Purdue University
Purdue e-Pubs

Purdue CIBER Working Papers | Krannert Graduate School of Management

1-1-1993

Central Bank's Expected Profits from Intervention

John A. Carlson
Purdue University

Insook Kim
Purdue University

Follow this and additional works at: http://docs.lib.purdue.edu/ciberwp

Carlson, John A. and Kim, Insook, "Central Bank's Expected Profits from Intervention" (1993). *Purdue CIBER Working Papers.* Paper 77.
http://docs.lib.purdue.edu/ciberwp/77

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact epubs@purdue.edu for additional information.

Central Banks' Expected Profits from Intervention

John A. Carlson and Insook Kim

93-110

Center for International Business Education and Research
Purdue University
Krannert Graduate School of Management
1310 Krannert Building
West Lafayette, IN 47907-1310
Phone: (317) 494-4463
Fax: (317) 494-9658

Central Banks' Expected Profits from Intervention

John A. Carlson and Insook Kim
Purdue University

Abstract

A leaning-against-the-wind intervention that has only a temporary effect on the exchange rate and that is not too aggressive can be shown analytically to yield positive expected profits to a central bank even when the exchange-rate process is non-stationary. These profits arise if there are some transitory shocks to the exchange rate. Furthermore, very aggressive intervention will yield positive expected profits eventually when there is a tendency for exchange rates to return to a long-run equilibrium level.

February 1993

Central Banks' Expected Profits from Intervention

Central banks often justify interventions in foreign exchange markets on the grounds that they are countering "disorderly markets". A sudden drop in the price of foreign exchange may be slowed or stopped by a central-bank purchase of the foreign exchange. If the decrease is only transitory, the bank can later sell the foreign exchange at a profit when the price has gone back up. This line of reasoning led Milton Friedman (1953) to argue that stabilizing intervention should be profitable for a central bank. A policy of buying when low and selling when high moderates the extent to which exchange rates vary and at the same time is profitable for the central bank. Conversely, if the central bank loses money from its interventions, Friedman would view that as an indication of destabilizing intervention in the sense of keeping the exchange rate away from its equilibrium level.

In a provocative paper, Dean Taylor (1982) examines whether central banks did or did not make profitable interventions. Using data from nine countries from the beginning of general floating in the early 1970's until the end of 1979, he found that most central banks lost money from their interventions and for some the losses were significant. Taylor, following up on Friedman's ideas, argues that when a permanent economic shock causes the equilibrium exchange rate to shift, a "leaning-against-the-wind" policy will lose money. For example, if the equilibrium price of foreign exchange is falling a central-bank purchase of foreign exchange will keep the price from falling as fast as it would without intervention, but the central bank in effect loses on the transaction when the price later falls below the purchase price. The bank may have succeeded in moderating the rate of change in the exchange rate but, in the sense that the price has been kept away from its equilibrium, the loss sustained by the central bank

signals destabilizing intervention. In light of the losses by central banks, Taylor suggests that private speculators could make money and serve a stabilizing role by betting against the central bank.

The estimated profits or losses sustained by central banks turn out to be sensitive to the time period chosen for the calculations. To avoid the problems of measuring inventory gains or losses, other researchers have restricted attention to periods in which net intervention is small. Victor Argy (1982), for example, restricts his calculations to periods in which purchases of foreign exchange are approximately equal to sales. A Bank of England study (1983) provides evidence that the intervention by the Bank of England tended to yield profits, most convincingly in periods of nearly zero net intervention. Lawrence Jacobsen (1983) calculates the profitability of US intervention in DM. While the intervention appears unprofitable over periods in which cumulative net intervention is substantial, the results are more positive if one examines the entire period and subperiods in which net intervention was near zero.

Charles Corrado and Dean Taylor (1986) are critical of the studies that restrict the sample to periods over which cumulative net intervention is zero and argue that this introduces a positive bias to expected profits. By a similar line of reasoning, one could argue that studies of almost any realization of profits from intervention are subject to the charge of bias relative to expected profits. Taylor's calculations of profits, for example, were over a period in the 1970's in which the dollar realized substantial depreciation. The central banks by buying dollars had accumulated reserves with a market value at the end of the sample period well below the purchase price. For most patterns that might have reasonably been anticipated *ex ante*, this particular end-of-sample realization surely fell below the expected price of the dollar.

What is neat about Corrado and Taylor's contribution is that they calculate expected profits based on assumptions about (a) the movements in exchange rates that would have occurred in the absence of intervention, (b) the intervention rule utilized, and (c) the effects of intervention. Specifically, they assume that the exchange rate would have followed a random walk in the absence of intervention and prove that a leaning-against-the-wind intervention policy can expect negative profits.

We follow up this analysis by considering two other possibilities. The first begins with the observation that there can be both permanent and transitory changes in the exchange rate. The bank would want to offset the transitory changes and ignore the permanent ones, but if the bank cannot immediately tell one type of change from the other, should it intervene or not? To provide an analytic answer, we can introduce a first-order moving-average process for the exchange rate in the absence of intervention The moving-average parameter represents the extent to which exchange-rate changes are transitory. Although this is, like a random walk, a nonstationary process, we show that there is scope for profitable intervention.

