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C. D. McGillem

M. Svedlow

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IMAGE REGISTRATION ERROR VARIANCE
AS A MEASURE OF OVERLAY QUALITY

C. D. McGillem and M. Svedlow

Laboratory for Applications of Remote Sensing, Purdue University, West Lafayette, Indiana; and Laboratory for Applications of Remote Sensing, Purdue University, West Lafayette, Indiana

I. ABSTRACT

When one image (the signal) is to be registered with a second image (the signal plus noise) of the same scene, one would like to know the accuracy possible for this registration. This paper derives an estimate of the variance of the registration error that can be expected via two approaches. The solution in each instance is found to be a function of the effective bandwidth of the signal and the noise, and the signal-to-noise ratio. Application of these results to ERTS data indicates that for most cases registration variances will be significantly less than the diameter of one picture element.

II. INTRODUCTION

Many instances arise in which one would like to register two different images of the same scene. When one attempts to accomplish this overlay of images, several problems are encountered. An important question that arises is, given images of a particular scene, to within what tolerance can the two images be aligned? This is the problem with which this paper deals.

Two models for the variance of the error in the registration of two different images of the same scene are developed. The method of solution employed is analogous to that used for the determination of the error in the measured delay time in a radar system. For purposes here the radar system model assumes that the returned signal is a delayed version of the original signal corrupted by additive noise. As adapted to the registration of two images, the noise is defined as the difference between the two images at the correct registration position, and is therefore additive. The time delay corresponds to a spatial translation or displacement.

Several analyses of the radar problem have been carried out based upon different premises.[1,2,3] These approaches may be categorized as those which use the probability density function of the noise directly and those which do not. The first case utilizes maximum a posteriori, maximum likelihood, or minimum mean square error estimates. All three estimators are based upon knowledge of the noise probability density function. The second case is based only upon the output of a filter which gives a maximum output at the correct time delay when the input is noise free.

The solution to the problem of the first case, in which the probability density function of the noise is directly involved, depends upon the cost function which is assigned to the error and the a posteriori distribution, $p_f[m(\tau)]$, of the signal as a function of a parameter, $m(\tau)$, given the received signal, f. A minimum mean square error estimate is the mean of $p_f[m(\tau)]$; an absolute value cost function gives the median of the probability function; the maximum a posteriori estimate yields the maximum of $p_f[m(\tau)]$. The maximum likelihood estimate may be viewed as the same as the maximum a posteriori estimate when there is no prior knowledge of the density function of the parameter, $p[m(\tau)]$, or $p[m(\tau)]$ is assumed uniform over the entire range of interest. All four of the above cost functions will yield the same solution of $p[m(\tau)]$ uniform over the range of interest, and a symmetric, unimodal density function,

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$p_f[m(\tau)]$.[3] A Gaussian distribution which has been assumed for $p_f[m(\tau)]$ in several analyses, is a member of this latter class. The reason for the use of the Gaussian distribution is the availability of a closed form analytical solution.

An analysis of this sort should prove useful in several respects. The results should give an indication of the best possible registration of two images given the models of the data and noise. Once the models of the parameters involved have been found or assumed, an optimum processor to implement the overlaying procedure may be developed. Comparison of existing registration systems with the results obtained herein may also be performed. However, one must keep in mind the assumptions the entire analysis will be based on, for different assumptions may yield different results.

It is assumed in the following investigation that the useful signal is present, reducing the problem to one of estimation only rather than detection as well as estimation. It is further assumed that the signal shape is known and nonrandom, although the parameter that is to be measured is a random variable. Since the original signal is known, it does not have a probability density function. However, the second signal does contain noise and possibly other perturbations and is therefore a sample function of a random process. The problem will be approached with this in mind.

III. METHOD 1

This derivation of the variance of the registration error is an adaptation of the solution obtained by Zubakov and Wainstein.[4] In this problem one assumes that the additive noise is jointly Gaussian with zero mean. It is also assumed that the density function of the parameter (i.e. the misregistration or displacement of the images) is uniform in the range of interest.

