of real contrast between equational
logic and first-order logic; in the latter, theories of single models are very special.

The quotient $\mathfrak{m}/\mathfrak{S}[\Gamma]$ is always denumerable unless $\mathfrak{S}[\Gamma] = \text{Eq}_I$. The inconsistent theory $\text{Eq}_I$ of all equations has only trivial, 1-element models, but we see that every consistent theory has a denumerable model, in fact, it is the theory of some denumerable algebra. Thus in equational logic the Löwenheim-Skolem theorem assumes a very strong form.

The quotient algebra $\mathfrak{m}/\mathfrak{S}[\Gamma]$ is a free algebra over the model class of $\Gamma$ and is freely generated by the equivalence classes of variables $v_0/\mathfrak{S}[\Gamma], v_1/\mathfrak{S}[\Gamma], \ldots$. This particular construction of the free algebra is an almost direct algebraic paraphrasing of an essentially metamathematical construction: in this latter construction the free algebra over a class $K$ of algebras is defined as one generated by a set of elements among which no relations are satisfied that are not identically satisfied in all algebras of $K$. Historically free algebras, in particular, free groups, where first constructed in this way, although the true metamathematical nature of the construction was apparently not perceived. Later on the more familiar characterization of free algebras by means of the universal mapping property came in vogue. It is not difficult to see however that these two definitions lead to the same algebra.
For any theory $\theta$ of type I, $\mathfrak{m}/\theta$ is called the free algebra over $\theta$ (with $w$ free generators) and is denoted by $\mathfrak{m}_w$. For any class $K$ of algebras the free algebra over $K$, in symbols $\mathfrak{m}_w K$, is defined to be the free algebra over $\text{Th} K$.

Let $K$ be any class of algebras of type I. For each equation $\epsilon \notin \text{Th} K$ there exists by definition of $\text{Th} K$ an algebra $\mathfrak{m}_\epsilon \in K$ and a homomorphism $f_\epsilon : \mathfrak{m}_I \to \mathfrak{m}_\epsilon$ such that $f_\epsilon \notin f_\epsilon$. Consider the Cartesian product

$$\mathfrak{m} = \prod_{\epsilon \in \text{Eq}_I \sim \text{Th} K} \mathfrak{m}_\epsilon.$$

For each $\epsilon$ let $p_\epsilon$ be the projection of $\mathfrak{m}$ onto $\mathfrak{m}_\epsilon$ and let $h$ be the unique homomorphism from $\mathfrak{m}_I$ onto $\mathfrak{m}$ such that $p_\epsilon \circ h = f_\epsilon$ for each $\epsilon \in \text{Eq}_I \sim \text{Th} K$. Clearly $\text{Th} K$ is the congruence relation on $\mathfrak{m}_I$ induced by $h$. Hence $\mathfrak{m}_w K = \mathfrak{m}_I/\text{Th} K$ is isomorphic to a subalgebra of $\mathfrak{m}$. Thus for every class $K$, $\mathfrak{m}_w K$ can be represented isomorphically as a subalgebra of a Cartesian product of a system of algebras of $K$.

As a final comment on the completeness theorem we remark that it can be obtained as corollary of various classical results of first-order logic. For example applying Herbrand's Theorem one can easily deduce that whenever $\Gamma \not\models \epsilon$ there exists a derivation of $\epsilon$ from $\Gamma$ in the second deductive system described above.
Section 1.4. The characterization of varieties.

The completeness theorem of Birkhoff can be thought of as providing a purely syntactical characterization of the set ThMor, namely,

\[ \text{ThMor} = \emptyset[\Gamma]. \]

The second major theorem of Birkhoff [35] is in a sense the dual result; it provides a purely algebraic characterization of the class MoThK for any class K of algebras.

Given any class K of algebras (of the same type) we take H(K) and S(K) to be respectively the classes of all homomorphic images of members of K and all algebras isomorphic to subalgebras of members of K. P(K) is the class of all algebras isomorphic to a Cartesian product of an arbitrary system of algebras in K. We write \( H^\ast \) for \( H\{\ast\} \) and \( S^\ast \) for \( S\{\ast\} \).

**Theorem 1.4.1. For any class K of algebras,**

\[ \text{MoThK} = \text{HSP}(K). \]

It is not difficult to show \( \text{HSP}(K) \subseteq \text{MoThK} \). For the inclusion in the opposite direction consider any \( \mathcal{U} \in \text{MoThK} \). The class \( \text{HSP}(K) \) is closed under each of the operations H, S, and P so in order to prove \( \mathcal{U} \in \text{HSP}(K) \) it suffices to
prove that \( \mathcal{U} \in \mathcal{Q}(K) \) where \( \mathcal{Q} \) is some sequence of the \( H, S, \) and \( P. \)

It is well known that \( \mathcal{U} \) can be isomorphically represented as an ultraproduct, and thus as a homomorphic image of a Cartesian product, of the finitely generated subalgebras of \( \mathcal{U}. \) Hence we may assume that \( \mathcal{U} \) is finitely generated. Then \( \mathcal{U} \) is a homomorphic image of \( \exists x K. \) But we have seen that \( \exists x K \in \mathcal{SP}(K) \) so that \( \mathcal{U} \in \mathcal{HSP}(K) \).

As an immediate corollary of this result we get that \( K \) is a variety just in case it is closed under the formation of homomorphic images, subalgebras, and arbitrary Cartesian products.

There is a corresponding characterization theorem for elementary classes due to H. J. Keisler and it is interesting to observe the similarity between their proofs.

For any class \( K \) of similar relational structures let \( \mathcal{U}pK \) be the class of all structures isomorphic to an ultraproduct of an arbitrary system of structures in \( K. \) Let \( \mathcal{Sp}K \) be the class of all structures \( \mathcal{U} \) such that some ultrapower of \( \mathcal{U} \) is isomorphic to an ultrapower of a structure in \( K. \) Then using the generalized continuum hypothesis (GCH) Keisler [61] showed that, for any class \( K \) of structures, \( \mathcal{SpUp}K \) is the elementary class generated by \( K, \) and later Shelah [71] showed how the assumption of the \( GCH \) may be eliminated.
There is an interesting parallel between the proof of this result and that of Birkhoff. Let $\mathcal{U}$ be an arbitrary structure in the elementary class generated by $K$. By a construction quite analogous to that used in the familiar proof of the compactness theorem via ultraproducts one obtains an ultraproduct $\mathcal{V}$ of members of $K$ such that $\mathcal{V}$ is elementarily equivalent to $\mathcal{U}$. By a result of Keisler [61] $\mathcal{V}$ and $\mathcal{U}$ have isomorphic ultrapowers. Thus $\mathcal{U} \in \mathrm{Sp}\mathcal{U}$ and hence $\mathcal{V} \in \mathrm{SpUpK}$.

The ultraproduct $\mathcal{V}$ in this proof plays roughly the same role as the free algebra $\bigwedge_{(\omega)} K$ played in the proof of Birkhoff's theorem. In many other respects free algebras and ultraproducts play parallel roles.

Section 1.5. Definitional equivalence of theories

The notion of interpreting one equational theory in another is completely analogous to the corresponding notion of first-order logic. Roughly speaking an interpretation of a theory $\mathcal{A}$ in a theory $\mathcal{B}$ is given by defining the fundamental operations of $\mathcal{A}$ in terms of the fundamental operations of $\mathcal{B}$ in such a way that an equation $\epsilon^\mathcal{A}$ is a law of $\mathcal{A}$ just in case the equation $\epsilon'$ obtained from $\epsilon^\mathcal{A}$ by replacing each operation of $\epsilon^\mathcal{A}$ by the term defining it is a law of $\mathcal{B}$. To make this notion precise requires a precise definition of the syntactical trans-
formation that takes € to €'.

Let I and J be arbitrary types. By a possible definition of I in J we mean any function ρ: I → Te_J such that, for every Q ∈ I of positive rank κ, the variables actually occurring in ρQ are included among the first κ variables v_0, v_1, ..., v_{κ-1} in the natural sequence of variables; in case rank Q = 0, ρQ is either a constant term, that is, contains no variables, or contains only the variable v_0. With any possible definition ρ we associate a certain syntactical transformation Te_I to Te_J called elimination by ρ; in symbols el_ρ. It is defined by induction on the length of terms by the conditions: 

el_ρ v = v
for each λ < ω, el_ρ Q = ρQ for each Q ∈ I of rank 0, and

el_ρ Qτ_0⋯τ_{κ-1} = sup ρQ
for each Q ∈ I of rank κ > 0 and all τ_0⋯τ_{κ-1} ∈ Te_I where

φv_λ = el_ρ τ_λ
for each λ < κ. (Notice that since ρ is a possible definition the value of φ at v_λ for λ ≥ κ is irrelevant.)

For example let I = {Q, P} where Q and P are of rank 3 and 0, respectively. Let J = {R, S} with R, S of rank 2 and 1. Let ρQ = R^2_v_0v_1v_0 and ρP = Sv_0. Then

el_ρ (Q^2_vxyP) = R^4_vxyR^2_vSv_0R^2_vxySv_0.
Let $p$ be any possible definition of $I$ in $J$. By the constant set of $p$, in symbols $\Omega_p$, we mean the set of all equations $pQ \approx (pQ)'$ where $Q \in I$, rank $Q = 0$, $pQ$ is not a constant term, and $(pQ)'$ is the result of substituting $v_1$ for each occurrence of $v_0$ in $pQ$. It is easy to prove by induction on the length of terms that, for any $\varphi: \forall a \rightarrow T_{e_I}$ and $\sigma \in T_{e_I}$,

\[ \Omega_p \subseteq \hat{\xi}. \]

We are now in a position to define precisely the notions of interpretation and definitional equivalence with regard to theories.

Let $I$ and $J$ be any types and $\xi$ a theory of type $J$. A possible definition $p$ of $I$ in $J$ is called a definition of $I$ in $\xi$ if

\[ \text{el} \sup_{\varphi} \sigma \equiv \sup_{p} \text{el}_{p} \varphi \cdot \text{el} \sigma. \]

We have immediately

\[ \text{el} \sup_{\varphi} \sigma \equiv \sup_{\varphi} \text{el} \sigma. \]

whenever $p$ is a definition of $I$ in $\xi$, $\varphi: \forall a \rightarrow T_{e_I}$, and $\sigma \in T_{e_I}$.

The elimination function $\text{el}_{p}: T_{e_I} \rightarrow T_{e_J}$ is called an interpretation of $\theta$ in $\xi$ if $p$ is a definition of $I$ in $\xi$ and, for each $\sigma \approx \tau \in \text{Eq}_I$, $\sigma \approx \tau \in \theta$ iff $\text{el} \sigma \approx \text{el} \tau \in \hat{\xi}$.
Let \( p \) and \( \pi \) be definitions of \( I \) in \( \mathcal{F} \) and \( J \) in \( \Theta \), respectively. \( \Theta \) and \( \mathcal{F} \) are definitionally equivalent by \( p \) and \( \pi \), in symbols \( \Theta \equiv_{p,\pi} \mathcal{F} \), if all the following conditions hold:

1. \( \sigma \equiv \tau \in \Theta \) implies \( \exists_{p} \sigma \equiv \exists_{p} \tau \in \Theta \) for each \( \sigma \equiv \tau \in \text{Eq}_{I} \);
2. \( \sigma \equiv \tau \in \mathcal{F} \) implies \( \exists_{\pi} \sigma \equiv \exists_{\pi} \tau \in \Theta \) for each \( \sigma \equiv \tau \in \text{Eq}_{\mathcal{F}} \);
3. \( \forall_{\pi} \exists_{\mathcal{F}} \sigma \equiv \exists_{\mathcal{F}} \tau \in \Theta \) for all \( \sigma \equiv \tau \in \text{Eq}_{I} \);
4. \( \forall_{\mathcal{F}} \exists_{\pi} \sigma \equiv \exists_{\pi} \tau \in \Theta \) for all \( \sigma \equiv \tau \in \text{Eq}_{\mathcal{F}} \);
5. \( \forall_{\mathcal{F}} \exists_{\pi} \sigma \equiv \exists_{\pi} \tau \in \Theta \) for all \( \sigma \equiv \tau \in \text{Eq}_{\mathcal{F}} \).

Example. Let \( I = \{ ., \cdot^{-1} , e \} \) and \( \Theta \) be the theory of groups with composition, inverse, and the identity as fundamental operations. Let \( J = \{ : \} \) and \( \mathcal{F} \) be the theory of groups with right-hand division as the only fundamental operation. Let \( \rho(\cdot) = v_{0} : ((v_{0} : v_{0}) : v_{1}) \), \( \rho(\cdot^{-1}) = (v_{0} : v_{0}) : v_{0} \), and \( \rho(e) = v_{0} : v_{0} \); let \( \pi(\cdot) = v_{0} : v_{1}^{-1} \). Notice that \( \rho \) is actually a definition of \( I \) in \( \mathcal{F} \) since \( v_{0} : v_{0} \approx v_{1} : v_{1} \in \mathcal{F} \). It is well known that conditions (2) and (3) hold and (3) and (4) are easy to check. For example

\[
\exists_{\mathcal{F}} \exists_{\pi} \rho(v_{0} \cdot v_{1}) = v_{0} : ((v_{0} \cdot v_{1}^{-1}) \cdot v_{1}^{-1})^{-1},
\]

but of course \( v_{0} \cdot v_{1} \approx v_{0} : ((v_{1} \cdot v_{1}^{-1}) \cdot v_{1}^{-1})^{-1} \) is a law of groups.

**Theorem 1.5.1.** Let \( \Theta \) and \( \mathcal{F} \) be theories of type \( I \) and
J, respectively, \( \rho \) a definition of \( I \) in \( \psi \), and \( \pi \) a definition of \( J \) in \( \theta \). If \( \theta = \rho, \pi \), then \( \epsilon_\rho \) is an interpretation of \( \theta \) in \( \psi \) and \( \epsilon_\pi \) is an interpretation of \( \phi \) in \( \theta \).

We will prove that \( \epsilon_\pi \) is an interpretation of \( \phi \) in \( \theta \). Consider any \( \sigma \approx \tau \in \text{Eq}_J \). If \( \sigma \approx \tau \in \phi \), then \( \epsilon_\pi \sigma \approx \epsilon_\pi \tau \in \theta \) by (3). Now assume that \( \epsilon_\pi \sigma \approx \epsilon_\pi \tau \in \theta \); then

\[
\epsilon_\rho \epsilon_\pi \sigma \approx \epsilon_\rho \epsilon_\pi \tau \in \psi
\]

by (2). We now prove by induction of the length of \( \sigma \) that \( \epsilon_\rho \epsilon_\pi \sigma \approx \sigma \in \psi \). This is obviously true if \( \sigma \in \text{Va} \). If \( \sigma = P \epsilon_\pi \tau \) and \( P \) is of rank 0, then \( \epsilon_\rho \epsilon_\pi \sigma = \epsilon_\rho \epsilon_\pi \tau \approx \epsilon_\phi \sigma \)

by (5). Finally, let \( \sigma = P \tau_0 \cdots \tau_{\lambda-1} \). Take \( \varphi_\kappa = \tau_\kappa \) for each \( \kappa < \lambda \). Then we have

\[
\epsilon_\rho \epsilon_\pi \sigma = \epsilon_\rho \sup_\psi \epsilon_\pi \varphi \epsilon_\pi P
\]

by (1) with \( \varphi = \epsilon_\pi \varphi \) and \( c = \epsilon_\pi \varphi \)

by induction hypothesis

\[
\approx \psi \sup_\psi \epsilon_\rho \epsilon_\pi \varphi \epsilon_\pi P
\]

\[
\approx \psi \sup_\psi \epsilon_\rho \epsilon_\pi \varphi \epsilon_\pi P
\]

\[
\approx \psi \sup_\psi \epsilon_\rho \epsilon_\pi \varphi \epsilon_\pi P
\]

Thus \( \epsilon_\rho \epsilon_\pi \sigma \approx \sigma \in \psi \), and by the same proof we have \( \epsilon_\rho \epsilon_\pi \tau \approx \tau \in \psi \).
Combining this with (6) we get \( \sigma \preceq \tau \in \theta \). Thus \( \mathfrak{e}_{|\pi} \) is an interpretation of \( \hat{\theta} \) in \( \theta \).

We observe that conditions (2) and (3), and even the stronger conditions that \( \theta \) and \( \hat{\theta} \) be interpretable one in another by \( \mathfrak{e}_{\rho} \) and \( \mathfrak{e}_{|\pi} \), are not in themselves sufficient for \( \theta \equiv_{\rho, |\pi} \hat{\theta} \). For instance, let \( Q, P \), and \( R \) all be binary operation symbols; let \( I = \{Q\} \) and \( J = \{P, R\} \). Let

\[ \hat{\theta} = \operatorname{Ta}_I (= \{ \tau \rightleftharpoons \tau : \tau \in \text{Te}_I \}) \]

Similarly, let \( \hat{\theta} = \operatorname{Ta}_J \). \( \theta \) is clearly interpretable in \( \hat{\theta} \) and, if we define \( \pi \) so that

\[ \pi_P = QQv_0 v_1 v_0 \quad \text{and} \quad \pi_R = Qv_0 Qv_1 v_0, \]

then it is easy to see that \( \mathfrak{e}_{|\pi} \) is an interpretation of \( \hat{\theta} \) in \( \theta \); in fact using the rather obvious fact that no substitution instance of \( \pi_P \) can equal a substitution instance of \( \pi_R \) one can prove by induction on the number of occurrences of \( P \) and \( R \) in \( \sigma \in \text{Te}_J \) that, for any \( \tau \in \text{Te}_J \), \( \mathfrak{e}_{|\pi} \sigma = \mathfrak{e}_{|\pi} \tau \) iff \( \sigma = \tau \).

On the other hand, \( \theta \) and \( \hat{\theta} \) cannot be definitionally equivalent since it is obvious that condition (5) can hold for no definitions \( \rho, |\pi \).

We shall discuss another characterization of definitional equivalence which proves to be useful in theoretical considerations. It involves the important notion of an extension of a theory.

Let \( \theta \) and \( \hat{\theta} \) be theories of type \( I \) and \( J \), respectively.
\( \theta \) is called a **generalized extension** of \( \phi \) if \( I \supseteq J \) and \( \theta \supseteq \phi \). We distinguish several kinds of generalized extensions:

\( \theta \) is called a **conservative extension** or an **expansion** if

\( \theta \cap \text{Eq}_J = \emptyset \); \( \theta \) is a (simple) extension, and \( \phi \) is a subtheory of \( \theta \), if \( I = J \). If \( \theta \) is an extension of \( \phi \), a set \( \Gamma \) of equations such that \( \theta = \theta[\phi \cup \Gamma] \) is called a **base** for \( \theta \) relative to \( \phi \), or a \( \phi \)-**base** for \( \theta \); we shall often write

\[
\theta[\Gamma] = \theta[\phi \cup \Gamma].
\]

Finally, \( \theta \) is a **definitional extension** of \( \phi \) if it is a conservative extension and there exists a definition \( \rho \) of \( I \) in \( \phi \) such that \( \rho P = P \cup \cdots \cup \lambda - 1 \) for each \( P \in J \) and

\[
Q \cup \cdots \cup \lambda - 1 \equiv \rho Q \in \theta
\]

for each \( Q \in I \sim J \); in this event \( \theta \) is called a **definitional extension** of \( \phi \) by \( \rho \). It is easy to prove by induction on the length of \( \sigma \) that \( e_\rho \sigma \) is \( \theta \)-equivalent to \( \sigma \) for each \( \sigma \in \text{Te}_I \). Hence, since \( \theta \) is a conservative extension of \( \phi \), we conclude immediately that \( e_\rho \) is an interpretation of \( \theta \) in \( \phi \) and, furthermore, that \( \theta \) and \( \phi \) are actually definitionally equivalent by \( \rho, \pi \) where \( \pi P = P \cup \cdots \cup \lambda - 1 \) for each \( P \in J \).

Suppose now that \( I \) and \( J \) are arbitrary types, that \( \theta \) and \( \phi \) are theories of type \( I \) and \( J \), respectively, and that
there exists a theory $\tau$ that is at the same time a definitional extension of both $\theta$ and $\hat{\phi}$; we may assume that $\tau$ is of type $I \cup J$. Let $\xi$ and $\eta$ be the possible definitions of $I \cup J$ in $I$ and $J$, respectively, that establish $\tau$ as a definitional extension of $\theta$ and $\hat{\phi}$. Define $\rho$ to be the restriction of $\eta$ to $I$ and $\pi$ the restriction of $\xi$ to $J$. The equations $e_1 \rho \approx \sigma$ and $e_1 \pi \approx \tau$ are contained in $\tau$ for each $\sigma \in Te_I$ and $\tau \in Te_J$, and $\tau$ is a conservative extension of both $\theta$ and $\hat{\phi}$. Hence we conclude immediately that conditions (2)--(5) hold. Thus the existence of a common definitional extension of $\theta$ and $\hat{\phi}$ implies their definitional equivalence. The converse also holds, at least in the case that the types are disjoint. Assume that $\theta \equiv_{\rho, \pi} \hat{\phi}$ and $I \cap J = 0$. Then using the conditions (2)--(5) it is easy to show that

$$\theta_{I \cup J}[\theta U \{pv_0 \cdots v_{\lambda-1} \approx \pi P : P \in J\}] = \theta_{I \cup J}[\theta U \hat{\phi}]$$

$$= \theta_{I \cup J}[\hat{\phi} U \{qv_0 \cdots q_{\kappa-1} \approx \rho Q : Q \in I\}],$$

and thus that $\theta_{I \cup J}[\theta U \hat{\phi}]$ is a common definitional extension of $\theta$ and $\hat{\phi}$.

In order to fully characterize the notion of definitional equivalence in terms of that of extension we must consider the relation of isomorphism between theories.

Let $\theta, \hat{\phi}$ be theories of type $I, J$. An interpretation
el of \( \theta \) in \( \phi \) is called an **isomorphism** if for each \( Q \in I \) there exists a \( P \in J \) of the same rank \( k \) such that
\[ P^Q = P v_0 \ldots v_{k-1}, \]
every \( P \in J \) occurs in this way. In this case \( \theta \) and \( \phi \) are said to be **isomorphic**. Isomorphic theories are clearly definitionally equivalent and it should now be obvious that arbitrary theories \( \theta \) and \( \phi \) are definitionally equivalent just in case there exist theories \( \theta', \phi' \) isomorphic respectively to \( \theta, \phi \) such that \( \theta' \) and \( \phi' \) have a common definitional extension.

We now discuss how a given base for \( \theta \) can be transformed into a base for \( \phi \) when the two theories are definitionally equivalent.

**Theorem 1.5.2.** Let \( \theta \) and \( \phi \) be theories of type \( I \) and \( J \), respectively, and let \( \rho \) and \( \pi \) be definitions of \( I \) in \( \phi \) and \( J \) in \( \theta \). Assume \( \theta \equiv_{\rho, \pi} \phi \) and \( \Gamma \) is a base for \( \theta \). Then:

(i) \[ \phi = \theta[el^*_{\rho} \cup \left\{ P v_0 \ldots v_{k-1} \equiv_{\rho} el_{\rho} P : P \in J \right\}] ; \]

(ii) if \( \theta \) is a definitional extension of \( \phi \) by \( \rho \), then
\[ \phi = \theta[el^*_{\rho} \cup \left\{ P v_0 \ldots v_{k-1} \equiv_{\rho} el_{\rho} P : P \in J \right\}] . \]

We first prove (ii). Observe that \( el_{\rho} \varepsilon = \varepsilon \) for every \( \varepsilon \in EQ_I \) since \( P^P = P v_0 \ldots v_{k-1} \) for each \( P \in J \). Consider any \( \varepsilon \in \phi \) and let \( \delta_0, \ldots, \delta_{k-1} \) be a derivation of \( \varepsilon \) from \( \Gamma \). Then it follows from (i) that \( el_{\rho} \delta_0, \ldots, el_{\rho} \delta_{k-1} \) is a subsequence
of a derivation of $\varepsilon$ from $\ell^*_\rho \Gamma \cup \Omega \rho$. On the other hand, since $\ell^* \sigma \approx \sigma \in \Theta$ for each $\sigma \in T \varepsilon$, we have that $\ell^*_\rho \Gamma \subseteq \Psi$. This gives (ii).

If $I \cap J \neq \emptyset$ in (i) we can replace $\Theta$ by a isomorphic image so assume $I$ and $J$ are disjoint. Extend $\rho$ to $I \cup J$ by taking $\rho P = Pv_0 \cdots v_{\lambda-1}$ for each $P \in J$. Then $\Theta_{I \cup J}[\Gamma \cup \{Pv_0 \cdots v_{\lambda-1} \approx \pi P: P \in J\}]$ is a definitional extension of $\Psi$ by $\rho$. Hence

$$\ell^*_\rho \Gamma \cup \Omega \rho \cup \{\ell^*_\rho P = P \cap (y \cdot x) \approx x, x \cdot e \approx x, x \cdot x^{-1} \approx e.$$}

Then using 1.5.2 a base for $\Phi$ consisting of the following six equations is constructed.

(7) $[x:((x:x):y)]:\lfloor ((x:(x:x):y)):(x:((x:x):y))\rfloor \cdot z\approx x:((x:x):(y:(y:y):z)))$

(8) $x:((x:x),(x:x)) \approx x$
(9) \[(x:x) : ((x:x):(x:x)) : x \approx x\]
(10) \[x : ((x:x):(x:x)) : x : x \approx x : x\]
(11) \[x : x \approx y : y\]
(12) \[x : y \approx x : ((x:x) : (y:y) : y)\]

The set \(1 \cdot \Gamma \cup \Omega \rho \pi \) alone does not in general form a base for \(\xi\) in 1.5.2(i); in particular (7)--(11) alone do not form a base for the theory of groups with right-hand inverse. A 1-element base for this theory is given in the introduction.

**Corollary 1.5.3.** If \(\theta = \rho, \pi \xi\) and \(\Delta \subseteq \text{Eq}_r\), then

\[\xi^r[\Delta] = \rho, \pi \xi[1 \cdot \Gamma \cup \Omega \rho \pi \Delta].\]

It follows that if \(\theta = \rho, \pi \xi\), then for every extension \(\theta'\) of \(\theta\) there exists an extension \(\xi'\) of \(\xi\) such that \(\theta' = \rho, \pi \xi'\).

Actually the relation \(\equiv \rho, \pi\) establishes a one-one inclusion preserving correspondence between the set of extensions of \(\theta\) and the set of extensions of \(\xi\).

We define theories \(\theta\) and \(\xi\) to be definitionally equivalent, in symbols \(\theta \equiv \xi\), if there exist definitions \(\rho, \pi\) such that \(\theta = \rho, \pi \xi\). This relation is clearly an equivalence relation.

Every theory is equivalent with itself by the identity definition \(\rho Q = Q \cdot v_0 \cdots v_{x-1}\) for each \(Q\) in its type. However, a theory may be equivalent to itself by non-identity definitions. Take, for example, \(\theta\) to be the theory of groups with composition and
inverse as fundamental operations. Let $\rho^{-1} = v_0^{-1}$, but take $\rho(x) = v_1 \cdot v_0$ (rather than $v_0 \cdot v_1$). Taking $\rho$ for both $\rho$ and $\pi$ conditions (2)--(5) are easily checked. (It is clear that in general (2) need only be checked for equations $\sigma = \tau$ chosen from a fixed set of axioms of $\theta$.) Thus $\theta \equiv \rho, \pi$.

Earlier in this section we showed that the theory $T_{\alpha}(Q)$ of tautologies in a single binary operation symbol $Q$ is interpretable in the theory of tautologies in two binary operations and vice-versa, but that the two theories are not definitionally equivalent. From the discussion there it is seen that, if $\tau$ is any term of type $[Q]$ containing occurrences of exactly the two variables $v_0$ and $v_1$, and $\rho \tau = \tau$, then $\rho_\tau$ is an interpretation of $T_{\alpha}(Q)$ in itself. We know of only four other theories of groupoids with this properties: these are four semigroup theories $\Omega^e_\lambda, \Omega^e_x, \Omega^l_\lambda$, and $\Omega^e_1, 1$ defined in Section 1.7. It would be interesting to know if there are any others.

We see that examples of theories interpretable in themselves by non-identity definitions are not difficult to find. In view of this it is natural to ask if theories exist which are definitionally equivalent to a proper subtheory of themselves. We now describe such a theory.

Let $I = J = \{Q, R, P_1, P_2\}$ where $Q, R$ are binary and $P_1, P_2$ are unary operation symbols. Let
\[ \theta = \exists \{ R P_{1} v_{0}^{p_{2} v_{0}} \cong v_{0}, P_{1} R v_{0}^{v_{1}} \cong v_{0}, P_{2} R v_{0}^{v_{1}} \cong v_{1} \} . \]

An algebra \( \mathcal{U} = (A, Q^{(\mathcal{U})}, R^{(\mathcal{U})}, P_{1}^{(\mathcal{U})}, P_{2}^{(\mathcal{U})}) \) is a model of \( \mathcal{U} \) just in case \( Q^{(\mathcal{U})} \) is a one-one correspondence between \( A \) and \( A \times A \) and \( P_{1}^{(\mathcal{U})}, P_{2}^{(\mathcal{U})} \) are the associated projection functions; notice that \( \theta \) has only trivial (i.e., one-element) finite models.

Let \( \rho Q = R Q v_{0}^{v_{1}} Q v_{0}^{v_{1}} \)

and \( \rho \) be the identity definition on \( R, P_{1}, P_{2} \). Let

\[ \pi Q = P_{1} Q v_{0}^{v_{1}} \]

and \( \pi \) also be the identity definition on \( R, P_{1}, P_{2} \). Finally, let

\[ \phi = \exists \{ Q v_{0}^{v_{1}} \cong R P_{1} Q v_{0}^{v_{1}} P_{1} v_{0}^{v_{1}} \} . \]

Conditions (2)--(5) are easy to check so that \( \theta \equiv \rho, \pi \phi \). But \( \phi \) is clearly a proper subtheory of \( \theta \). The theory \( \theta \) has no finite non-trivial models but starting with \( \theta \) one can construct a theory definitionally equivalent to a proper subtheory of itself which has models of every finite cardinality. On the other hand we will see later on that any theory which is the theory of its finite models (i.e., whose model class is generated by its finite members) fails to be definitionally equivalent to any proper subtheory; cf. Theorem 1.6.3 below.
Section 1.6. Definitional equivalence of classes of algebras.

Let $K$ and $L$ be varieties of type $I$ and $J$. If $\rho$ and $\tau$ are any possible definitions of $I$ in $J$ and $J$ in $I$, respectively, then $K$ and $L$ are said to be **definitionally equivalent by $\rho$ and $\tau$**, in symbols $K \equiv_{\rho, \tau} L$, if their corresponding theories $\text{Th}_K$ and $\text{Th}_L$ are definitionally equivalent in the sense of Section 1.5. This definition can be extended however to apply to other classes of algebras besides varieties as we shall presently see.