The second possibility considered is based on the observation that exchange rates appear to move in long waves, with periodic returns to what may be considered long-run equilibrium levels. Consequently, we shall also explore the expected profitability of a leaning-against-the wind intervention policy when the exchange rate follows a first-order autoregressive process in the absence of intervention. Again, profitable intervention is possible.

1. The Model

To set the stage for our own analysis, we initially reproduce several elements of the specification used by Corrado and Taylor (1986). Assume first that the intervention rule is the following leaning-against-the-wind policy:

(1) $$I_t = -\lambda (S_t - S_{t-1})$$

where

S_t = the observed spot exchange rate, and

I_t = net purchases of foreign exchange by the central bank.

The parameter λ indicates the extent of the intervention in response to a change in the spot exchange rate. Assume $\lambda > 0$. A positive value of I_t denotes a purchase of foreign exchange and a negative value a sale. So the central bank buys foreign currency when the price falls and sells when the price rises. A substantial body of empirical work finds that the major explanatory variable for official intervention is the change in the exchange rate. Some of this is cited by Corrado and Taylor. A recent survey by Geert Almekinders and Sylvester Eijffinger (1991) suggests similar results. While evidence of asymmetry in responses under different situations emerge in some studies, interventions still frequently appear to lean against the wind.

Assume next that intervention has a temporary effect on the exchange rate:

(2) $$S_t = F_t + \alpha I_t$$

where

F_t = the "free" exchange rate in the absence of intervention.

According to equation (2), intervention changes the spot exchange rate from what it would have been, but there is no permanent effect. The parameter $\alpha \geq 0$ denotes the strength of the effect of intervention. When the central bank buys foreign exchange the price will be temporarily higher (assuming $\alpha > 0$) than it would have been without the intervention, and when the bank sells foreign exchange the price will be temporarily lower. Almekinders and Eijffinger (1991) also survey several empirical studies assessing the effects of intervention. There appears to be virtually no evidence that sterilized intervention can make a permanent change in the exchange rate through a portfolio balance effect, but there is some evidence of short term effects in daily data that die out in the course of a month.

The net profit π from central bank transactions in the foreign exchange market from time 1 to N is defined as the difference between the net accumulated reserves valued at the end-of-period rate and their initial cost:

$$\pi = F_N \sum_{t=1}^{N} I_t - \sum_{t=1}^{N} S_t I_t \tag{3}$$

When the sum of the interventions is zero, profits are positive if the purchases (with I_t positive) occur at a lower average price than the sales (when I_t is negative). When the sum of the accumulations is non-zero, then one must also take into account the valuation gains or losses by valuing the stocks at the end-of-period free rate relative to the purchase or sale prices at which the interventions took place. Equation (3) taken from Corrado and Taylor accomplishes this valuation.

To express profits in terms of the free exchange rate , which will be assumed to be given exogenously, substitute for S_t from equation (2) into the profit equation (3) and note that $F_N - F_t = \sum_{j=t+1}^{N} \Delta F_j$ for $t < N$, where $\Delta F_t = F_t - F_{t-1}$. The result is:

$$(4) \qquad \pi = \begin{cases} \sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j - \alpha \sum_{t=1}^{N} I_t^2 & \text{if } N > 1 \\ -\alpha I_1^2 & \text{if } N = 1 \end{cases}$$

With successive substitutions of equations (1) and (2), I_t can be written as:

$$(5) \qquad I_t = \frac{-\phi}{\alpha} \sum_{k=0}^{\infty} \phi^k \Delta F_{t-k}$$

where

$$\phi = \frac{\alpha\lambda}{1 + \alpha\lambda} < 1.$$

Equation (5) can then be substituted into (4) to get profits that will result from the intervention policy as a function of changes in the free exchange rate.

Corrado and Taylor assume that the free exchange rate follows a random walk:

$$(6) \qquad F_t = F_{t-1} + u_t$$

where the u_t are independently and identically distributed random variables with a mean of zero and a variance of σ^2. They then prove that the <u>expected</u> profit $E(\pi)$ is always negative as long as the central bank's intervention can affect the exchange rate ($\alpha > 0$). If

$\alpha = 0$ and if the exchange rate follows a random walk, the central bank's expected profits are zero.

We will replace the random walk assumption of equation (6) with two alternatives, both of which contain the random walk as a special case.

II. Intervention with a Moving Average Process

Consider the following moving average process:

$$F_t = F_{t-1} + u_t - \theta u_{t-1} \tag{7}$$

where u_t is an unobserved exogenous shock, with the properties previously assumed. The parameter θ indicates the degree to which the shocks are transitory. If $\theta = 0$, the level of F is permanently changed by u. This would be the case of a random walk. If $\theta = 1$, the effect of a shock disappears fully after one period and the shocks are entirely transitory. For any θ in between there is a mixture of permanent and transitory shocks. θu_t is transitory and $(1 - \theta) u_t$ is permanent.

