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The monadic second-order theory of ordinals $\omega_2$

J. Richard Buchi and Charles Zaiontz

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0. Introduction

We will present here a method for deciding the truth of monadic second-order sentences in \([\alpha, <]\), for all \(\alpha < \omega_2\). (Theorem 3.8); we also show that the MT of all well-orders \(< \omega_2\) is decidable (Corollary 3.9). These results are cited in the Notices of the American Math Society and in [Bü]. A discussion of some properties of the MT[\(\alpha, <\)] (the monadic second-order theory of \([\alpha, <]\)) for \(\alpha < \omega_2\) is included, e.g. which are MT-categorical, finitely MT-axiomatizable, and which ones are MT-equivalent (Theorems 3.6 and 3.7). We use the ideas, techniques and notation of [Büchi]. Shelah in [Sh] has subsequently obtained similar results using a nondeterministic method. As our method uses a deterministic approach we get added information.

To show that MT[\(\alpha, <\)] is decidable for \(\alpha < \omega_2\), it is natural to mimic the proof of the decidability of countable well-orderings ([Bü] Section 4). It is then desirable to start with the same transition system as in [Bü] Section 4, adding \(\chi_1[\sup Z, Zz]\) as a transition condition for \(\omega_1\)-limits, and use the techniques of [Bü] Section 6 to extend the subset-construction uniformly to all \(\alpha < \omega_2\). However, we will show (Section 1) that for any such

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transition system there is no deterministic system of the same form which accepts the same input. To remedy the situation we show how to construct a deterministic system of equal behavior but with a more complicated transition condition at \( w_1 \)-limits involving relativized filters of cofinal-closed sets. In order to prove a complementation lemma we then must get rid of the relativized quantifiers—thus yielding a non-deterministic system. As we use splicing and the fact that \( \mathcal{G}_{w_1} \) is atomless for any \( w_1 \)-limit \( z \) (Corollary 1.10), we will require the axiom of choice, \( AC_{w_1}^\gamma (\gamma = 2^\omega) \).
1. The relativized filters of cofinal-closed sets

In this section we discuss the filters $\mathcal{F}_1^\alpha$ of sets cofinal-closed in $\alpha$, where $\alpha$ is $\kappa_1$-accessible (i.e. $\omega_1$-accessible) if $\text{AC}^\omega_1$, and in particular we will be interested in the relativized filters, $\mathcal{F}_1^U,\alpha$ where $U$ is a set cofinal in $\alpha$. These latter filters will be useful for technical reasons when we extend our decision procedure up to $\omega_2$.

In the sequel we will assume $\text{AC}^\omega_1$—thus $\kappa_1 = \omega_1$, and for any $\omega_0$-inaccessible limit ordinal $\alpha$, let $\mathcal{F}_1^\alpha$ be the set of all those subsets of $\alpha$ which contain a cofinal-closed set, i.e.

\[(1) \quad x \in \mathcal{F}_1^\alpha: \exists Q \subseteq \alpha \land Q \subseteq x \land (\forall x)\big[ (\exists x^t)Qt \supset Qx \big] \land (\exists x^t)Qt\]

The following are proved exactly as in [Bü] Section 5.

**Lemma 1.1:** Suppose $\alpha$ is an $\omega_0$-inaccessible limit ordinal. If $X_i, i < \omega_0$ are cofinal-closed subsets of $\alpha$, then so is $X = \bigcap_{i < \omega_0} X_i$.

**Theorem 1.2:** For any $\omega_0$-inaccessible limit $\alpha$, the class $\mathcal{F}_1^\alpha$ is an $\omega_0$-filter on $\mathcal{P}\alpha$. Equivalently, $\mathcal{C}_I^\alpha(X; x \in \mathcal{F}_1^\alpha)$ is an $\omega_0$-ideal in the Boolean algebra of all subsets of $\alpha$.

**Remark 1.3:** If $\alpha \notin \omega_0$, then $X$ is cofinal $\alpha$ if and only if the derivative $X'$ (set of all limits of members of $X$) is cofinal, and therefore, cofinal-closed $\alpha$. Also $X$ is
cofinal \( \alpha \) just in case \( X' \in f_1^\alpha \).