To extend this definition to arbitrary classes we need to define precisely the notion of a **polynomial operation**. Let $\mathcal{A}$ be any algebra of type $I$. Let $\mathcal{A}^n$ be the set of all $n$-sequences of elements of the universe of $\mathcal{A}$ and let $\mathcal{B}^n = \mathcal{A}^n$ be the $n$th Cartesian power of $\mathcal{A}$. Then the universe $B$ of $\mathcal{B}$ is the set $\mathcal{B}^n$ of all functions $f: \mathcal{A}^n \to A$, and, for any $Q \in I$ of rank $\kappa$ and all $f_0, \ldots, f_{\kappa-1} \in B$, $Q^{(\mathcal{B})}$ is the function from $\mathcal{B}^n$ into $A$ such that, for any $\omega$-sequence $a = \langle a_0, a_1, a_2, \ldots \rangle$ of elements of $A$,

$$Q^{(\mathcal{B})}(f_0, \ldots, f_{\kappa-1})(a) = Q^{(\mathcal{A})}(f_0 a, \ldots, f_{\kappa-1} a).$$

Let $\text{re}_{\mathcal{B}}$ be the unique homomorphism from the term algebra $\Sigma_{\mathcal{T}}$ into $\mathcal{B}$ such that $\text{re}_{\mathcal{B}, \kappa}$ is the $\kappa$th projection of $\mathcal{A}$ onto $A$; thus
for all $v \in Va$, $a \in wA$. Then for each term $\tau$ of type $I$, $\operatorname{re}_{\mathcal{U}} \tau$ is a well-defined function from $\omega$-sequences with terms in $A$ into $A$, i.e., an operation on $A$ of infinite rank. $\operatorname{re}_{\mathcal{U}} \tau$ is called the polynomial (operation) of $\mathcal{U}$ represented by $\tau$. Although $\operatorname{re}_{\mathcal{U}} \tau$ is technically an operation of infinite rank it is clearly independent of all coordinates $\kappa$ such that $v_\kappa$ does not occur in $\tau$. Let $\mathcal{O}$ be an arbitrary operation on $A$ or rank $\leq \omega$. If rank $\mathcal{O} = \omega$, let $\overline{\mathcal{O}} = \mathcal{O}$, and, if rank $\mathcal{O} = \kappa < \omega$, let $\overline{\mathcal{O}}$ be operation of rank $\omega$ such that

$$\overline{\mathcal{O}}a = \mathcal{O}(\langle a_0, a_1, \ldots, a_{\kappa-1} \rangle)$$

for every $a \in wA$. The operation $\mathcal{O}$ is called a polynomial (operation) of $\mathcal{U}$ if $\overline{\mathcal{O}}$ is the polynomial operation represented by some $\tau \in T\mathcal{U}$; in this case we also say that $\mathcal{O}$ is represented by $\tau$. The set of all polynomial operations of $\mathcal{U}$ (of infinite rank) forms a subuniverse of $\operatorname{wA}_{\mathcal{U}}$ and is in fact the subuniverse generated by the set of projection functions.

Thus, while the notion of a polynomial is by nature of its definition a metamathematical one, it also has a purely algebraic characterization.

Consider any algebra $\mathcal{V}$ of type $J$. If $\rho$ is a definition of $I$ in the theory of $\mathcal{V}$, then there is an algebra $\mathcal{W}$ of type
I associated with \( \mathfrak{U} \) in a natural way: the universe of \( \mathfrak{U} \) coincides with the universe of \( \mathfrak{U} \) and for each \( Q \in I \), of rank \( \kappa \), say, the fundamental operation \( Q(\mathfrak{U}) \) of \( \mathfrak{U} \) corresponding to \( Q \) is just the polynomial operation of \( \mathfrak{U} \) of rank \( \kappa \) represented by \( \rho Q \). (The fact that \( \rho \) is a definition of \( I \) in Th\( \mathfrak{U} \) guarantees that \( \rho Q \) represents an operation of rank \( \kappa \).)

We shall call \( \mathfrak{U} \) the \( \rho \)-transform of \( \mathfrak{U} \) and denote it by \( \mathfrak{T}_\rho \mathfrak{U} \). Thus \( \mathfrak{T}_\rho \mathfrak{U} \) is a function from algebras of type \( J \) into algebras of type \( I \).

Let \( K \) and \( L \) be arbitrary classes of algebras of type \( I \) and \( J \), respectively; let \( \rho \) be a definition of \( I \) in Th\( K \) and \( \pi \) a definition of \( J \) in Th\( L \). Then \( K \) and \( L \) are said to be definitionally equivalent by \( \rho \) and \( \pi \), in symbols \( K \equiv_{\rho, \pi} L \), if the following conditions hold:

1. \( \mathfrak{T}_\rho \) maps \( L \) one-one onto \( K \);
2. \( \mathfrak{T}_\pi \) maps \( K \) one-one onto \( L \);
3. \( \mathfrak{T}_\rho \mathfrak{T}_\pi \mathfrak{U} = \mathfrak{U} \) for all \( \mathfrak{U} \in K \);
4. \( \mathfrak{T}_\pi \mathfrak{T}_\rho \mathfrak{B} = \mathfrak{B} \) for all \( \mathfrak{B} \in L \).

As usual we say that \( K \) and \( L \) are definitionally equivalent and write \( K \equiv L \) if there exist \( \rho, \pi \) such that \( K \equiv_{\rho, \pi} L \). For individual algebras \( \mathfrak{U} \) and \( \mathfrak{B} \), \( \mathfrak{U} \equiv_{\rho, \pi} \mathfrak{B} \) and \( \mathfrak{U} \equiv \mathfrak{B} \) respectively mean \( \{\mathfrak{U}\} \equiv_{\rho, \pi} \{\mathfrak{B}\} \) and \( \{\mathfrak{U}\} = \{\mathfrak{B}\} \). It is not difficult to see
that when $K$ and $L$ are varieties the two notions of definitional equivalence considered define the same relation.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be any algebras of type $I$ and $J$, respectively, and suppose that $\mathfrak{A} = \mathfrak{A}_\rho$ for some definition $\rho$ of $I$ in $\text{Th}_\mathfrak{A}$. Then each fundamental operation of $\mathfrak{A}$ is a polynomial of $\mathfrak{B}$. Conversely, suppose each fundamental operation of $\mathfrak{A}$ is a polynomial of $\mathfrak{B}$. Consider $Q \in I$ of rank $\kappa$ and choose $\tau \in \text{Te}_J$ such that $\tau$ represents $Q(\mathfrak{A})$; we may assume that $\tau$ contains no variable different from $v_0, \ldots, v_{\kappa-1}$, or different from $v_0$ if $\kappa = 0$, since otherwise we could substitute $v_0$ for all such variables and still have a term that represents $Q(\mathfrak{A})$. Set $\rho Q = \tau$. Clearly, then, $\rho$ is a definition of $I$ in $\text{Th}_\mathfrak{B}$, and $\mathfrak{A} = \mathfrak{A}_\rho$. Thus we have established that, for any pair of algebras $\mathfrak{A}$ and $\mathfrak{B}$,

(5) $\mathfrak{A} = \mathfrak{A}_\rho$ for some definition $\rho$ iff every polynomial of $\mathfrak{A}$ is a polynomial of $\mathfrak{B}$.

This gives a purely algebraic criterion for the definitional equivalence of two algebras:

(6) $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ have the same polynomial operations.

**Theorem 1.6.1.** Let $K$ and $L$ be any classes of algebras of type $I$ and $J$ and $\rho, \pi$ definitions of $I$ in $\text{Th}_L$ and $J$ in $\text{Th}_K$. If $K \equiv_{\rho, \pi} L$, then
We shall only establish the middle equivalence, the proofs of the other two being similar.

It follows immediately from (5) that for any pair of algebras $\mathfrak{A}, \mathfrak{A}'$,

\[(7) \quad \text{every homomorphism from } \mathfrak{A} \text{ into } \mathfrak{A}' \text{ is a homomorphism from } \mathfrak{x} \mathfrak{A} \text{ into } \mathfrak{x} \mathfrak{A}'.\]

Consider any $\mathfrak{A}' \in H(L)$. Let $\mathfrak{A} \in L$ and

\[(8) \quad h: \mathfrak{A} \rightarrow \mathfrak{A}'.\]

Then, by (7), $h: \mathfrak{x} \mathfrak{A} \rightarrow \mathfrak{x} \mathfrak{A}'$ so that $\mathfrak{x} \mathfrak{A}' \in H(K)$; thus $\mathfrak{x}_\rho$ maps $H(L)$ into $H(K)$. Applying (7) with $\mathfrak{A}, \mathfrak{A}'$, and $\rho$ replaced by $\mathfrak{x}_\rho \mathfrak{A}, \mathfrak{x}_\rho \mathfrak{A}'$, and $\rho$, respectively, we get

\[h: \mathfrak{x}_\rho \mathfrak{x} \mathfrak{A} \rightarrow \mathfrak{x}_\rho \mathfrak{x} \mathfrak{A}'.\]

But $\mathfrak{x}_\rho \mathfrak{x}_\rho \mathfrak{A} = \mathfrak{A}$ by (4), so

\[h: \mathfrak{A} \rightarrow \mathfrak{x}_\rho \mathfrak{x}_\rho \mathfrak{A}'.\]

From this together with (8) we conclude that $\mathfrak{x}_\rho \mathfrak{x}_\rho \mathfrak{A}' = \mathfrak{A}'$; thus $\mathfrak{x}_\rho$ is the inverse of $\mathfrak{x}_\rho$ on $H(L)$ and in a similar way we get that $\mathfrak{x}_\rho$ is the inverse of $\mathfrak{x}_\rho$ on $H(K)$. This establishes $H(K) \equiv_{\rho, \pi} H(L)$. 
As a corollary of this result we have that, if $K \equiv_{\rho, \pi} L$ and $K$ is a variety, then $L$ must also be a variety. Thus the relation $\equiv_{\rho, \pi}$ determines a one-one correspondence between subvarieties of $K$ and subvarieties of $L$; this correspondence is dual to the one between extensions of definitionally equivalent theories discussed in the remarks following Corollary 1.5.3.

Just for the purpose of formulating the next theorem conveniently we write $\mathfrak{U} \sim_{\rho, \pi} \mathfrak{S}$ if there exists an algebra $\mathfrak{S}'$ such that $\mathfrak{U} =_{\rho, \pi} \mathfrak{S}'$ and $\mathfrak{S}' = \mathfrak{S}$.

**Theorem 1.6.2.** If $K$ and $L$ are varieties and $\rho, \pi$ are definitions of the proper kind, then the following three conditions are equivalent:

(i) $K \equiv_{\rho, \pi} L$;

(ii) $\forall \mathfrak{U} \mathfrak{K} \sim_{\rho, \pi} \forall \mathfrak{U} \mathfrak{L}$;

(iii) there exists a one-one function $\mathfrak{f}$ from $K$ onto $L$ such that $\mathfrak{U} =_{\rho, \pi} \mathfrak{f} \mathfrak{U}$ for each $\mathfrak{U} \in K$.

The equivalence of (i) and (iii) follows immediately from the definition of $\equiv_{\rho, \pi}$ and does not depend on the hypothesis that $K$ and $L$ are varieties.

To show that (i) implies (ii) assume that $K \equiv_{\rho, \pi} L$. Let $x_0, x_1, \ldots$ be the free generators of $\forall \mathfrak{U} \mathfrak{K}$ and $y_0, y_1, \ldots$ the free generators of $\forall \mathfrak{U} \mathfrak{L}$. Let $f_{x} = y_{x}$ for each $x < w$; since
x₀, x₁, … freely generate \( \mathfrak{Y}_w K \) there exists a unique extension of \( f \) to a homomorphism

\[
h: \mathfrak{Y}_w K \rightarrow \mathfrak{Y}_w L.
\]

Then we have

\[
h: \mathfrak{Y}_w K \rightarrow \mathfrak{Y}_w L = \mathfrak{Y}_w L.
\]

By an analogous argument we can prove there exists a homomorphism \( g \) from \( \mathfrak{Y}_w L \) onto \( \mathfrak{Y}_w K \) which extends \( f^{-1} \); then \( g \) must be the inverse of \( h \) so that \( h \) is an isomorphism. Thus

\[
\mathfrak{Y}_w K \cong \mathfrak{Y}_w L.
\]

Conversely, assume that \( \mathfrak{Y}_w K \cong \mathfrak{Y}_w L \). Then, by 1.6.1,

\[
K = HSP(\mathfrak{Y}_w K) \cong HSP(\mathfrak{Y}_w L) = L.
\]

Thus (ii) implies (i).

Again under the assumption that \( K \) and \( L \) are varieties it is obvious that conditions (i) and (ii) of this theorem remain equivalent when \( =_{\rho, \pi} \) is replaced by \( \approx \). However it is an open problem whether or not (iii) remains equivalent to (i) in this case. In this connection the discussion of the last section concerning theories definitionally equivalent to a proper extension of themselves proves to be relevant.
Suppose condition (iii) of 1.6.2 holds with \( \equiv \) in place of \( \equiv_{\rho, \pi} \) but \( K \) is not definitionally equivalent to \( L \). Then by 1.6.1 we have

\[ K = \text{HSP}_w \neq K = \text{HSP}_w \neq L \subseteq L. \]

Thus \( K = L' \) where \( L' \) is a proper subvariety of \( L \). Similarly, \( K' = L \) where \( K' \) is a proper subvariety of \( K \). But then in view of the remark following the proof of Theorem 1.6.1 we have that there exists a proper subvariety \( K'' \) of \( K \) definitionally equivalent to \( L' \) and hence also to \( K \). Consequently, we see that, if 1.6.2(iii) with \( \equiv_{\rho, \pi} \) replaced by \( \equiv \) holds for varieties \( K \) and \( L \), then either \( K = L \) or both \( K \) and \( L \) are definitionally equivalent to proper subvarieties of themselves.

In the last section we constructed a theory \( \Theta \) that was definitionally equivalent to a proper extension of itself whence the variety \( \text{Mo}(\Theta) \) is definitionally equivalent to a proper subvariety; thus varieties with this property exist. It turns out however that there is an important class of varieties no member of which has this property.

Theorem 1.6.3. Let \( I \) be a finite type. Assume \( K, K' \) are varieties of type \( I \), \( K \equiv K' \), and \( K' \subseteq K \). Then \( K \) and \( K' \) must contain the same finite algebras; consequently, if \( K \)
is generated by its finite members, then we must have $K' = K$.

Let $K \equiv_{\rho,\pi} K'$. Consider any $\mathfrak{A} \in K$. Then $\operatorname{Tr}_\pi^\lambda \mathfrak{A} \in K'$ for each positive $\lambda < \omega$ where $\operatorname{Tr}_\pi^\lambda \mathfrak{A}$ is the result of applying $\operatorname{Tr}_\pi^\lambda$ to $\mathfrak{A}$ $\lambda$-times. Assume now that $\mathfrak{A}$ is finite. Then for some $\kappa, \lambda > 0$ we must have

$$\operatorname{Tr}_\pi^\kappa \mathfrak{A} = \operatorname{Tr}_\pi^{\kappa + \lambda} \mathfrak{A}$$

since $I$ is also finite by hypothesis. Applying $\operatorname{Tr}_\rho^\kappa$ to both sides of this equation we get

$$\mathfrak{A} = \operatorname{Tr}_\rho^\kappa \mathfrak{A} \in K'. $$

Thus every finite algebra of $K$ is contained in $K'$.

It follows directly from this theorem and the remarks immediately preceding it that conditions 1.6.6(i),(iii) remain equivalent when $\equiv_{\rho,\pi}$ is replaced by $\equiv$ whenever $K$ and $L$ are varieties at least one of which is generated by its finite members.

We have one final remark to make on this subject. The variety with the property that it is definitionally equivalent to a proper subvariety of itself which comes out of the construction of the last section turns out not to have any finite members at all; varieties with this property can be found however that have a great many finite members.
We see from (6) that when $K$ and $L$ are singleton classes of algebras we have a purely algebraic criterion for their definitional equivalence. With the aid of 1.6.2(i), (ii) this can be used to give a purely algebraic criterion for the definitional equivalence of two varieties. We now consider other criterion of this kind which moreover apply to more general classes of algebras.

Let $\mathfrak{g}$ be a function whose domain and range are classes of algebras of fixed types. $\mathfrak{g}$ is said to be functorial if the following conditions hold for all algebras $\mathfrak{A}$ and $\mathfrak{B}$ in the domain.

$\mathfrak{g}$ and $\mathfrak{B}$ have the same universe;

(10) every homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ is a homomorphism from $\mathfrak{g}(\mathfrak{A})$ into $\mathfrak{g}(\mathfrak{B})$ and vice-versa.

If in (10) we drop the phrase "and vice-versa", then $\mathfrak{g}$ is said to be weakly functorial. Finally, $\mathfrak{g}$ is a functorial equivalence if it is one-one and both $\mathfrak{g}$ and $\mathfrak{g}^{-1}$ are functorial (or, equivalently, weakly functorial).

\textbf{Theorem 1.6.4.} Let $K, L$ be arbitrary classes of algebras and $\mathfrak{g}$ a weakly functorial map from $K$ onto $L$. If $\mathfrak{g} \upharpoonright K \in K$, then $\mathfrak{g} = \mathfrak{g} \upharpoonright \pi$ on $K$ for some definition $\pi$ of the type of $L$ in the theory of $K$. 
We shall show that, for each \( \mathfrak{U} \in K \),

\[(11) \quad \text{every polynomial of } \mathfrak{U} \text{ is a polynomial of } \mathfrak{U} \, . \]

Let \( h \) be the unique homomorphism of \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) into \( \mathfrak{w} \mathfrak{A} \mathfrak{u} \) (the \( \mathfrak{w} \mathfrak{A} \)th Cartesian power of \( \mathfrak{U} \)) such that the value of \( h \) at the \( x \)'th free generator of \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) is the \( x \)'th projection function of \( \mathfrak{w} \mathfrak{A} \) on \( A \). Then the image of \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) under \( h \) is just the set of all polynomial operations of \( \mathfrak{U} \). For each \( a \in \mathfrak{w} \mathfrak{A} \) let \( F_a : \mathfrak{w} \mathfrak{A} \rightarrow A \) be defined by \( F_a(f) = f(a) \) for each \( f \in \mathfrak{w} \mathfrak{A} \); clearly \( F_a \circ h : \mathfrak{U} \mathfrak{r} \mathfrak{K} \rightarrow \mathfrak{U} \) for each \( a \in \mathfrak{w} \mathfrak{A} \). Then since \( \mathfrak{U} \) is weakly functorial we conclude that \( F_a \circ h \) is a homomorphism from \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) into \( \mathfrak{U} \) for each \( a \in \mathfrak{w} \mathfrak{A} \). It then follows by a well known property of Cartesian powers that \( h \) is a homomorphism from \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) into \( \mathfrak{w} \mathfrak{A} \mathfrak{U} \). Thus the image of this homomorphism, which we know to be the set of all polynomial of \( \mathfrak{U} \), is a subuniverse of \( \mathfrak{w} \mathfrak{A} \mathfrak{U} \) including all projection functions; hence it includes all polynomials of \( \mathfrak{U} \). This establishes \( (11) \).

Applying \( (11) \) with \( \mathfrak{U} = \mathfrak{U} \mathfrak{r} \mathfrak{K} \) we conclude by \( (5) \) that \( \mathfrak{U} \mathfrak{r} \mathfrak{K} = \mathfrak{U} \mathfrak{r} \mathfrak{K} \mathfrak{r} \mathfrak{K} \) for some definition \( \pi \). \( \mathfrak{U} \mathfrak{r} \mathfrak{K} \) is clearly a weakly functorial map so that the theorem will be established once we have proved the following lemma.

\[(12) \quad \text{Let } \mathfrak{U}, \mathfrak{U}' \text{ be any pair of weakly functorial maps on } K \text{ such that } \mathfrak{U} \mathfrak{r} \mathfrak{K} = \mathfrak{U}' \mathfrak{r} \mathfrak{K} \text{. Then } \mathfrak{U} \mathfrak{U} = \mathfrak{U}' \mathfrak{U} \text{ for every } \mathfrak{U} \in K \, . \]
Consider any $\mathcal{M} \in \mathcal{K}$. Then by the functorial property of $\mathcal{G}$ we have that any homomorphism $h$ from $\mathcal{G}\mathcal{R}_w \mathcal{K}$ into $\mathcal{M}$ is also a homomorphism from $\mathcal{G}\mathcal{R}_w \mathcal{K}$ into $\mathcal{G}\mathcal{M}$; hence its image is a subalgebra of $\mathcal{G}\mathcal{M}$ which we denote by $h^*\mathcal{G}\mathcal{R}_w \mathcal{K}$.

Let $M$ be the set of all $h^*\mathcal{G}\mathcal{R}_w \mathcal{K}$ where $h$ is an arbitrary homomorphism from $\mathcal{G}\mathcal{R}_w \mathcal{K}$ into $\mathcal{M}$. $M$ is a class of subalgebras of $\mathcal{G}\mathcal{M}$ which is directed by the relation $\subseteq$ of inclusion between algebras and such that each element of $\mathcal{G}\mathcal{M}$ included in a member of $M$. Thus $\mathcal{G}\mathcal{M}$ is equal to the algebraic union of $M$. On the other hand, since $\mathcal{G}\mathcal{R}_w \mathcal{K} = \mathcal{G}^\prime \mathcal{R}_w \mathcal{K}$ by hypothesis we also have

$$h^*\mathcal{G}\mathcal{R}_w \mathcal{K} = h^*\mathcal{G}^\prime \mathcal{R}_w \mathcal{K}$$

for every homomorphism from $\mathcal{G}\mathcal{R}_w \mathcal{K}$ into $\mathcal{M}$. Thus $\mathcal{G}^\prime \mathcal{M}$ also equals the algebraic union of $M$. This proves (12) and thus also the theorem.

A slightly weaker version of this theorem (with "weakly functorial" replaced by "functorial") together with the following corollary is due to Felcher [68].

**Corollary 1.6.5.** Let $\mathcal{K}, \mathcal{L}$ be classes of algebras and $\mathcal{G}$ a functorial equivalence from $\mathcal{K}$ onto $\mathcal{L}$. If $\mathcal{G}\mathcal{R}_w \mathcal{K} \in \mathcal{K}$ and $\mathcal{G}\mathcal{R}_w \mathcal{L} \in \mathcal{L}$, then $\mathcal{K} \equiv \mathcal{L}$; in particular, definitions $\rho, \pi$ exist.
such that $K \equiv_{\rho, \pi} L$, $\mathfrak{J} = \mathfrak{X}_{\pi}$ on $K$ and $\mathfrak{J}^{-1} = \mathfrak{X}_{\rho}$ on $L$.

A weaker version of the corollary with the condition "$\mathfrak{X}_W K \in K$ and $\mathfrak{X}_W L \in L$" replaced by "$\text{SP}(K) = K$ and $\text{SP}(L) = L$" was first proved by Mal'cev [58].

We close this section with one more algebraic characterization of definitional equivalence.

**Theorem 1.6.6.** Let $K$ and $L$ be classes of algebras with $\mathfrak{X}_W K \in K$ and $\mathfrak{X}_W L \in L$. Let $\mathfrak{J}$ be a one-one function from $K$ onto $L$ satisfying the following conditions:

(i) the universe of $\mathfrak{J} U$ coincides with that of $U$ for each $U \in K$;

(ii) $U \leq V$ iff $\mathfrak{J} U \leq \mathfrak{J} V$ for all $U, V \in K$;

(iii) $\mathfrak{J}(U \times V) = \mathfrak{J} U \times \mathfrak{J} V$ for all $U, V \in K$.

Then $K \equiv L$; in particular, there exists definitions $\rho, \pi$ such that $K \equiv_{\rho, \pi} L$, $\mathfrak{J} = \mathfrak{X}_{\pi}$ on $K$, and $\mathfrak{J}^{-1} = \mathfrak{X}_{\rho}$ on $L$.

Consider any $U, V \in K$ and any $h: A \to B$. Then it is an easy matter to check that $h$ is a homomorphism from $U$ into $V$ iff $h$ is a subuniverse of $U \times V$. In view of this the conditions (i)--(iii) tell us that $\mathfrak{J}$ is a functorial equivalence. The theorem now follows immediately from 1.6.5.
Section 1.7. Theories of groups and other familiar algebraic structures.

Now that we have discussed definitional equivalence we are in a position to define precisely what we mean by a theory of groups. By the standard theory of all groups we shall intend the specific theory $G$ of type $I = [\cdot, -1, e]$ where $\cdot$ and $-1$ are respectively binary and unary operation symbols and $e$ a constant symbol and

$$G = \mathcal{L} \left[ (x \cdot y) \cdot z \approx x \cdot (y \cdot z), \ e \cdot x \approx x, \ x \cdot e \approx x, \ x \cdot x^{-1} \approx e \right]$$

We will generally not bother to distinguish between a theory and its isomorphic images so the operation symbols $\cdot$, $-1$, $e$ are not to be considered completely determined. Throughout this entire paper $G$ will be used exclusively to denote the standard theory of groups and the same applies to other bold-faced symbols denoting theories that are introduced below.

By a theory of all groups we mean any theory definitionally equivalent to $G$. Examples of theories of all groups different from the standard theory of all groups are obtained by conceiving of groups as algebras whose fundamental operations are respectively composition and inverse only, right-hand division, and left-hand division. In the case of composition and inverse the resulting theory has a base consisting of the following three
equations:

\[(x \cdot y) \cdot z = x \cdot (y \cdot z),
\quad x \cdot (x^{-1} \cdot y) = y,
\quad (y \cdot x) \cdot x^{-1} = y.\]

The six equations (7)–(12) of Section 1.5 form a base for the theory of all groups with right-hand division. These equations were obtained the base for \( G \) given above, using 1.5.2(i). A base for the theory of all groups with left-hand division can be obtained in the same manner using the following definitions:

\[p(x) = (x : (x : x)) : y, \quad p^{-1}(x) = x : (x : x), \quad \rho(e) = x : x, \quad \tau(x) = x^{-1} : y.\]

In the case of right-hand division the theory has a base consisting of single equation; one such equation was given in the introduction. The theory in the case of left-hand division is also obviously \( 1 \)-based, and, although it is far from being obvious, it turns out that the theory of all groups with composition and inverse is also \( 1 \)-based. On the other hand, the theory \( G \) fails to be \( 1 \)-based. These results are all special cases of a comprehensive result concerning the cardinalities of bases of group theories which is due to Thomas Green and Tarski and which will be discussed in Chapter 3. We only remark that these results show that the cardinality of the smallest possible base for a theory need not be preserved by definitional equivalence.

By a **standard theory of groups** we mean any extension of \( G \) and, finally, by a **theory of groups** we mean any theory defini-
tionally equivalent to a standard theory of groups. Important examples of theories of groups which we shall consider are:

the theory of all Abelian groups

\[ AG = \bigcirc_G [y \cdot y \approx y \cdot x]; \]

for each positive \( x < \omega \), the theory of all Burnside groups of exponent \( x, 1 < x < \omega \),

\[ BG_x = \bigcirc_G [x^x \approx e]; \]

and the corresponding theory of all Abelian groups of exponent \( x, 1 < x < \omega \),

\[ AG_x = \bigcirc_G [x^y \approx y^x, x^x \approx e]. \]

Notice that \( BG_1 = G, AG_1 = AG \), and \( BG_2 = AG_2 \).

For any pair of words of the type of the standard theory of groups the commutator of \( \tau \) and \( \sigma \) is defined to be the word

\[ [\tau, \sigma] = \tau^{-1} \sigma^{-1} \tau \sigma, \]

and for any sequence \( \tau_0, \ldots, \tau_{k-1} \) of words with \( k \geq 2 \) the (left-normed) commutator of \( \tau_0, \ldots, \tau_{k-1} \) is defined recursively by the condition
Then other important theories of groups are:

The theory of all nilpotent groups of class \( \gamma \): \( \overline{N}_0 = \emptyset \) and for \( \gamma \geq 1 \)
\[
\overline{N}_\gamma = \emptyset \mathcal{C}[x_0, x_1, \ldots, x_\gamma] = e.
\]

The theory of all solvable groups of length \( \kappa \). Let \( \tau_1 = [v_0, v_1] \) and, for each \( \tau > 1 \), \( \tau_\kappa = \lbrack \tau_{\kappa-1}, \sup_{x} \tau_{\kappa-1} \rbrack \) where \( \kappa_\lambda = v_2 \kappa-1 + \lambda \) for each \( \lambda < 2^{\kappa-1} \). Then \( \overline{S}_\kappa = \emptyset \mathcal{C} \) and for each \( \kappa \geq 1 \)
\[
\overline{S}_\kappa = \emptyset \mathcal{C} [\tau_\kappa = e].
\]

Notice that \( \overline{N}_1 = \overline{S}_1 = \overline{A}_G \).

By a commutator word we mean any word that is contained in every set \( \Gamma \) of words with the properties \( x, x^{-1} \in \Gamma \) for every \( x \in \mathcal{G} \) and \( [\tau, \sigma] \in \Gamma \) whenever \( \tau, \sigma \in \Gamma \). A commutator equation is any equation \( \tau \approx e \) where \( \tau \) is a product of commutator words.

The following theorem is due to B. H. Neumann [37].

**Theorem 1.7.1.** Every standard theory of groups has a base consisting of set of commutator equations together with a single equation of the form \( x^\kappa \approx e \) with \( \kappa \leq \omega \).

To see this consider an arbitrary equation \( \epsilon \). Observe first of all that every equation \( \tau \approx \sigma \) is \( \mathcal{G} \)-interderivable with \( \tau \sigma^{-1} \approx e \) so that we may assume that \( \epsilon \) is of the form
\( \tau \approx e \). By induction on the length of \( \tau \) we can prove that every word \( \tau \) is \( G \)-equivalent to a word of the form

\[
x_0^{\eta_0} x_1^{\eta_1} \cdots x_{\mu-1}^{\eta_{\mu-1}} \sigma
\]

where \( \eta_0, \eta_1, \ldots, \eta_{\mu-1} \) are integers and \( \sigma \) is a product of commutator words; hence \( \epsilon \) is \( G \)-derivable from the pair of equations

\[
(1) \quad x^k \approx e, \quad \sigma \approx e.
\]

where \( k \) is the greatest common divisor of the \( \eta_0, \ldots, \eta_{\mu-1} \).
Conversely, by leaving \( x_v \) fixed and substituting \( e \) for each variable different from \( x_v \) in the equation

\[
(2) \quad x_0^{\eta_0} \cdots x_{\mu-1}^{\eta_{\mu-1}} \sigma \approx e,
\]

we see that \( x_v^{\eta_v} \approx e \) is \( G \)-derivable from \( \epsilon \) for each \( v < \mu \); hence so is \( x^k \approx e \). But \( \sigma \approx e \) is \( G \)-derivable from \( x^k \approx e \) and \( \epsilon \) and thus from \( \epsilon \) alone. Therefore \( \epsilon \) is \( G \)-interderivable with the pair of equations (1). It is now an easy step to the proof of the theorem.