In a discussion of empirical regularities in the behavior of exchange rates, Mussa (1979, p. 11) notes that typically for monthly data a "first-order moving average process gives a slightly better explanation of the data than the model that assumes that u_t is serially uncorrelated." According to Mussa, estimates of θ are generally between 0 and .2. Using extremely high-frequency data, Ito and Roley (1986) and Goodhart and Figliuoli (1991) report that large exchange rate jumps are partially reversed within a day. These studies suggest one should not dismiss the existence of both permanent and transitory shocks in exchange rate movements An analytically tractible way to handle this is to assume a first-order moving average process.

With $\theta < 1$, the process specified by equation (7) is non-stationary over time. If only the net effects of the shocks are considered, then the process can be viewed as a random walk, as if there were a series of cumulative shocks of $(1 - \theta)\, u_t$. We examine whether or not leaning-against-the-wind intervention, as assumed in equation (1), can be expected to generate positive profits if the free rate follows this moving-average process.

The calculation of expected profits involves fairly messy algebra, which has been relegated to an appendix. As shown in the appendix, the expected profits are:

$$E(\pi) = \frac{N\phi(\theta - \phi)(1 - \theta\phi)\sigma^2}{\alpha(1 - \phi^2)} - \frac{\theta\phi\sigma^2}{\alpha} \tag{8}$$

Note that for $\theta = 0$, which is the case of a random walk, $E(\pi) = \dfrac{-N\phi^2\sigma^2}{\alpha(1 - \phi^2)} < 0$. This is the result obtained by Corrado and Taylor. More generally, if $\phi \geq \theta$ then $E(\pi) < 0$ for any N. If the extent of the intervention is too great relative to the moving-average parameter, the central bank can expect losses from the intervention.

If $\phi < \theta$, the first term in (8) increases with N. Therefore, $E(\pi)$ will be positive for any

$$N > \frac{\theta(1 - \phi^2)}{(\theta - \phi)(1 - \theta\phi)} = N^*.$$

Assuming θ is positive, a central bank can always expect a positive profit by choosing the coefficient of the leaning against the wind (λ) in equation (1) so that $\phi < \theta$ for any $N > N^*$. From the definition of ϕ, this holds if $\dfrac{\alpha\lambda}{1 + \alpha\lambda} < \theta$ or if

$$\lambda < \frac{\theta}{\alpha(1 - \theta)}. \tag{9}$$

The central bank has more flexibility in its choice of a profitable λ when the effect of the intervention (α) is smaller and when θ is larger, i.e., when the transitory shocks are larger relative to the permanent shocks.

To help get an intuitive grasp on this result, consider Figure 1. Based on information through observations available up to and including time t-1, there is an expected value of the free exchange rate in period t. This is depicted by the point denoted by $E_{t-1}F_t$ in the figure. Suppose in the absence of intervention, the exchange rate would fall to the point F_t. Given the assumed moving average process, the expected value at time t for F_{t+1} is given by the following adaptive-expectations formula:

$$E_tF_{t+1} = \theta E_{t-1}F_t + (1 - \theta) F_t$$

The upward sloping line in Figure 1 represents equation (2) and shows the assumed temporary tradeoff between the amount of intervention and the observed exchange rate. As long as S_t is below E_tF_{t+1}, the central bank is buying foreign exchange at a price below its expected value next period and this is essentially how it can make an expected profit. The scope for profitable intervention is increased by a larger θ which would move E_tF_{t+1} farther from F_t, and by a lower α which would mean a flatter line representing equation (2). This is consistent with the analytic results discussed above in conjunction with inequality (9).

Our analysis has proceeded as if all the parameters are known and the draws of the free exchange rate always come from the process assumed in equation (7). If this were generally known and the central bank did not intervene to the extent that $S_t = E_tF_{t+1}$, then private speculators would presumably have an incentive to buy the currency when $S_t < E_tF_{t+1}$, and sell the currency when $S_t > E_tF_{t+1}$. If speculative trades drive S_t