Remark 1.4: If \( \alpha = \omega_1 \), there is an \( \omega_1 \)-sequence \( P \) of \( \omega_0 \)-limits which is cofinal-closed \( \alpha \).

Proof: If \( \alpha = \omega_1 \), there is an \( \omega_1 \)-sequence \( Q \) cofinal \( \alpha \). Take \( P = Q' \cap \alpha \) and the result follows by Remark 1.3. Q.E.D.

Remark 1.5: Let \( Z \) be a map from an \( \omega_0 \)-inaccessible limit \( \alpha \), to a finite set of states, \( K \). At most one state of \( Z \) occurs cofinal-closed \( \alpha \). Exactly one set \( D \) of states occurs cofinal-closed as \( \sup \alpha Z \), namely \( D = \sup \alpha Z \). In fact (using Remark 1.4) there is a cofinal-closed \( Q \) such that \( Qx \subseteq x = \omega_0 \wedge \sup \alpha Z = \sup \alpha Z \).

To facilitate subsequent discussion we make the following convention:

Convention: Throughout this paper it is assumed that \( x \) ranges over \( \omega_0 \)-limits and unless stated otherwise that \( z \) ranges over \( \omega_1 \)-limits. We can do this as \( x = \omega_0 \) and \( z = \omega_1 \) are definable in our language by \( \text{Acc}_0(x) \) and \( \text{Acc}_1(z) \) (see [Bü] Section 1).

Ideally, we would like to have a subset-construction for ordinals \( < \omega_2 \) which is exactly like that for countable ordinals ([Bü] Lemma 4.4). Namely,

For any transition-system \( \Gamma = [H, \kappa_0, \kappa_1] \) of form
(2) \( \Gamma_u(X,Z) : (\forall t) u H[Xt,Zt,Zt'] \land (\forall x) u 'X_0[\sup_0 XZ, Zx] \land (\forall y) u 'Y[\sup_1 YZ, Zz] \)

with initial condition \( E[\cdot] \) and terminal condition \( L[\cdot] \) one can construct an automata \( A = [a,F,\mathcal{L}_0,\mathcal{L}_1] \) of form

(3) \( Y_0 = a \land Yt' = F[Xt,Yt] \land Yx = \mathcal{L}_0[\sup_0 XY] \land Yz = \mathcal{L}_1[\sup_1 YZ] \)

with terminal condition \( L'[\cdot] \) such that, for any \( \alpha < \omega_2 \) and any input \( X[0,\alpha] \), \( [E,T,L] \) accepts \( X \) if and only if \( [A,L'] \) accepts \( X \), i.e. \( (\exists Z) E[Z0] \land \Gamma_0(X,Z) \land L[Z\alpha] \) holds just in case the recursion \( A \) applied to \( X \) yields a final state \( Z\alpha \) such that \( L'[Z\alpha] \).

The following simple counterexample shows that this is only possible in a very strange set theory:

Counterexample: Clearly \( X \in \mathcal{J}_1^\omega = (\exists Y). (\forall t) Zt = Xt \land (\forall t^1) Zt. \)

Thus if the above claim were true we could find \( a,F,\mathcal{L}_0,\mathcal{L} \) such that \( X \in \mathcal{J}_1^\omega = (\exists Y). Y_0 = a \land Yt' = F[Xt,Yt] \land Yx = \mathcal{L}_0[\sup_0 XY] \land \mathcal{L}[\sup_1 YZ]. \)

For such \( Y \), let \( D = \sup_0 Y \). By remark 1.5, \( \{x; \sup_0 X = D\} \in \mathcal{J}_1^{\omega_1} \) and so \( \{x; Yx = \mathcal{J}_0[D]\} \in \mathcal{J}_1^{\omega_1} \). Consequently \( \sup_1 Y = \{\mathcal{J}_0[D]\} \).