With these assumptions one may construct the likelihood function and then find its peak to determine the optimum estimator.

$$(1)\quad \Lambda(\tau_x,\tau_y) = p_m(\tau_x,\tau_y)\,\frac{p_{m(\tau_x,\tau_y)}(f)}{p_o(f)}$$

where,

$\Lambda(\tau_x,\tau_y)$ = likelihood function of the displacement parameters, τ_x and τ_y

$p_m(\tau_x,\tau_y)$ = density function of the parameters, τ_x and τ_y, given the known signal

$p_{m(\tau_x,\tau_y)}(f)$ = conditional density of $f(x,y)$ given $m(x,y,\tau_x,\tau_y)$ is present

$p_o(f)$ = conditional density of $f(x,y)$ given $m(x,y,\tau_x,\tau_y)$ is absent

$m(x,y,\tau_x,\tau_y)$ = known signal as a function of the spatial coordinates and the displacement parameters

$f(x,y)$ = $m(x,y) + n(x,y)$ = received signal

$n(x,y)$ = additive noise; assumed independent of the signal

Since the data that is being analyzed is discrete, it is convenient to use integer subscripts rather than continuous spatial coordinates. A further notational savings is realized by combining the double subscripts into a single subscript. A two dimensional array m_{ij}, $i = 1,\ldots,p$; $j = 1,\ldots,q$, is converted to a one dimensional data set m_h, $h = 1,\ldots,pq$. This conversion loses nothing from the standpoint of the results to be derived.

In the discrete case a continuous function has been sampled and may be denoted,

$$m_h = m(x_i,y_j)$$
$$n_h = n(x_i,y_j)$$
$$f_h = f(x_i,y_j) = m_h + n_h$$
$$h = 1,\ldots,H$$
$$H = pq = \text{total number of samples}$$

To arrive at an analytical result, the probability density function of the noise must be known. Because of the many independent contributions to the differences between images being registered, it is reasonable to approximate the density function as being Gaussian. The probability density function of the noise is therefore given by

$$(2)\quad p_{\underline{n}}(\underline{n}) = \frac{1}{(2\pi)^{H/2}|\underline{R}|^{1/2}}\exp\left(-\frac{1}{2}\,\underline{n}^T\underline{R}^{-1}\underline{n}\right)$$

where $\underline{R}$ is the covariance matrix of the noise, $R_{gh} = E[n_g n_h]$. The density functions in the likelihood equation then become,

$$p_{m(\tau_x,\tau_y)}(f) = p_{\underline{n}}(\underline{f} - m(\tau_x,\tau_y))$$

$$p(f) = p_{\underline{n}}(f)$$

$$\underline{f}^T = (f_1,\ldots,f_H)$$

$$\underline{m}^T = (m_1,\ldots,m_H)$$

The likelihood function can be reduced to

$$(3)\quad \Lambda(\tau_x,\tau_y) = p_m(\tau_x,\tau_y) \exp\left\{\sum_g^H \sum_h^H Q_{gh} f_g m_h(\tau_x,\tau_y) - \frac{1}{2}\sum_g^H \sum_h^H Q_{gh} m_g(\tau_x,\tau_y) m_h(\tau_x,\tau_y)\right\}$$

$Q_{gh} = gh^{th}$ element of $\underline{R}^{-1}$

Since it is only the maximum of $\Lambda(\tau_x,\tau_y)$ which is desired, the problem can be reduced even further. Let $p_m(\tau_x,\tau_y)$ be a uniform distribution over a given area. This is a reasonable assumption since there is no a priori knowledge about the actual distribution. The question in point here is concerned only with a spatial delay, so that the summation term,

$$(4)\quad \mu = \sum_g^H \sum_h^H Q_{gh} m_g(\tau_x,\tau_y) m_h(\tau_x,\tau_y)$$

will also be a constant function of τ_x and τ_y. The only factor which is not a constant with respect to τ_x and τ_y is,

$$(5)\quad \phi = \sum_g^H \sum_h^H Q_{gh} f_g m_h(\tau_x,\tau_y)$$

Therefore the maximum of $\Lambda(\tau_x,\tau_y)$ is determined solely by the maximum of ϕ. The optimum receiver is then the one which finds the maximum of ϕ. This type of receiver may be viewed as a correlator which is weighted according to the inverse noise covariance function, Q_{gh}. For the case in which the noise is white with spectrum $N_o/2$, the covariance matrix becomes $\frac{2}{N_o}\underline{I}$ ($\underline{I}$ = identity matrix), and the optimum receiver is simply a correlator.