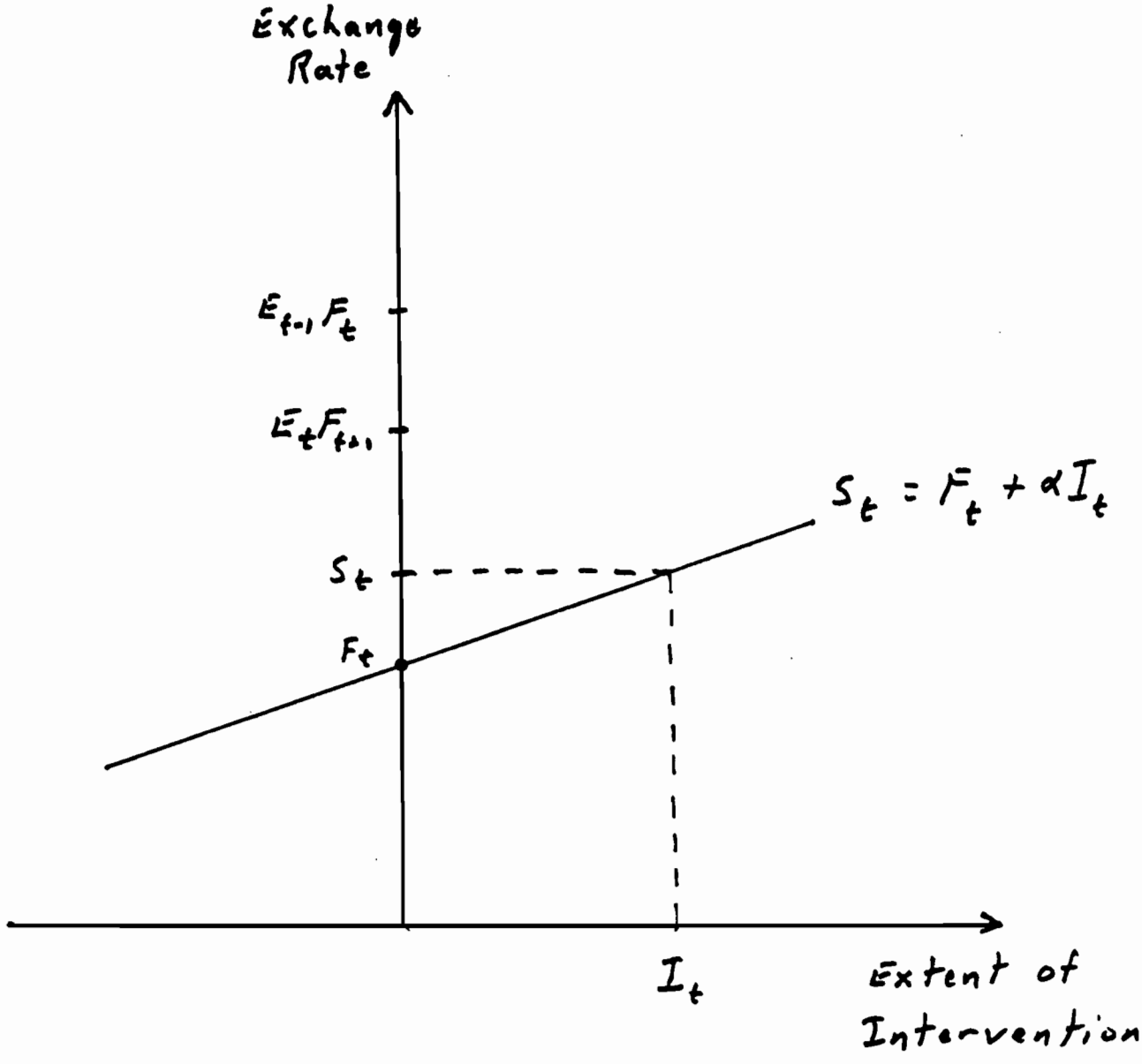
Exchange Rate
$E_{t-1}F_t$
E_tF_{t+1}
$S_t = F_t + \alpha I_t$
S_t
F_t
I_t
Extent of Intervention

Figure 1

into equality with E_tF_{t+1}, then the best forecast of the next value of the free exchange rate is the current exchange rate. This sort of informed speculation leads to the familiar theoretical claim that the actual exchange rate will follow a random walk.

What we have in mind is addressing a different question. Suppose central banks mechanically follow a frequently assumed leaning-against-the-wind intervention rule and amid the noise in exchange rate movements there is a mixture of permanent and transitory shocks, represented by a moving-average process which is not clearly perceived. Could the mechanical intervention rule yield positive expected profits and hence satisfy Friedman's criterion for stabilizing intervention? The answer is yes if the strength of the intervention satisfies inequality (9).

Furthermore, as shown in the appendix:

$$\text{(10)} \qquad \mathrm{Var}(\Delta S_t) = \frac{1-\phi}{1+\phi}(1+\theta^2-2\phi\theta)\sigma^2 \leq (1+\theta^2)\sigma^2 = \mathrm{Var}(\Delta F_t)$$

A strict inequality holds if $\phi > 0$. Since $\phi > 0$ when the central bank leans against the wind ($\lambda > 0$) and the intervention has some effect on the exchange rate ($\alpha > 0$), the central bank can moderate the short-run variability of the exchange rate and have positive expected profits for $0 < \phi < \theta$.

III. Intervention with an Autoregressive Process

Plots of exchange rates over time between major currencies indicate clearly that there have been long swings in the data. How this can be best modelled formally is still an open question, but some form of mean reversion is frequently acknowledged. For example, Huizinga (1987, p. 208) writes: "...the long-run movements of real exchange

rates differ from those implied by a random walk by having a notable mean reverting compontent." More recently, Glen (1992) using variance ratio tests suggested by Lo and MacKinlay (1988) rejects random walk behavior in favor of mean reversion with annual data at lags greater than two years. Over shorter periods of time, however, these tests indicate that exchange rates have what may be called mean aversion, a tendency to move away from some long-run equilibrium significantly faster than a random walk would predict. This is reported by Glen with monthly data and by Liu and He (1991) with weekly data. A similar interpretation can be taken from the analysis by Engle and Hamilton (1990), who argue that long swings in exchange rates can be modeled by an underlying two state Markov chain. These and other empirical studies provide scope for a variety of hypotheses one might choose to analyze the expected profitability of leaning-against-the-wind intervention by central banks.