From this we have,

\( X \in \mathcal{J}_1^{\omega_1} = (\exists Y). Y_0 = a \land Yt' = F[Xt,Yt] \land Yx = \mathcal{L}_0[\sup_0 XY] \land [\sup_0 \mathcal{L}\sup_1 Y] \)

where \( \mathcal{L}'[\sup_0 Y] \) is \( \mathcal{L}'[\mathcal{J}_0[\sup_0 Y]] \). Thus \( \sup_1 Y \) is expressible by a \( \sup_0 \)-transition system. If we assume that \( \mathcal{J}_1^{\omega_1} \) is not
prime which is the case by [Bü] Theorem 5.6 if AC_{\omega_1}^\gamma (\gamma = 2^\omega)
then by [Bü] Remark 5.5 we have a contradiction.

To remedy the situation we replace the condition \( Yz = \&_1[\sup_1^2 Y] \)
in (3) by the more complicated \( Yz = \&_1[\sup_0^2 Y, \sup_1^2 Y, \ldots, \sup_1^n Y] \)
where the \( \&_1[\cdot] \) are matrices and the \( \sup_1^i Y \) are the sups
which correspond to the relativized filters, \( \&_1^i \). So what is
called for now is some discussion of the relativized filters.

Let \( \alpha \) be an \( \omega_0 \)-inaccessible limit and let \( U \) be cofinal \( \alpha \).
We define \( \&_1^{U, \alpha} \) to be the filter of cofinal-closed sets relativized
to \( U \), i.e. \( x \in \&_1^{U, \alpha} \) is defined by relativizing formula (1) to
\( U \) in the manner described in [Bü] Section 1.

We now give some definitions useful in discussing \( \&_1^{U, \alpha} \). If
\([x_i; i < \lambda] \) is an increasing sequence in \( U \), then the \( U \)-limit of
the \( x_i \)'s is defined in the obvious way, \( \lim_{i < \lambda} U x_i = (\mu x). Ux \land x \geq \lim_{i < \lambda} x_i \)
(such an \( x \) exists as \( U \) is cofinal \( \alpha \)). If \( V \subseteq U \), then the
\( U \)-derivative of \( V \) at \( \alpha, V^{U, \alpha} \), is the set of all \( U \)-limits of
members of \( V \cap \alpha \). We also use the notation \( V^{\alpha} \) for \( V^{\alpha, \alpha} \), i.e.
\( V \cap \alpha \). The following remark is an obvious consequence of the
above definitions.

Remark 1.6: If \( V \subseteq U, U \) cofinal \( \alpha \), and \( \alpha \neq \omega_0 \), then

(a) \( V^{\alpha} \cap U \subseteq V^{U, \alpha} \)

(b) \( V^{\alpha} \subseteq U \Rightarrow V^{\alpha} = V^{U, \alpha} \)
Note that if \( U \) is cofinal \( \alpha \), then \( \mathcal{g}^{U,\alpha}_1 \) is the set of all those subsets of \( U,\alpha \) which contain a set that is cofinal-closed in \( U,\alpha \); i.e. cofinal in \( U \) and closed with respect to \( U \)-limits. As usual, \( \mathcal{C}^{U,\alpha}_1 \) is the ideal which corresponds to \( \mathcal{g}^{U,\alpha}_1 \), i.e. \( \mathcal{C}^{U,\alpha}_1 = \{ \tilde{X} \mid X \in \mathcal{g}^{U,\alpha}_1 \} \). We also make the convention that if \( U \) is not cofinal \( \alpha \), then \( \mathcal{g}^{U,\alpha}_1 \) is the empty filter. In the usual way we shall make use of quantification over these filters, where:

\[
\begin{align*}
(\forall^{U,\alpha}_1 \mathcal{Z}) & : \quad z \cap U \cap \alpha \in \mathcal{g}^{U,\alpha}_1 \\
(\exists^{U,\alpha}_1 \mathcal{Z}) & : \quad z \cap U \cap \alpha \in \mathcal{C}^{U,\alpha}_1 \\
\sup^{U,\alpha}_1 \mathcal{Z} & : \quad \{ c ; (\exists^{U,\alpha}_1 \mathcal{Z}) \mathcal{Z} = c \}
\end{align*}
\]

It should be noted that these satisfy the properties stated in [EÜ] Lemma 5.8. The following lemma is just what is needed to handle the technical details of the next section.