Every commutator equation is contained in \( \overset{\sim}{AG} \). Hence we have

**Corollary 1.7.2.** Every standard theory of Abelian groups is isomorphic to \( \overset{\sim}{AG} \) for some \( \kappa, \ 1 \leq \kappa \leq \omega \).

In parts of the previous discussion of group theories we
used a special abbreviated notation for expressing terms. The binary operation symbol \( \cdot \) is omitted entirely and an arbitrary term is represented as a string of subterms with no indication as to how they are associated. For example, we write \( \tau^{-1} \sigma^{-1} \tau \sigma \) as an abbreviation for \((\tau^{-1} \cdot \sigma^{-1}) \cdot \tau \cdot \sigma \) and \(x^n\) for 
\(((\cdots(x \cdot x) \cdot x) \cdots) \cdot x\) where \(x\) occurs \(n\)-times in the latter expression. (If \(n\) is negative \(x^n\) stands for \(((\cdots(x^{-1} \cdot x^{-1}) \cdot x^{-1}) \cdots) \cdot x^{-1}\) with \(x^{-1}\) occurring \(-n\) times; if \(n = 0\) then \(x^n\) is \(e\).) This notation is universally employed in the literature of group theory; it together with similar special notation will be used in the sequel whenever convenient and usually without explanation. In every case the notation is, as in the group case, completely standard and no confusion is likely to occur.

The various varieties of groups are catalogued in the same way as theories and the terminology is essentially the same. Thus \(\text{MoG}\) is called the standard variety of all groups, or the standard full variety of groups. A variety of all groups, or a full variety of groups, is the model class of any theory of all groups. Finally by a variety of groups we mean the model class of any theory of groups. When there is no reason for being more specific we shall often refer to the standard theory of all groups simply as the theory of groups, or, even more simply, as
group-theory and similarly for the standard full variety of group. Also for reasons of simplicity, we refer to any term or equation of the language of the type of group theory as a group-term or a group-equation. It should be emphasized that in the case of an equation \( \in \) the use of this terminology is not intended to imply that \( \in \) is actually a member of the theory of groups; the terms group-identity or group-law are used for this purpose. However, when we want to emphasize that \( \in \) is not a group law we refer to it as a special group-equation. Similar terminology is used in the case of the other theories we deal with such as those of rings, lattices, semigroups, etc.

The theories and varieties of each of these familiar classes of algebraic structures are catalogued in a manner similar to the group case: a particular theory is chosen and designated as the standard theory and the other theories are defined in its terms.

The fundamental operations of the standard ring are taken to be addition, multiplication, additive inverse, and the additive identity. The standard theory of all rings, \( P \), is defined by the following laws:

\[
(x+y)+z \approx x+(y+z) \\
x+y \approx y+x \\
0+x \approx x
\]
The standard ring with unit has the multiplicative identity as an additional fundamental distinguished element. The defining equations of its standard theory are those of the standard theory of all rings together with the two additional laws

\[ 1 \cdot x \approx x, \ x \cdot 1 \approx x. \]

The standard theory of all lattices, \( \mathcal{L} \), is defined by the equations

\[
\begin{align*}
(x \lor y) \lor z & \approx x \lor (y \lor z), \quad x \land (y \land z) \approx (x \land y) \land z \\
x \lor x & \approx x \\
x \land x & \approx x \\
x \lor (x \land y) & \approx x \\
x \land (x \lor y) & \approx y.
\end{align*}
\]

We also have the theory of all distributive lattices

\[
\mathcal{D} = \mathcal{L} \left[ (x \lor (y \land z)) \approx (x \lor y) \land (x \lor z) \right]
\]

and the theory of all modular lattices

\[
\mathcal{M} = \mathcal{D} \left[ (x \lor (x \land y) \lor z) \approx (x \lor y) \land (x \lor z) \right].
\]

The standard theory \( \mathcal{S} \) of all semigroups is of course defined by the associative law alone:
Some standard theories of semigroups are: the theory of all commutative semigroups

$\Sigma = \theta[(x \cdot y) \cdot z \approx x \cdot (y \cdot z)]$.

the theories of all left-zero and of all right-zero semigroups

$\sim_{L} \cdot \sim \cdot [x \cdot y \approx x], \sim_{R} \cdot \sim \cdot [x \cdot y \approx y]$;

the theory of all constant semigroups

$\varepsilon = \theta_{\varepsilon} [x \cdot y \approx z \cdot w]$;

the theory of all Burnside semigroups of order $\lambda$ and exponent $\kappa$ ($1 \leq \kappa, \lambda < w$)

$B_{\kappa, \lambda} \Sigma = \theta_{\kappa, \lambda} [x^{\kappa+\lambda} \approx x^{\lambda}]$;

the theory of all commutative semigroups of order $\lambda$ and exponent $\kappa$ ($1 \leq \kappa, \lambda < w$)

$A_{\kappa, \lambda} \Sigma = \theta_{\kappa, \lambda} [x \cdot y \approx y \cdot x, x^{\kappa+\lambda} \approx x^{\lambda}]$.

$B_{1, 1} \Sigma$ and $A_{1, 1} \Sigma$ coincide respectively with the theories of all idempotent semigroups and all semilattices.

In addition to the semigroup theories just described we have two standard theories of semigroups that are also the varieties of groups. The theory of all Burnside groups of exponent $\kappa$

$B_{\kappa} \Sigma = \theta_{\kappa} [x^{\kappa} \cdot y \approx y, y \cdot x^{\kappa} \approx y]$.
and the theory of all Abelian groups of exponent \( x \)

\[ \cong \Sigma_x \subseteq \cong \Sigma_x [x \cdot y \equiv y \cdot x, x^x \cdot y \equiv y]. \]

Clearly, \( \cong \Sigma_x \subseteq \cong \Sigma_x \) and \( \cong \Sigma_x \subseteq \cong \Sigma_x \), respectively.

In the course of our work we shall study theories and varieties of special algebraic structures other than the ones considered above, for example, quasigroups and loops, but we shall defer defining these theories until we have occasion to consider them.

There are a number of interesting problems having to do with determining for a given theory \( \Theta \) and a given property of theories all theories definitionally equivalent to \( \Theta \) that have the property. For example, it is clear that every theory of all semigroups with a single binary operation symbol is isomorphic to \( \Sigma \). There exist four known non-isomorphic theories of all groups which are also theories of groupoids. They result from treating groups as an algebra with a single fundamental operation corresponding to right-hand division, left-hand division, and their duals:

\[ x:_{R} y = x \cdot y^{-1}, \quad x:_{L} y = x^{-1} \cdot y, \quad x:_{d}^{R} y = y \cdot x^{-1}, \quad x:_{d}^{L} y = y^{-1} \cdot x. \]

It is an open problem raised by Hieman--Neumann [52] whether these
exhaust all such theories; more precisely:

(1) Let $\theta$ be a theory with a single binary operation symbol $Q$. Let $\rho, \tau$ be definitions of $Q$ in $G$ and the operation symbols of $G$ in $\theta$, respectively, such that $\theta \equiv_{\rho, \tau} G$. Is it then true that $\rho Q$ must be $G$-equivalent to one of the four terms

$$v_0^{-1} \cdot v_1^{-1}, v_0^{-1} \cdot v_1, v_1^{-1} \cdot v_0, v_1 \cdot v_0^{-1}?$$

The corresponding problem for the theory $AG$ of all Abelian groups has been solved and the solution sheds some light on the possible form of $\rho Q$ in the group case.

**Theorem 1.7.3.** Let $\theta$ be a theory with one binary operation symbol $Q$ and let $\rho, \tau$ be definitions of $Q$ in $AG$ and the operation symbols of $AG$ in $\theta$ such that $\theta \equiv_{\rho, \tau} AG$. Then either

$$\rho Q \equiv_{AG} v_0^{-1} v_1^{-1} \quad \text{or} \quad \rho Q \equiv_{AG} v_0 v_1^{-1}.$$ 

Clearly we must have

(2) $$\rho Q \equiv_{AG} v_0 v_1^{-1}$$

for some integers $n$ and $m$. For each $\kappa$, $2 \leq \kappa < \omega$, we have by Corollary 1.5.3 that $AG_{\kappa}$ is definitionally equivalent by $\pi$ and $\rho$ to some extension of $\theta$. Hence by condition (2) of
Section 1.5

(3) \( v_0 \cdot v_1 \equiv_{AG} \text{el}_p \pi(\cdot) \).

If either \( n \) or \( m \) in (2) were divisible by \( \kappa \), then \( \rho(Q) \) would be \( AG \)-equivalent to a power of a single variable. But this is obviously impossible in view of (3). Hence neither \( n \) nor \( m \) in (2) is divisible by any \( \kappa \geq 2 \). The theorem follows easily from this observation.

As a corollary we have that there are exactly two non-isomorphic theories of groupoids that are also theories of all Abelian groups. By an argument similar to that used in the proof of 1.7.3 we can show that, in the problem stated in (1), in order for \( \theta \equiv_{\rho, \pi} G \) it is necessary that

(4) \( \rho Q \equiv_G v_0 \cdot v_1 \cdot v_0 \cdot v_1 \cdots v_0 \cdot v_1 \)

where \( 0 < \kappa < \omega \) and \( n_0, m_0, \cdots, n_{\kappa-2}, m_{\kappa-2} \) are integers such that

(5) \( (\sum_{\lambda < \kappa} n_\lambda)(\sum_{\lambda < \kappa} m_\lambda) = -1 \).

The only other result bearing on problem (1) that we know of is that the corresponding problem for the theory \( NG_{2} \) has a positive solution; see Fajtlowicz [72a]. It is also known that there is no theory of groupoids definitively equivalent to
either the theory of all rings or the theory of all lattices.

Problem (1) has some obvious generalizations.

(6) For any given \( x, \ 2 \leq x < \omega \), find all theories \( \theta \) with a single operation symbol of rank \( x \) such that \( \theta \) is a theory of all groups.

(7) Find all finite types \( I \) with at least one operation symbol of rank \( \geq 2 \) such that there exists a theory of all groups of type \( I \) which is not a proper definitional extension of any theory.

This last condition is clearly necessary for the problem to be non-trivial.

A partial solution of problem (7) is contained in unpublished work of Thomas Green. He has shown that for each \( x < \omega \) there is a theory \( \theta \) with \( x \) operation symbols (or rank \( \leq 4 \)) such that \( \theta \) is a theory of all groups and is not a proper definitional extension of any theory.

Let us say that the fundamental operations of an arbitrary class \( K \) of similar algebras, or a single algebra \( \mathfrak{A} \), are independent if \( \operatorname{Th} K \), or \( \operatorname{Th} \mathfrak{A} \), is not a proper definitional extension of any theory. Then Green's result says that there is a full variety of groups with any given positive finite number of independent operations. Post [41] has made a systematic investigation
the independence of the operations of an arbitrary 2-element algebra \( M \). He proved that \( M \) can have at most five independent operations and, if \( M \) is operationally complete in the sense that every possible operation on \( A \) is a polynomial operation of \( M \), then \( M \) can have at most four independent operations. In the same paper Post shows that there exist 3-element algebras with any finite number of independent operations; if \( M \) is operationally complete however and contains three elements, then there can exist at most a finite number of independent operations.

Recall that \( \theta \) is defined to be a theory of groups if it is definitionally equivalent to some standard theory \( \bar{\theta} \) of groups. It is interesting that there exist theories of groups that are not extensions of a theory of all groups; in fact, \( \bar{BG} \) is such a theory for each finite \( K \geq 3 \).

To show this suppose \( \bar{BG} \) is an extension of \( \theta \) where \( \bar{\theta} = \rho, \pi \). To avoid confusion denote the unique operation of \( \bar{BG} \) by \( * \); also, for simplicity assume \( K = 3 \). In view of the remark following the proof of 2.7.3 we may assume that conditions (4) and (5) hold. Let

(8) \[ a = \sum_{\lambda < K} n_\lambda \quad \text{and} \quad b = \sum_{\lambda < K} m_\lambda. \]

By Corollary 1.5.3 we have that \( \bar{BG} \cong \rho, \pi \) where \( \phi \) is the extension of \( \theta \) defined by the three equations
\[
\begin{align*}
\text{el}_p [(x^o y)^o z] &= x^o (y^o z)] \\
\text{el}_p [(x^o x)^o y] &= y \\
\text{el}_p [y^o ((x^o x)^o x)] &= y.
\end{align*}
\]

Using (4) and (8) it is easy to check that \( \text{el}_p (x^o x)^o x \sim \phi \)
\( x (a^2 + ab + b) \) and \( \text{el}_p (x^o (x^o x)) \sim \phi \)
\( x^{a + ab + b^2} \). Hence from (5) we can conclude that \( x = x^{-1} \) so that \( \phi \) would be a theory of Burnside groups of exponent 2. This is impossible however since it is clear that \( \text{BG}_3 \) cannot be definitionally equivalent to such a theory.

The theory \( \theta \) of Boolean algebras is an example of a theory of rings that is not an extension of any theory of all rings. This is so since it is well known that \( \theta \) is definitionally equivalent to a theory of groupoids, but, as was previously mentioned, the theory of all rings fails to have this property.

The problem stated in (7) for groups can also be formulated for other algebraic structures. Some partial results along these lines which are easy to establish are the following: there is a theory of all rings with a single ternary operation symbol and a theory of lattices with a single operation symbol of rank 4.
Problems

Problem 1.1. Is it true that the class of Boolean algebras treated as algebras $\mathcal{A} = \langle A, \vee, \wedge \rangle$ where $\vee$ and $\wedge$ have the usual meaning can be characterized by the following identities:

\[
x \lor y \sim y \lor x
\]
\[
x \lor (y \lor z) \sim (x \lor y) \lor z
\]
\[
((x \lor y)' \lor (x \lor y)')' \sim x'
\]

Cf. the Introduction for the history of this problem.

Problem 1.2. Let $\theta$ be a theory of groupoids with the property that, whenever $\tau$ is a term containing exactly the two variables $v_0$ and $v_1$ and $\rho\theta = \tau$ where $\rho$ is the operation symbol of $\theta$, then $e_\rho$ is an interpretation of $\theta$ in itself. Must $\theta$ necessarily be isomorphic to one of the theories $\Theta_1$, $\Theta_2$, $\Theta_3$, and $\Theta_4$?

Cf. the remarks following 1.5.3.

Problem 1.3. Let $K$ and $L$ be arbitrary varieties and assume that there exists a one-one function $\mathcal{J} : K \rightarrow L$ such that $\mathcal{U} = \mathcal{J}\mathcal{W}$ for each $\mathcal{U} \in K$. Is it necessarily true that $K \approx L$?

Cf. theorem 1.6.2 and the following remarks.
Problem 1.4. Does there exist a variety $K$ of semi-groups, quasigroups, groups, or lattices which is definitionally equivalent to a proper subvariety of itself?

By theorem 1.6.3 any such $K$ must fail to be generated by its finite members, varieties with this property are known for all four kinds of algebras. Cf. Baker [69], Evans [71], and H. Neumann [67], p. 19.

Problem 1.5. Let $\theta$ be a theory with a single binary operation symbol $Q$. Let $\rho, \pi$ be definitions of $Q$ in $G$ and the operation symbols of $\sim$ in $\theta$, respectively, such that $\theta \subseteq_{\rho, \pi} G$. Is it true that $\rho Q$ must be $\sim$-equivalent to one of the four terms $v_0 v_1^{-1}, v_0^{-1} v_1, v_1^{-1} v_0, v_1 v_0^{-1}$?

This problem originates with Higman--Neumann [52]. Cf. (1) of section 1.7 and the following remarks.

Problem 1.6. For any given $x$, $2 \leq x < \omega$, find all theories $\theta$ with a single operation symbol of rank $x$ such that $\theta$ is a theory of all groups.

This problem and the following one generalize Problem 1.4.

Problem 1.7 Find all finite types $I$ with at least one operation symbol of rank $\geq 2$ such that there exists a theory
of all groups of type I which is not a proper definitional extension of any theory.

Thomas Green has obtained a partial solution to this problem; see (7) of Section 1.7 and the following remarks.
Chapter 2

The lattices of theories

If we disregard certain difficulties having to do with the foundations of set theory we can consider Mo and Th as operations connecting classes of algebras with sets of equations. These operations are the polarities associated with the relation of consequence \( \models \) (cf. Birkhoff [67], p. 122); hence the composite operations MoTh and ThMo are closure operations on the space of algebras and space of equations respectively. The corresponding families of closed sets, i.e., the classes of all varieties and of all equational theories of type I, then become complete lattices under set theoretical inclusion. We shall denote these lattices respectively by \( \mathfrak{V}_I \) and \( \mathfrak{Th}_I \).

The intersection of any set of varieties is again a variety and similarly for any set of equational theories so that the meet operation in both lattices is set-theoretical intersection and is denoted by \( \cap \). On the other hand the union of even two varieties, or theories, is not in general a variety, or theory, and for this reason the join operation in both lattices is denoted by \( \vee \). Therefore the lattices of varieties and theories (of type I) are written respectively

\[
\mathfrak{V}_I = \langle \mathfrak{V}_I, \subseteq, \cap, \vee \rangle, \quad \mathfrak{Th}_I = \langle \mathfrak{Th}_I, \subseteq, \cap, \vee \rangle.
\]
Notice that \( V_a \) and \( \text{Th}_I \) denote respectively the class of all varieties of type \( I \) and the set of all theories of type \( I \); keep in mind also that for any set \( X \) of varieties we have

\[
V(X) = \text{MoTh} \cup X = \text{HSP}(\cup X)
\]

and for any set \( \mathcal{L} \) of theories

\[
V(\mathcal{L}) = \text{ThMo}(\cup \mathcal{L}) = \mathcal{E}(\cup \mathcal{L})
\]

The zero and unit elements of \( \text{Th}_I \) are the theories \( Ta_I \) and \( Eq_I \), respectively. The corresponding elements of \( m_a \) are the varieties of 1-element algebras and the variety of all algebras.

For any pair of theories \( \Theta, \Phi \) of type \( I \) such that \( \Theta \subseteq \Phi \), the set of all theories \( V \) such that \( \Theta \subseteq V \subseteq \Phi \) forms a sublattice of \( \text{Th}_I \). This lattice is called the interval (sublattice) of \( \text{Th}_I \) determined by \( \Theta \) and \( \Phi \) and is denoted by

\[
\text{Th}_I[\Theta, \Phi]
\]

We write

\[
\text{Th}_I[\Theta] = \text{Th}_I[\Theta, Eq_I]
\]

Thus \( \text{Th}_I[\Theta] \) is the principal dual ideal of \( \text{Th}_I \) determined by \( \Theta \); its universe \( \text{Th}_I[\Theta] \) is the set of all simple extensions of \( \Theta \). The interval \( m_a[I, K] \) determined by varieties \( L, K \)
is defined analogously. In this case however $\mathfrak{M}_1[K]$ is used to denote the principal ideal $\mathfrak{M}_1[0,K]$ where $0$ is the variety of all 1-element algebras.

Observe that $\mathfrak{M}_0$ is a dual isomorphism from $\mathfrak{X}_1[\emptyset,\emptyset]$ onto $\mathfrak{M}_1[\emptyset,\emptyset]$ and from $\mathfrak{X}_1[\emptyset]$ onto $\mathfrak{M}_1[\emptyset,\emptyset]$.

Since the lattices $\mathfrak{X}_1$ and $\mathfrak{M}_1$ are dually isomorphic, any result concerning the structure of one lattice automatically entails the dual result for the other. In this paper we concentrate on the lattice of theories, only occasionally considering the lattice of varieties when it is especially convenient to do so; the same applies to the lattice of extensions of a given theory $\emptyset$ and the lattice of subvarieties of the variety of models of $\emptyset$.

Finally, we observe that in view of 1.5.3 and the remark following it, the lattices of extensions of any two definitionally equivalent theories are isomorphic. For this reason we make no attempt in this Chapter to distinguish between such theories.

Section 2.1. The characterization problem.

A general problem in equational logic is the one of finding for various natural classes of lattices of theories intrinsic properties that characterize the class. Of particular interest are the classes
(1) $T = \{ \emptyset : \emptyset \cong \mathfrak{T}_I(\emptyset) \text{ for some type } I \text{ and theory } \emptyset \}$

and

(2) $T' = \{ \emptyset : \emptyset \cong \mathfrak{T}_I \text{ for some type } I \}.$

By an intrinsic property of a lattice $\mathfrak{B}$ we understand, loosely speaking, a property that can be expressed entirely in terms of symbols denoting the fundamental lattice operations, the membership relation $\in$, and variables ranging exclusively over elements of the universe $L$ of $\mathfrak{B}$, subsets of $L$, relations between elements of $L$, sets of such subsets and relations, etc. Notice that the definitions of $T$ and $T'$ given in (1) and (2) are clearly not intrinsic. The problem of finding intrinsic characterizations of these classes remains open. The problems of characterizing were raised respectively by Mal'cev [68] and Grätzer [68].

Some light is shed on this problem by the fact that theories can be construed as completely invariant congruence relations on the term algebra. This gives another non-intrinsic but useful characterization of $\mathfrak{T}_I$ as the lattice of all congruence relations on the term algebra $\mathfrak{T}_I$ with its set of fundamental operations augmented by all substitution operations. Consequently, the lattices $\mathfrak{T}_I(\emptyset)$ can be characterized (up to isomorphism) as the congruence lattices of arbitrary quotients of augmented term algebras. Grätzer--Schmidt [63] have succeeded in giving
an intrinsic characterization of a class of closely related algebras—the congruence lattices of arbitrary quotients of unaugmented term algebras. These are of course just the congruence lattices of arbitrary algebras. In order to describe this characterization we introduce some special terminology.

An element of an arbitrary lattice $\mathcal{G}$ is **compact** if, whenever it is included in the join of a set of elements of $\mathcal{G}$, it must already be included in the join of a finite subset. $\mathcal{G}$ is called **algebraic** if it is complete and each element is the join of all compact elements included in it. According to Gratzer–Schmidt [63] a lattice is algebraic just in case it is isomorphic to the congruence lattice of some algebra. Thus in particular each of the lattices $\mathcal{ZT}_I[\emptyset]$ is algebraic. Moreover, since the inconsistent theory can be axiomatized by the single equation $x \approx y$, we see that the greatest element of $\mathcal{ZT}_I[\emptyset]$ is compact. This is the only special property lattices of theories are known to have and it has been conjectured by Ralph McKenzie that this is the only such property, i.e., that $T$ coincides with the class of algebraic lattices with compact unit.

The problem of characterizing the classes $T$ and $T'$ appears to be quite difficult. In light of this it is interesting to observe that the corresponding problems for lattices of first-order theories have been to a large extent solved.
In the case of a first-order theory $\theta$ the compact members of the lattice $\mathcal{D}_\theta$ of theories extending $\theta$ form a Boolean algebra, the so-called Lindenbaum–Tarski algebra of $\theta$, and the class of all Lindenbaum–Tarski algebras is just the class of all Boolean algebras (up to isomorphism). Any algebraic lattice is isomorphic to the lattice of ideals of the upper semilattice of its compact elements. Hence $\mathcal{D}_\theta$ is isomorphic to the lattice of ideals of the Lindenbaum–Tarski algebra of $\theta$. This leads to the characterization of the first-order analogues of the lattices $\mathcal{I}_I[\theta]$ as ideal lattices of Boolean algebras. These in turn have been intrinsically characterized by Tarski [37] as algebraic Brouwerian lattices with compact unit; a lattice $\mathcal{B}$ is Brouwerian if it satisfies the special distributive law

$$x \land \bigvee_{y \in Y} y = \bigvee_{y \in Y} x \land y$$

for all $x \in L$ and $Y \subseteq L$. On the other hand, no intrinsic characterization of the first order analogues of the lattices $\mathcal{I}_I$ is known. But in case $I$ is finite and contains relation symbols exclusively, at least one of which is of rank $\geq 2$, Hanf [62] has obtained the remarkable result that these lattices are all isomorphic. Specifically what Hanf shows is that for every two types $I$ and $J$ of the kind described, the Lindenbaum–Tarski algebras of the theories $\theta_I$ and $\theta_J$ of logically true sentences are isomorphic. We shall see in Section 2.4 that
by a result of Ralph McKenzie there is no equational analogue of Hanf's result.

The basic characterization problems seem difficult and at this point in the development of equational logic a long way from being completely solved. Much work has been done however investigating the structure of various specific lattices of equational theories. This chapter is devoted to reporting on these investigations.

The investigation has centered on the lattice \( \mathcal{T} \) of all theories and on the particular lattices \( \mathcal{T}[\Sigma] \), \( \mathcal{T}[\mathcal{G}] \), \( \mathcal{T}[\Lambda] \), and the lattice of loop theories. (We suppress the type designation here in accordance with the convention mentioned in Chapter 1.) The technique of the investigations vary depending on which lattice is being studied. In the case of \( \mathcal{T} \), \( \mathcal{T}[\Sigma] \), and the lattice of loop theories, combinatorial or proof-theoretical methods predominate, while in the case of \( \mathcal{T}[\mathcal{G}] \) and \( \mathcal{T}[\Lambda] \) model-theoretical methods, with emphasis on the structure theory of the particular algebras involved, seem to be more useful. The techniques in each particular care are well illustrated by the solution of the most basic problem: How many different arbitrary theories, semigroup theories, group theories, lattice theories, loop theories are there? It turns out that in each case the answer is continuum many but the methods used to obtain these results vary considerably.
These results imply that when $\theta$ is $\mathcal{T}_a$, $\Sigma$, $\mathcal{G}$, $\Lambda$, or the theory of loops, there exists at least one extension of $\theta$ that fails to be finitely based over $\theta$. There is a stronger condition on $\theta$ that is often considered which is not implied by the uncountability of the extensions of $\theta$. A set $\Gamma$ of equations is said to be irredundant over $\theta$, or simply irredundant if $\theta = \mathcal{T}_a$, if $\neg\theta[\Delta] = \neg\theta[\Delta']$ implies $\Delta = \Delta'$ for all sets $\Delta, \Delta' \subseteq \Gamma$. If there exists an infinite irredundant set over $\theta$, then $\theta$ has a non-finitely based extension.

(3) Let $\theta$ be any one of the theories $\mathcal{T}_a$, $\Sigma$, $\mathcal{G}$, $\Lambda$, or the theory of loops. Does there exist an infinite irredundant set of equations over $\theta$?

In each particular case $\theta$ a positive answer to (3) would imply that there are continuum extensions of $\theta$. As we have already observed the set of compact elements of the lattice of all first-order theories forms a sublattice. It would be interesting to know for which of the theories $\theta$ of (3) this condition fails for $\mathcal{Z}[\theta]$, i.e.,

(4) for $\theta = \mathcal{T}_a$, $\Sigma$, $\mathcal{G}$, $\Lambda$, or the theory of loops, does there exist a pair of finitely based extensions of $\theta$ whose intersection fails to be finitely based?

The answer to both (3) and (4) is known to be positive for each
of the five theories mentioned except for (4) in case \( \theta = \mathcal{G} \) or \( \delta \) is the theory of loops; the question is still open in these cases.

In the study of the structure of the lattices \( \mathcal{H} \) and \( \mathcal{H}[\Sigma] \), a new characterization of the relation \( \vdash \) of consequence plays a very important role. As we shall see this new characterization is quite different in spirit from the two discussed in Section 1.3; while both of the latter took the form of derivability relations in certain deductive systems the present characterization can be put in this form only with difficulty.

Consider any set \( \Gamma \) of equations (of a fixed type \( \mathcal{T} \)). Let us denote by \( \sigma \overset{=}{\sim} \tau \Gamma \) the relation that holds between two terms \( \sigma, \tau \) just in case they are identical or there exists an equation \( \rho \approx \pi \) and an assignment \( \varphi : \mathcal{V} \rightarrow \mathcal{T}_e \) such that \( \rho \approx \pi \) or \( \pi \approx \rho \) is contained in \( \Gamma \), \( \text{suf } \varphi \rho \) is a subterm of \( \varphi \sigma \), and \( \tau \) is obtained by replacing this subterm by \( \text{suf } \varphi \pi \).

It turns out that the relation \( \overset{=}{\sim} \Gamma \) is just the transitive closure of \( \overset{=}{\sim} \Gamma \); more precisely we have

**Theorem 2.1.1.** For any \( \sigma, \tau \in \mathcal{T}_e \) the following two conditions are equivalent:

(i) \( \sigma \approx \tau \Gamma \);

(ii) there exists a finite sequence \( \rho_0, \rho_1, \ldots, \rho_{n-1} \) of
terms such that \( \sigma = \rho_0 \vdash \rho_1 \vdash \cdots \vdash \rho_{K-1} = \tau \).

Let \( R \) be the set of all equations \( \sigma \approx \tau \) such that condition (ii) holds. Clearly \( R \) is closed under the inference rules of transitivity, symmetry, equality, and substitution described in Section 1.3. Thus, since \( \nu_0 \approx \nu_0 \in R \), we have \( \theta[\Gamma] \subseteq R \) and hence (i) implies (ii); the implication in the opposite direction is obvious.

We remark incidentally that it is also possible to construe this new characterization of \( \models \) as the relation of derivability in a certain deductive system. There is a single logical axiom schema, the set of all tautologies, and each non-logical axiom \( \gamma \in \Gamma \) is interpreted as the schema

\[
\{ \sigma \models \tau : \sigma, \tau \in T_\Gamma, \sigma \models [\gamma]\tau \}.
\]

The system has one rule of inference: transitivity. Notice that this system satisfies a fundamental condition required of all reasonable deductive systems: given \( \gamma \) there is an effective procedure for determining whether or not an arbitrary equation is actually an instance of the axiom schema (5). As we shall have the opportunity to see this property of the axiom schemas together with the primitive character of the single rule of inference are just what account for the usefulness the characterization of consequence given in 2.1.1.
For any pair of terms \( \sigma, \tau \) we say that \( \sigma \) is a substitution-subterm of \( \tau \), in symbols \( \sigma \prec_s \tau \), just in case some substitution instance of \( \sigma \) is a subterm of \( \tau \); thus \( \sigma \prec_s \tau \) iff there exist a \( \varphi : \forall a \to \mathcal{T} \) and expressions \( p, \pi \) such that \( \tau = p(\varphi(\sigma)) \pi \).

The following theorem is an immediate consequence of 2.1.1.

**Theorem 2.1.2.** Let \( \sigma, \tau \in \mathcal{T} \) and \( \Gamma \subseteq \text{Eq} \). If \( \sigma \sim \tau \in \varnothing(\Gamma) \) and \( \varepsilon \in \tau \backslash \sigma \) for every \( \varepsilon \in \Gamma \), then \( \sigma = \tau \).

By a simple application of this theorem we can show that there are a continuum number of different groupoid theories.