In what follows, we have worked out the results if the mean reverting tendencies are modeled by a first-order autoregressive process in the absence of intervention. Assume therefore that the free exchange rate follows the following autoregressive process:

(11) $$F_t = \delta F_{t-1} + u_t$$

where $0 \leq \delta \leq 1$.

If $\delta = 1$, this would be a random walk. With $0 < \delta < 1$, there is a gradual return to a long-run value, which has been normalized to 0 in this formulation. As in the previous section, we investigate whether a mechanical leaning-against-the-wind policy can be profitable and if so, under what conditions. The algebra has again been relegated to the appendix. Expected profits are given by:

$$(12) \qquad E(\pi) = \begin{cases} \dfrac{\phi\sigma^2}{\alpha(1+\delta)(1-\phi\delta)} \left\{ N\left(\dfrac{1-\phi}{1+\phi}\right) - \dfrac{1-\delta^N}{1-\delta} \right\} & \text{if } 0 \le \delta < 1 \\ -\dfrac{N\phi^2\sigma^2}{\alpha(1-\phi^2)} & \text{if } \delta = 1 \end{cases}$$

Note that when $\delta = 1$, the Corrado-Taylor result emerges again and leaning against the wind cannot achieve positive expected profits.

For $0 \le \delta < 1$, however, $E(\pi)$ is positive if

$$(13) \qquad \frac{1-\phi}{1+\phi} > \frac{1-\delta^N}{N(1-\delta)}$$

The right side of (13) asymptotically approaches zero as N gets larger. Therefore, no matter how aggressively the central bank intervenes, i.e., no matter how close ϕ is to one, its intervention can eventually achieve positive expected profits if the exchange rate follows a stable autoregressive process in the absence of intervention.

In the appendix it is also shown that

$$(14) \qquad \mathrm{var}(\Delta S_t) = \frac{2(1-\phi)^2\sigma^2}{(1+\phi)(1+\delta)(1-\phi\delta)}$$

Recall that $\phi = \alpha\lambda/(1 + \alpha\lambda)$ where λ is the strength of the intervention and α is the effect of the intervention. By choosing a large λ that puts ϕ close to one, the central bank can eliminate almost all variability in the exchange rate, if it has sufficient reserves, and still expect long-run profitability from its intervention assuming $\delta < 1$, i.e., assuming a long-run mean-reverting tendency.

IV. Conclusions

We have shown that under a mechanical rule for intervention, positive expected profits are possible if the free exchange rate in the absence of intervention would have followed a first-order moving-average process or a first-order autoregressive process. These assumed processes are not meant to represent precisely how exchange rates will move, and interventions are often sporadic rather than following a mechanical rule. The analyses do, however, provide an antidote to the impression left by Corrado and Taylor that intervention can necessarily be expected to lose money in the long run, with the implied interpretation that central banks should have known better than to have intervened. The analysis with the moving average process suggests that central-bank profits from responding to what are at least in part transitory changes in the exchange rate can outweigh losses from the permanent changes even if the exchange-rate process is non-stationary. Furthermore, the analysis with an autoregressive process suggests that even very aggressive leaning against the wind by the central bank will yield positive expected profits eventually when there is a tendency for exchange rates to return to a long-run equilibrium level.

We should note that other analytical issues arise when central banks intervene to try to defend exchange rate target-zone systems, such as the Bretton Woods System or the European Exchange Rate Mechanism. In those cases, the issue becomes whether or for how long the central banks can weather speculative attacks on pegged exchange rates when there are divergent domestic policies. Our analysis applies more to regimes of managed floating in which the central bank is reacting to exogenous events in an attempt to moderate the severity of observed exchange rate changes.

References

Almekinders, Geert J. and Sylvester C.W. Eijffinger, "Empirical Evidence on Foreign Exchange Market Intervention: Where Do We Stand?" *Weltwirstschaftliches Archiv*, 127, 1991, 645-677.

Argy, Victor, "Exchange Rate Management in Theory and Practice," *Princeton Studies in International Finance*, No. 50, 1982.

Bank of England, "Intervention, Stabilization and Profits," *Bank of England Quarterly Review*, September 1983.

Corrado, Charles J. and Dean Taylor, "The Cost of a Central Bank Leaning Against a Random Walk," *Journal of International Money and Finance*, 5, 1986, 303 - 314.

Engle, Charles and James D. Hamilton, "Long Swings in the Dollar: Are They in the Data and Do Markets Know It?" *American Economic Review*, 80,September 1990, 689-713.

Friedman, Milton, "The Case for Flexible Exchange Rates," *Essays in Positive Economics*, University of Chicago Press, 1953, 157 - 203.

Glen, Jack D., "Real exchange rates in the short, medium, and long run," *Journal of International Economics*, 33 (August 1992), 147-166.

Goodhart, Charles, and L. Figliuoli, "Every Minute Counts in Financial Markets," *Journal of International Money and Finance*, 10, 1991, 23-52.