**Lemma 1.7:** Let \( \alpha \) be an \( \omega_0 \)-inaccessible limit and suppose \( X \subseteq Y \in \mathcal{g}^\alpha_1 \). Then \( X \in \mathcal{g}^{Y,\alpha}_1 \). \( \Rightarrow X \in \mathcal{g}^\alpha_1 \).

**Proof:** First suppose \( X \in \mathcal{g}^{Y,\alpha}_1 \). Thus, we have a \( U \subseteq X \) with \( U \) cofinal-closed \( Y,\alpha \). As \( U \) is cofinal \( Y \) and \( Y \) is cofinal \( \alpha \), it follows that \( U \) is cofinal \( \alpha \). Hence \( U^\alpha \) is cofinal closed.
α, and since \( Y \in \mathcal{J}_1^\alpha \) it follows that

(a) \( U'^\alpha \cap Y \in \mathcal{J}_1^\alpha \). By Remark 1.6 we know that \( U'^\alpha \cap Y \subseteq U'^Y,\alpha \). As \( U \) is closed in \( Y,\alpha \) we get \( U'^Y,\alpha \subseteq U \subseteq X \), and so

(b) \( U'^\alpha \cap Y \subseteq X \). By (a) and (b) we have \( X \in \mathcal{J}_1^\alpha \). This argument shows that \( X \in \mathcal{J}_1^Y,\alpha \supseteq X \in \mathcal{J}_1^\alpha \). In the next paragraph we complete the proof by showing \( X \in \mathcal{J}_1^Y,\alpha \). Suppose \( X \in \mathcal{J}_1^\alpha \). Therefore, there is a \( U \subseteq X \) with \( U \) cofinal-closed \( \alpha \). As \( U \) is cofinal \( \alpha \), and \( Y \) is cofinal \( \alpha \) it follows that \( U \) is cofinal \( Y,\alpha \). Since \( U \) is closed in \( \alpha \) we know that

(c) \( U'^\alpha \subseteq U \). As \( U \subseteq X \subseteq Y \) we have \( U'^\alpha \subseteq Y \) and so by Remark 1.6, (d) \( U'^\alpha = U'^Y,\alpha \). From (c) and (d) we get \( U'^Y,\alpha \subseteq U \), and consequently \( U \) is cofinal-closed in \( Y,\alpha \).

Since \( U \subseteq X \), it follows that \( X \in \mathcal{J}_1^Y,\alpha \). Q.E.D.

**Corollary 1.8:** Let \( \alpha \) be an \( \omega_0 \)-inaccessible limit and suppose \( Y \in \mathcal{J}_1^\alpha \). Then \( X \cap Y \in \mathcal{O}_1^Y,\alpha = X \in \mathcal{O}_1^\alpha \).

The proof of the complementation lemma which we will give uses the assumption that for any \( \omega_1 \)-accessible \( \alpha \) \( \Theta_\alpha/\mathcal{J}_1^\alpha \) has no atoms. Consequently we assume \( AC_{\omega_1}^\gamma (\gamma = 2^\omega) \) and use the Ulam argument of [Bü] Theorem 5.6 to get the following result.

**Theorem 1.9. (AC):** Let \( \alpha \) be an \( \omega_1 \)-accessible limit. Every non-null element of \( \Theta_\alpha/\mathcal{J}_1^\alpha \) has breadth \( \omega_1 \). That is, in any \( U \in \mathcal{O}_1^\alpha \) there are \( \omega_1 \)-many disjoint \( U_1 \in \mathcal{O}_1^\alpha \).
Proof: By Remark 1.4 there is an $\omega_1$-sequence $P$ of $\omega_0$-limits. Thus $[P, <] \cong [\omega_1, <]$ (Note: $i \rightarrow \omega \cdot i$ is an isomorphism $[\omega_1, <] \cong [\{\lambda; \lambda < \omega_1\}, <]$). Let $U \in \mathcal{O}_1^\alpha$. Then by Corollary 1.8, $U \cap P \in \mathcal{O}_1^{P, \alpha}$. Using the above isomorphism and [Büchi] Theorem 5.6, there are $\omega_1$-many disjoint $U_i$ in $U \cap P$ such that $U_i \in \mathcal{O}_1^{P, \alpha}$. Since $U_i = U_i \cap P$, by Corollary 1.8 these $U_i \in \mathcal{O}_1^\alpha$. Q.E.D.