Let \( \sigma_0, \sigma_1, \sigma_2, \ldots \) be any infinite sequence of terms such that \( \sigma_\kappa \not\prec_s \sigma_\lambda \) for all \( \kappa, \lambda < \omega \) with \( \kappa \neq \lambda \); for instance we can take \( \sigma_\kappa = Qv_0(Qv_1)^\kappa v_0 \) for each \( \kappa < \omega \). It follows immediately from 2.1.2 that

\[ (6) \{ \sigma_\kappa \sim \sigma_0 : 0 < \kappa < \omega \} \text{ is an infinite irredundant set of equations.} \]

It is now a trivial matter to construct a sequence \( \sigma_0, \sigma_1, \ldots \) of terms of type I satisfying (6) for any type I which contains at least one operation symbol of rank \( \geq 2 \), and it is only slightly more difficult to do the same when I contains no operation symbols of rank \( \geq 2 \) but at least two unary symbols.

However, if I is finite and contains a unary operation symbol as its only symbol of positive rank, then no infinite irredundant set of equations can be constructed; in fact, in this case \( \mathcal{H}_1 \)
is denumerable as we shall see in Section 2.3.

In many instances in equational logic there is a natural trichotomy of results corresponding to whether the type \( I \) contains at least one operation symbol of rank \( \geq 2 \), no symbol of rank \( \geq 2 \) but at least two of rank 1, or one unary symbol as the only one of positive rank. For this reason we shall refer to types satisfying the respective conditions as binary types, bi-unary types, and unary types.

In some cases 2.1.2 also proves useful in the construction of an infinite irredundant set of equations over a given theory \( \theta \). For this purpose a relativized version of 2.1.2 proves convenient.

We say that \( \sigma \) is a substitution-subterm of \( \tau \) over \( \theta \), in symbols \( \sigma \leq_{S}^{(\theta)} \tau \), whenever there exist terms \( \rho, \pi \) such that

\[
\sigma \leq_{\theta}^{\rho} \leq_{\theta}^{\pi} \leq_{\theta}^{\tau}.
\]

We remark that in this definition we may require \( \rho \) to coincide with \( \sigma \) without loss of generality.

**Theorem 2.1.3.** Let \( \theta \in \text{Th}, \sigma, \tau \in \text{Te}, \) and \( \Gamma \subseteq \text{Eq} \). If

\[
\sigma \leq \tau \in \theta_{\theta}[\Gamma] \quad \text{and} \quad \xi_{\in} \xi_{\in} \leq_{\theta}^{(\theta)} \sigma \quad \text{for every} \quad \xi \in \Gamma, \text{then}
\]

\[
\sigma \leq \tau \in \theta.
\]

This theorem proves to be very useful in constructing infinite irredundant sets of equations over \( \Sigma \). In this regard
The irredundant set of equations constructed by Isbell has the novel feature that each member contains just two distinct variables. We shall present a more involved construction which gives at the same time a positive solution to both problems (3) and (4) when \( \theta = \Sigma \).

Let \( \theta \) be an arbitrary theory and \( \Omega \) any extension of \( \theta \). A theory \( \phi \) in the interval \( \text{Th}[\theta, \Omega] \) is said to be essential for \( \Omega \) over \( \theta \), or \( \theta \)-essential for \( \Omega \), if every \( \theta \)-base for \( \Omega \) includes as a subset a \( \theta \)-base for \( \phi \). If \( x < w \) and \( \text{Th}[\theta, \Omega] \) contains at least \( 2^x \) distinct theories \( \theta \)-essential for \( \Omega \), then clearly every \( \theta \)-base for \( \Omega \) must contain at least \( x \) members. Furthermore, if \( \xi \leq w \) and there are at least \( \xi \) \( 1 \)-based theories \( \theta \)-essential for \( \Omega \), then it is easy to see there is an irredundant set of equations over \( \theta \) of cardinality \( \xi \).

Theorem 2.1.4. There exist finitely based extensions \( \Omega_0 \) and \( \Omega_1 \) of \( \Sigma \) such that \( \text{Th}[\Sigma, \Omega_0 \cap \Omega_1] \) contains an infinite number of distinct \( 1 \)-based theories each of which is \( \Sigma \)-essential for \( \Omega_0 \cap \Omega_1 \).

As already mentioned this theorem provides a positive answer to both (3) and (4) for \( \theta = \Sigma \).

As far as we know the first example of any kind to appear
in the literature of a pair of finitely based theories whose
intersection fails to have this property is due to Perkins [69].
The particular example we give here and the proof of Theorem 2.1.3
are due to Joel Karnofsky but based on Perkins' ideas.

Let \( \Omega_0 \) be the theory of semigroups generated by the
following pair of equations (together with the associative law)

\[
(xyz)^2 \approx x^2 y^2 z^2, \quad x^3 y^3 z^3 w^2 \approx y^3 z^3 w^2.
\]

Let \( \Omega_1 \) be generated over \( \Sigma \) by the single equation

\[
x^3 y^3 \approx y^3 x^3.
\]

Let

\[
R = \{ \varepsilon(\kappa) : \kappa < w \}
\]

where, for each \( \kappa < w \),

\[
\varepsilon(\kappa) = (x^2 y^2 z^2 w^3)^{\kappa} \approx y^2 z^2 w^3.
\]

Clearly \( \varepsilon(\kappa) \) is not \( \Sigma \)-derivable from \( \varepsilon(\lambda) \) if \( \kappa \neq \lambda \) and
it is easy to show that each \( \varepsilon(\kappa) \) is contained in \( \Omega_0 \cap \Omega_1 \).

We now prove that \( \varepsilon(\kappa) \) is \( \Sigma \)-essential for \( \Omega_0 \cap \Omega_1 \). Consider
any \( \Sigma \)-base \( \Delta \) for \( \Omega_0 \cap \Omega_1 \). Then there exists terms
\( \tau_0, \ldots, \tau_\lambda \) such that

\[
\varepsilon(\kappa) = \tau_0 \triangleq \Delta \tau_1 \triangleq \Delta \cdots \triangleq \Delta \tau_\lambda = \varepsilon(\kappa).
\]
where \( \xi^{(\kappa)} = (\xi^{(\kappa)}_L \equiv \xi^{(\kappa)}_R) \). Since \( \Delta \subseteq \Omega_0 \) we must have, for each \( \mu < \lambda \),

\[
\tau_{\mu} \overset{\equiv}{=} \delta_{\mu}^{T_{\mu}} \overset{\equiv}{=} \delta_1 \cdots \overset{\equiv}{=} \delta_{\mu+1}
\]

where \( \delta \) is the equation (9). Any variable appearing in a substitution instance of either side of \( \delta \) must have at least three distinct occurrences. Thus if such a substitution instance occurs as a subterm of \( \tau_0 (= \xi^{(\kappa)}_L) \) it must be as the initial segment \( x^3 y^3 \). Hence \( \tau_\mu' \) must either equal \( \tau_0 \) or differ from it by a replacement of this initial segment by \( y^3 x^3 \). By induction we prove that \( \tau_\lambda \) is in the same relation to \( \tau_0 \). Hence

\[
\xi^{(\kappa)}_L \overset{\equiv}{=} \delta \overset{\equiv}{=} \xi^{(\kappa)}_R.
\]

Therefore there must exist terms \( \sigma, \tau \), expressions \( \xi, \eta \), and an assignment \( \phi : V_a \rightarrow T_{\text{e}_I} \) such that either \( \sigma \equiv \tau \) or \( \tau \equiv \sigma \) is contained in \( \Delta \) and

\[
(9) \quad \xi^{(\kappa)}_L = \xi (s u \phi \sigma) \eta, \quad \xi^{(\kappa)}_R = \xi (s u \phi \tau) \eta.
\]

Because the two sides of \( \xi \) begin with different variables, \( \xi \) must be empty. Let \( \rho, \tau \) be the first and second equations of (7), respectively. It is an easy matter to prove that no proper initial segment of \( \xi^{(\kappa)}_L \) is \( [\rho, \tau] \)-equivalent to a term which ends in an expression of the form \( \zeta^3 \). Hence no
proper initial segment of $\xi^{(N)}_{L}$ is $[\rho, \pi]$-equivalent to an initial segment of $\xi^{(N)}_{E}$. This implies that in (9) $\eta$ as well as $\xi$ is empty. Finally, observe that since $\sigma \approx \tau$ is a non-tautological consequence of $[\rho, \pi]$ every variable appearing in $\sigma$ must occur at least twice. Thus $\varphi$ cannot take a non-variable term as value at any variable appearing in $\sigma$ since no single variable occurs at more than three distinct places in $\xi^{(N)}_{L}$. This implies that $\sigma \approx \tau$ differs from $\xi^{(N)}_{E}$ only by a possible change of variables. In other words, $\sigma \approx \tau$ is $\Sigma$-inter- derivable with $\xi^{(N)}_{E}$. Since $(\sigma \approx \tau) \in \Delta$ this completes the proof of 2.1.4.

There are various proper extensions of the theory of semigroups relative to which infinite irredundant sets of equations are known to exist, for example, the theory $\varnothing[\varnothing^2 \equiv \varnothing^3]$; cf. Burris-Nelson [71] and the discussion of Section 2.5.

Each of the equations (7) and (8) has the property that every variable appearing in it occurs the same number of times on each side. An equation of arbitrary type having this property is said to be balanced. An equation satisfying the weaker condition that every variable appearing in it occurs at least once on each side is called variable-uniform. The set of balanced equations and the set of variable-uniform equations of any type are both closed under substitution and replacement.
and hence both form theories. In the case of groupoids these theories correspond respectively to $\mathcal{AE}$, the theory of all commutative semigroups, and $\mathcal{AE}_{1,1}$, the theory of all commutative idempotent semigroups (cf. Section 2.2).

Consequently, the theories $\Omega_0$ and $\Omega_1$ in the proof of 2.1.4 both consist exclusively of balanced equations and a careful analysis of that proof will show that this fact plays an important role. Indeed the usefulness of Theorem 2.1.3 for the purpose of constructing $\Theta$-irredundant sets of equations is greatly reduced if $\Theta$ does not consist entirely of balanced or at least variable-uniform equations. In particular, 2.1.3 as it is formulated is completely useless when $\Theta$ is $G_A$, or the theory of loops since in all of these cases $\Theta$ contains an equation of the form $x \approx \tau$ where $\tau$ contains a variable different from $x$. Thus $\rho \prec (\Theta) \pi$ for every pair $\rho, \pi$ of terms. For these theories other methods of proving the existence of infinite relatively irredundant sets of equations must be found.

Combinatorial methods still play a useful role in the case of loops however. An infinite irredundant set of equations over the theory of loops was constructed in Neumann-Evans [53]. Later Evans [71] exhibited an infinite irredundant set over a theory of loops that is anti-finite in the sense that it has no non-trivial finite model. Hence there are uncountably many
anti-finite loop theories. Model-theoretical methods were used to obtain these results but it is not hard to see how these may be replaced by purely combinatorial ones. Actually, the general approaches to problems of this kind for semigroups and loops seem very closed in spirit; the role played by the properties of balanced equations in the semigroup case is taken in the loop case by consequences of the failure of the associative law. These remarks will be illustrated in the next section where some results on loop theories are presented in detail.

In the case of lattices and group theories the situation is entirely different and combinatorial methods play a much smaller role. This seems to be especially true in the lattice case where the systematic study of the structure of the lattice of lattice theories has only recently begun—given impetus by the discovery by Jónsson [67] of a powerful model-theoretical tool for this purpose.

Let us call a variety $K$ congruence-distributive if each member of $K$ has a distributive lattice of congruence relations; a theory will be congruence-distributive if $\text{Mo}\emptyset$ has the property. Obviously every extension of a congruence-distributive theory is again congruence-distributive. In particular, every theory of lattices is congruence-distributive.

It is a well known result of the general theory of algebras
that, if \( \mathfrak{U} \) is a homomorphic image of an algebra \( \mathfrak{B} \), then the lattice of congruence relations on \( \mathfrak{U} \) is (isomorphic to) a sublattice, in fact a principal dual ideal, of the congruence lattice of \( \mathfrak{B} \). Thus the distributivity of the latter implies the distributivity of the former. It is also well known that the congruence lattice of \( \mathfrak{U} \) is distributive iff every finitely generated subalgebra of \( \mathfrak{U} \) has the property. It follows from these two observations that a theory \( \mathfrak{T} \) is congruence-distributive iff \( \mathfrak{U} \) has a distributive congruence lattice. (Actually, as a consequence of work of Jónsson [67] \( \mathfrak{T} \) is congruence-distributive if the free algebra over \( \mathfrak{T} \) with only three generators has a distributive congruence lattice; cf. Section 2.5.)

The congruence-distributivity of \( \mathfrak{T} \) implies the distributivity of the lattice \( \mathfrak{Xh}[\mathfrak{T}] \) of extensions of \( \mathfrak{T} \) since, as we observed in Section 1.3, \( \mathfrak{Xh}[\mathfrak{T}] \) can be construed as a sublattice of the lattice of congruence relations on \( \mathfrak{Yr} \mathfrak{U} \). The reverse implication does not hold in general however. For example, it will be seen from the work of Sections 2.2 and 2.6 that \( \mathfrak{Xh}[\Gamma_{\mathfrak{U} \mathfrak{W}}] \) is a 2-element lattice and hence distributive but \( \mathfrak{Yr} \mathfrak{U} \mathfrak{W} \mathfrak{W} \) satisfies no special lattice equation.

The link between congruence-distributivity and the structure of lattices of theories proves to be the notion of a subdirect product of algebras and a fundamental result of Garrett Birkhoff concerning this notion. Let \( \langle \mathfrak{N}_i : i \in I \rangle \) be
any system of similar algebras. By a subdirect product of the \( \mathfrak{A}_i \) we mean any subalgebra \( \mathfrak{B} \) of the Cartesian product 
\[ \Pi_{i \in I} \mathfrak{A}_i \] such that for each \( i \in I \) the projection of \( \mathfrak{B} \) onto the \( i \)-th coordinate maps \( \mathfrak{B} \) onto all of \( \mathfrak{A}_i \); in particular, each \( \mathfrak{A}_i \) is a homomorphic image of \( \mathfrak{A} \) and thus the set of all the \( \mathfrak{A}_i \) is included in every variety that contains \( \mathfrak{A} \) and vice-versa.

An algebra \( \mathfrak{B} \) is subdirectly irreducible if, whenever it is isomorphic to a subdirect product of a system \( \{ \mathfrak{A}_i : i \in I \} \) of algebras, it is necessarily isomorphic to one of the components \( \mathfrak{A}_j \). Alternately, \( \mathfrak{B} \) is subdirectly irreducible if its lattice of congruence relations contains a smallest proper member, i.e., a congruence relation \( R \) different from the identity relation which includes all other congruence relations with this property.

By a well known result of Birkhoff [44] every algebra can be isomorphically represented as a subdirect product of subdirectly irreducible algebras. It follows from this result and the remark at the end of the previous paragraph that every variety is generated by its subdirectly irreducible members.

We are now ready to state the basic theorem of Jónsson mentioned earlier; see Jónsson [67] for its proof. Recall that \( \text{UpK} \) denotes the class of all algebras isomorphic to an ultraproduct of a system of algebras of \( K \).
Theorem 2.1.5. Let $K$ be any class of models of some congruence-distributive theory. Then the subdirectly irreducible members of $\text{HSP}(K)$ are already contained in $\text{HSUP}(K)$.

Corollary 2.1.6. Let $K$ and $L$ be varieties that are both included in some congruence-distributive variety. Then every subdirectly irreducible member of $\text{HSP}(K \cup L)$ is already a member of either $K$ or $L$.

Corollary 2.1.7. Let $K$ be an arbitrary set of models of some congruence-distributive theory.

(i) Every finite subdirectly irreducible member of $\text{HSP}(K)$ is already contained in $\text{HS}(K)$.

(ii) If $K$ is a finite set of finite algebras, then every subdirectly irreducible member of $\text{HSP}(K)$ is finite and hence, by (i), already included in $\text{HS}(K)$.

Both these corollaries are obvious consequences of the theorem and well known properties of ultraproducts. For example, to prove 2.1.6 we make use of the fact that every ultraproduct of a system of algebras with members in $K \cup L$ is isomorphic to the Cartesian product of an ultraproduct of the members in $K$ and an ultraproduct of the members in $L$.

The second corollary has a number of important consequences for the structure the lattice of extensions of a congruence-
distributive theory. Let \( K \) be as in the statement of 2.1.7(ii). Then \( \text{HSP}(K) \) can include only finitely many non-isomorphic subdirectly irreducible algebras. Since every subvariety of \( \text{HSP}(L) \) is generated by its subdirectly irreducible members, we have that \( \text{HSP}(L) \) can have only a finite number of sub-varieties. Dually, if \( \Theta \) is a congruence-distributive theory of a finite set of finitely subdirectly irreducible algebras, then \( \mathbb{Z}\Theta[\Theta] \) is necessarily a finite lattice. It also follows from 2.1.7(i) that, whenever \( \mathfrak{A}, \mathfrak{B} \) are finite subdirectly irreducible algebras such that \( \text{HSP}(\mathfrak{A}) = \text{HSP}(\mathfrak{B}) \) and this theory is congruence-distributive, we must have \( \text{HS}(\mathfrak{A}) = \text{HS}(\mathfrak{B}) \); but this last equality can hold only in case \( \mathfrak{A} = \mathfrak{B} \). Stated in terms of theories this says that two finite subdirectly irreducible algebras with congruence-distributive theories can have identical theories iff they are isomorphic.

Some of these consequences of Jonsson's theorem can be put in sharper focus using the notion of an irredundant set of algebras. In analogy to an irredundant set of equations a set \( K \) of algebras is said to be \textit{irredundant} if \( \text{HSP}(L) = \text{HSP}(L') \) implies \( L = L' \) for any \( L, L' \subseteq K \). Obviously, the existence of an infinite irredundant set of models of any given theory \( \Theta \) implies that \( \Theta \) has uncountably many extensions.

By a \textit{factor} of an algebra \( \mathfrak{A} \) we shall mean any member of \( \text{HS}(\mathfrak{A}) \). Part of the content of Corollary 2.1.7(i) can now
be re-formulated in the following terms:

Let \( K \) be an arbitrary set of finite models of a congruence-distributive theory such that \( \mathfrak{A} \) is not a factor of \( \mathfrak{B} \) for any distinct \( \mathfrak{A}, \mathfrak{B} \in K \). Then \( K \) is irredundant.

In this form 2.1.7(i) is an efficient means of proving that a congruence-distributive theory \( \mathfrak{A} \) has a continuum number of extensions. For each prime number \( p \) let \( \mathfrak{B}_p \) be the finite modular lattice of all subspaces of the Desarguesian projective plane coordinatized by the Galois field of order \( p \). Baker [69] shows that \( \mathfrak{B}_p \) is subdirectly irreducible and fails to be a factor of \( \mathfrak{B}_p \) if \( q \neq p \). Thus \( \mathfrak{M} \mathfrak{A} \) and \textit{a fortiori} \( \mathfrak{A} \) have continuum many extensions. Notice however that the existence of an infinite irredundant set of lattice theories does not imply the existence of an infinite irredundant set of equations over \( \mathfrak{A} \); the existence of such a set is proved in McKenzie [70] thus demonstrating, independently of Baker [69], that there are continuum many lattice-theories. The existence of finitely-based lattice-theories \( \mathfrak{A} \) and \( \mathfrak{B} \) such that \( \mathfrak{A} \cap \mathfrak{B} \) fails to be finitely based is mentioned in Baker [71].

The existence of a continuum number of distinct group theories proved to be the most difficult problem of all and, in fact, a positive solution has only recently been obtained by Ol'šanskii [70]. Ol'šanskii's method is somewhat similar to the one for lattices outlined above in that it involves the construction
of an infinite set \( K \) of finite subdirectly irreducible groups where \( \Upsilon \) fails to be a factor of \( \Theta \) for all distinct \( \Upsilon, \Theta \in K \). However, since \( G \) is not congruence-distributive, the use 2.1.7(i) to establish the irredundance of \( K \) must be replaced by a much more sophisticated argument relying heavily on the structure theory of groups and special properties of the members of \( K \); we shall discuss this in more detail in Chapter 3.

A short time after Ol'šanskij obtained his result Vaughn-Lee [70] using a different method, constructed an infinite irredundant set of equations over \( G \).

Section 2.2. Equationally complete theories

A theory \( \Theta \) is called (equationally) complete if it is consistent and has no proper consistent simple extension. We shall see that the existence of an equationally complete extension of any given consistent theory follows from simple set-theoretical considerations. The first systematic study of the structure of lattices of theories was concerned with the number of equationally complete extensions of various theories \( \Theta \) and with their characterization.

It is clear that \( \Theta \) is equationally complete iff it is a dual atom of the lattice \( \mathcal{L} \). Let \( \Theta_\xi, \Theta_\xi \prec \alpha \), be any chain of theories ordered by inclusion, i.e., \( \Theta_\xi \subseteq \Theta_\eta \) whenever \( \xi \leq \eta < \alpha \).
Then from the completeness theorem, 1.3.1, we conclude that

\[ V_{\xi}^{< \alpha \theta_\xi} = \bigcup_{\xi} V_{\xi}^{< \alpha \theta_\xi}. \]

Thus \( V_{\xi}^{< \alpha \theta_\xi} \) is consistent if each \( \theta_\xi \) is consistent since the inconsistent theory \( Eq \) is finitely based. We can now apply Zorn's lemma to conclude that every consistent theory is included in at least one equationally complete theory.

In first-order logic complete theories coincide with the theories of individual relational structures. This of course is not the case in equational logic where every theory is, as we have seen, the theory of an individual algebra. However it is easy to see that \( \theta \) is equationally complete iff it coincides with the theory of each of its non-trivial models. The following theorem is an immediate consequence of this fact.

**Theorem 2.2.1.** Let \( \theta \) be a consistent theory. Assume there exists an algebra \( M \) such that for every non-trivial \( \mathcal{B} \in M_0 \theta \) we have

\[ M \in HSP\mathcal{B} \quad \text{and} \quad \mathcal{B} \in HSPM, \]

then \( \theta \) is equationally complete.

This theorem proves useful in demonstrating equational completeness in various cases.
For every type $I$ with at least one operation symbol of positive rank there are at least two complete theories: the theory of all variable-uniform equations and the theory 

$\{\sigma \approx \tau; \sigma, \tau \in T_{\text{eq}} \sim \text{Va}\}$ which is called the theory of all constant algebras (of type $I$). The latter theory is obviously complete.

Let $\theta$ be the former theory. Consider any $\xi \in \text{Eq}_I \sim \theta$. One side of $\xi$, say $\xi_l$, contains a variable $x$ not contained in the other side. Choose $y \in \text{Va} \sim \{x\}$ and let $\varphi x = x$ and $\varphi z = y$ for all $z \in \text{Va} \sim \{x\}$; also let $\varphi' y = y$ and $\varphi' z = x$ for all $z \in \text{Va} \sim \{y\}$. If $\xi_r$ contains at least one variable, then

$$y = \text{sup}_\theta \varphi r \{\xi\} \text{ sup}_\theta \varphi \xi \sim \text{sup}_\theta \varphi' \xi \sim \text{sup}_\theta \varphi \xi \sim \text{sup}_\theta y;$$

on the other hand, if $\xi_r$ fails to contain a variable then

$$x = \text{sup}_\theta \varphi r \{\xi\} \text{ sup}_\theta \varphi \xi \sim \text{sup}_\theta \varphi' \xi \sim \text{sup}_\theta \varphi \xi \sim \text{sup}_\theta y.$$

in either case we get $x \equiv y \in \theta[\Gamma \cup \{\xi\}]$ is $\theta$ is complete.

In Theorem 2.3.1 of the next chapter we shall see that if $I$ is a unary type then these are the only complete theories. For binary and bi-unary types the situation is radically different as the next theorem shows.

Equationally complete theories were first studied in a systematic way in Kalicki [55] and Kalicki-Scott [55]. The central result of the first paper is the following.

**Theorem 2.2.2.** For each binary type $K$ there are $2^w$ equationally complete theories of type $I$.

An arbitrary class $K$ of algebras is called (equationally)
complete if \( \text{Th}K \) has this property. Thus a variety is equationally complete if and only if it contains at least one non-trivial algebra but does not have any proper subvariety with this property, i.e., if and only if it is an atom of the lattice \( \mathfrak{A} \).

Kalicki's approach to the proof of 2.2.2 is to construct \( 2^\omega \) mutually disjoint varieties of type I. Then any two different varieties must include distinct equationally complete subvarieties and this gives rise to \( 2^\omega \) distinct equationally complete theories. Since it is clear there can be no more, 2.2.2 is established.

To fix ideas we assume that I is a type of groupoids; to obtain a proof for the general case only minor modifications of the following argument are required. Let \( Q \) be a binary operation symbol and consider the infinite sequence of terms \( \tau_k, 1 \leq k < \omega \), such that \( \tau_1 = Qxx \) and \( \tau_k = Q \tau_{k-1} \tau_k \) for each \( k, 1 \leq k < \omega \); let \( \tau_k(y) \) denote the result of substituting \( y \) for \( x \) in \( \tau_k \). For any \( \Delta \subseteq \omega \sim \{0,1\} \) let \( B_\Delta \) be the set consisting of the following equations:

(1) \( Q\tau_1(x)x \approx \tau_1(y)y \)
(2) \( Q\tau_k(x)x \approx Q\tau_{k-1}(x)x \) for every \( k \in \Delta \)
(3) \( Q\tau_\lambda(x)x \approx x \) for every \( \lambda \in \omega \sim (\Delta \cup \{0,1\}) \).

If \( \Delta \neq \Delta' \), then \( B_\Delta \) and \( B_{\Delta'} \) have no common non-trivial models, i.e.,

\[ \delta[B_\Delta \cup B_{\Delta'}] = \text{Eq.} \]

For if \( \lambda \in \Delta \sim \Delta' \), say, then \( Q\tau_\lambda(x)x \approx B_\Delta Q\tau_1(x)x \) and \( Q\tau_\lambda(x)x \approx B_{\Delta'} x \).
thus

\[ Q T_1(x) \Delta = B_\Delta \cup B_{\Delta'}, \]

and by (1) this implies \((x \approx y) \in \Theta[B_\Delta \cup B_{\Delta'}].\) It now only remains to show that \(B_\Delta\) is consistent for all \(\Delta\). Kalicki demonstrates this by constructing a non-trivial model of \(B_\Delta\).

For each \(\Delta \subseteq \omega \sim \{0,1\}\) let

\[ \mathcal{U}_\Delta = \langle \omega, Q(\Delta) \rangle \]

where \(Q(\Delta)\) is defined in the following way: for all \(x < \omega\)

\[ Q(\Delta) (x+1, x) = 0 \]
\[ Q(\Delta) (x+\lambda, x) = 0 \quad \text{for all } \lambda \in \Delta \]
\[ Q(\Delta) (x+\mu, x) = \kappa \quad \text{for all } \mu \in \omega \sim (\Delta \cup \{0,1\}) \]
\[ Q(\Delta) (x, x) = x + 1. \]

It is easy to see that \(\mathcal{U}_\Delta\) is a model of \(B_\Delta\).

The construction of the model \(\mathcal{U}_\Delta\) of \(B_\Delta\) is a simple and natural way of showing that \(B_\Delta\) is consistent but it should be pointed out that the purely combinatorial methods discussed in the last section in connection with Theorem 2.1.1 can also be used for this purpose as we now show.

For the purposes of the present remarks only we shall call a term \(\sigma\) symmetric if \(\sigma = Q\rho\rho\) for some \(\rho \in \mathcal{T}_\omega\). Obviously any substitution instance of a symmetric term is again symmetric.
Notice that each \( \kappa, 2 \leq \kappa < \omega \), and all of its non-variable subterms are symmetric. Consequently,

\[ QT_\kappa(x)x \] and all of its substitution instances fail to be symmetric while any substitution instance of a proper non-variable subterm of \( QT_\kappa(x)x \) is symmetric.

Set

\[ w \sim (\Delta \cup \{0, 1\}) = \sim \Delta \] and let

\[ \theta = \theta[[QT_\lambda(x)x \sim x : \lambda \sim \Delta]] \]

and consider any \( \sigma \) such that \( x \sim_\theta \sigma \). Using 2.1.1 it is an easy matter to prove that for any \( \rho \in T_e \), if \( \rho \not\sim \sigma \), then \( \rho \) must be either a variable or a substitution instance of a non-variable subterm of one of the \( QT_\lambda(x)x \) with \( \lambda \sim \Delta \). Hence in view of (4) with \( x = \lambda \), if \( \rho \) fails to be symmetric, then it is either a variable or a substitution instance of one of the \( QT_\lambda(x)x \) with \( \lambda \sim \Delta \). However, again with the aid of (4) we see that neither side of any of the equations (1) or (2) fits this description. Thus we have proved that, if \( \Gamma \) is the set consisting of all the equations (1) and (2), then \( \xi _1, \xi _r \sim_\theta \xi \) for every \( \xi \in \Gamma \). We now apply 2.1.3 to conclude that

\[ (x \sim y) \not\sim_\theta \xi \sim \xi _1, \xi _r \sim_\theta \xi, \xi _1 (\gamma) = \xi \sim \xi _1 (\gamma) \]
in other words, that $B_\Delta$ is consistent.

The two different proofs of the consistency of $B_\Delta$ outlined above illustrate rather well the basic differences between model-theoretical and proof-theoretical methods in equational logic. The former are usually simpler and more direct while the latter often give a deeper insight into the nature of the problem at hand and consequently lend themselves more readily to generalization. Thus in Chapter 4 we shall see how an analysis of the nature of a formal proof in equational logic, like that which led Theorems 2.1.2 and 2.1.3 but deeper, leads also to solutions, mostly negative, of a large variety of decision problems which do not seem susceptible to a model-theoretical attack; compare the remarks following the proof of Theorem 2.1.4. Similar remarks apply to the discussion in Section 2.4.

It is not difficult to see that the definition of the model $\mathcal{M}_\Delta$ of $B_\Delta$ constructed in the proof of 2.2.2 can be modified so as to make $\mathcal{M}_\Delta$ commutative while remaining a model of $B_\Delta$. Alternatively, it can be checked that the proof of (5) outlined above remains valid with only minor changes when $B_\Delta$ is extended to include the commutative law. Thus, as was shown in Kalicki [55], there are $2^w$ equationally complete theories of commutative groupoids.

The number of equationally complete theories of type $I$ have been determined for all $I$. If $I$ contains a single unary
operation symbol, $F$, then from the complete description of the lattice $\mathfrak{L}_I$ given in Theorem 2.3.1 of the next section we shall see that it contains exactly two dual atoms:

$$\sim[Fx \approx x] \quad \text{and} \quad \sim[Fx \approx Fy];$$

actually it is not hard to see that every unary type has exactly two complete theories. If $I$ is either binary or bi-unary with $\alpha$ operation symbols of positive rank, then $\mathfrak{L}_I$ has $2^\alpha$ dual atoms if $\alpha \geq \omega$ and $\omega$ otherwise. The result for unary types is due to Kalicki-Scott [55] and for arbitrary types to Burris [71a] and Ježek [70]; for binary types the result was also obtained by Fred Backer and Richard Thompson independently of Burris and Ježek.