Huizinga, John, "An Empirical Investigation of the Long-Run Behavior of Real Exchange Rates," in K. Brunner and A.H. Meltzer (eds), *Empirical Studies of Velocity, Real Exchange Rates, Unemployment and Productivity*, North Holland, 1987.

Ito, Takatoshi and V. Vance Roley, "News from the U.S. and Japan: Which Moves the Yen/Dollar Exchange Rate?" NBER Working Paper No. 1853, March 1986.

Jacobson, Laurence R., "Calculations of Profitability for U.S. Dollar-Deutschemark Intervention," Board of Governors of the Federal Reserve System, *Staff Study* No. 131, September 1983.

Liu, Christina Y. and Jia He, "A Variance-Ratio Test of Random Walks in Foreign Exchange Rates, *The Journal of Finance*, 1991, 773-785.

Lo, Andrew W. and Craig MacKinlay, "Stock Market Prices Do Not Follw Random Walks: Evidence from a Simple Specification Test,: *Review of Financial Studies*, 1, 1988, 41-66.

Mussa, Michael, "Empirical Regularities in the Behavior of Exchange Rates and Theories of the Foreign Exchange Market," in K. Brunner and A.H. Meltzer (eds), *Policies for Employment, Prices and Exchange Rates*, North Holland, 1979.

Taylor, Dean, "Official Intervention in the Foreign Exchange Market, or, Bet against the Central Bank," *Journal of Political Economy*, 90, April 1982, 356 - 368.

Central Banks' Expected Profits from Intervention, Appendix

Derivations of Expected Profits

Taking the expected value of the profit function equation (4) in the text gives the following expression, assuming N > 1.

$$\text{(A.1)} \qquad E(\pi) = E\left\{\sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j - \alpha \sum_{t=1}^{N} I_t^2\right\}$$

The intervention rule and intervention effect imply as shown in equation (5):

$$\text{(A.2)} \qquad I_t = \frac{-\phi}{\alpha} \sum_{k=0}^{\infty} \phi^k \Delta F_{t-k}$$

<u>A Moving-Average Process</u>

Assume that the free exchange rate follows the moving average process:

$$\text{(A.3)} \qquad F_t = F_{t-1} + u_t - \theta\, u_{t-1}$$

with $Eu = 0$ and $Eu^2 = \sigma^2$. This implies the following covariance structure:

$$\text{(A.4)} \qquad E(\Delta F_t \Delta F_{t-k}) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } k = 0 \\ -\theta\sigma^2 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

$E(\pi)$ from equation (A.1) can be evaluated in two parts. The first term is:

$$\text{(A.5)} \quad E\left(\sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j\right) = E\left\{\left(\frac{-\phi}{\alpha}\sum_{t=1}^{N-1}\sum_{j=t+1}^{N} \Delta F_j\right)\left(\sum_{k=0}^{\infty} \phi^k \Delta F_{t-k}\right)\right\}$$

$$= \frac{-\phi}{\alpha}\sum_{t=1}^{N-1} E(\Delta F_{t+1}\Delta F_t) = \sum_{t=1}^{N-1} \frac{\theta\sigma^2\phi}{\alpha} = \frac{(N-1)\theta\phi\sigma^2}{\alpha}$$

For the second term in $E(\pi)$:

$$\text{(A.6)} \qquad E(I_t^2) = E\left\{\sum_{k=0}^{\infty} \frac{-\phi}{\alpha}\phi^k \Delta F_{t-k}\right\}^2 \qquad \text{from (A.2)}$$

$$= \frac{\phi^2}{\alpha^2}\sum_{k=0}^{\infty}\left[\phi^{2k}(1+\theta^2)\sigma^2 - 2\phi^{2k+1}\theta\sigma^2\right]$$

$$= \frac{\phi^2}{\alpha^2}\left\{\frac{(1+\theta^2)\sigma^2}{1-\phi^2} - \frac{2\theta\phi\sigma^2}{1-\phi^2}\right\}$$

$$= \frac{(1+\theta^2-2\theta\phi)\phi^2\sigma^2}{\alpha^2(1-\phi^2)}$$

From (A.1), (A.5) and (A.6),

$$\text{(A.7)} \qquad E(\pi) = N\left\{\frac{\theta\phi\sigma^2}{\alpha} - \frac{(1+\theta^2-2\theta\phi)\phi^2\sigma^2}{\alpha(1-\phi^2)}\right\} - \frac{\theta\phi\sigma^2}{\alpha}$$

$$= \frac{N\phi\sigma^2}{\alpha(1-\phi^2)}\left[\theta(1-\phi^2) - (1+\theta^2-2\theta\phi)\phi\right] - \frac{\theta\phi\sigma^2}{\alpha}$$

$$= \frac{N\phi(\theta-\phi)(1-\theta\phi)\sigma^2}{\alpha(1-\phi^2)} - \frac{\theta\phi\sigma^2}{\alpha}$$

This is reproduced as equation (8) in the text.