Corollary 1.10 (AC): For any $\omega_1$-accessible limit $\alpha$, $\mathcal{O} / \mathcal{J}_1^\alpha$ is an atomless Boolean algebra.
8. The modified subset-construction extended up to $w_2$.

The subset-construction we present here will be modeled after that of [Bü] Section 4 with changes as indicated in the last section. Once again our proof will use McNaughton's construction, as well as the ideas of [Bü] Section 6. We will be using $AC^\gamma_{\omega_1}$, so $\omega_1 = \kappa_1$, and for any $\omega_1$-accessible $\alpha$, $\mathcal{Q}\alpha/\mathcal{Q}_1$ has no atom.

Throughout this section individual variables range over $w_2$, and all lemmas are about the following transition-system (without initial and terminal conditions).

(1) $\Gamma^v_{u}(X,Z) : (\forall t)_{u} H[x_t, z_t, z'_t] \land (\forall x)_{u} \chi_{0}[\sup x Z, z_x] \land (\forall z)_{u} \chi_{1}[\sup z Z, z_z]$

An $X[u, v]$-run of $\Gamma$ is a sequence $Z[u, v]$ such that $\Gamma^v_{u}(X,Z)$; it is from $a$ to $b$ if $Z_u = a$ and $Z_v = b$; it is $C$-exact if $[Z_u; u \leq t < v] = C$. For a given input $X$ we are interested in the sets $S^c_{u,v}$, of all states $b$ which can be reached from $c$ by a $C$-exact run. Thus, for $u \leq v$ we have,

(2) $S^c_{u,v} : \{b; \exists Z[u, v]. Z_u = c \land \Gamma^v_{u}(X,Z) \land Z_v = b \land [Z_t; u \leq t < v] = C\}$
and the merging relations $u \equiv v(t), u \equiv v(-x), u - v(t)$ are defined as in [Bu].

Our main task is to find for each $\omega_2$-sequence $X$ a finite-state recursion which uniformly defines the sequence $S_u$. The recursion (8) of [Bu] Section 4 tells us how to define $S_u t$ for $t = 0$, $t$ a successor, and $t = \omega_0$. We need to add to these, recursions for $S_z$ in case $z = \omega_1$. These will be modeled after the first formula of (6) in [Bu] Section 6. Later in this section we will construct a finite state recursion of the form:

\begin{align*}
Y_0 &= a \\
Y_t' &= F[X_t, Y_t] \\
(Y_t, Y_x) &= \begin{cases}
\sup_0 Y & \text{if } t = 0 \\
F[X_t, Y_t] & \text{if } t > 0
\end{cases} \\
Y_x &= \begin{cases}
\sup_0 Y & \text{if } x < 0 \\
L_1 x & \text{if } 0 \leq x < \omega_2 \\
\sup_1 Y & \text{if } x = \omega_2
\end{cases}
\end{align*}

where $L_1 [Z_t], \ldots, L_n [Z_t]$ are matrices and $\sup_1 L, z Y$ is defined to be $\sup_1 U, z Y$ where $U = \{t; L[Z_t]\}$. Our object is to show that the first coordinate of $Y, Y_0$, defines the sequence $S_0$.

We make the following inductive assumption:

**Inductive Hypothesis:** For all $t < a$ we have $Y^t_0 t = S_0 t$ where $Y$ is defined by (3).

We will henceforth assume that $a < \omega_2$ is fixed and satisfies the inductive hypothesis, and then show $Y^a_0 = S_0 a$. This then proves that $Y^t_0 t = S_0 t$ for all $t < \omega_2$. We start by stating a new version of [Bu] Lemma 6.1. The proof is the same as for [Bu] Lemma and uses the inductive hypothesis and Remark 1.5.