Using the same basic approach as Kalicki [55] Evans [71] has been able to show that there is an uncountable number of equationally complete quasigroup and loop theories. A quasigroup is usually defined to be a groupoid $(G, \cdot)$ in which for all $a, b \in G$ each of the equations

$$(5) \quad x \cdot a = b \quad \text{and} \quad a \cdot y = b$$

have a unique solution (while the associative law is not required to hold). In this conception the class of quasigroups do not form a variety but they do if the type is extended to include binary operation symbols to denote the unique solutions.
of the equations (5). We now make this definition formal.

Let \( I = \{ Q, D_L, D_R \} \) where \( Q, D_L, \) and \( D_R \) are all binary operation symbols. By the standard theory of all quasigroups we mean the theory \( QG \) of type \( I \) generated by the following equations:

\[
\begin{align*}
QD_R xyy & \approx x \\
D_R xyy & \approx x \\
QyD_L xy & \approx x \\
D_L yxy & \approx x
\end{align*}
\]

Thus for any quasigroup \( G = (G, Q^{(G)}, D_R^{(G)}, D_L^{(G)}) \) and any \( a, b \in G, \)
\( D_R^{(G)}(a,b) \) and \( D_L^{(G)}(a,b) \) are the results of dividing \( a \) by \( b \) on the right and on the left, respectively. The standard theory of all loops, in symbols \( LQ \), is the simple extension of \( QG \) obtained by adjoining the single equation:

\[ D_R xx \approx D_L yy \]

to the 4-equation base for \( QG \) given above. It can be shown with little difficulty that the theories of all associative quasigroups and all associative loops are definitionally equivalent to \( G \) by the same definition: \( \rho Q = v_0 \cdot v_1, \rho D_R = v_0 \cdot v_1^{-1}, \rho D_L = v_1^{-1} \cdot v_0. \)

Recall from Section 2.1 that a theory of loops or quasigroups is said to be anti-finite if it has no non-trivial finite model.

Theorem 2.2.3. \( QG \) and \( LQ \) both have \( 2^w \) equationally
complete extensions; in fact, both have \(2^\omega\) anti-finite equationally complete extensions.

We shall prove this theorem only for \(\mathcal{Q}_0\); the argument need only be slightly modified to obtain the result for \(\mathcal{Q}_1\).

In a given loop \(\langle G, \cdot \rangle\) there may be little relation between the left powers

\[
(3) a = a \cdot (a \cdot a), \\
(4) a = a \cdot (a \cdot (a \cdot a)), \ldots
\]

and the right powers

\[
(3) a^r = (a \cdot a) \cdot a, \\
(4) a^r = ((a \cdot a) \cdot a) \cdot a, \ldots
\]

of an element \(a\). This is the fact exploited by Evans [71] in his proof of the theorem.

For any subset \(\Delta\) of the set \(E = \{4, 6, 8, \ldots\}\) of all even positive integers \(\geq 4\) let \(C_\Delta \subseteq \mathcal{E}_G\) consist of all the equations

\[
(6) (Qx)^{x-1} x \sim Q^x x^{x+1} \quad \text{for all } x \in \Delta \\
(7) (Qx)^{x-1} x \sim Q^x x^x \quad \text{for all } x \in E \sim \Delta.
\]

As in the proof of Theorem 2.2.2 it is not hard to prove that

\[
\theta = \theta(C_\Delta \cup C_{\Delta'}) = \mathcal{E}_G \quad \text{whenever } \Delta \neq \Delta'. \quad \text{In fact, if } \Delta \neq \Delta',
\]

then \(Q^{x-1} x \sim \theta Q^x x^{x+1}\) for some \(x \in E\) and hence

\[
D^x_x x \sim \theta D^x_x Q^{x-1} x x + Q^x x^{x+1} \sim \theta x.
\]
Since $D_{RX} \approx D_{RY}$ is a loop identity we conclude that $x \approx y \in \theta$.

In order to prove that $\mathcal{C}_\Delta$ is consistent for every $\Delta \subseteq E$, Evans formulates and proves an embedding theorem for partial loops. For any non-empty set $A$ and $\kappa < \omega$, by a *partial operation* $O$ of rank $\kappa$ on $A$ we mean any function with domain a subset of $A^\kappa$ and range included in $A$. By a *partial algebra of type* $I$ we will mean any system $\mathfrak{A} = (A, \{Q^{(\xi)}\}_{\xi \in I})$ such that $A$ is a non-empty set and $Q^{(\xi)}$ is a partial operation on $A$ of the same rank as $Q$ for each $\xi \in I$. In the context of partial algebras we shall often refer to ordinary algebras as *total algebras* for emphasis. Let $\cdot$ be any partial binary operation on $\omega \sim \{0\}$ and let $\kappa, \lambda, \mu, \nu$ range over $\omega \sim \{0\}$. Consider the following conditions on $\cdot$ and $\kappa$:

\begin{align*}
(8) & \quad 1 \cdot \kappa = \kappa \cdot 1 = \kappa; \\
(9) & \quad \text{if } \lambda \neq \mu \text{ and } \kappa \cdot \lambda \text{ and } \kappa \cdot \mu \text{ are both defined, then } \kappa \cdot \lambda \neq \kappa \cdot \mu; \\
(10) & \quad \text{if } \lambda \neq \mu \text{ and } \lambda \cdot \kappa \text{ and } \mu \cdot \kappa \text{ are both defined, then } \\
& \quad \lambda \cdot \kappa \neq \mu \cdot \kappa; \\
(11) & \quad \text{if } \kappa \neq 1, \text{ then } \\
& \quad |\{\lambda: \kappa \cdot \lambda \text{ is undefined}\}| = |\{\mu: \mu \cdot \kappa \text{ is undefined}\}| = \omega; \\
(12) & \quad \text{if } \kappa \neq 1, \text{ then } \\
& \quad |\{\lambda \neq \kappa \cdot \mu \text{ for every } \mu \text{ such that } \kappa \cdot \mu \text{ is defined}\}| = \omega \\
& \quad |\{\nu: \nu \neq \nu \cdot \kappa \text{ for every } \nu \text{ such that } \nu \cdot \kappa \text{ is defined}\}| = \omega; \\
(13) & \quad |\{\langle \lambda, \mu \rangle: \lambda \cdot \mu \text{ is defined and } \lambda \cdot \mu = \kappa\}| < \omega;
\end{align*}
The following theorem is proved in Evans [71].

**Theorem 2.2.4.** Let \( \cdot \) be any partial operation on \( w \sim \{0\} \) such that conditions (8)--(14) hold for all \( \kappa \in w \sim \{0\} \).

Then there exists a loop

\[
\mu = \langle w \sim \{0\}, Q^{( WI)}, \mathcal{D}_R^{( WI)}, \mathcal{D}_L^{( WI)} \rangle
\]

such that \( Q^{( WI)}(\kappa, \lambda) = \kappa \cdot \lambda \) for all \( \kappa, \lambda \in w \sim \{0\} \) for which \( \kappa \cdot \lambda \) is defined, and \( \mathcal{D}_R^{( WI)}(\kappa, \kappa) = \mathcal{D}_L^{( WI)}(\lambda, \lambda) = 1 \) for all \( \kappa, \lambda \in w \sim \{0\} \).

To prove this theorem take any \( w \)-ordering \( \langle \kappa_0, \lambda_0 \rangle, \langle \kappa_1, \lambda_1 \rangle, \cdots \) of \( (w \sim \{0\}) \times (w \sim \{0\}) \). We define by recursion a sequence of partial operations \( \cdot_0, \cdot_1, \cdot_2, \cdots \) on \( w \sim \{0\} \) such that

\[
\cdot = \cdot_0,
\]

conditions (8)--(14) with \( \cdot = \cdot_0 \) hold for each \( \mu < w \) and \( \kappa \in w \sim \{0\} \), and the domain of \( \cdot_\mu \) is included in that of \( \cdot_\nu \) whenever \( \mu \leq \nu < w \). Furthermore, for each \( \mu < w \) we have that

\[
\mu \cdot \mu \cdot \mu \text{ is defined},
\]

there exist \( \rho, \pi \in w \sim \{0\} \) such that \( \mu \cdot \pi \) and \( \rho \cdot \mu \cdot \mu \) are both defined and equal to \( \lambda \).

It is clear how this sequence of partial operations is to be defined.

Now define \( Q^{( WI)}(\pi, \rho) \) for all \( \pi, \rho \in w \sim \{0\} \) by setting it equal
to \( \mu \cdot \lambda \), where \( (\pi, \rho) = (\mu, \lambda) \). From the fact that conditions (9) and (10) (for all \( \mu \in \omega \sim \{0\} \) and with \( \cdot = \cdot \)) and (7) and (18) hold for all \( \mu < \omega \), we conclude that operations \( D^{(W)}_R \) and \( D^{(W)}_L \) exist that make \( \omega \) in (15) a loop; in view of (16) this proves Theorem 2.2.4.

The equations (6) and (7) are one-variable equations and because of this their satisfaction in a given loop \( \omega \) depends on the value of \( Q^{(W)}(a, b) \) for a relatively limited number of pairs \( a, b \in A \). It is just for this reason that 2.2.4 is useful in constructing a non-trivial model \( \omega \Delta \) of \( C \Delta \) and Evans' construction, which we now describe, illustrates this fact very well.

Let \( p_1, p_2, p_3, \ldots \) be the sequence of all positive primes in their natural order. For a given \( \Delta \subseteq E = \{4, 6, 8, \ldots\} \) define the partial operation \( \cdot \) on \( \omega \sim \{0\} \) by the following conditions:

\[
\begin{align*}
(19) & \quad 1 \cdot \kappa = \kappa \cdot 1 = \kappa \text{ for every } \kappa \in \omega \sim \{0\}; \\
(20) & \quad \text{for every } \kappa \in \omega \sim \{0\} \text{ and } \lambda \in \omega \sim \{0, 1\} \\
& \quad \kappa(\lambda) = p_\kappa^{\lambda - 1} \\
& \quad (\lambda)_\kappa = \kappa p_\kappa^{\lambda - 1} \text{ if } \lambda \text{ is odd} \\
& \quad \quad \quad p_\kappa^\lambda \text{ if } \lambda \in \Delta \\
& \quad \quad \quad p_\kappa^{\lambda - 1} \text{ if } \lambda \in E \sim \Delta.
\end{align*}
\]

In this last expression \( \kappa(\lambda) \) and \( (\lambda)_\kappa \) represent the left- and
right-associated powers of $\kappa$ under the operation $\cdot$ while $p_\kappa^\lambda$ and $p_\kappa^{\lambda-1}$ denote powers of $p_\kappa$ under ordinary multiplication of integers. It is easily checked that $\cdot$ satisfies conditions (8) -- (14) for all $\kappa \in \omega \sim \{0\}$. Hencely by 2.2.4 there exists a loop $\mathcal{Y}_\Delta$ with universe $\omega \sim \{0\}$ such that $Q$ agrees with $\cdot$ whenever the latter is defined. $\mathcal{Y}_\Delta$ is easily seen to be a model of (6) and (7) which shows $C_\Delta$ is consistent for all $\Delta \subseteq E$ and completes the proof of the first part of 2.2.3. To prove the existence of a continuum number of anti-finite equationally complete loop theories one need only observe that there are $2^\omega$ distinct subsets $\Delta$ of $E$ which contain $\kappa$: for each $\kappa$, $2 \leq \kappa < \omega$, and that for each such $\Delta$ the variety $\mathcal{C}_\Delta$ is anti-finite; this last observation follows immediately from the easily established fact that the identity $(Qx)^{\kappa!-1}_x \approx Q^{\kappa!-1}_x \kappa!$ holds in every finite loop of order $\kappa$.

The model-theoretical demonstration of the consistency of $C_\Delta$ given above can be replaced by a purely combinatorial one similar to the alternate proof of the consistency of $B_\Delta$ outlined in the remarks following the proof of 2.2.2. In this case however the argument is much more complicated; in particular, as was pointed out in Section 2.2, Theorem 2.1.2 can be of no direct use in proving that $x \approx y$ fails to be a consequence of the equations (6) and (7) since, for instance, $x \approx D_R Qxy$ is a loop identity
and hence \( \tau \preceq \sigma \) for all \( \tau, \sigma \in T_e \). In Chapter 4 we shall present certain results that are established by purely combinatorial methods and from which Theorem 2.2.3 follows as an easy corollary.

A result stronger than the first part of 2.2.3 was obtained earlier by Bol'bot [67]. The theory of totally symmetric quasigroups is defined relative to \( \mathbb{QG} \) by the equations

\[
D_R xy = Qxy \quad \text{and} \quad D_L xy = Qyx;
\]

this theory definitionally equivalent to the theory of groupoids defined by the two equations

\[
x \cdot (x \cdot y) = y, \quad (x \cdot y) \cdot y = x.
\]

Bol'bot [67] shows that there are even a continuum number of equationally complete theories of totally symmetric quasigroups.

Evans [71] defines a theory of loops to be anti-associative if it has no non-trivial group for a model. Since every consistent variety of groups contains a finite non-trivial group we see that every anti-finite loop theory is also anti-associative. On the other hand, Evans [71] shows that equationally complete anti-associative loop theories exist which fail to be anti-finite. Since every such theory is the theory of a finite loop there can be at most denumerably many of them; if is still an open question, raised in Evans [71], whether or not there are infinitely many such theories.
Equationally complete associative loop theories coincide with equationally complete group theories and will be described below.

Embedding theorems for partial algebras, like Theorem 2.2.4, play an important role in the theory of varieties of non-associative systems. Mal'cev [66] used an embedding result quite similar to that of 2.24 to construct a quasigroup (considered as a groupoid) with an finitely-based but undecidable equational theory; cf. Chapter 4. More recently, Bol'bot [70], [72] has obtained some interesting results which show that for certain theories $\mathfrak{T}$ there is a very close connection between partial algebras and extensions of $\mathfrak{T}$. In order to present these results conveniently we introduce some terminology.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras of type $I$. $\mathfrak{B}$ is said to be an extension of $\mathfrak{A}$ if $A = B$ and $Q^{(\mathfrak{A})} \subseteq Q^{(\mathfrak{B})}$ for every $Q \in I$, i.e., $\mathfrak{A}$ and $\mathfrak{B}$ have the same universe and $Q^{(\mathfrak{B})}(a_0, \ldots, a_{n-1})$ is defined and equals $Q^{(\mathfrak{A})}(a_0, \ldots, a_{n-1})$ whenever the latter is defined. $\mathfrak{B}$ is a total extension of $\mathfrak{A}$ if it is a total algebra. The relation that holds between partial algebras $\mathfrak{A}$ and $\mathfrak{B}$ when $\mathfrak{B}$ is an extension of $\mathfrak{A}$ is easily seen to be a complete lower semi-lattice ordering but is, in general, not even an ordinary lattice ordering since it is a trivial matter to construct two partial algebras with no common extension. If we adjoin an arbitrary element, which we denote
by \( l \), to the set of extensions of \( U \) and arbitrary specify that \( l \) is an extension of all extensions of \( U \), then the extension-ordering does become a complete lattice ordering. The corresponding lattice will be called the lattice of extensions of \( U \). The total algebras constitute the dual atoms of this lattice.

If \( U \) is a partial algebra, then by the theory of \( U \), in symbols \( \text{Th}_U \), we shall understand the theory of the set of all possible total extensions of \( U \).

The entire discussion of polynomial operations which appears at the beginning of Section 1.6, including the terminology and notation introduced there, can be carried over to partial algebras virtually without change of course, in the present situation polynomial operations are only partial operations.

We will be particularly interested in those partial algebras \( U \) of arbitrary binary type which satisfy the following two conditions:

(21) For each \( a \in A \) there exists a polynomial operation \( P_a \) such that \( P_a(x) = a \) for all \( x \in A \).

(22) There exists a \( \bar{A} \subseteq A \times A \) such that, for all \( a, b \in A \) such that \( a \neq b \), either \( (a, b) \in \bar{A} \) or \( (b, a) \in \bar{A} \), and for every \( (a, b) \in \bar{A} \) there is a binary polynomial operation \( P_{a,b} \) such that

\[
P_{a,b}(a,x) = P_{a,b}(a,y) \quad \text{for all} \quad x, y \in A
\]

while

\[
P_{a,b}(b,x) = x \quad \text{for all} \quad x \in A.
\]
The following theorem generalizes a result implicit in Bol'bot [70].

**Theorem 2.2.5.** Let \( \mathfrak{A} \) be any partial algebra of binary type satisfying conditions (21) and (22), and let \( \mathfrak{M} \) be an extension of \( \mathfrak{A} \).

(i) If \( \mathfrak{M}' \) is any other extension of \( \mathfrak{A} \), then \( \mathfrak{M}' \) is an extension of \( \mathfrak{M} \) iff \( \text{Th} \mathfrak{M} \subseteq \text{Th} \mathfrak{M}' \); thus, in particular, \( \mathfrak{M} = \mathfrak{M}' \) iff \( \text{Th} \mathfrak{M} = \text{Th} \mathfrak{M}' \).

(ii) \( \text{Th} \mathfrak{M} \) is equationally complete iff \( \mathfrak{M} \) is a total extension of \( \mathfrak{A} \).

Let the polynomials functions \( P_a \) and \( P_{ab} \) be represented by the terms \( \tau_a \) and \( \sigma_{ab} \), respectively. From (21) it is clear that for all \( Q \in I \) and \( a_0, \ldots, a_{k-1}, b \in A \),

\[
Q^{\mathfrak{M}}(a_0, \ldots, a_{k-1}) = b \iff (Q \tau_{a_0}(x) \cdots \tau_{a_{k-1}}(x) = \tau_b(x)) \in \text{Th} \mathfrak{M}
\]

and that the same equivalence holds when \( \mathfrak{M} \) is replaced by \( \mathfrak{M}' \). Thus we have that \( \text{Th} \mathfrak{M} \subseteq \text{Th} \mathfrak{M}' \) implies \( \mathfrak{M}' \) is an extension of \( \mathfrak{M} \); the implication in the opposite direction follows from the fact that if \( \mathfrak{M}' \) is an extension of \( \mathfrak{M} \) then the set of total extensions of \( \mathfrak{M}' \) is included in the set of total extensions of \( \mathfrak{M} \). Thus (i) holds.

To prove (ii) consider any non-trivial \( \mathfrak{G} \in \text{HSP} \mathfrak{A} \). For each \( a \in A \) let \( h_a \) be the unique element in the range of the constant polynomial operation over \( \mathfrak{G} \) that is represented by the term
τₐ. In view of (23) and the fact that $\mathcal{G}$ is a model of $\text{Th}_\mathfrak{B}$ we conclude immediately that $h$ is a homomorphism from $\mathfrak{B}$ into $\mathcal{G}$. Suppose $a,b \in A$ (which is also the universe of $B$) such that $a \neq b$ but $ha = hb$. Then we have

$$\tau_a \approx \tau_b \in \text{Th}_\mathcal{G}. \tag{24}$$

Without loss of generality we may assume $\langle a, b \rangle \in \mathcal{A}$. Then by (22) the equations $\sigma_{a,b}(\tau_a, x) \approx \sigma_{a,b}(\tau_a, y)$ and $\sigma_{a,b}(\tau_b, y) \approx y$ both hold in $\text{Th}_\mathfrak{B}$ and hence also in $\text{Th}_\mathcal{G}$ which together with (24) imply that $x \approx y \in \text{Th}_\mathcal{G}$. Thus $\mathcal{G}$ is trivial which is contrary to assumption. Thus we have shown that $h$ is an isomorphism from $\mathfrak{B}$ into $\mathcal{G}$ and hence that $\mathfrak{B} \in \text{HSP}_\mathcal{G}$ for every non-trivial $\mathcal{G} \in \text{HSP}_\mathfrak{B}$. We now conclude from Theorem 2.2.1 that $\text{Th}_\mathfrak{B}$ is equationally complete; and by an analogous argument one can show that every equationally complete extension of $\text{Th}_\mathfrak{B}$ can be obtained in this way and so (ii) is established.

Bol'bot [70] uses this theorem to obtain the following generalization of Theorem 2.2.2; a somewhat weaker version of the theorem was independently obtained by Ježek [70].

**Theorem 2.2.6.** Let $I$ be any binary type and $\mathcal{K}$ any finite set of non-trivial theories of type $I$. Then there are $2^{|\mathcal{K}|}$ equationally complete theories $\Theta$ which fail to be extensions of any theory in $\mathcal{K}$.
To fix ideas we assume that $I$ is a groupoid type consisting of the single binary operation $Q$. For each $t \in I$ choose a non-trivial $e \in I$ and let $E$ be the set of all of them. Let $\Gamma$ be the set of all terms which occur as a subterm of either side of at least one member of $\Gamma$. Choose any one-one function $\kappa: \text{Te}_{I} \rightarrow \mathbb{w} \sim [0,1]$ which satisfies the following two conditions:

(25) $\kappa$ maps $\Gamma$ onto an initial segment of $\mathbb{w} \sim [0,1]$ under the natural ordering of integers;

(26) whenever $\tau$ and $Q\tau\tau$ are both contained in $\Gamma$ where $\tau$ is any term, then $\kappa_{Q\tau\tau} = \kappa_{\tau} + 1$.

We now define a partial binary operation $\cdot$ on $\mathbb{w}$ by stating a series of conditions it is required to satisfy.

(27) whenever $Q\tau\sigma \in \Gamma$ then $\kappa_{\tau} \cdot \kappa_{\sigma}$ is defined and is equal to $\kappa_{Q\tau\sigma}$;

(28) $\lambda \cdot \lambda = \lambda + 1$ for each $\lambda \in \mathbb{w} \sim [0,1]$;

(29) $\lambda \cdot (\lambda + |\Gamma| + 2) = \begin{cases} \lambda - 1 & \text{if } 0 < \lambda < \mathbb{w} \\ 0 & \text{if } \lambda = 0; \end{cases}$

(30) $\lambda \cdot (\lambda + |\Gamma| + 3) = 0$ for every $\lambda < \mathbb{w}$

(31) $0 \cdot \lambda = 0$ for every $\lambda < \mathbb{w}$

(32) $1 \cdot (\lambda + |\Gamma| + 4) = \lambda$ for every $\lambda < \mathbb{w}$

It is easy to see that conditions (27)--(32) taken together are.
consistent so that the partial operation actually exists.
For instance, condition (26) assures the consistency of (27) and (28).

Let \( \mathcal{U} \) be the partial algebra \((\mathcal{U}, \cdot)\). We will show that \( \mathcal{U} \) satisfies conditions (21) and (22). The sequence \( \tau_\lambda, \lambda < \omega \), is defined by recursion by the conditions
\[
\tau_0 = x \quad \text{and} \quad \tau_{\lambda+1} = Q \tau_\lambda \tau_\lambda \quad \text{for each} \quad \lambda < \omega.
\]

For each \( \lambda < \omega \) let
\[
\rho_\lambda (x) = \tau_\lambda (Qx \tau |\Gamma|+3 (x)).
\]

The sequence \( \sigma_\lambda, \lambda \in \omega \sim \{0\}, \) is defined by recursion
\[
\sigma_1 (x) = Qx \tau |\Gamma|+2 (x), \quad \sigma_{\lambda+1} (x) = Q \sigma_\lambda (x) \tau |\Gamma|+2 (x).
\]

Finally for each \( \mu, 0 < \mu < \omega \), let
\[
\pi_\mu (x, y) = Q \sigma_{\mu-1} (x) \tau |\Gamma|+4 (y).
\]

For each \( \lambda \in \omega \) let \( \rho_\lambda \) be the partial polynomial operation over \( \mathcal{U} \) represented by \( \rho_\lambda \). Let \( \overline{\lambda} = \{ \langle \mu, \lambda \rangle : \lambda < \mu < \omega \} \) and for each \( \langle \mu, \lambda \rangle \in \overline{\lambda} \) and let \( \rho_{\mu \lambda} \) be the polynomial operation represented by \( \pi_\mu \). It is now an easy matter to check that conditions (21) and (22) hold in \( \mathcal{U} \). Thus the conclusions of
Theorem 2.2.6 hold for \( \mathfrak{M} \) and we have that the theory of each total extension of \( \mathfrak{M} \) is complete and that distinct extensions of \( \mathfrak{M} \) give distinct theories. Since there are obviously \( 2^w \) distinct extensions we have \( 2^w \) distinct equationally complete theories. Moreover, it is easily seen that from condition (27) that none of these theories extends any theory in \( \mathfrak{X} \). This completes the proof of the theorem.

In [72] Bol'bot Theorem 2.2.6 is relativized to loops thus generalizing Theorem 2.2.3. In particular he proves that if \( \mathfrak{S} \) is any member of a certain special non-empty class of loop-theories, then 2.2.6 continues to hold when \( \mathfrak{X} \) is taken to be a finite set of non-trivial extensions of \( \mathfrak{S} \) and \( \mathfrak{G} \) is required to be an extension of \( \mathfrak{S} \). Roughly speaking Bol'bot proves this theorem by combining Theorem 2.2.4 and Lemma 2.2.5 but the actual construction is quite complicated.

In constrast to what, in view of the previous discussion, seems to be the common situation for non-associative systems there are only countably many equationally complete semigroup theories and the same is true for most of the familiar associative systems. See Section 1.6 for definition of the special semigroup and group theories referred to in the following theorem which is due to Kalicki-Scott [55].
Theorem 2.2.7. (i) There are $\omega$ equationally complete theories of semigroups. In particular they are: $\Omega_{\mu}^{\Sigma}, \Omega_{\mu}^{\Sigma_{L}}, \Gamma_{\Sigma_{L}}^\mu, A_{\Sigma_{11}}^\mu$, and $A_{\Sigma_{X}}^\mu$ for each prime $\mu < \omega$.

(ii) There are $\omega$ equationally complete theories of groups. In particular they are the theories $\Lambda_{\Sigma_{X}}^\mu$ for each prime $\mu < \omega$.

Let $\Sigma$ be the set of all semigroup theories listed part (i) of the theorem.

We first prove that each theory in $\Sigma$ is actually equationally complete. This can be done in either a model-theoretical or combinatorial way; in the former approach one finds for each theory $\theta \in \Sigma$ a semigroup $\mathcal{U}_{\theta} = \langle A_{\theta}^{\Sigma}, Q(\theta) \rangle$ satisfying the hypothesis of Theorem 2.2.1. If $\theta$ is either $\Omega_{\mu}^{\Sigma_{L}}, \Omega_{\mu}^{\Sigma_{T}}, \Gamma_{\Sigma_{L}}^\mu$, or $A_{\Sigma_{11}}^\mu$ we can take $A_{\theta}^{\Sigma}$ to be $[0,1]$ and the operation $Q(\theta)$ is given in the following table:

<table>
<thead>
<tr>
<th>$\Omega_{\mu}^{\Sigma_{L}}$</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<th>$\Omega_{\mu}^{\Sigma_{T}}$</th>
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<table>
<thead>
<tr>
<th>$\Gamma_{\Sigma_{L}}^\mu$</th>
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<tr>
<th>$A_{\Sigma_{11}}^\mu$</th>
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and, for each prime $\mu < \omega$, $\mathcal{U}_{\theta}^{\Sigma_{X}}$ is taken to be the cyclic group of order $\mu$. It is now an easy matter to check that, for each of the theories $\theta \in \Sigma$, every non-trivial model of $\theta$ has a subalgebra isomorphic to $\mathcal{U}_{\theta}$; for instance, let $\mathcal{B}$ be a model of
with at least two distinct elements, say a and b. Then either \( a \cdot b \neq a \) or \( a \cdot b \neq b \). Thus either \([a, a \cdot b]\) or \([b, a \cdot b]\) forms the universe of a subalgebra of \( \mathcal{A} \) isomorphic to \( \mathcal{A}_{\Sigma_{11}} \).

Therefore we conclude from 2.2.1 that each of the theories in \( \mathcal{K} \) is equationally complete.

Another way of showing a theory \( \mathcal{A} \) is complete is to obtain a characterization of the equations of \( \mathcal{A} \) that is so explicit that it allows you to prove that \( \mathcal{A} \) together with any equation not in \( \mathcal{A} \) generates the inconsistent theory. It turns out that this can be done for each of the theories in \( \mathcal{K} \). We have already observed in the remarks following the proof of 2.1.4 that \( \mathcal{A}_{\Sigma_{11}} \) consists of all variable-uniform equations, i.e., \( \sigma \sim \tau \in \mathcal{A}_{\Sigma_{11}} \) iff every variable occurring in \( \sigma \) also occurs in \( \tau \) and vice-versa. We also have

\[
\mathcal{L}_{\Sigma_{11}} = \{ x \sigma \sim x \tau : x \in V_a, \sigma, \tau \in T_e \}
\]

(as explained in Section 1.6, when writing terms here we delete all occurrences of the operation symbol in accordance with the usual convention when dealing with semigroup theories. Thus \( \mathcal{L}_{\Sigma_{11}} \) consists of all equations both sides of which begin with the same variable);

\[
\mathcal{L}_{\Sigma_{x}} = \{ x \sigma \sim x \tau : x \in V_a, \sigma, \tau \in T_e \};
\]

\[
\mathcal{L}_{\Sigma_{x}} = \{ \sigma \sim \tau : \sigma, \tau \in T_e - V_a \};
\]
and for each prime \( \kappa \leq \omega \), \( A_{\kappa} E \) consists of all equations \( \sigma \sim \tau \) where for each variable \( x \) the number of occurrences of \( x \) in \( \sigma \) is congruent modulo \( \kappa \) to the number of times it occurs in \( \tau \).