We next consider how the intervention affects the variance of changes in the exchange rate. To do this, use equations (1) and (2) in the text to relate S to F and take first differences. The result is:

$$\Delta S_t = \phi\,\Delta S_{t-1} + (1-\phi)\,\Delta F_t$$

This implies:

(A.8) $$\Delta S_t = (1 - \phi) \sum_{k=0}^{\infty} \phi^k \Delta F_{t-k}$$

It follows from (A.8) and (A.4) that:

(A.9) $$\begin{aligned} Var(\Delta S_t) &= (1 - \phi)^2 \left[\sum_{k=0}^{\infty} \phi^{2k} E(\Delta F_{t-k}{}^2) + \sum_{k=0}^{\infty} \phi^{2k+1} E(\Delta F_{t-k} \Delta F_{t-k-1}) \right] \\ &= \frac{(1 - \phi)\sigma^2}{(1 - \phi^2)} [(1 + \theta^2) - 2\phi\theta] \\ &= \frac{1 - \phi}{1 + \phi} (1 + \theta^2 - 2\phi\theta)\sigma^2 \leq (1 + \theta^2)\sigma^2 = Var(\Delta F_t) \end{aligned}$$

This is reproduced as (10) in the text.

An Autoregressive Process

Now assume that the free exchange rate follows a first-order autoregressive process:

(A.10) $$F_t = \delta F_{t-1} + u_t$$

where $0 \leq \delta \leq 1$ and $Eu = 0$, $Eu^2 = \sigma^2$. We first need to obtain the covariance structure for ΔF_t.

For $0 < \delta < 1$, $F_t = \sum_{j=0}^{\infty} \delta^j u_{t-j}$ and $\Delta F_t = F_t - F_{t-1} = (\delta - 1)F_{t-1} + u_t$. So

$$\Delta F_t = u_t + (\delta - 1) \sum_{j=0}^{\infty} \delta^j u_{t-j-1} \text{ and}$$

$$E(\Delta F_t \Delta F_t) = \sigma^2 + (\delta - 1)^2 \sum_{j=0}^{\infty} \delta^{2j}\sigma^2$$

$$= \sigma^2 + \frac{(1 - \delta)^2\sigma^2}{1 - \delta^2}$$

$$= \sigma^2 + \frac{(1 - \delta)\sigma^2}{1 + \delta} = \frac{2\sigma^2}{1 + \delta}$$

For $k \geq 1$, $E(\Delta F_t \Delta F_{t-k})$

$$= E\left[\left\{u_t + (\delta - 1)\sum_{j=0}^{\infty} \delta^j u_{t-j-1}\right\}\left\{u_{t-k} + (\delta - 1)\sum_{j=0}^{\infty} \delta^j u_{t-k-j-1}\right\}\right]$$

$$= E\left[\left\{(\delta - 1)\delta^{k-1}u_{t-k} + (\delta - 1)\sum_{j=k}^{\infty} \delta^j u_{t-j-1}\right\}\left\{u_{t-k} + (\delta - 1)\sum_{j=0}^{\infty} \delta^j u_{t-k-j-1}\right\}\right]$$

$$= E\left[\left\{(\delta - 1)\delta^{k-1}u_{t-k} + (\delta - 1)\sum_{j=k}^{\infty} \delta^j u_{t-j-1}\right\}\left\{u_{t-k} + (\delta - 1)\sum_{j=k}^{\infty} \delta^{j-k} u_{t-j-1}\right\}\right]$$

$$= \delta^{k-1}(\delta - 1)\sigma^2 + (\delta - 1)^2 \sum_{j=k}^{\infty} \delta^{2j-k}\sigma^2$$

$$= \left[\delta^{k-1} + (\delta - 1)\delta^k \sum_{j=0}^{\infty} \delta^{2j}\right](\delta - 1)\sigma^2$$

$$= \left[1 + \frac{\delta(\delta - 1)}{1 - \delta^2}\right](\delta - 1)\delta^{k-1}\sigma^2$$

$$= -\frac{(1 - \delta)\delta^{k-1}\sigma^2}{1 + \delta}$$

In summary, for the autoregressive terms, we have:

$$\text{(A.11)} \qquad E(\Delta F_t \Delta F_{t-k}) = \begin{cases} \dfrac{2\sigma^2}{1+\delta} & \text{if } k = 0 \\[2ex] -\dfrac{(1-\delta)\delta^{k-1}\sigma^2}{1+\delta} & \text{if } k \geq 1 \end{cases}$$

Before deriving expected profits, we first obtain the variance of the observed change in the exchange rate.

$$\mathrm{Var}(\Delta S_t) = (1-\phi)^2 E\left[\sum_{k=0}^{\infty} \phi^k \Delta F_{t-k} \sum_{k=0}^{\infty} \phi^k \Delta F_{t-k}\right] \quad \text{from (A.8)}$$

$$= 2(1-\phi)^2\sigma^2 \left\{ \sum_{k=0}^{\infty} \frac{\phi^{2k}}{1+\delta} - \sum_{j>k}^{\infty} \phi^{j+k} \left(\frac{1-\delta}{1+\delta}\right) \delta^{j-k-1} \right\}$$