**Lemma 2.1:** Let $a < \omega$ be any ordinal which satisfies the inductive hypothesis. Let $X[0, z]$ be any input, $\omega_1 \leq z < a$, and $u < z$. There is a unique state $S_u$ such that $(y^z_1)S_u Y = S_u$.
Remark 2.2: Let \( X \) be arbitrary and let \( w_1 = z \). There is a finite set \( W \) such that \( \forall w \in W \left( z \approx w(-z), (y \mid w) \right) \) and \( v, w \in W \land v \neq w \implies v \not\approx w(-z) \). In fact for any such \( W \) there is an \( w_1 \)-sequence \( P \) of \( w_0 \)-limits and sets \( P_w \) such that \( P = \bigcup_{w \in W} P_w \) is cofinal-closed in \( z \), \( w \in W \implies P_w \in \mathcal{O}_1^z \), \( v, w \in W \land v \neq w \implies P_v \cap P_w = 0 \).

Proof: We know \( \approx (-z) \) is an equivalence of finite index. Some of the classes will occur non-null in the sense of \( \mathcal{O}_1^z \), i.e. \( \in \mathcal{O}_1^z \). Let \( W \) be a system of representatives of these classes. Clearly \( W \) satisfies the first set of properties stated in the remark.

Thus there is a set \( P \) and for each \( w \in W \) there are sets \( P_w \) such that (a) \( P_w \notin \mathcal{O}_1^z \), (b) \( y \in P_w \implies y \approx w(-z) \), and (c) \( P = \bigcup_{w \in W} P_w \in \mathcal{O}_1^z \). By (c) and Remark 1.4 we may assume (d) \( P \) is a cofinal-closed \( w_1 \)-sequence of \( w_0 \)-limits. As \( v, w \in W \land v \neq w \implies v \not\approx w(-z) \), by (b) we may further assume (e) \( v, w \in W \land v \neq w \implies P_v \cap P_w = 0 \). Now define \( y_i, w_i, t_i \) by the following induction over \( i < \omega_1 \):

\[
\begin{align*}
    Y_0 &= (\mu y).Py \\
    w_i &= (\mu w).w \in W \land P_w y_i \\
    t_i &= (\mu t).y_i \approx w_i(t) \\
    y_{i+1} &= (\mu y).Py \land t_i < y \\
    y_\lambda &= \lim_{i<\lambda} y_i \\
    y_0 &= \lim_{i<\lambda} y_i \\
    P_{w_0} &\forall y_i, w_i \in W, \text{ by (b) } y_i \sim w_i(-z) \\
    y_i &\approx w_i(t_i) \\
    y_i &\approx w_i(y_{i+1}) \\
    P_{y_\lambda} &\text{ as } P_{y_i} \text{ and (d)}
\end{align*}
\]
As \( \{y_i; i < \omega_1\} \) is a cofinal closed subset of \( P \) we may assume \( P = \{y_1; i < \omega_1\} \). Suppose that \( w \in W, v < y, P_wv \) and \( Py \). So there are \( i < j \) such that \( v = y_i \) and \( y = y_j \).

As \( y_i \simeq w_i(y_{i+1}) \) and \( y_{i+1} \leq y_j \), we have \( v \simeq w_i(y) \). Since \( v = y_i \in P_{w_i} \) and \( v \in P_w \), we have by (e) that \( w_i = w \). Hence \( v \simeq w(y) \).

Q.E.D.

Let \( W \) be an index set and let \( s_0 \), and for each \( w \in W \), let \( s_w \) be states of the \( S_u \) type; we define the following set of states of \( r \).

\[
\begin{align*}
b \in B^c, C[s_0, \{s_w; w \in W\}] \iff & \bigvee \text{D_0} \subseteq C \land \bigwedge \chi_0[D_0, e] \land \chi_1[D_1, b] \\
& \land D_1 \subseteq s_0^{\omega, C} \land Q \subseteq D_1 \times W \land Q \text{ onto } D_1 \\
& \land Q \text{ onto } W \land \bigwedge \text{D_1} \subseteq s_w^{e, D_0}
\end{align*}
\]

(4)

We have now come to the main step in the proof of the subset-construction. Let \( \alpha < \omega_2 \) satisfy the induction hypothesis, \( w_1 = z < \alpha \), \( u < z \), and \( X \) be any input. Then,

\[
\begin{align*}
b \in S_u^{\alpha, C}z \iff & \bigvee \text{w_\in W (g^z_1)y}_w \simeq w(-z) \land (g^z_1y)_w \bigvee y \simeq w(-z) \land \\
& (g^z_1y)b \in B^c, C[s_u, y, \{s_w; y; w \in W\}]
\end{align*}
\]
Proof: Suppose first that \( b \in S_{u}^{C_{1}} z \). There is a run \( Z[u, z] \) such that \( Zu = c, \Gamma_{u}^{z}(X, Z), Zz = b, \) and \( \{Zt; u \leq t < z\} = C \).