Using these characterizations it is a trivial matter to show that each of the theories \( A_{\kappa} E, \Omega_{\kappa} E, \Gamma_{\kappa}, \) and \( A_{\kappa}E_{11} \) cannot be properly extended to a consistent theory. It is somewhat more difficult to do this in the case of the group theories \( A_{\kappa} E \) as we now see. Let \( \kappa \) be any prime and consider any equation \( \sigma \sim \tau \) such that for some variable \( x \) the numbers \( \mu \) and \( v \) of occurrence of \( x \) in \( \sigma \) and \( \tau \), respectively, are not congruent modulo \( \kappa \). Let

\[
\Gamma = A_{\kappa} E \cup \{ \sigma \sim \tau \}.
\]

We want to show \( \Gamma \) is inconsistent. By substituting \( y^\kappa \) for \( y \) in \( \sigma \sim \tau \) for each variable \( y \neq x \) we see that \( x^\mu \sim \Gamma x^v \); clearly we can assume that \( \mu < \kappa < x \). Combining both sides of this equivalence with \( x^{\kappa-v} y \) and setting \( \pi = \mu + (\kappa-v) \) we conclude that

\[
(34) \quad x^\pi y \sim \Gamma y.
\]

Because \( 0 < \pi < \kappa \) and \( \kappa \) is prime, \( \pi \) is congruent to 1 modulo \( \kappa \). Thus from (34) we conclude that \( xy \sim \Gamma y \). This implies \( \Gamma \) is inconsistent since it contains the commutative law.
We now turn to the proof that $\mathcal{K}$ contains every equationally complete theory of semigroups. It is obvious that this will be the case if we can prove that, for every non-trivial semigroup $\mathfrak{B}$,

\begin{equation}
\forall \theta \in \text{HS}\mathfrak{B}
\end{equation}

for some $\theta \in \mathcal{K}$. Let $\mathfrak{B}$ be a non-trivial semigroup. Assume for the time being that $\mathfrak{B}$ is not idempotent. If every element of $\mathfrak{B}$ is of infinite order, then clearly (35) holds with

$\theta = \mu_\Sigma$ for every prime $\mu$. So we may assume that $\mathfrak{B}$ contains a non-idempotent element $b$ of finite order. Let $\lambda, \mu$ be the smallest pair of numbers $\rho, \nu$ such that $0 < \rho < \nu < \omega$ and $b^\rho = b^\nu$, and set $\nu = \mu - \lambda$. Then

$b^{\lambda + \nu} = b^\lambda$

and either $\lambda > 1$ or $\nu > 1$. If $\nu > 1$ then, for each prime divisor $\kappa$ of $\nu$, $\mu_\Sigma$ is a homomorphic image of the subalgebra of $\mathfrak{B}$ whose universe is $\{b^\lambda, b^{\lambda+1}, \ldots, b^{\lambda+\nu-1}\}$. We can assume therefore that $\nu = 1$. In this case, $\{b^{\lambda-1}, b^\lambda\}$ forms the universe of a subalgebra of $\mathfrak{B}$ which is isomorphic to $\mu_\Sigma$. Therefore, we can assume that $\mathfrak{B}$ is idempotent. Since $\mathfrak{B}$ is non-trivial it contains two distinct elements $a$ and $b$. Either $ab \neq a$ or $ab \neq b$. We assume the first non-equality holds; the argument is similar in the other case. If $aba \neq ab$, 
then (using the idempotency of \( \mathcal{S} \)) \( \{ab, aba\} \) forms the universe of a subalgebra isomorphic to \( \mathcal{A}_{\mathcal{S} \Sigma} \). On the other hand, if \( aba = ab \), then \( \{a, ab\} \) forms a subalgebra isomorphic to \( \mathcal{A}_{\mathcal{S} \Sigma} \).

Hence for every non-trivial semigroup \( \mathcal{S} \), (36) holds for some \( \theta \in \mathcal{K} \). This completes the proof of part (i). The proof of part (ii) is very similar but less complicated. We omit the details. This completes the proof of 2.2.7.

Recall that \( \mathcal{A}_{\mathcal{G} \Sigma} \) is definitionally equivalent to \( \mathcal{A}_{\mathcal{S} \Sigma} \) for each positive \( \kappa < \omega \). Thus in a sense every equationally complete group-theory is also an equationally complete semigroup theory.

Comparing Theorems 2.2.3 and 2.2.7 we have another, and even more striking example of the fundamental difference between the equational logic of associative and non-associative system. This difference will be constantly manifested throughout our work.

Tarski [56] has shown that there are also a denumerable number of equationally complete theories or rings. They naturally fall into two classes: (i). For each prime \( \kappa < \omega \), the theory of the additive cyclic group of order \( \kappa \) with trivial multiplication. This theory is generated relative to the theory of all commutative rings by the two equations

\[ \kappa x \approx x \text{ and } x \cdot y \approx 0. \]
(ii). For each prime \( p < \omega \) the theory of the finite field of order \( p \). This theory is generated over the theory of commutative rings by

\[ px = x \quad \text{and} \quad x^{p+1} = x. \]

In the same paper Tarski also characterizes the equationally complete theories of relation algebras. Finally, we remark that it is easy to see that the only equationally complete theory of lattices is \( \Delta \), the theory of all distributive lattices.

Section 2.3. The structure of relativized lattices of theories and the covering relation.

We will be chiefly concerned in this section with those lattices of theories \( \mathcal{H}[\theta] \) whose structure has been systematically investigated with the idea of describing it in detail. However, a complete description of \( \mathcal{H}[\theta] \) has actually been obtained in only a few cases. Most of the information we have about relativized theory-lattices is of a general character, like the nature of the sublattices embeddable as sublattices in \( \mathcal{H}[\theta] \), rather than more specific information that might lead to an intrinsic characterization of the isomorphism class of \( \mathcal{H}[\theta] \). The investigations of these general properties will be reported on the latter sections of the chapter.
The only types \( I \) for which the lattice \( \text{Th}_I \) of all theories of type \( I \) have been described in detail are unary types, and in this case the description is complete. This was first done by Jacobs-Schwabauer [64] in the case \( I \) consists of a single unary operation symbol, and their result is based on the following theorem.

**Theorem 2.3.1.** Let \( I = \{F\} \) with \( F \) unary and \( \theta \in \text{Th}_I \). Then \( \theta \) is \( 1 \)-based and if \( \theta \not\in \text{Eq}_I, T_{\alpha I} \) then it is generated by an equation in one of the two forms

\[
\begin{align*}
(i) & \quad F^{\#\#}x \approx F^x \\
(ii) & \quad F^{\#}x \approx F^y
\end{align*}
\]

where \( \# < \omega, \# > 0 \) in (i) and \( x > 0 \) in (ii). Furthermore any two distinct equations of form (i) or (ii) generate distinct theories.

Consider any consistent \( \varepsilon \in \text{Eq}_I \). If \( \varepsilon \) is variable uniform then it is clearly interderivable with one of the equations (i). If \( \varepsilon \) is not variable-uniform, then it is interderivable with an equation of the form \( F^{\#\#}x \approx F^y \) with \( 0 < x < \omega \) and \( \# < \omega \). But then by substituting \( F^y \) for \( y \) we easily derive (ii) while on the other hand \( F^{\#\#}x \approx F^y \) is obviously derivable from (ii). Thus any \( 1 \)-based consistent theory is generated by one of the equations (i) or (ii) and it is easily seen that distinct equations generate distinct theories. Thus it only remains to show that every theory is \( 1 \)-based.
Let $\Gamma$ be an arbitrary finite or infinite set of equations of the form (i) or (ii). We can assume without loss of generality that $\lambda > 0$ for all equations of type (i) contained in $\Gamma$. Let $\kappa$ be the least ordinal among all the $\kappa$ and let $\overline{\lambda}$ be the greatest common divisor of all the $\lambda$ (if $\Gamma$ contains exclusively equations of type (ii) take $\overline{\lambda} = 1$). To prove the theorem it is sufficient to prove that $\mathfrak{R}[\Gamma]$ is generated by

$$\delta = F^{\kappa + \overline{\lambda}}_x \approx F^{\kappa}_x \text{ or } \gamma = F^{\kappa + 1}_x \approx F^{\kappa}_y$$

depending on whether $\Gamma$ consists exclusively of equations of type (i) or not. It is clear we can assume $\Gamma$ is finite and hence that it has just two members $\varepsilon_0, \varepsilon_1$; for the time being assume $\varepsilon_0, \varepsilon_1$ are both of type (i) so that

$$\varepsilon_0 = (F^{\kappa_0 + \lambda_0}_x \approx F^{\kappa}_x), \quad \varepsilon_1 = (F^{\kappa_1 + \lambda_1}_x \approx F^{\kappa}_x).$$

Furthermore, we can assume without loss of generality that $\overline{\kappa} = \kappa_0$. Let $\mu_0, \mu_1 > 0$ such that $\mu_0 \lambda_0 = \mu_1 \lambda_1 + \overline{\lambda}$ and choose any $\nu$ such that $\kappa_0 + \nu \lambda_0 \geq \kappa_1$. Then

$$F^{\overline{\kappa}_x} \approx \begin{cases} [\varepsilon_0] & F^{\overline{\kappa} + \nu \lambda_0 + \mu_0 \lambda_0}_x \\ [\varepsilon_1] & F^{\overline{\kappa} + \nu \lambda_0 + \mu_1 \lambda_1 + \overline{\lambda}}_x \\ [\varepsilon_0] & F^{\overline{\kappa} + \nu \lambda_0 + \overline{\lambda}}_x \\ [\varepsilon_0] & F^{\overline{\kappa} + \lambda}_x \end{cases}.$$
Thus

\[(1) \quad \Gamma \vdash \delta;\]

on the other hand, it is obvious that each member of \( \Gamma \) is a consequence of \( \delta \), hence \( \mathcal{Q}[\Gamma] = \mathcal{Q}[\delta] \).

Assume now that at least one of the two equations of \( \Gamma \) is of type (ii), say

\[\xi_0 = (F^{x_0+1}x \approx F^{x_0}y).\]

We first observe that by substituting \( x \) for \( y \) in \( \xi_0 \) and in all the other equations of \( \Gamma \) of type (ii) we can prove

\[(2) \quad \Gamma \vdash \overline{\mathcal{Q}}^{x_0+1}x \approx \overline{\mathcal{Q}}^x\]

in exactly the same way we proved (1). Clearly \( \xi_0 \vdash F^{x_0+1}x \approx F^{x_0+1}y \)

and hence \( \xi_0 \vdash F^y x \approx F^y y \) for every \( y \geq x_0+1 \). Now choosing \( y \)

such that \( \overline{x_0} \geq x_0+1 \) we have by (2)

\[F^{\overline{x_0}+1}x \vdash \overline{\mathcal{Q}}^{\overline{x_0}+1}x \vdash \xi_0 \vdash \overline{\mathcal{Q}}^{\overline{x_0}+1}y \vdash \Gamma \overline{\mathcal{Q}}^y .\]

Thus \( \Gamma \vdash \gamma \) and it is the obvious that \( \mathcal{Q}[\Gamma] = \mathcal{Q}[\gamma] \). This completes the proof of the theorem.

It is now an easy matter to describe exactly the structure of the lattice \( \mathbb{Th}_x \). The members are of \( \mathbb{Th}_x \) are \( Eq_x \), \( Ta_x \),
\[(3)\] 
\[\theta_{k\lambda} = \theta[F_{k+1}^x \approx F_k^x]\] 

for all \((k, \lambda) \in \omega \times (\omega \setminus \{0\})\), and

\[\phi_{k\lambda} = \phi[F_{k+1}^x \approx F_k^y]\] 

for all \(k \in \omega \setminus \{0\}\). These theories are all distinct, and for all \((k, \lambda), (k', \lambda') \in \omega \times (\omega \setminus \{0\})\) and \(\mu, \mu' \in \omega \setminus \{0\}\),

\[\theta_{k\lambda} \subseteq \theta_{k'\lambda'} \iff k \geq k' \text{ and } \lambda' \text{ divides } \lambda,\]
\[\phi_{\mu} \subseteq \phi_{\mu'} \iff \mu' \geq \mu,\]
\[\phi_{\mu} \subseteq \theta_{k\lambda} \iff \mu \leq k,\]

and

\[\theta_{k\lambda} \subseteq \phi_{\mu} \text{ never holds.}\]

This completely describes the lattice ordering of \(\mathcal{H}_l\). Every theory is finitely based, in fact 1-based, and the lattice is denumerable. The only two equationally complete theories are \(\theta_0\) and \(\phi_1\): these are respectively the theories of all variable-uniform equations and of all constant algebras.

An obvious extension of the above analysis leads to a complete description of \(\mathcal{H}_l\) for any unary type \(I\). For instance, 
\[|\mathcal{H}_l|\] is always the maximum of \(\omega\) and \(|I|\) and, if \(|I| < \omega\), then every \(\theta \in \mathcal{H}_l\) is finitely based. Also, for every unary \(I\),
there exist just two equationally complete theories of type I. For all bi-unary and binary types I the structure of $\text{Th}_I$ is very complicated as the discussion of the previous two sections shows. No description of any of these lattices is available that resembles that of the lattice of unary theories. Most of the facts known about these lattices are trivial consequences of results obtained for various relativized theory-lattices about which we shall report on in this and the remaining sections of the chapter. The one important except to this rule is the work of McKenzie [71] discussed in some detail in Section 2.4 below.

It turns out that there are very few theories $\theta$ for which the lattice $\text{Th}[\theta]$ has been completely described, and in most of these cases the theory lies "high up" in the lattice $\text{Th}$ in a sense we now describe.

A theory $\hat{\theta}$ will be called semi-complete if

$$\hat{\theta} = \psi_0 \cap \psi_1 \cap \cdots \cap \psi_k$$

where $\{\psi_0, \ldots, \psi_k\}$ is any finite set of complete theories. Let $\theta$ be any theory such that $\text{Th}[\theta]$ is distributive. Suppose $\hat{\theta}, \psi_0, \ldots, \psi_k \in \text{Th}[\theta]$ where the $\psi_\lambda$ are complete and suppose that

$$\psi_0 \cap \psi_1 \cap \cdots \cap \psi_k \subseteq \hat{\theta}.$$
Then by the distributivity of $\mathfrak{Th}[\emptyset]$ we have

$$\xi = (\xi \lor \psi_0) \land (\xi \lor \psi_1) \land \cdots \land (\xi \lor \psi_\lambda).$$

Since the $\psi_\lambda$ are complete we have either $\xi \lor \psi_\lambda = \mathbb{E}q$ or $\xi \lor \psi_\lambda = \psi_\lambda$ for each $\lambda \leq \kappa$. It follows therefore that, if $\mathfrak{Th}[\emptyset]$ is distributive, then $\xi$ is semi-complete whenever $\xi$ includes a semi-complete theory. Also, a semi-complete theory can have only a finite number of complete extensions and any two semi-complete theories are identical just in case they have the same complete extensions. These observations can be summed up in the following way: the set of all semi-complete extensions of $\emptyset$ forms a dual ideal of the lattice $\mathfrak{Th}[\emptyset]$; furthermore, the function which assigns to each finite set $\kappa$ of complete extensions of $\emptyset$ the semi-complete theory $\cap \kappa$ is a dual isomorphism between the Boolean set ring of all finite subsets of the set of complete theories in $\mathfrak{Th}[\emptyset]$ and the dual ideal of all semi-complete theories of $\mathfrak{Th}[\emptyset]$. Any lattice $\mathfrak{Th}_1[\emptyset]$ of theories which satisfies the conditions just described will be called an (upper) partial Boolean ring.

Observe that, if $\mathfrak{Th}[\emptyset]$ is a partial Boolean ring and $\xi$ is any semi-complete extension of $\emptyset$, then $\mathfrak{Th}[\xi]$ is a finite Boolean algebra of cardinality $2^\kappa$ where $\kappa'$ is the number of complete extensions of $\xi$. Thus, if $\emptyset$ has an infinite number
of complete extensions, then \( \text{Th}[\Theta] \) has a principal dual ideal isomorphic to any given finite Boolean algebra.

As we have seen, if \( \text{Th}[\Theta] \) is distributive, in particular, if \( \Theta \) is congruence-distributive, then \( \text{Th}[\Theta] \) is a partial Boolean ring; hence \( \text{Th}[\Delta] \) is a partial Boolean ring but of course there is a very little content in this result since \( \text{Th}[\Delta] \) contains only one complete theory. Although the condition of distributivity cannot be eliminated it can be replaced by modularity in certain circumstances.

**Theorem 2.3.2.** Let \( \Theta \) be any theory satisfying the following condition:

(i) for every finite set \( \kappa \) of complete extensions of \( \Theta \) and for every complete extension \( \delta \) of \( \Theta \),

\[
\bigcap \kappa \subseteq \delta \iff \delta \in \kappa.
\]

Then \( \text{Th}[\Theta] \) is a partial Boolean ring when it is modular.

Assume \( \delta \in \text{Th}[\Theta] \sim \{\text{Eq}\} \) and \( \kappa \) is a finite set of complete theories in \( \text{Th}[\Theta] \). In order to prove the theorem it suffices to prove that \( \delta \) is semi-complete whenever

\[
\bigcap \kappa \subseteq \delta.
\]

Let \( \mathcal{L} = \{ \gamma : \delta \subseteq \gamma \in \kappa \} \) and \( \mathcal{L}' = \kappa \sim \mathcal{L} \). Then

\[
\bigcap \kappa \subseteq \delta \subseteq \bigcap \mathcal{L} \subseteq \text{Eq}.
\]
By definition of $\mathcal{L}$

(5) \[ \mathcal{L} \cap \mathcal{L}' \notin \mathcal{V} \]

for each $\mathcal{V} \in \mathcal{L}'$, and, by (i), (5) holds for every complete theory not included in $\mathcal{L}'$. Thus $\mathcal{L} \cap \mathcal{L}' = \mathcal{E}_q$; this equality together with (4) would imply that $\mathcal{E}_q$ includes a 5-element non-modular sublattice unless $\mathcal{L} = \bigcap \mathcal{L}$. Hence the modularity of $\mathcal{E}_q$ implies that $\mathcal{L}$ is semi-complete.

Condition 2.3.2(i) seems to hold for a variety of theories $\Theta$ including the theories $\mathcal{G}$ or groups and $\mathcal{S}$ of semigroups. In fact, $\mathcal{G}$ and $\mathcal{S}$ both satisfy a somewhat stronger condition as we now see.

**Theorem 2.3.3.** Let $\Theta$ be either $\mathcal{G}$ or $\mathcal{S}$. Then for any finite set $X \subseteq \Theta$ and any (not necessarily complete) $\mathcal{V} \in \Theta$, $\mathcal{V} \subseteq \mathcal{V}$ iff $\exists \, \mathcal{V}_0 \in \Theta$ for some $\mathcal{V}_0 \in X$.

We consider first the case $\Theta = \mathcal{S}$. Let $\mathcal{V}$ be an arbitrary complete semigroup-theory. To prove the theorem it is obviously sufficient to consider any $\mathcal{V}_0, \mathcal{V}_1 \in \Theta$ and to prove that

(6) $\mathcal{V}_0, \mathcal{V}_1 \notin \mathcal{V}$ implies $\mathcal{V}_0 \cap \mathcal{V}_1 \notin \mathcal{V}$.

This proof will depend on the simple structural characterization of the equations in $\mathcal{V}$ that was described in the proof of 2.2.7.

As in the first case take $\mathcal{V} = \bigcap_{\mathcal{V}_0, \mathcal{V}_1 \notin \mathcal{V}}$. The hypothesis of (6)
implies the existence of terms \( \sigma_0, \sigma_1, \tau_0, \tau_1 \) such that

\[
(7) \quad \sigma_0 x \approx \tau_0 y \in \mathcal{V}_0 \quad \text{and} \quad \sigma_1 x \approx \tau_1 y \in \mathcal{V}_1.
\]

By replacing these equations by substitution instances if necessary we may assume that they do not contain occurrences of any variables other than \( x \) and \( y \). Consider the equation \( \mathcal{E} \) that results from substituting \( \sigma_1 x \) for \( x \) and \( \tau_1 y \) for \( y \) in the equation

\[
(8) \quad \tau_0 y \sigma_0 x \approx \sigma_0 x \tau_0 y.
\]

Then \( \mathcal{E} \in \mathcal{E}_0 \), since it is of form \( \rho x \approx \sigma y \) for some \( \rho, \tau \in \mathcal{T}_e \). On the other hand, \( \mathcal{E} \in \mathcal{V}_0 \), since it is a substitution instance of (8) which is obviously contained in \( \mathcal{V}_0 \) in view of the first formula of (7). From the second formula of (7) we get that \( \mathcal{E} \) is a \( \mathcal{V}_1 \)-consequence of equation (8) with \( x \) substituted for \( y \); since this last equation is in \( \mathcal{E}_0 \), we have \( \mathcal{E} \in \mathcal{V}_1 \). This proves (6) when \( i = \mathcal{E}_0 \), and a symmetric argument proves (6) with \( i = \mathcal{E}_1 \). Similar arguments work in all the remaining cases also, and we omit details. We shall just briefly indicate in each case the construction of the equation \( \mathcal{E} \in (\mathcal{V}_0 \cap \mathcal{V}_1) \sim \mathcal{E} \) from given equations \( \mathcal{E}_0 \in \mathcal{V}_0 \sim \mathcal{E} \) and \( \mathcal{E}_1 \in \mathcal{V}_1 \sim \mathcal{E} \).

\[
\mathcal{E} = A \mathcal{E}_0 \mathcal{E}_1 : \text{then } \mathcal{E}_0 = (\sigma_0 (x,y) \approx \tau_0 (x)) \text{ and } \mathcal{E}_1 = (\sigma_1 (x,y) \approx \tau_1 (x)) \text{ where } \sigma_0 (x,y), \sigma_1 (x,y) \text{ contain}
\]
occurrences of $x,y$ but $\tau_0(x)$, $\tau_1(x)$ contain only occurrences of $x$. Observe that $\sigma_0(x,x) \approx \tau_0(x)$ is a consequence of $\delta_0$; thus by replacing $\tau_0(x)$ by $\sigma_0(x,x)$ in $\delta_0$ if necessary we can assume that both sides of $\delta_0$ are of the same length. In this case we can take

$$\xi_0 = (\sigma_0(\tau_1(x), \sigma_1(x,y)) \approx \tau_0(\tau_1(x))).$$

$\hat{\delta} = \sum_{\mu \in \mu}$: In this case $\delta_0 = (\sigma_0 \approx x)$ and $\delta_1 = (\sigma_1 \approx x)$ where $\sigma_0, \sigma_1$ are non-variables. By substituting $x$ for all variables occurring in $\delta_0, \delta_1$, we can assume that for some positive $\lambda_0, \lambda_1 < \omega$, $\delta_0 = (x^{\lambda_0 + 1} \approx x)$, $\delta_1 = (x^{\lambda_1 + 1} \approx x)$. Now take $\xi = (x^{\lambda_0 + 1} \approx x)$.

$\hat{\delta} = \sum_{\mu \in \mu}$ for arbitrary prime $\mu$: then $\delta_0, \delta_1$ are equations with the property that for some variable $x$ the number of occurrences of $x$ in $(\delta_0)_x$ fails to be congruent modulo $\mu$ to the number of occurrences of $x$ in $(\delta_0)_x$, and similarly for $\delta_1$. By first substituting the proper power of $x$ for $x$ in $\delta_0, \delta_1$ and then substituting $x$ for all the other variables we can assume that $\delta_0 = (x^{\kappa_0 + \lambda_0} \approx x^{\kappa_0})$ and $\delta_1 = (x^{\kappa_1 + \lambda_1} \approx x^{\kappa_1})$ where $\lambda_0, \lambda_1$ fail to be divisible by $\mu$. Take

$$\xi = (x^{\kappa_0 + \kappa_1 + \lambda_0 \lambda_1} \approx x^{\kappa_0 + \kappa_1}).$$
This completes the proof of the theorem for \( \theta = \Sigma \). The proof for \( \theta = \mathcal{G} \) is obtained from the preceding paragraph by replacing \( \Sigma_\mu \) by \( \mathcal{G}_\mu \).

Evans [71a] has obtained a related result. If \( \theta \) is any theory which is not included in any one of the four non-group complete semigroup-theories, then \( \theta \) is an extension of some theory \( \mathcal{B}_\kappa \) of Burnside groups. This implies a special case of Theorem 2.4.3: the set of semigroup-theories which fail to include any one of the non-group complete semi-group theories forms a dual ideal of \( \mathcal{H}[\Sigma] \).

It is well-known that \( \mathcal{H}[\mathcal{G}] \) is modular (see also the remarks preceding Theorem 2.3.4). Thus by 2.3.2 and 2.3.3 \( \mathcal{H}[\mathcal{G}] \) is a partial Boolean ring. This fact is however also an easy consequence of Corollary 1.7.2 and the obvious fact that the set of complete group-theories coincides with the set of \( \mathcal{A}_\kappa \) where \( \kappa \) is a square-free positive integer; the meet of this set is \( \mathcal{A}_\kappa \).

We do not know if \( \Sigma \) is a partial Boolean algebra although it seems very likely that it is. Theorem 2.3.2 cannot be applied directly since \( \mathcal{H}[\Sigma] \) is known not to be modular. However, Evans [71a] and Tamura [66] have shown that the dual ideal generated by the semi-complete theory

\[
(9) \quad \mathcal{O}_{\Sigma} \cap \mathcal{O}_{\mathcal{E}} \cap \mathcal{G} \cap \mathcal{A}_{\Sigma_{11}}
\]
formed by the intersection of the four non-group complete semigroup theories is a 16-element Boolean algebra. They have also explicitly exhibited a finite base for each of these sixteen theories. For example, the theory (9) turns out to have a base relative to consisting of the equations \(xyzw \approx xzyw\), \(xy \approx x^2y\), and \(xy \approx xy^2\). Using this result it is not hard to show that the meet of all complete semigroup theories is \(\vee \{xyzw \approx xzyw\}\).

The lattice \(\mathcal{L} \{P\}\) of ring-theories also turns out to be a partial Boolean ring; this is a consequence of a more general result we now discuss.

In analogy to motion of a congruence-distributive variety defined in Section 2.1 we call a variety \(K\) **congruence-modular** if every member of \(K\) has a modular lattice of congruence relations. A variety is called **congruence-permutable** if for every \(W \in K\) and congruence relations \(R,S\) on \(W\) we have

\[
(10) \quad R|S = S|R
\]

where the relative product \(R|S\) of an arbitrary pair \(R,S\) of congruence relations on \(W\) is defined by the formula

\[
R|S = \{(x,z): xRy \text{ and } ySz \text{ for some } y \in A\}.
\]

It follows almost immediately from the definitions of the notions involved that the join \(R \vee S\) of \(R\) and \(S\) in the
lattice of congruence relations on \( \mathfrak{M} \) is the set-theoretical union of the relations \( R|S, R|S|R, R|S|R|S, \ldots \). Hence (10) is equivalent to the condition

\[(11) \quad R \lor S = R|S.\]

Thus an arbitrary variety \( K \) is congruence-permutable iff for each \( \mathfrak{M} \in K \) the equality (11) holds for all congruence relations \( R,S \) on \( \mathfrak{M} \). Using this fact it is not difficult to prove that \( K \) is congruence-modular whenever it is congruence-permutable; cf. Birkhoff [67], p.162, Theorem 4.

An arbitrary theory \( \Theta \) is congruence-modular, or congruence-permutable, if \( M_{\Theta} \) is. Recall that \( \mathfrak{Xh}[\Theta] \) can be construed as a sublattice of the congruence lattice of \( \mathfrak{M}_{\Theta} \). As in the case of congruence-distributivity it follows immediately from this that, if \( \Theta \) is congruence-modular, then \( \mathfrak{Xh}[\Theta] \) is modular, and if \( \Theta \) is congruence-permutable, then we must have

\[(12) \quad \Phi \lor \Psi = \Phi|\Psi\]

for all \( \Phi, \Psi \in \mathfrak{Th}[\Theta] \).

**Theorem 2.3.4.** Theorem 2.3.3 continues to hold when \( \Theta \) is taken to be any congruence-permutable theory.

Suppose \( \Phi \) is any consistent extension of \( \Theta \) and \( \Psi_0, \Psi_1 \) are arbitrary members of \( \mathfrak{Th}[\Theta] \) such that \( \Psi_0, \Psi_1 \not\in \Phi \). Then taking \( \Psi \) to be \( \Psi_0 \) and \( \Psi_1 \) in (12) we get
Thus there exist terms \( \tau_0(x,y), \tau_2(x,y) \) in which only the variables \( x, y \) may occur such that

\[
x \overset{\varphi}{\sim} \tau_0(x,y) \overset{\psi_0}{=} y \quad \text{and} \quad x \overset{\varphi}{\sim} \tau_1(x,y) \overset{\psi_1}{=} y.
\]

Thus

\[
x \overset{\varphi}{\sim} \tau_0(\tau_1(x,y),y) \overset{\psi_0 \cap \psi_1}{=} \tau_0(y,y) \overset{\varphi}{=} y
\]

Therefore \( \psi_0 \cap \psi_0 \not\subseteq \varphi \). This proves the theorem.

Combining this result with 2.3.2 and using the fact that every congruence-permutable theory is congruence-modular we get

Corollary 2.3.5. If \( \Theta \) is congruence-permutable, then

\( \mathfrak{Z}[\Theta] \) is a partial Boolean ring,

It is well known that the theories of groups and of rings are congruence-permutable. Thus \( \mathfrak{Z}[\mathcal{G}] \) is a partial Boolean ring and we have a new proof that \( \mathfrak{Z}[\mathcal{G}] \) is partial-Boolean.

There is one more theory that we want to discuss whose lattice of extensions turns out to be a partial Boolean algebra. Let \( I \) be any type containing only operation symbols of positive rank. A theory \( \Theta \) of type \( I \) will be called a theory of idem-
potent algebras (of type I) if it contains the equation

$$Qx \ldots x \sim x$$

for each $Q \in I$.

**Theorem 2.3.6.** Let $I$ be any type which contains only operation symbols of positive rank. Let $\theta$ be the theory of all idempotent algebras of type I. Then $\mathcal{T}h[\theta]$ is a partial Boolean algebra.

We first prove the following lemma.

(13) Consider any $\phi, \psi \in Th_I$ such that $\phi \lor \psi = Eq_I$ then for all $\theta' \in Th[\theta]$ we have

$$\theta' = (\theta' \land \phi) \lor (\theta' \land \psi).$$

There exists a finite sequence $\tau_0(x,y), \ldots, \tau_x(x,y)$ of terms including only the variables $x$ and $y$ such that

(14) $x \prec_\phi \tau_0(x,y) \prec_\psi \tau_1(x,y) \prec_\phi \cdots \prec_\psi \tau_{x-1}(x,y) \prec_\phi \tau_x(x,y) \prec_\psi y$.

Consider any $\sigma \approx \rho \in \theta'$ and observe that for each $\lambda < x$ we have

$$\tau_\lambda(\sigma, \rho) \prec_\theta \sigma \sim_\theta \tau_{\lambda+1}(\sigma, \rho).$$

Thus by substituting $\sigma$ for $x$ and $\rho$ for $y$ in (14) we get $\sigma \approx \rho \in (\theta' \land \phi) \lor (\theta' \land \psi)$. Hence $\theta' \subseteq (\theta' \land \phi) \lor (\theta' \land \psi)$. 

and since the inclusion in the opposite direction is obvious we have (13).

It follows easily from (13) that Condition 2.3.2(i) holds. In fact let $\mathfrak{A}$ be a complete extension of $\theta$ and $\psi_0, \psi_1$ arbitrary extensions of $\theta$ such that $\psi_0, \psi_1 \not\in \mathfrak{A}$. By (13) we have $\psi_0 = (\psi_0 \cap \psi_1) \lor (\psi_0 \cap \mathfrak{A})$; thus

$$\text{Eq} = \psi_0 \lor \mathfrak{A} \subseteq (\psi_0 \cap \psi_1) \lor \mathfrak{A}$$

Therefore, $\psi_0 \cap \psi_1 \not\in \mathfrak{A}$.

We cannot now apply 2.3.2 directly to conclude that $\mathfrak{Lh}[\theta]$ is a partial Boolean algebra since as far as we know $\mathfrak{Lh}[\theta]$ is not modular. However, by examining the proof of 2.3.2 we see that the theorem remains true if the condition that $\mathfrak{Lh}[\theta]$ be modular is replaced by the weaker condition that $\mathfrak{Lh}[\theta]$ fail to contain any 5-element non-modular sublattice whose largest element is $\text{Eq}$. But from (13) it is easily seen that this latter condition is satisfied when $\theta$ is the theory of idempotent algebras; we omit details.