$$(6)\quad \phi = \frac{2}{N_o}\sum_h^H f_h m_h(\tau_x,\tau_y)$$

Given that the maximum point (this translation position is denoted by $(\hat{\tau}_x,\hat{\tau}_y)$) of the likelihood function has been found, a measure of the accuracy of the estimate is necessary so that the performance of the estimator may be evaluated. One such measure is the variance of the estimate about the maximum point of $\Lambda(\tau_x,\tau_y)$. For this analysis it is convenient to use $\ln[\Lambda(\tau_x,\tau_y)]$ which is a monotonic function of $\Lambda(\tau_x,\tau_y)$.

The logarithm of the likelihood function is expanded in a second order Taylor series as a function of the delay parameters about its peak in the x-axis and y-axis directions separately. It is assumed that $\ln[\Lambda(\tau_x,\tau_y)]$ can be approximated by a second order polynomial around its peak.

Only the results in the x-axis direction are given since the y-axis direction results are completely analogous.

$$(7)\quad \ln\Lambda(\tau_x,\hat{\tau}_y) \approx \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y) + \frac{\partial \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial \tau_x}(\tau_x - \hat{\tau}_x) + \frac{1}{2}\frac{\partial^2 \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial \tau_x^2}(\tau_x - \hat{\tau}_x)^2$$

A necessary condition for the maximum point of $\ln\Lambda(\tau_x,\tau_y)$ is that,

$$(8)\quad \frac{\partial \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial \tau_x} = 0 = \frac{\partial \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial \tau_y}$$

The Taylor series expansion may then be reduced to,

$$(9)\quad \ln\Lambda(\tau_x,\hat{\tau}_y) \approx \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y) + \frac{1}{2}\frac{\partial^2 \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial \tau_x^2}(\tau_x - \hat{\tau}_x)^2$$

Rearranging this equation one obtains,

$$(10)\quad \Lambda(\tau_x,\hat{\tau}_y) = \Lambda(\hat{\tau}_x,\hat{\tau}_y)\exp\left\{-\frac{1}{2}\frac{(\tau_x-\hat{\tau}_x)^2}{\Delta_x^2}\right\}$$

where,

$$(11)\quad \Delta_x^2 = -\left[\frac{\partial^2 \ln\Lambda(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x^2}\right]^{-1} = \text{variance}$$

in the x-direction

Assuming $p_m(\tau_x,\tau_y)$ to be uniformly distributed,

$$(12)\quad \frac{1}{\Delta_x^2} = \sum_g^H \sum_h^H Q_{gh}[m_g(\hat{\tau}_x,\hat{\tau}_y)-f_g]\,\frac{\partial^2 m_h(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x^2} + \sum_g^H \sum_h^H Q_{gh}\,\frac{\partial m_g(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}\,\frac{\partial m_h(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}$$

If one further assumes a large signal-to-noise ratio, then

$$(13)\quad \frac{1}{\Delta_x^2} = \sum_g^H \sum_h^H Q_{gh}\,\frac{\partial m_g(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}\,\frac{\partial m_h(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}$$

since $[m_g(\hat{\tau}_x,\hat{\tau}_y) - f_g]$ is dependent only upon the noise and is small compared to $m_g(\hat{\tau}_x,\hat{\tau}_y)$.