Note that $\displaystyle\sum_{j>k}^{\infty} \phi^{j+k}\delta^{j-k-1} = \left(\sum_{k=0}^{\infty} \phi^{2k+1}\right)\left(\sum_{j=k+1}^{\infty} (\phi\delta)^{j-k-1}\right)$

$$= \phi\left(\sum_{k=0}^{\infty} \phi^{2k}\right)\left(\sum_{j=0}^{\infty} (\phi\delta)^{j}\right)$$

$$= \frac{\phi}{(1-\phi\delta)(1-\phi^2)}$$

$$\text{(A.12)} \quad \therefore \mathrm{var}(\Delta S_t) = \frac{2(1-\phi)^2\sigma^2}{1+\delta} \left\{ \frac{1}{1-\phi^2} - \frac{(1-\delta)\phi}{(1-\phi\delta)(1-\phi^2)} \right\}$$

$$= \frac{2(1-\phi)\sigma^2}{(1+\phi)(1+\delta)} \left\{ 1 - \frac{(1-\delta)\phi}{1-\phi\delta} \right\} = \frac{2(1-\phi)^2\sigma^2}{(1+\phi)(1+\delta)(1-\phi\delta)}$$

This is reproduced as (14) in the text.

Since $I_t = -\lambda(\Delta S_t)$, $E(I_t^2) = \lambda^2 \operatorname{var}(\Delta S_t)$

Also since $\phi = \dfrac{\alpha\lambda}{1+\alpha\lambda}$, $\lambda^2 = \dfrac{\phi^2}{\alpha^2(1-\phi^2)}$, and it follows that

$$\text{(A.13)} \quad E(I_t^2) = \frac{2\phi^2\sigma^2}{\alpha^2(1+\delta)(1+\phi)(1-\phi\delta)}$$

This will be used for the second term in equation (A.1) for expected profits.

To obtain the first term for expected profits in equation (A.1), we need to consider various values of the parameter δ. If $\delta = 0$, then $\Delta F_t = \Delta u_t$ from (A.10) and

$$\text{(A.14)} \quad E\left(\sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j\right) = E\left\{\left(\frac{-\phi}{\alpha}\sum_{t=1}^{N-1}\sum_{j=t+1}^{N}\Delta F_j\right)\left(\sum_{k=0}^{\infty}\phi^k \Delta F_{t-k}\right)\right\} \qquad \text{from (A.2)}$$

$$= E\left\{\left(\frac{-\phi}{\alpha}\sum_{t=1}^{N-1}\sum_{j=t+1}^{N}\Delta u_j\right)\left(\sum_{k=0}^{\infty}\phi^k \Delta u_{t-k}\right)\right\} \qquad \text{from (A.10) and } \delta = 0$$

$$= \frac{(N-1)\phi\sigma^2}{\alpha}$$

For $\delta = 1$, $\Delta F_t = u_t$ and:

$$\text{(A.15)} \qquad E\left(\sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j\right) = 0.$$

Now for $0 < \delta < 1$, we have:

$$E(\Delta F_j I_t) = \frac{-\phi}{\alpha}\sum_{k=0}^{\infty}\phi^k E(\Delta F_j \Delta F_{t-k})$$

$$= \quad \frac{\phi}{\alpha} \sum_{k=0}^{\infty} \phi^k \frac{1-\delta}{1+\delta} \delta^{j+k-t-1} \sigma^2 \qquad \text{from (A.11)}$$

$$= \quad \frac{\phi(1-\delta)\delta^{j-t-1}\sigma^2}{\alpha(1+\delta)(1-\phi\delta)}$$

Since for $0 < \delta < 1$, $\sum_{j=t+1}^{N} \delta^{j-t-1} = \frac{1-\delta^{N-t}}{1-\delta}$, we have:

$$\text{(A.16)} \qquad E\left(\sum_{t=1}^{N-1} I_t \sum_{j=t+1}^{N} \Delta F_j\right) = \sum_{t=1}^{N-1} \frac{\phi(1-\delta^{N-t})\sigma^2}{\alpha(1+\delta)(1-\phi\delta)}$$

$$= \frac{\phi\sigma^2}{\alpha(1+\delta)(1-\phi\delta)} \left(N - 1 - \frac{\delta - \delta^N}{1-\delta}\right)$$

$$= \frac{\phi\sigma^2}{\alpha(1+\delta)(1-\phi\delta)} \left(N - \frac{1-\delta^N}{1-\delta}\right)$$

Therefore, using (A.13) through (A.16) in (A.1)

$$\text{(A17)} \qquad E(\pi) = \begin{cases} \dfrac{\phi\sigma^2}{\alpha(1+\delta)(1-\phi\delta)} \left\{N\left(\dfrac{1-\phi}{1+\phi}\right) - \dfrac{1-\delta^N}{1-\delta}\right\} & \text{if } 0 \le \delta < 1 \\ -\dfrac{N\phi^2\sigma^2}{\alpha(1-\phi^2)} & \text{if } \delta = 1 \end{cases}$$

This is reproduced as (12) in the text.