It is an open question whether or not Theorem 2.3.2 remains true when the condition that $\mathfrak{Lh}[\theta]$ is modular is simply omitted. Also, in view of the preceding discussion, it seems natural to ask if every lattice of theories is a partial Boolean algebra. Although this seems highly unlikely to us we do not
know of any counterexample. In particular we do not know if $L_P$ is a partial Boolean algebra.

Bol'bot [70] and Jezek [70] have independently shown that if $I$ is any binary type then the intersection of all complete theories of type $I$ is $T_I$; see Theorem 2.2.6. Moreover, Jezek [70] shows that if $I$ is, bi- unary then this intersection coincides with the set of all equations which are either tautologies of contain no occurrences of variables. In this connection see also McNulty [73].

We close our discussion of the structural characteristics of theory-lattices relating to complete theories by observing that not every theory $\emptyset$ is determined in $\mathcal{H}$ by the complete theories which include it. In fact we have already seen that for each prime $\kappa A \Sigma_\kappa$ is the only complete theory including $B \Sigma_\lambda$ for every $\lambda$ which is a power of $\kappa$. Also each of the continuum number of distinct lattice-theories in included in only one complete theory, $\Delta A$. A theory is defined to be pre-complete if it is included in a unique equationally complete theory. Thus every lattice-theory and every theory of Burnside groups of prime-power exponent is pre-complete; in the case of groups it is easy to see that this exhausts all pre-complete theories. In Subramanian-Sundararaman [71] and Sundararaman [73] the pre-complete theories of rings, modules (over a given ring), and linear algebras are characterized.
Apart from the more-or-less trivial cases where $\theta$ is semi-complete there are very few theories $\theta$ for which a detailed description of the structure of $\mathbb{X}h[\theta]$ is available. As we have seen such a description is available for the theory $\mathbb{A}G$ of Abelian groups. The lattices $\mathbb{X}h[\mathbb{N}G_2]$ and $\mathbb{X}h[\mathbb{N}G_3]$ have also completely described; see Jónsson [66] and Remeslennikov [65]. For the other group varieties defined in Section 1.7 the problem is open and appears to be formidable. Recently there has been a considerable amount of research done on this problem, particularly in connection with the theory $\mathbb{S}G_2$ of solvable groups of length 2. The results of this work will be discussed in Section 2.7 below.

We do not know of any results which give a complete characterization of any non-trivial lattice $\mathbb{X}h[\theta]$ where $\theta$ is a theory of rings. However a number of different semi-group-theories have been extensively studied.

One of these is the lattice $\mathbb{X}h[\mathbb{A}^\mathbb{E}]$ of all commutative semigroup theories. The structure of this lattice has not been completely described and in light of available information it is clear that this structure is complicated enough so that no description will ever be obtained comparable, say, to that which is available for $\mathbb{X}h[\mathbb{A}G]$ or $\mathbb{X}h[\mathbb{N}G_2]$. On the other hand, there are several pieces of evidence available to suggest that the structure of $\mathbb{X}h[\mathbb{A}^\mathbb{E}]$ is relatively simple. For example,
Perkins [69] has proved that every \( \theta \in \text{Th}[\Sigma] \) is finitely-based, cf. Chapter 3. Also, as we have previously noted, \( \Sigma \) is the intersection of the set of Abelian group-theories \( \Sigma \) for all positive \( \kappa < \omega \), and a fortiori, the set of commutative semigroup theories \( \Sigma \) for all positive \( \kappa, \lambda < \omega \). In the latter case however we can say even more. Let \( \theta \) be any proper extension of \( \Sigma \); since \( \Sigma \) is the set of all balanced equations this means that \( \theta \) contains an equation \( \tau \approx \sigma \) such that some variable \( x \) occurs more times in \( \tau \) than in \( \sigma \). By substituting \( x^\nu \) for \( x \) for large enough \( \nu \) we may suppose that \( \tau \) is of greater length than \( \sigma \). Finally by substituting \( x \) for all variables we conclude that \( \theta \) contains an equation of the form \( x^{\kappa+\lambda} \approx x^\lambda \) with \( 1 \leq \kappa, \lambda < \omega \). Thus \( \theta \supseteq \Sigma \). Therefore we have proved that

\[
\text{Th}[\Sigma] \sim [\Sigma] = \bigcup_{1 \leq \kappa, \lambda < \omega} \text{Th}[\Sigma \kappa, \lambda].
\]

The semigroup theories \( \Sigma \kappa, \lambda \) for all positive \( \kappa, \lambda < \omega \) form a sublattice of \( \text{Th}[\Sigma] \). Moreover, it is not difficult to show that the function which for each \( (\kappa, \lambda) \in (\omega \sim \{0\}) \times (\omega \sim \{0\}) \) takes the theory \( \theta \kappa, \lambda \) of unary algebras defined in (3) into \( \Sigma \kappa, \lambda \) is a lattice isomorphism. The leads to a complete description of the sublattices of \( \Sigma \kappa, \lambda \)'s. In view of this and (15) we would know a lot about the structure of \( \text{Th}[\Sigma] \) if we could describe...
The structure of the lattices (16) have been investigated by a number of different authors. The structure of (16) for \( \kappa = 1 \) and all positive \( \lambda \) has been completely described by Nelson \[ \]; for \( \kappa = 2 \) and all positive \( \lambda \) the same thing has been done by Carlisle [70]. Nelson \[ \] has also investigated the structures of the interval lattices \( Z_\kappa^{\lambda \sim_{\lambda'}}, \lambda' \) with \( \kappa \leq \kappa' \) and \( \lambda \) dividing \( \lambda' \). See also Schwabauer [69], [69a], and \[ \]. For a survey of all these results see Evans [71a]; also some of them will be described in more detail in Section 2.5.

The lattices of extensions of some closely related theories have been described: the theory of all commutative monoids by Head [68] and the theory of all semigroups with zero by Carlisle [70] and Nelson [71].

The situation changes radically when semigroup-theories which fail to satisfy the commutative law are considered. However, one result does carry over from the commutative case with only slight modifications: by the same argument that led to (15) we can show that

\[
(17) \quad Th[\Sigma] \sim Th[Ta, \Lambda \Sigma] = \bigcup_{1 \leq \kappa, \lambda \leq \omega} Th[\Lambda \Sigma, \kappa, \lambda].
\]

Thus, in analogy with the commutative case, the structure of
\( \mathcal{L} \) is intimately related to the structure of the lattices
\( \mathcal{L}[B_{\Sigma,\lambda}, \kappa] \); however in opposition to the commutative case the
structure of these lattices are very complicated even for small
\( \kappa \) and \( \lambda \). Among them only the structure of the simplest,
\( \mathcal{L}[B_{\Sigma,1}, 1] \), has been completely determined; recall that the
corresponding lattice in the commutative case, \( \mathcal{L}[A_{\Sigma,1}, 1] \), is
a trivial 2-element lattice. Also while \( \mathcal{L}[A_{\Sigma,1}, 2] \) has been
completely described as noted above, we shall see in Section 2.5
that \( \mathcal{L}[B_{\Sigma,1}, 2] \) is in a sense as complex as any lattice can be.

The complete description of the lattice \( \mathcal{L}[B_{\Sigma,1}, 1] \) of
idempotent semi-group theories was carried out independently
by Biryukov [67], [70], Fennemore [69], and Gerhard [70]. Earlier
Kimura [58] had described all theories of idempotent semigroups
generated by equations with three variables; see also Kimura [58a] and
Yamada [62]. The description is based on a detailed analysis of the very
complex relation of \( B_{\Sigma,1} \)-derivability between equations. The
argument is too complicated for us to give in detail, and we shall
only attempt to describe the main ideas. The proofs of the three
authors are basically the same but we shall follow Gerhard's
most closely.

Although much more complicated in detail the basic approach
to the problem is the same as for determining the structure of
the lattice of unary theories employed in the proof of 2.3.1.
In particular every theory of idempotent semigroups is shown to
be \( l \)-based relative to \( \Sigma_{\sim 1,1} \) and the relation of \( \Sigma_{\sim 1,1} \)-derivability between individual equations is completely described. What results is a denumerable distributive lattice whose diagram is given on an accompanying page.

At the top of the lattice is the 8-element Boolean algebra generated by the three complete theories \( \Omega_{\sim k}, \Omega_{\sim k}, \) and \( \Sigma_{\sim 1,1} \). Each of these is listed below together with the single equation which generates it relative to \( \Sigma_{\sim 1,1} \):

\[
\begin{align*}
\text{Eq} & : x \approx y \\
\Omega_{\sim k} & : xy \approx x \\
\Omega_{\sim k} & : xy \approx y \\
\Sigma_{\sim 1,1} & : xy \approx yx \\
\Omega_{\sim k} \cap \Omega_{\sim k} & : xyz \approx xz \\
\Omega_{\sim k} \cap \Sigma_{\sim 1,1} & : xyz \approx xzy \\
\Omega_{\sim k} \cap \Sigma_{\sim 1,1} & : xzy \approx zxy \\
\Omega_{\sim k} \cap \Omega_{\sim k} \cap \Sigma_{\sim 1,1} & : xyz \approx xzyx \\
\end{align*}
\]

Next come the two theories

\[
\begin{align*}
\hat{\phi}_2 & : xy \approx xyx \\
\hat{\phi}_2^* & : xy \approx yxy .
\end{align*}
\]

For each \( \kappa, \ 2 \leq \kappa < \omega \), we define a binary relation \( \sim_{\kappa} \).
Eq: $x \approx y$

- $\Omega \approx_i \hat{x}$
- $xy \approx x$
- $\Omega \approx_i \hat{y}$
- $xy \approx y$
- $\Omega \approx_i \hat{z}$
- $xyz \approx xzy$
- $\phi_2$
- $xy \approx xyx$
- $E_3$
- $xyz \approx xyxz$
- $xyz \approx xzxyz$
- $\phi_3$
- $E_4$
- $\phi_4$
- $E_4$
- $\phi_5$
between terms such that two terms $\sigma, \tau$ are in the relation just in case we have $\epsilon \in B_{\sim \ll_1} \ll_1$ whenever $\epsilon$ is obtained from the equation $\sigma \approx \tau$ by identifying variables in such a way that $\epsilon$ contains fewer than $\kappa$ distinct variables. More precisely,

$$\sigma \sim \kappa \tau \text{ iff } \sup_{\phi} (\sigma \approx \tau) \in B_{\sim \ll_1} \ll_1 \text{ for every } \phi : \forall a \rightarrow [v_0, \ldots, v_{\kappa-1}]$$

It is not difficult to see that $\sim \kappa$ is an equivalence relation on terms and that $\sigma \sim_2 \tau$ for all $\sigma, \tau \in T_e$, and

$$\sigma \sim_3 \tau \text{ iff } \sigma \approx \tau \text{ is variable-uniform and } \sigma \text{ and } \tau \text{ begin with the same variable and end with the same variable.}$$

Observe that all of the theories listed above with the exception of $\sim_1 \ll_1$ are generated (relative to $B_{\sim \ll_1} \ll_1$) by an equation $\sigma \approx \tau$ such that $\sigma \not\sim_3 \tau$ (and $\sigma \sim_2 \tau$). Tamura [66] showed that these nine theories constitute all the theories with this property. Gerhard carries on in this direction to characterize for each $\kappa$, $3 \leq \kappa < \omega$, all theories with the property that they can be generated by an equation $\sigma \approx \tau$ such that

$$(18) \quad \sigma \sim_{\kappa+1} \tau \text{ and } \sigma \sim_\kappa \tau.$$ 

It turns out that for each $\kappa \geq 3$ there are exactly eight such theories and each of them is generated (relative to $B_{\sim \ll_1} \ll_1$) by an equation which contains exactly $\kappa$ distinct variables in addition to satisfying (18); such equations are called essential in $\kappa$-variables. Let us call a theory generated by such an equation
an $E_k$-theory. The partial-ordering structures obtained by restricting for each $k$ the ordering of $\Theta(\Sigma_k)$ to the eight $E_k$-theories are all isomorphic. Furthermore, relative to this isomorphism the relation between the $E_k$-theories and the $E_{k+1}$-theories are the same for all $k \geq 3$. This is illustrated on the diagram where we have labeled the $E_k$-theories for $k = 3,4$. In addition to the $E_k$-theories there is associated with each $k \geq 3$ a pair of theories which fail to be generated by any essential equation; these positions in the lattice are denoted on the diagram by $\xi$, $\xi^*$. A generating equation for each $E_3$-theory is also indicated on the diagram. We now describe the general structural properties characterizing these equations. Let $R_3$, or $R_k^*$, be the set of all variable-uniform equations $\sigma \approx \tau$ such that the ordering of variables by first, or last, occurrence in $\sigma$ is the same as the corresponding ordering in $\tau$. Take $S_3$ to be the set of all variable-uniform $\sigma \approx \tau \in \operatorname{Eq}$ such that $\sigma_i \approx \tau_i \in \Sigma_3$ where $\sigma_i$ is the shortest initial segment of $\sigma$ which contains an occurrence of every variable occurring in $\sigma$ and $\tau_i$ is the corresponding segment of $\tau$. Finally, $S_k^*$ is the set of variable-uniform $\sigma \approx \tau$ with $\sigma_f \approx \tau_f \in \Sigma_k$ where $\sigma_f, \tau_f$ are the corresponding final segments of $\sigma, \tau$, respectively.

It can be shown that as a base for $\Theta(\Sigma_k) \cap \Theta(\Sigma_k) \cap \Sigma_k$
you can take any member of

\[ \sim R_3 \cap \sim R_3^* \cap (V \sim R_3) \cap (V \sim R_3^*) \text{ where } V \text{ is the set of all variable-uniform equations} \]

that is essential in \( \kappa \) variables; for example, \( xyzx \sim xzyx \).

The corresponding set for each of the other seven \( E_3 \)-theories is listed below together with a representative member.

\[
\begin{align*}
R_3 \cap \sim R_3^* \cap \sim S_3 & : \ xyz \approx xyxz \\
R_3^* \cap \sim R_3 \cap \sim S_3^* & : \ xyz \approx xzyz \\
R_3 \cap R_3^* = \sim S_3 \cap \sim S_3^* & : \ xyz \approx xyxz3 \\
\sim R_3^* \cap S_3 & : \ xyz \approx xzyxz3 \\
\sim R_3 \cap S_3^* & : \ xyz \approx xzxyz3 \\
R_3^* \cap S_3 \cap \sim S_3^* & : \ xyz \approx xzyxz3 \\
R_3 \cap S_3^* \cap \sim S_3 & : \ xyz \approx xyxzxyz3.
\end{align*}
\]

It is easy to check that \( S_3 \subseteq R_3 \) and \( S_3^* \subseteq R_3^* \); also it can be shown that \( \sigma \sim_4 \tau \iff \sigma \approx \tau \in S_3 \cap S_3^* \). Hence the eight sets listed above together include all equations essential in \( \kappa \) variables. This is how it is proved that there are no more than eight \( E_3 \)-theories.

The structural properties of the equations characterizing

the equations which generate the theories \( \xi_\kappa, \xi_\kappa^* \) for \( \kappa > 3 \) and

the \( E_\kappa \)-theories for \( \kappa > 3 \) are similar to those defining \( R_3, R_3^*, S_3, \) and \( S_3^* \) but become progressively more complicated as \( \kappa \).
increases. We will not describe them here and only remark that they are based on the solution of the decision problem for the theory $\mathcal{B}_{\sim_{11}}$ given by Green-Rees [52] and Biryukov [63]; this will be discussed in Chapter 4.

This completes our description of the 1-based members of $\text{Th}[\mathcal{B}_{\sim_{11}}]$. It only remains to show that every extension $\theta$ of $\mathcal{B}_{\sim_{11}}$ is 1-based. It is easy to show that $\theta$ is finitely based, for suppose it was not. If it has an irredundant base, then there would be an infinite set of pair-wise incomparable 1-based members of $\text{Th}[\mathcal{B}_{\sim_{11}}]$; on the other hand, if $\theta$ does not have an irredundant base, then it is easy to see that $\text{Th}[\mathcal{B}_{\sim_{11}}]$ must have an infinite ascending change of 1-based theories. But one sees immediately from the diagram that neither of these two conditions hold. Thus $\theta$ is finitely based; the proof that it is 1-based depends on the analysis of the $\mathcal{B}_{\sim_{11}}, 1$-derivability relation between equations described above.

We have seen from the discussion of this and the preceding section that a considerable amount of research has been done on properties of equationally complete theories, i.e., on dual atoms in the lattices $\text{Th}[\theta]$ for various theories $\theta$. More recently there has been considerable interest in the properties of dual atoms in the interval $\text{Th}[\theta, \bar{\psi}]$ where $\bar{\psi}$ is a proper extension of $\theta$; if $\psi$ is a dual atom of this lattice we say that $\psi$ is
covered by \( \hat{\theta} \) and write \( \forall \theta \). By the same argument used to show that every theory \( \theta \) has at least one complete extension we can prove that, for every theory \( \theta \) and every proper extension \( \hat{\theta} \) of \( \theta \) that is finitely based relative to it, there exists an extension of \( \theta \) covered by \( \hat{\theta} \). McKenzie [71] has shown that, if \( \theta = \text{Ta} \), then the condition that \( \hat{\theta} \) be finitely based (relative to \( \theta \)) can be dropped, i.e., that every theory different from \( \text{Ta} \) covers at least one theory. This result will be proved in the next section, and in Section 2.7 we will present some general results concerning the number of theories covered by a given theory. It turns out also that the condition that \( \hat{\theta} \) be finitely based can also be dropped in case \( \theta \) is any of the theories \( \mathcal{G}, \mathcal{P}, \) or \( \mathcal{A} \), i.e., any proper extension of the theory of all groups, all rings, or all lattices covers a theory of the same kind. For \( \theta = \mathcal{A} \) this was proved by Jonsson [67].

**Theorem 2.3.7.** Let \( \theta \) be any theory satisfying the following conditions:

(i) \( \text{Th}[\theta] \) is modular;
(ii) \( \theta = \text{Th}[\{\mathcal{U} \in \mathcal{MO}\theta, \mathcal{U} \text{ finite}\}] \);
(iii) \( \text{Th}[\text{Th}[\mathcal{U}]] \) is finite for every finite \( \mathcal{U} \in \mathcal{MO}\theta \).

Then for each \( \hat{\theta} \in \text{Th}[\theta] \sim \{\theta\} \) there exists a \( \forall \in \text{Th}[\theta] \) such that \( \forall \prec \hat{\theta} \).
Assume \( \theta \in \text{Th}[\emptyset] \sim \{ \emptyset \} \). Then by (ii) there exists a finite \( \mathfrak{U} \) such that

\[
(19) \quad \mathfrak{U} \in M\emptyset \quad \text{and} \quad \mathfrak{U} \notin M\emptyset.
\]

Consider the lattice intervals

\[
\text{Th}[\emptyset \cap \text{Th} U, \emptyset], \quad \text{Th}[\text{Th} U, \emptyset \vee \text{Th} U].
\]

These intervals are transposes and thus isomorphic by (i); cf. Birkhoff [67, p.14. By (ii) and the second formula of (19) the second interval is finite and contains at least two members. Thus the first interval contains a dual atom \( \mathfrak{v} \), i.e., \( \mathfrak{v} < \emptyset \). Finally the first interval is included in \( \text{Th}[\emptyset] \) because of the first formula of (19); in particular \( \emptyset \in \text{Th}[\emptyset] \).

**Theorem 2.3.8.** Assume \( \theta \) is either \( G, P, \) or \( \Lambda \). Then the conclusion of Theorem 2.3.7 holds.

\( \text{Th}[\Lambda] \) is congruence-distributive and \( \text{Th}[G] \) and \( \text{Th}[P] \) are both congruence-modular. Thus 2.3.7(i) holds in all three cases. That 2.3.7(ii) holds in all three cases is well known; for instance, see Magnus-Karrass-Solitar [66, p.116, Problem 24C) in the case of groups.

We now consider 2.3.7(iii). Let \( \mathfrak{U} \) be a finite model of \( \emptyset \). If \( \emptyset = \Lambda \), then by 2.1.7(ii) \( \text{HSP}\mathfrak{U} \) contains only finitely many
subdirectly irreducible lattices and, since every subvariety of $HSP^\wedge$ is generated by its subdirectly irreducible members, 2.3.7(iii) is established for $\theta = \Lambda$. In case $\theta = \Sigma$ the proof is similar except that the role of a subdirectly irreducible group is played by that of a critical group. A group is called critical if it is finite and does not belong to the variety generated by it proper factors (see the remarks following 2.1.7). A variety $K$ is called locally finite if every finitely generated member of $K$ is finite. It is not difficult to show that every locally finite variety of groups is generated by its critical members; cf. H. Neumann [67], p. 149, 51.41. But Oates-Powell [64] have shown that every variety generated by a single finite group can contain only a finite number of critical groups; see also H. Neumann [67], p. 151, 52.11. Since such a variety and every subvariety is obviously locally finite we get 2.3.7(iii) for the group case; the same result for ring follows by an analogous argument; see Kruse [73], [ ]. This gives the theorem.

In Chapter 4 we will treat some of the group and ring-theoretical results quoted about in more detail.

It is a open problem, first raised by Evans [71a], whether or not the conclusion of 2.3.7 holds when $\theta = \Sigma$. On the other hand, we do not know of any example of a consistent theory $\theta$ for which the conclusion of 2.3.7 fails to hold. Related to
this is the problem of which theories $\theta$ satisfy condition 2.3.7(iii). We have observed that as a consequence of 2.1.7(ii) it holds whenever $\theta$ is congruence-distributive; it is an open problem whether this condition on $\theta$ can be eliminated entirely or whether it can be replaced by the weaker condition that $\theta$ is congruence-modular. The only general result along these lines is the following theorem of Scott [56]:

**Theorem 2.3.9.** If $\mathcal{M}$ is a finite algebra, then $\text{Th}\mathcal{M}$ has only finitely many complete extensions.

Each complete extension of $\text{Th}\mathcal{M}$ is the theory of some $\mathfrak{B} \in \text{HSP}\mathcal{M}$ generated by two elements. But $\mathfrak{B}$ is a homomorphic image of the free algebra $\mathfrak{F}$ over $\{\mathcal{M}\}$ with two free generators. It is easy to see that $\mathfrak{F}$ is isomorphic to a finite Cartesian power of $\mathcal{M}$ and hence is itself finite; cf. the remarks at the end of Section 1.3. Since $\mathfrak{F}$ can have only finitely many non-isomorphic homomorphic images, $\text{Th}\mathcal{M}$ can have only finitely many complete extensions.

It is obvious how the above argument can be generalized to show that, if $\mathcal{M}$ is finite and $\theta$ is an extension of $\text{Th}\mathcal{M}$ such that $\theta$ is the theory of those of its models which are generated by $\leq \kappa$ elements for some fixed $\kappa < \omega$, then $\theta$ can cover at most finitely many extensions of $\text{Th}\mathcal{M}$; it is open whether
not every extension of \( \theta \) has this property. This would of course be the case if 2.3.7(iii) held for every theory \( \theta \).

As mentioned in Section 2.1 the result of Jónsson [67] given here as Theorem 2.1.5 has given strong impetus to the study of the structure of the lattice of lattice-theories. One consequence of it we have already seen: every lattice-theory different from \( \mathcal{A} \) covers another lattice-theory. This suggests that one might approach the study of \( \mathcal{A} \) starting at the top of the lattice and working down by determining at each level the theories covered by the theories at that level, and in this way passing to the next level. One is encouraged in this approach by the fact that in opposition to the case for most other familiar theories, of groups, rings, and semigroups for example, each of the first three levels is already known to include only finite many theories (Eq is counted as the first level).

As we have previously observed the theory \( \mathcal{D}_\mathcal{A} \) of distributive lattices is the only theory covered by \( \mathcal{E} \). \( \mathcal{D}_\mathcal{A} \) in turn covers just two theories, the theory \( \mathcal{M}_3 \) of the non-distributive modular lattice \( \mathcal{M}_3 \) of height 2 and width 3 which is often referred to in the literature as the diamond, and the theory \( \mathcal{N}_5 \) of the 5-element non-modular lattice \( \mathcal{N}_5 \).
This is an immediate consequence of the well known result that every non-distributive lattice includes either $M_3$ or $M_5$ as a sublattice; see Birkhoff [67], p. 13, Theorem 12 and p. 39, Theorem 13.

Because of the distributivity of $\mathfrak{A}$ both $M_3$ and $N_5$ are covered by $M_5 \cap N_5$. Also, since $M_5$ is a sublattice of every non-modular lattice, $M \cap N_5$ is covered by $M_\mathfrak{A}$ where $M_\mathfrak{A}$ is the theory of all modular lattices. For the same reason every theory covered by $M_3$ that is different from $M_3 \cap N_5$ is a theory of modular lattices, i.e., is a dual atom of the interval $\mathfrak{A}[M_\mathfrak{A}, M_3]$. Jónsson [68] showed that there are exactly two of these theories: the theory $M_4$ of the 5-element modular lattice $M_4$ of height 2 and width 4 and the theory $M_{3,3}$ of the 8-element modular lattice $M_{3,3}$ of height 3 and width 4. This result is a consequence of the following theorem. By the height of a lattice $\mathfrak{L}$ we mean the least upper bound of the lengths of all chains in $\mathfrak{L}$; also for any terms $\tau, \sigma$ we write $\tau \leq \sigma$ as an abbreviation for the equation $\tau + \sigma = \sigma$.

**Theorem 2.3.10.** For any variety $K$ of modular lattices the following conditions are equivalent:

(i) $M_{3,3} \notin K$;

(ii) every member of $K$ is a subdirect product of sublattices of height $\leq 2$;
(iii) \text{the inclusion} \ x(y+w)(z+w) \leq y+xz+xw \ \text{holds in each member of} \ K.

The core of the proof is contained in the proof that (i) implies (ii). Let \( \mathcal{L} \) be any subdirectly irreducible modular lattice of height 3. To establish the implication it clearly suffices to prove that \( \mathbb{M}_3 \) is a homomorphic image of some sublattice \( \mathcal{L}' \) of \( \mathcal{L} \). Let \( a_0 < a_1 < a_2 < a_3 \) be any chain of length 3 in \( \mathcal{L} \). It is well known that any two (non-trivial) intervals of a subdirectly irreducible modular lattice possess subintervals that are projective. It follows easily from this that some subinterval \([x,y]\) of \([a_1,a_2]\) is projective to subintervals of both \([a_0,a_1]\) and \([a_2,a_3]\). Starting from this point Jónsson [68] is able to prove by an intricate analysis of the relation of projectivity between intervals in subdirectly irreducible modular lattices that \([x,y]\) transposes down into an upper interval of some diamond and up onto a lower interval of some other diamond, i.e., that \( \mathcal{L} \) contains \( \mathcal{L}' \) as a sublattice. Clearly \( \mathbb{M}_3 \) is a homomorphic image of \( \mathcal{L} \).

That (ii) implies (iii) is proved by simply checking that in every modular lattice of height \( \leq 2 \) the inclusion of (iii)
is identically satisfied. Finally, that (iii) implies (i) is obvious since the inclusion of (iii) fails to hold in $\mathbb{T}_{3,3}$.

This theorem says that the set $\text{Th}[\mathbb{M}_A]$ of all modular lattice-theories can be decomposed into the disjoint union of a principal ideal and a principal dual ideal of $\text{Th}[\mathbb{M}_A]$:

$$\text{Th}[\mathbb{M}_A, M_{3,3}] \text{ and } \text{Th}[\mathbb{M}_w]$$

where

$$M_w = 2_{\mathbb{M}_A} \times (y+xz+zw) \leq y+xz+xw].$$

Furthermore, 2.3.10 tells us that $M_w$ is the theory of all subdirectly irreducible modular lattices of height $\leq 2$. Now these lattices are easy to describe. They are all finite and form an infinite sequence

by 2.1.7(ii) (see in particular the remarks following) these lattices determine a strictly decreasing chain of theories

$$\mathbb{E}_1 \supset \mathbb{D}_1 \supset \mathbb{M}_3 \supset \mathbb{M}_4 \supset \mathbb{M}_5 \supset \ldots$$
where $M_w$ coincides with the intersection of the chain. Furthermore, it follows from 2.1.7(i) that these theories constitute all extensions of $M_w$ so that the lattice $\mathcal{Th}[M_w]$ is an infinite chain of type $1+w^*$. The modular lattice-theories covered by $M_3$ are now easily determined. It is easy to see that the only subdirectly irreducible members of $\mathcal{HS}M_3,3$ are $M_2$ and $M_3$; thus $M_3$ covers $M_3,3$ by 2.1.7(ii). By 2.3.10 any other theory covered by $M_3$ must be contained in $\mathcal{Th}[M_w]$ and we have already seen that $M_4$ is the only such theory.

The weaker result that $M_{3,3}$ and $M_4$ are the only theories of a single finite algebra that are covered by $M_3$ was first obtained by Grätzer [66].

Jónsson [68] gives some other consequences of 2.3.10; for instance, for each $\kappa$ with $3 < \kappa < w$, $M_\kappa$ covers precisely two members of $\mathcal{Th}[M_\kappa]$, $M_{\kappa+1}$ and $M_\kappa \cap M_{3,3}$.

The problem of which lattice theories are covered by $N_5$ has been attacked by a number of authors including, Stephen Comer, K.B. Lee, and Ralph McKenzie. So far sixteen distinct theories
covered by $N_5$ have been found and each of these theories is the theory of a finite subdirectly irreducible lattice; these lattices are described in McKenzie [72] and in Section 2.6 below. It is still an open question whether or not any other lattice-theories are covered by $N_5$; it is even open whether $N_5$ covers only finitely many lattice theories. McKenzie [72] however has obtained results that bear considerably on this problem. For example, he has shown how the problem of proving that no lattice-theory other than one of the sixteen mentioned above is covered by $N_5$ can be reduced to that of proving that two well specified equations are derivable from a certain infinite but effectively generated list of equations. This result is but one consequence of a profound investigation undertaken by McKenzie of the phenomena, which we have already observed in the case of $\mathcal{L}_h[\mathcal{M}_\Lambda]$, of a lattice of lattice-theories being split into the disjoint union of a principal ideal and a principal dual ideal. These investigations will be reported on in detail in Section 2.6 below.

As a consequence of Jónsson's Theorem, 2.1.5, the structure of the lattice of extensions of any lattice theory which is generated by its finite subdirectly indecomposable models is effectively determined as soon all these particular models are; this is the main reason Jónsson's theorem is such a powerful tool in the investigation of the structure of $\mathcal{L}_h[\mathcal{A}]$. We saw how this
result was applied in the discussion following 2.3.10 to completely describe the lattice $\mathbb{Th}[M]$. For other results along these lines see Day [28], Lakser [71], Lee [70], Monk [17], and Nemitz-Whaley [71].