Greater insight into the solution may be obtained by looking at the result in the frequency domain as opposed to the spatial domain. This transformation yields an interesting answer. The variance becomes,

$$(14)\quad \Delta_x^2 = \frac{1}{\Delta W_x^2 \mu}$$

where

ΔW_x^2 = effective bandwidth in the x-axis direction

μ = signal-to-noise ratio

$$(15)\quad \mu = \sum_u^p \sum_v^q \frac{|M(u,v)|^2}{S_R(u,v)}$$

$$(16)\quad \Delta W_x^2 = \frac{4\pi^2 \sum_u^p \sum_v^q \frac{u^2|M(u,v)|^2}{S_R(u,v)}}{\sum_u^p \sum_v^q \frac{|M(u,v)|^2}{S_R(u,v)}}$$

$M(u,v)$ = Fourier transform of the known signal

$S_R(u,v)$ = noise spectrum

In the spatial domain,

$$(17)\quad \mu = \sum_g^H \sum_h^H Q_{gh} m_g(\hat{\tau}_x,\hat{\tau}_y)\, m_h(\hat{\tau}_x,\hat{\tau}_y)$$

$$(18)\quad W_x^2 = \frac{\sum_g^H \sum_h^H Q_{gh}\,\frac{\partial m_g(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}\,\frac{\partial m_h(\hat{\tau}_x,\hat{\tau}_y)}{\partial\tau_x}}{\sum_g^H \sum_h^H Q_{gh}\, m_g(\hat{\tau}_x,\hat{\tau}_y)\, m_h(\hat{\tau}_x,\hat{\tau}_y)}$$

With the above assumptions the variance has been reduced to a function of the effective bandwidth and signal-to-noise ratio. This implies that if one can estimate the effective bandwidth and the signal-to-noise ratio in the x-axis and y-axis directions, then the variance of the registration error can be estimated.

Now consider the second derivation for the variance which is based upon different assumptions.

IV. METHOD 2

A second derivation of the variance of the registration error is developed in this section. In this case, the only assumption about the signal and processor is that in the absence of noise, the output of the processor will be a maximum at the correct time delay.[2] No assumptions about the probability distribution of the noise are needed. As will be seen, the results of this derivation are similar to those obtained in the previous derivation, even though the two approaches are quite unalike.

The signal corresponding to the image to be overlayed is comprised of two components, the desired signal and additive noise. This signal is passed through a filter and the position where the maximum of the output signal occurs is taken to be the correct registration position. However, the filter is designed to yield a maximum at the correct delay only in the noise free case. The discrepancy between these two positions is the registration error.

First consider the parameters involved.

$f(x,y)$ = signal

$m(x,y)$ = additive noise

$f(x,y) + m(x,y)$ = data set to be registered

$h(x,y)$ = filter impulse response

$g(x,y) = f(x,y)*h(x,y)$ = output signal in the absence of noise

$n(x,y) = m(x,y)*h(x,y)$ = output due to the noise input

$z(x,y) = g(x,y) + n(x,y)$ = composite output signal used to estimate the correct registration position

$(\tilde{x},\tilde{y})$ = true registration position

$(\bar{x},\bar{y})$ = estimated registration position

The derivation proceeds as follows. First expand $g(x,y)$ in a second order Taylor series about (x,y).

$$(19)\quad g(x,y) \simeq g(\tilde{x},\tilde{y}) + g_x(\tilde{x},\tilde{y})[x-\tilde{x}] + g_y(\tilde{x},\tilde{y})\,[y-\tilde{y}] + g_{xy}(\tilde{x},\tilde{y})\,[x-\tilde{x}]\,[y-\tilde{y}] + \frac{1}{2}\,g_{xx}(\tilde{x},\tilde{y})[x-\tilde{x}]^2 + \frac{1}{2}g_{yy}(\tilde{x},\tilde{y})[y-\tilde{y}]^2 + \ldots$$

where

$$g_x(\tilde{x},\tilde{y}) = \left.\frac{\partial g(x,y)}{\partial x}\right|_{x=\tilde{x},\ y=\tilde{y}};$$

this subscript notation is used for the remainder of this section

Assume that $(x-\tilde{x})$ and $(y-\tilde{y})$ are small enough so that all higher order terms may be neglected.