We have seen that for arbitrary theories $\mathfrak{e}$ the property of an extension $\mathfrak{f}$ of $\mathfrak{e}$ that it cover another extension of $\mathfrak{e}$ has been extensively studied. The dual property of $\mathfrak{f}$ that it be covered by any other theory (necessarily an extension of $\mathfrak{e}$) has been studied very little. The only results in this area seem to be that practically all the familiar theories $\mathfrak{e}$ fail to have this dual property, i.e., that the lattice $\mathbb{Th}[\mathfrak{e}]$ is atomless. This is true in particular when $\mathfrak{e}$ is the theory of all semigroups, groups, rings, loops, and lattices, and when $\mathfrak{e} = \text{Ta}_{\mathfrak{I}}$ for any type $\mathfrak{I}$ containing at least an operation symbol of positive rank. Actually, for every one of these theories $\mathfrak{e}$ it turns out that $\mathbb{Th}[\mathfrak{e},\mathfrak{f}]$ is infinite for each proper extension $\mathfrak{f}$ of $\mathfrak{e}$. These results will be discussed in more detail in Section 2.6.

Section 2.4. Definability in the Lattices $\mathbb{Th}_{\mathfrak{I}}$.

From the discussion of the last two sections, in particular Theorem 2.2.1, it would appear that for any two binary or bi-unary types $\mathfrak{I}$ and $\mathfrak{J}$ the two lattices $\mathbb{Th}_{\mathfrak{I}}$ and $\mathbb{Th}_{\mathfrak{J}}$ are
undistinguishable on the basis of how the covering relation behaves near the top of the lattices. However, by looking at a stronger covering relation, which he calls hereditary covering, McKenzie [71] has been able to distinguish these lattices in all cases except when $I$ and $J$ differ only in a trivial sense. In the course of his investigations McKenzie also obtains some very interesting results on first-order definable sets of theories.

We now state the main result of McKenzie [71]. Recall that the two lattices $Th_I$ and $Th_J$ are elementary equivalent iff they have the same first-order theories; obviously, isomorphic lattices are elementarily equivalent.

**Theorem 2.4.1.** Let $I$ and $J$ be any two types (possibly infinite and not necessarily binary or bi-unary). Then the following three conditions are equivalent:

(i) $Th_I$ and $Th_J$ are isomorphic;

(ii) for each $\kappa < \omega$ the sets of $\kappa$-ary operation symbols in $I$ and $J$ respectively are of the same cardinality.

Furthermore, if either $I$ or $J$ is finite, then both (i) and (ii) are equivalent to

(iii) $Th_I$ and $Th_J$ are elementary equivalent.

Observe that this result is in exact opposition to the result of Hanf for the lattices of first-order theories discussed in Section 2.1.
McKenzie's method of proof is strictly combinatorial in nature, using 2.1.1 and ideas similar to those found in the proofs of 2.1.2 and 2.1.3. But his argument is much more complex than any of the combinatorial arguments we have seen so far and we will not be able to present it in its entirety. However we will give an outline of the proof and treat some parts of it in detail; in this way we hope to give the reader at least some appreciation of the ingenuity of McKenzie's method. At several points in a careful proof of 2.4.1 the argument is complicated by the fact that operations symbols of very small rank must be treated separately. In order not to obscure the main ideas we shall systematically avoid such distinctions.

Unless otherwise noted throughout the following discussion I will be assumed to be a fixed but arbitrary type.

Consistent with the remarks following the proof of 2.1.4 variable-uniform equations play an important part in the proof. For any term $\tau$ let $\Gamma(\tau)$ be the set of all variable-uniform equations of type $I$ of the form

$$su_{\varphi}\tau \approx su_{\psi}\tau$$

where $\varphi$ and $\psi$ are any two permutations of the set $Va$ of all variables; let $\Delta(\tau)$ be the set of all equations $\sigma \approx \rho$ such that either $\sigma = \rho$ or $\tau$ does not contain as a subterm any substitution instance of either $\sigma$ or $\rho$. Thus
(1) \[ \Delta(\tau) = \{ \sigma \cong \rho : \text{ or } \sigma, \rho \not\in S, \tau \}. \]

This set is a theory since it is obviously closed under substitution and replacement. Furthermore, the set

(2) \[ \theta(\tau) = \Delta(\tau) \cup \Gamma(\tau) \]

is also a theory. To see this suppose \( \Delta(\tau) \cup \Gamma(\tau) \models \rho \cong \pi \) and let \( \sigma_0, \sigma_1, \ldots, \sigma_n \) be a sequence of terms such that

\[ \rho = \sigma_0 \overset{\Delta(\tau)}{=} \sigma_1 \overset{\Delta(\tau)}{=} \sigma_1 \overset{\Delta(\tau)}{=} \ldots \overset{\Delta(\tau)}{=} \sigma_n = \pi. \]

If \( \sigma_\lambda \not\in S, \tau \) for any one of the \( \lambda \leq k \), then \( \sigma_\lambda \not\in S, \tau \) for all \( \lambda \leq k \) since this property is clearly preserved by \( \overset{\Delta(\tau)}{=} \); in particular, \( \rho \) and \( \pi \) would have the property and hence \( \rho \cong \pi \in \Delta(\tau) \subseteq \theta(\tau) \) by definition. We assume therefore that

(3) \[ \rho = \sigma_0 \overset{\Gamma(\tau)}{=} \sigma_1 \overset{\Gamma(\tau)}{=} \ldots \overset{\Gamma(\tau)}{=} \sigma_n = \pi. \]

If \( \tau \not\leq \sigma_\lambda \) for any one \( \lambda \leq k \), then it follows immediately from the definition of \( \overset{\Gamma(\tau)}{=} \) that all of the \( \sigma_\lambda \) are identical; in particular, \( \rho = \pi \) and hence \( \rho \cong \pi \in \Delta(\tau) \subseteq \theta(\tau) \). Therefore, we may assume finally that of the \( \sigma_\lambda \), and in particular \( \rho \) and \( \pi \), contain a substitution instance of \( \tau \) and vice-versa. But it is clear that this means that both \( \rho \) and \( \pi \) differ from \( \tau \) only by a change of variables. Thus, since
\( \rho \approx \pi \) is variable-uniform by (3), we finally obtain that 
\( \rho \approx \pi \in \Gamma(\tau) \subseteq \Theta(\tau). \) Hence \( \Theta(\tau) \) is indeed a theory.

As an example consider \( \tau = Qv_0Qv_1v_1 \). Then \( \Gamma(\tau) \) consists of all equations of the form

\[
QxQyy \approx QxQyx \quad \text{or} \quad QxQyy \approx QyQxx
\]

Where \( x \) and \( y \) are distinct variables; \( \Lambda(\tau) \) consists of all equations \( \sigma_0 \approx \sigma_1 \) where \( \sigma_\lambda \) with \( \lambda = 0,1 \) either contains more than two occurrences of \( Q \) or is one of the following forms:

\[
QxQxx, \quad QQxyz, \quad QQxxz, \quad QQxyy, \quad QQxxx.
\]

Observe that, if \( \tau \) is a variable, then

\[
\Theta(\tau) = \Delta(\tau) = [x \approx x : x \in V] \cup \{\sigma \approx \rho : \sigma, \rho \in T \cap V\}.
\]

Let \( \tau \) be an arbitrary term again and let \( \varnothing \) be any subgroup of the group of all permutations of the variables occurring in \( \tau \); we consider each \( \varphi \in G \) a transformation of \( V \) by setting \( \varphi(x) = x \) for each variable \( x \) not occurring in \( \tau \).

Let

\[
(4) \quad \Gamma_\varnothing(\tau) = \{\sigma \varphi \tau \approx \sigma \tau \varphi : \sigma, \tau \in G\};
\]

Notice that \( \Gamma_\varnothing(\tau) = \Gamma(\tau) \) when \( \varnothing \) is the group of all permutations of the variables of \( \tau \). By an argument entirely
analogous to that used above to show $\theta(\tau)$ is a theory we establish

**Theorem 2.4.2.** $\theta_\omega(\tau) = \Gamma_\omega(\tau) \cup \Delta(\tau)$ is a theory for every $\tau \in \Theta$ and every group $\omega$ of permutations of the variables of $\tau$. Conversely, for every $\hat{\omega} \in \text{Th}[\Delta(\tau), \theta(\tau)]$ we have $\hat{\omega} = \theta_\omega(\tau)$ for some group $\omega$ of permutations of the variables of $\tau$.

**Corollary 2.4.3.** For every $\tau \in \Theta$, $\text{Th}[\Delta(\tau), \theta(\tau)]$ is isomorphic to the symmetric group on $\kappa$ letters where $\kappa$ is the number of distinct variables occurring in $\tau$.

Thus two operation symbols $Q$ and $P$ of different ranks $\kappa$ and $\lambda$, respectively, are distinguishable by the fact that the associated lattice intervals

$$\text{Th}[\Delta(Qv_0 \ldots v_{\kappa-1}), \theta(Qv_0 \ldots v_{\kappa-1})], \quad \text{Th}[\Delta(Pv_0 \ldots v_{\lambda-1}), \theta(Pv_0 \ldots v_{\lambda-1})]$$

are not isomorphic and, since both are finite, are not even elementarily equivalent. It is now clear that this will imply the conclusion of Theorem 2.4.1 if it can be proved that, for each type $I$ and each $\kappa < \omega$, the binary relation on $\text{Th}_I$

$$(5) \quad \{ \langle \Delta(Qv_0 \ldots v_{\kappa-1}), \theta(Qv_0 \ldots v_{\kappa-1}) \rangle : Q \in I, Q \text{ of rank } \kappa \}$$

is definable in the lattice $\text{Th}_I$. By (5) being definable we
mean that there exists a first-order formula \( \varphi \), with two free variables such that \( \varphi \) is satisfied by theories \( \psi \), \( \psi \) in \( \mathfrak{H} \) just in case \( \langle \psi, \psi \rangle \) is contained in (5). McKenzie's proof of this fact is long and depends on a number of difficult lemmas. We shall state one of the key lemmas and give its proof; it is typical of the type of argument McKenzie uses in his paper.

A theory \( \theta \) is said to hereditarily cover another theory \( \psi \), in symbols \( \psi \prec_H \theta \), if \( \psi \prec \theta \) and \( \psi \cap \theta \prec \theta \cap \psi \) for every theory \( \psi \) such that \( \psi \cap \theta \neq \theta \cap \psi \); clearly the set of all pairs \( \psi, \theta \) such that \( \psi \prec \theta \) is first-order definable in \( \mathfrak{H} \).

For any term \( \tau \) let \( \Sigma_\tau \) be the set of all terms \( \sigma \) such that \( \sigma \prec \tau \) or \( \sigma = \sigma_\varphi \tau \) for some permutation \( \varphi \) of \( \mathfrak{V} \). Finally, let

\[
\Omega(\tau) = \{ \sigma \approx \rho : \sigma = \rho \text{ or } \sigma, \rho \in \Sigma_\tau \}
\]

It is easy to see that \( \Omega(\tau) \) is a theory. Observe that \( \theta(\tau) \subseteq \Omega(\tau) \) and that \( \xi \in \Omega(\tau) \approx \theta(\tau) \) if and only if one side of \( \xi \) differs from \( \tau \) only by a change of variables and the other side either has the same form, but contains a variable different from the first side, or has the property that no substitution instance of it occurs as a subterm of \( \tau \).

**Theorem 2.4.3.** \( \theta(\tau) \prec_H \Omega(\tau) \) for every \( \tau \in \mathfrak{T} \).

We must prove that, for any \( \psi \subseteq \Omega(\tau) \) such that \( \psi \notin \theta(\tau) \), we have \( \psi \cap \theta(\tau) \prec \psi \).
i.e., for every pair of equations $E, E' \in \mathcal{E} \sim \Theta(\pi)$,

\[(7) \quad E', E \in \Theta[\{E\} \cup (\Theta(\tau) \cap \Theta(\pi))].\]

Since $E \in \Omega(\tau) \sim \Theta(\tau)$ we must have $E = (\tau' \sim \tau)$ where either $\tau'$ or $\tau$, say $\tau'$, differs from $\tau$ only by a change of variables and either

\[(8) \quad \pi \not\subset \tau',\]

or $\pi$ also differs from $\tau$ only by a change of variables but contains a variable not in $\tau'$. By substituting for this variable in the latter case we can conclude that

\[(9) \quad \tau \sim \pi \in \Theta[\xi] \subseteq \Phi \quad \Phi\]

Where $\pi$ satisfies condition (8).

Since also $E' \in \Omega(\tau) \sim \Theta(\tau)$ we have by a similar argument that

\[(10) \quad \Theta[\tau \sim \pi'] = \Theta[E'] \subseteq \Phi\]

Where either (8) with $\pi$ replaced by $\pi'$ holds, or, for some permutation $\varphi$ of $\mathcal{V}$,

\[(11) \quad \pi' = su_{\varphi} \tau'.\]

Assume for the time being that the former case holds. Then
\[ \pi \approx \pi' \in \Delta(\tau) \subseteq \theta(\tau) \text{ and hence } \pi \approx \pi' \in \dot{\Phi} \cap \theta(\tau) \] since
\[ \tau \approx \pi, \tau \approx \pi' \in \dot{\Phi}. \] Then from (9) we obtain \( \tau \approx \pi' \in \dot{\theta}([\varepsilon]) \cup (\dot{\Phi} \cap \theta(\tau)) \)
and then finally, from (10), we get (7). Thus we assume now that (11) holds for some permutation \( \varphi \) of \( \text{Va} \).

From (10) and (11) we have \( \tau \approx \sup_{\varphi} \tau \in \dot{\Phi} \) and combining this with (9) gives \( \pi \approx \sup_{\varphi} \pi \in \dot{\Phi} \). Thus, by (8), \( \pi \approx \sup_{\varphi} \tau \in \dot{\Phi} \cap \theta(\tau) \).

Finally, this result together with (9) implies
\[ \tau \approx \sup_{\varphi} \tau \in \dot{\theta}([\varepsilon]) \cup (\dot{\Phi} \cap \theta(\tau)) \].

But the equation in this formula is just \( \varepsilon' \) by (11). Hence (7) holds and the theorem is proved.

Let \( Q \) be any operation symbol and let \( \kappa \) be rank of \( Q \) (\( \kappa = 0 \) is a possibility). Then it is easy to check that
\[ \Omega(Qv_0 \ldots v_{\kappa-1}) = \theta(v_0) = \{ \sigma \approx \rho : \sigma = \rho \text{ or } \sigma, \rho \in \text{Te} \sim \text{Va} \} \].

So by theorem 2.4.3, \( \theta(Qv_0 \ldots v_{\kappa-1}) \leq \Omega(Qv_0 \ldots v_{\kappa-1}) \) for every operation symbol \( Q \). Furthermore, \( \Omega(v_0) = \mathcal{E} \) so that, again by 2.4.3, \( \theta(v_0) \leq \mathcal{E} \). McKenzie completes the description of the hereditary covering relation at the top of \( \mathcal{Th} \) by proving the following: (I) besides \( \theta(v_0) \) the only theory hereditarily covered by \( \mathcal{E} \) is the theory \( \dot{\Phi} \) of all variable-uniform equations; (II) only \( \theta(v_0) \cap \dot{\Phi} \) is hereditarily covered by \( \dot{\Phi} \); and (III) \( \theta(v_0) \cap \dot{\Phi} \) is the only theory besides
the $\theta(Q_0,\ldots,Q_{n-1})$ that is hereditarily covered by $\Delta(Q_0)$.

Thus the top three levels of the hereditary covering relation in $\mathcal{H}$ are completely described; see the diagram where $I = \{Q, Q, Q, \ldots\}$ and for simplicity we write $\theta(Q)$ for $\theta(Q_0,\ldots,Q_{n-1})$, etc.

This result leads immediately to the definability of the set of all theories of the form $\theta(Q_0,\ldots,Q_{n-1})$ with $Q \in I$. McKenzie then proceeds to analyze the structure of the hereditary covering relation below each of the $\theta(Q_0,\ldots,Q_{n-1})$ and then uses the results to construct a first-order formula of the language of lattices that characterizes the relation (3). In this way Theorem 2.4.1 is established.

Using the same type of argument discussed above McKenzie [71a] proves the following remarkable result.

**Theorem 2.4.4.** $\Sigma, \Lambda$, and $\theta[Q_0, Q_1 \sim Q_1, Q_0]$ are all first-order definable elements of $\mathcal{H}_I$ where $I$ is the type of semigroups, lattices, and groupoids, respectively.

McKenzie also states without proof that the theories $\mathcal{E}$, $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are all first-order definable elements of $\mathcal{H}_I$. He then proceeds to analyze the structure of the hereditary covering relation below each of these theories and uses the results to construct a first-order formula of the language of lattices that characterizes the relation (3). In this way Theorem 2.4.1 is established.
and the theory of all Boolean algebras are definable.

By an argument different than that used by McKenzie, Jezek [71a] has been able to show that \( \subseteq \cap_a \subseteq \), is a definable element in the lattice of all groupoid theories; this will be discussed in Section 2.6.

McKenzie's success in being able to define all these theories has led him to make an interesting conjecture which we now describe.

Recall the notions of a definition \( \rho \) of a type \( I \) in a theory \( \Theta \) and of the definitional equivalence of two theories given in Section 1.5. Let \( I \) be a fixed type and let \( \rho \) and \( \tau \) be definitions of \( I \) in \( \Theta_I \) such that \( \Theta_I \) is definitionally equivalent to itself by \( \rho \) and \( \tau \). It is easy to see that there must exist a rank preserving permutation \( \varphi \) of \( I \) and a function \( \kappa \) from \( I \) into the set of permutations of finite ordinals such that for each \( Q \in I \)

\[
\rho(Q) = \varphi_Q \psi_{\kappa(Q)}(0) \cdots \psi_{\kappa(Q)}(\lambda-1)
\]

Where \( \lambda \) is the rank of \( Q \); compare the remark following the proof of 1.5.1. By 1.5.3 and the remark following it \( e_1 \rho \) induces an automorphism of \( \Theta_I \). McKenzie [71a] states that he has not been able to find an automorphism of \( \Theta_I \) that is not induced by some definition \( \rho \) of \( I \) in \( \Theta_I \) by which
$T_{a_1}$ is definitionally equivalent to itself and poses the existence of such an automorphism as an open problem.

Obviously a theory of type $I$ cannot be distinguished by an elementary property from any image of it under an automorphism of $\mathcal{X}_{a_1}$. It is conjectured in op. cit. that for finitely based theories (i.e., compact elements of $\mathcal{X}_{a_1}$) the converse is true: for each finitely based theory $\theta$ the set of all theories $\tilde{\theta}$ such that $\tilde{\theta}$ is an image of $\theta$ under some automorphism of $\mathcal{X}_{a_1}$ is definable; in particular, every finitely based theory that is fixed by each automorphism is definable. This conjecture has not yet been verified.

Theorems 2.4.1 and 2.4.2 have interesting consequences for categories of algebras and homomorphisms, some of which are new. For each type $I$ let $\mathcal{C}_I$ be the category whose objects are algebras of type $I$ and whose morphisms are homomorphisms between these algebras. W. Neumann \[\ldots\] proves that for any two types $I$ and $J$ the categories $\mathcal{C}_I$ and $\mathcal{C}_J$ are isomorphic if and only if $I$ and $J$ are essentially the same in the sense of 2.4.1(ii). Neumann’s results actually apply to types admitting operations of infinite rank but in their restriction to finitary types they can be easily inferred from Theorem 2.4.1.

The category-theoretical analogue of 2.4.3 is also of
interest. It is not difficult to show that, if $\theta$ is a first-order definable element of $\mathcal{Th}_I$, then $\text{Mo}\theta$ is a first-order definable set of objects of $C_I$. Thus from 2.4.3 it follows that the full varieties of semigroups, lattices, and commutative groupoids are all definable in $C_I$; by the remarks following 2.4.3 the same applies to the full varieties of commutative semigroups, distributive lattices, and groups.

In the final portion of this section we shall prove that every non-trivial theory covers at least one theory; Cf. the remarks preceding 2.3.7. The following theorem is due to McKenzie [71a].

**Theorem 2.4.5.** Let $I$ be an arbitrary type and $\psi \in \mathcal{Th}_I \sim [\mathcal{T}a_I]$. Then there exists a $\kappa < \omega$ and $\psi_0, \ldots, \psi_\kappa \in \mathcal{Th}_I$ such that

$$\psi_\kappa, \psi_\kappa-1 < \cdots < \psi_0 < \theta.$$

Let

$$\Gamma = \{\sigma : \sigma \in \mathcal{Te}_I, \sigma \equiv \rho \in \psi \text{ for some } \rho \in \mathcal{Te}_I \sim \{\sigma}\}$$

Choose any $\tau \in \Gamma$ with the property that, for every $\sigma \in \Gamma$, if $\sigma \leq \tau$, then $\sigma = \text{su}_\varphi \tau$ for some permutation $\varphi$ of $V_a$. It is easy to see that such a $\tau$ always exists; for example,
if \( \psi = \Sigma \), then we can take \( \tau \) to be any one of the terms \( QxQyz \) or \( QQxyz \).

Recall the definitions of \( \Delta(\tau) \), \( \theta(\tau) \) and \( \Omega(\tau) \) given in (1), (2), and (6), respectively. It is clear that

\[
\psi \subseteq \Omega(\tau) \quad \text{and} \quad \psi \not\subseteq \Delta(\tau).
\]

If \( \psi \not\subseteq \theta(\tau) \), then by 2.4.3 we have \( \psi \cap \theta(\tau) < \psi \). In this case we can take \( \kappa = 0 \) and \( \psi_0 = \psi \cap \theta(\tau) \). We assume therefore that \( \psi \subseteq \theta(\tau) \). In this case the following relativized version of 2.4.2 can be established in the same way the original version was; recall the definition of \( \Gamma_\theta(\tau) \) given in (4).

\[
\theta_0(\tau) = \Gamma_\theta(\tau) \cup (\psi \cap \Delta(\tau)) \quad \text{is a theory for every group} \quad \Theta \text{ of permutations of the variables of } \tau \text{ such that } \Gamma_\Theta(\tau) \subseteq \psi. \quad \text{Conversely, every theory in } \text{Th}[\psi \cap \Delta(\tau), \theta(\tau)] \text{ is of the form } \theta_\Theta(\tau) \text{ for some } \Theta.
\]

Let

\[
\Theta \subseteq \Theta_{\kappa-1} \subseteq \ldots \subseteq \Theta_0 \subseteq \Theta
\]

be a chain of maximal length such that \( \Gamma_\Theta(\tau), \ldots, \Gamma_\Theta(\tau), \Gamma_\Theta(\tau) \subseteq \psi \) and set

\[
\psi_{\lambda} = (\psi \cap \Delta(\tau)) \cup \Gamma_{\Theta_\lambda}(\tau)
\]

For \( \lambda \leq \kappa \). Notice that since (12) is maximal we must have
\[ \psi = (\psi \cap \Delta(\tau)) \cup \Gamma_{\lambda}(\tau) \quad \text{and thus} \]

\[ \hat{\psi}_{\kappa - 1} < \hat{\psi}_{\kappa - 2} < \ldots < \hat{\psi}_0 < \psi. \]

Observe also that \( \Theta_\kappa \) consists of the identity permutation only so that \( \hat{\psi}_\kappa = \psi \cap \Delta(\tau) \) and that \( \Theta_{\kappa - 1} \) is a minimal proper subgroup of the group of all permutations and hence is generated by each one of the non-identity permutations contained in it. But in view of (13) this means that for any pair of non-tautological equations \( \xi, \delta \in \Gamma_{\Theta_{\kappa - 1}}(\tau) \), \( \xi \) and \( \delta \) are \( (\psi \cap \Delta(\tau)) \)-interderivable; this implies of course that for any theories \( \Xi, \Xi' \in \text{Th}_1 \), if \( \Xi \subset \Xi' \) and \( (\Xi' \sim \Xi) \subseteq \Gamma_{\Theta_{\kappa - 1}}(\tau) \), then \( \Xi < \Xi' \). Thus \( \hat{\psi}_\kappa \not< \hat{\psi}_{\kappa - 1} \) and the theorem is proved.
Problems

Problem 2.1. Is it true that every algebraic lattice with a compact unit is isomorphic to a lattice of theories \( \mathcal{X}_I[8] \) for some type \( I \) and some theory \( \emptyset \) of type \( I \)?

The question of characterizing the class of lattices isomorphic to some \( \mathcal{X}_I[8] \) was raised by Mal'cev [68]. The particular conjecture formulated in the above problem is due to Ralph McKenzie. Cf. (1) of Section 2.1 and the following remarks.

Problem 2.2. Find an intrinsic characterization of the class of all lattices isomorphic to a lattice \( \mathcal{X}_I \) for some type \( I \).

This problem appears as problem 32, p. 194, in Grätzer [68]; Cf. (2) of Section 2.1 and the following remarks.

Problem 2.3. Do there exist finitely based theories \( \emptyset \) and \( \emptyset' \) with \( \emptyset, \emptyset' \in \text{Th}([G]) \) or \( \emptyset, \emptyset' \in \text{Th}([QG]) \) such that \( \emptyset \cap \emptyset' \) fails to be finitely based?

Cf. (4) of Section 2.1 and the remarks following it.

Problem 2.4. Does there exist a continuum number of equationally complete anti-associative theories of quasigroups?
This problem originates with Evans [71]. See the
remarks following Theorem 2.2.4.

**Problem 2.5.** Is $\mathfrak{H}[\Sigma]$ or $\mathfrak{H}[I]$ a partial Boolean
ring?

Compare Theorem 2.3.2 and the discussion following it.

**Problem 2.6.** Let $\Theta$ be a theory with the property that
every semi-complete extension of $\Theta$ has a unique representation
as the intersection of a finite set of complete theories. Is
$\Theta$ necessarily a partial Boolean Ring?

By Theorem 2.3.2 the answer to this question is positive
when $\mathfrak{H}[\hat{\Theta}]$ is modular. Notice that a positive solution to
this problem implies a positive solution to Problem 2.5.

**Problem 2.7.** Do semi-complete theories exist which fail
to have a unique representation as an intersection of a finite
set of complete theories? In particular, do complete theories
$\hat{\Theta}_0, \hat{\Theta}_1$ exist such that $\hat{\Theta}_0 \cap \hat{\Theta}_1$ has a complete extension different
from both $\hat{\Theta}_0$ and $\hat{\Theta}_1$?

**Problem 2.8.** Do there exist theories $\Theta, \Theta'$ such that
$\Theta \subset \Theta'$ and $\mathfrak{H}[\Theta, \Theta']$ fails to have any dual atoms; in particular,
can $\Theta$ be taken to be $\Sigma$?

This problem with $\Theta = \Sigma$ originates with Evans [71a].
Cf. the remarks following 2.3.8.

Problem 2.9. Does there exist a finite algebra whose theory has an infinite number of extensions? Does there exist a finite model of a congruence-modular theory with this property?

Problem 2.10. Does there exist a type $I$ and an automorphism of $Th_I$ that fails to be induced by $e_\rho$ for some definition $\rho$ of $I$ in $Ta_I$ by which $Ta_I$ is definitionally equivalent to itself?

See the remarks following 2.4.4. This problem and the next one originate with McKenzie [71].

Problem 2.11. Let $I$ be any type and $\Theta$ any finitely based theory of type $I$. Is it true that the set of all theories $\hat{\theta}$ such that $\hat{\theta}$ is an image of $\Theta$ under some automorphism of $Th_I$ is definable? In particular, is every finitely based theory that is fixed by each automorphism of $Th_I$ definable?
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Tarski, A.


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ERRATA FOR "EQUATIONAL LOGIC AND EQUATIONAL THEORIES OF ALGEBRAS"

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"This leads to an intrinsic characterization of the first-order analogues of the lattices ΣΣ_1[6] as algebraic lattices whose set of compact elements form a Boolean algebra. They have also been intrinsically characterized by Tarski [37]."
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<td>158</td>
<td>8</td>
<td>( \sigma_\lambda \not\subseteq \tau ) (in both places)</td>
<td>( \sigma_\lambda \not\subseteq \tau )</td>
</tr>
<tr>
<td>158</td>
<td>7B</td>
<td>( \sigma_\lambda \not\subseteq \tau )</td>
<td>( \sigma_\lambda \not\subseteq \tau )</td>
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<tr>
<td>158</td>
<td>4B</td>
<td>Insert after &quot;that&quot;</td>
<td>all</td>
</tr>
<tr>
<td>159</td>
<td>7</td>
<td>Insert after &quot;where&quot;</td>
<td>( \sigma_0 = \sigma_1 ) or</td>
</tr>
<tr>
<td>159</td>
<td>3B</td>
<td>( x ) (both places)</td>
<td>( \psi )</td>
</tr>
<tr>
<td>161</td>
<td>9</td>
<td>( \hat{\sigma} \theta )</td>
<td>( \hat{\sigma} \theta )</td>
</tr>
<tr>
<td>161</td>
<td>12</td>
<td>( \sigma \not\subseteq \tau )</td>
<td>( \sigma \not\subseteq \tau )</td>
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<td>PAGE</td>
<td>LINE</td>
<td>REPLACE</td>
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<tr>
<td>161</td>
<td>3B</td>
<td>2.4.3</td>
<td>2.4.4</td>
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<tr>
<td>162</td>
<td>9</td>
<td>Insert after &quot;case&quot; the following: &quot;and then changing variables if necessary&quot;</td>
<td></td>
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<tr>
<td>163</td>
<td>9</td>
<td>just $\xi'$ by (11) interderivable with $\xi'$ by (10) and (11)</td>
<td></td>
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<tr>
<td>163</td>
<td>8B</td>
<td>2.4.3</td>
<td>2.4.4</td>
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<td></td>
<td></td>
<td>$\Omega(\Omega v_0 \cdots v_{n-1})$</td>
<td>$\theta(v_0)$</td>
</tr>
<tr>
<td>163</td>
<td>6B</td>
<td>E</td>
<td>Eq</td>
</tr>
<tr>
<td>164</td>
<td>8B</td>
<td>(3)</td>
<td>(5)</td>
</tr>
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<td>164</td>
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<td>2.4.4</td>
<td>2.4.5</td>
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<td>165</td>
<td>3</td>
<td>$\Sigma \cap A\Sigma$</td>
<td>$\Sigma \cap A\Sigma_{11}$</td>
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<td>166</td>
<td>12</td>
<td>2.4.2</td>
<td>2.4.5</td>
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<tr>
<td>166</td>
<td>1B</td>
<td>2.4.3</td>
<td>2.4.5</td>
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<td>167</td>
<td>6</td>
<td>2.4.3</td>
<td>2.4.5</td>
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<td>12</td>
<td>2.4.5</td>
<td>2.4.6</td>
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<td>6</td>
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<td>2.4.4</td>
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<td>168</td>
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<td>(12)</td>
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<td>7</td>
<td>(13)</td>
<td>(12)</td>
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<td>169</td>
<td>4B</td>
<td>$\varnothing \cup A(\tau)$-</td>
<td>Delete</td>
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<tr>
<td>175</td>
<td>1</td>
<td>monoips</td>
<td>monoids</td>
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<td>175</td>
<td>6B</td>
<td>Prox.</td>
<td>Proc.</td>
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<tr>
<td>176</td>
<td>21</td>
<td>Delete lines 21 and 22</td>
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<tr>
<td>177</td>
<td>23</td>
<td>Izyestija</td>
<td>Izyestija</td>
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