Note that a necessary condition for a maximum is $\frac{\partial g(\tilde{x},\tilde{y})}{\partial x} = 0 = \frac{\partial g(\tilde{x},\tilde{y})}{\partial x}$. Substitute this result into the equation for $z(x,y)$.

$$(20)\quad z(x,y) = g(\tilde{x},\tilde{y}) + g_{xy}(\tilde{x},\tilde{y})\,[x-\tilde{x}]\,[y-\tilde{y}] + \frac{1}{2}\,g_{xx}(\tilde{x},\tilde{y})\,[x-\tilde{x}]^2 + \frac{1}{2}\,g_{yy}(\tilde{x},\tilde{y})\,[y-\tilde{y}]^2 + n(x,y)$$

Again use the necessary condition for an observed maximum, $\frac{\partial z(\bar{x},\bar{y})}{\partial x} = 0 = \frac{\partial z(\bar{x},\bar{y})}{\partial y}$,

$$(21)\quad z_x(\bar{x},\bar{y}) = 0 = g_{xy}(\tilde{x},\tilde{y})[\bar{y}-\tilde{y}] + g_{xx}(\tilde{x},\tilde{y})[\bar{x}-\tilde{x}] + n_x(\bar{x},\bar{y})$$

$$(22)\quad z_y(\bar{x},\bar{y}) = 0 = g_{xy}(\tilde{x},\tilde{y})[\bar{x}-\tilde{x}] + g_{yy}(\tilde{x},\tilde{y})[\bar{y}-\tilde{y}] + n_y(\bar{x},\bar{y})$$

Arrange these equations in terms of $(\bar{x}-\tilde{x})$ and $(\bar{y}-\tilde{y})$, the error in the registration.

$$(23)\quad (\bar{x}-\tilde{x}) = \frac{g_{xy}n_y - g_{yy}n_x}{g_{xx}g_{yy} - g_{xy}^2}$$

$$(24)\quad (\bar{y}-\tilde{y}) = \frac{g_{xy}n_x - g_{xx}n_y}{g_{xx}g_{yy} - g_{xy}^2}$$

where the arguments, $(\bar{x},\bar{y})$ and $(\tilde{x},\tilde{y})$ have been left out for notational convenience.

One can now find the variance of the error by taking the expectation of $(\bar{x}-\tilde{x})^2$ and $(\bar{y}-\tilde{y})^2$. It is assumed that $E[\bar{x}-\tilde{x}] = 0 = E[\bar{y}-\tilde{y}]$.

$$(25)\quad \mathrm{Var}[\bar{x}-\tilde{x}] = E[(\bar{x}-\tilde{x})^2] = \overline{(\bar{x}-\tilde{x})^2}$$

$$(26)\quad \mathrm{Var}[\bar{y}-\tilde{y}] = E[(\bar{y}-\tilde{y})^2] = \overline{(\bar{y}-\tilde{y})^2}$$

$$(27)\quad \overline{(\bar{x}-\tilde{x})^2} = \frac{g_{xy}^2\overline{n_y^2} - 2g_{xy}g_{yy}\overline{n_yn_x} + g_{yy}^2\overline{n_x^2}}{[g_{xx}g_{yy} - g_{xy}^2]^2}$$

$$(28)\quad \overline{(\bar{y}-\tilde{y})^2} = \frac{g_{xy}^2\overline{n_x^2} - 2g_{xy}g_{xx}\overline{n_yn_x} + g_{xx}^2\overline{n_y^2}}{[g_{xx}g_{yy} - g_{xy}^2]^2}$$

One may use these equations to calculate the variance of the error, but in doing so, it is found that a filter function must be specified first. This is intrinsic in the parameters in these equations. This is seen more clearly if one writes these terms as a function of the filter (stationarity is assumed).

$$(29)\quad \overline{n_y^2(\bar{x},\bar{y})} = \iiiint h_y(\bar{x}-\alpha,\bar{y}-\beta)\,h_y(\bar{x}-\gamma,\bar{y}-\lambda)\,R_m(\alpha-\gamma,\beta-\lambda)\,d\alpha\,d\beta\,d\gamma\,d\lambda$$

to equal zero it is sufficient that,

(44) $K(u,v) = K(-u,v)$

or necessary and sufficient that,

(45) $\int_0^\infty\int_0^\infty uv\, K(u,v)\, du\, dv = \int_0^\infty\int_0^\infty uv\, K(-u,v) du\, dv$

The expressions then become

(46) $\overline{(\bar{x}-\tilde{x})^2} = \frac{1}{B_x^2 SNR}$

(47) $\overline{(\bar{y}-\tilde{y})^2} = \frac{1}{B_y^2 SNR}$

An example of when these last assumptions might apply is the following situation. Let $F(u,v)$ and $S_m(u,v)$ be bandlimited to W_x and W_y in the respective axis directions. And let $\frac{|F(u,v)|^2}{S_m(u,v)}$ equal a constant. This would occur when the noise spectrum has a shape similar to the signal spectrum. In this case it might be advantageous to model the two spectra as differing only by a constant factor for simplicity in estimating the variance to be expected. This may be written,

(48) $\frac{|F(u,v)|^2}{S_m(u,v)} = c$, a constant

From equation (43),

(49) $SNR = c \int_{-W_x}^{W_x}\int_{-W_y}^{W_y} du\, dv$

So,

(50) $c = \frac{SNR}{4\, W_x\, W_y}$

Then from equations (41),(42) and (43),

(51) $B_x^2\, SNR = 4\pi^2\; c \left[\frac{2\, W_x^3}{3}\right] (2\, W_y)$

(52) $B_y^2\, SNR = 4\pi^2\; c(2\, W_x) \left[\frac{2\, W_y^3}{3}\right]$

Substituting in the expressions for c, the variances are,

(53) $\overline{(\bar{x}-\tilde{x})^2} = \frac{3}{4\pi^2\, W_x^2\, SNR}$

(54) $\overline{(\bar{y}-\tilde{y})^2} = \frac{3}{4\pi^2\, W_y^2\, SNR}$

The respective standard deviations then are,

(55) Standard deviation of $(\bar{x}-\tilde{x}) = \frac{1}{2\pi\, W_x}\sqrt{\frac{3}{SNR}}$

(56) Standard deviation of $(\bar{y}-\tilde{y}) = \frac{1}{2\pi\, W_y}\sqrt{\frac{3}{SNR}}$

One may obtain a quantitative feel for the values of these expressions by using the sampling intervals for the ERTS-1 data in this example. The sampling interval is about 60 meters along the columns and about 80 meters along the lines. Substituting these values in equations (55) and (56), one finds that,

(57) Standard deviation of error along the lines $= \frac{44.1}{\sqrt{SNR}}$ meters

(58) Standard deviation of error along the columns $= \frac{33.1}{\sqrt{SNR}}$ meters

These results indicate that with the chosen filter, the standard deviation of the registration error is quite small.

V. CONCLUSION

An evaluation of the quality of the registration of two images is possible via an estimate of the variance of the error. This should prove useful in several respects. It may be a basis for the analysis of different registration systems by giving a way to estimate the expected accuracy of the system. It also provides a straightforward way of estimating this error.

The two approaches used are quite different even though the solutions are similar. The variance in each case was found to be a function of the effective bandwidth of the signal and noise, and the signal-to-noise ratio.

As a final consideration the basic assumptions needed for the two methods are listed. These assumptions are important and must be realized fully to be sure that they apply to the situation in which they will be utilized. For the first method these assumptions are: the noise is additive and independent of the signal; the joint probability density function of the noise is Gaussian; the a priori distribution of the delay parameters is uniform over the range of interest; the variance may be modeled in the x-axis and y-axis directions separately; the final result is dependent upon a large signal-to-noise ratio [cf. step from equation (12) to (13)].

The basic assumptions for the second method are: the noise is additive and independent of the signal; the noise spectrum must be known; the chosen filter is the "matched filter"; to obtain results completely analogous to the first method there is one further assumption that must be made about the ratio $\frac{|F(u,v)|^2}{S_m(u,v)}$ [cf. equations (44) and (45)].

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