Let \( D_{0} = \text{sup}^{z}Z \), and \( D_{1} = \text{sup}^{z}Z \). Then clearly \( D_{0} \subseteq C \), and as \( \omega_{1} \rightarrow z \) and \( Z \) is a run of \( \Gamma \) ending in \( b, \chi_{1}[D_{1}, b] \). So we have (a) \( D_{0} \subseteq C \) and \( \chi_{1}[D_{1}, b] \). By the definition of \( D_{1} \), there are sets \( P_{e} \), such that (b) \( e \in D_{1} \supset P_{e} \in \delta^{z} \), (c)

\[ P = \bigcup_{e \in D_{1}} P_{e} \in \delta^{z} \] and (d) \( e \in D_{1} \supset P_{e} \vdash y = e \). Using Remark 1.4 we may assume that \( P \) is a cofinal-closed \( \omega_{1} \)-sequence of \( \omega_{0} \)-limits.

Now we define \( y_{i} \) by induction on \( i < \omega_{1} \).

\[ y_{0} = (\mu y).Py \land D_{0} \subseteq \{Zt; u \leq t < y\} \land (yt)^{Z}Zt \in D_{0} \] exists as \( P \) cofinal and \( D_{0} = \text{sup}^{z}Z \)

\[ y_{i+1} = (\mu y).Py \land y_{i} < y \land D_{0} = \{Zt; y_{i} \leq t < y\} \] exists for same reason

\[ y_{\lambda} = \lim_{i < \lambda} y \] exists as \( z \) is not an \( \omega_{0} \)-limit

Note that each \( y_{i} \in P \); in particular this holds at limits \( i \) because \( P \) is closed. As \( \{y_{i}; i < \omega_{1}\} \) is a cofinal-closed subset of \( P \), we may assume

\[ P = \{y_{i}; i < \omega_{1}\} \] without losing the above properties of \( P \);

the new \( P_{e} \)’s being \( \{y_{i}; i < \omega_{1}\} \cap P_{e} \). So we have the additional facts (e) \( v < y \land Py \land Py \vdash \{Zt; v \leq t < y\} = D_{0} \), and

(f) \( Py \vdash \{Zt; u \leq t < y\} = C \). Because of (e) we clearly have

\[ \text{sup}^{X}Z = D_{0} \] for every \( x \in P^{z} \). Since \( P^{z} \) is still cofinal-closed
and \( P' \subseteq P \) we may choose \( P' \) as the new \( P \) and redefine the \( P_e \). Without losing any of the old facts, we now have \( P_x \supseteq \sup^X Z = D_o \).

As \( Z \) is a run of \( \kappa_0 \), by (c,d) we therefore see that, (g)

\[ e \in D_1 \supseteq \kappa_0[D_0,e]. \]

By Remark 2.2 there is a finite set \( W \) and sets \( \overline{P},P_w \) such that (h) \( w \in W \supseteq P_w \subseteq \overline{G}^Z \), (i) \( \overline{P} = \bigcup_{w \in W} P_w \) is cofinal-closed in \( Z \),

(j) \( v < y \wedge P_w y \wedge \overline{P}_y \supseteq v \approx w(y). \)

By Lemma 2.1 we have an \( s_0 \), and for each \( w \in W \) an \( s_w \), such that \( (v^Z_y)s_u y = s_0 \) and

\[ (v^Z_y)s_w = s_w. \]

As \( W \) is finite, by (i) we may assume (k)

\[ \overline{P}_y \supseteq S_u y = s_0 \wedge \bigwedge_{w \in W} S_w y = s_w. \]

Suppose \( e \in D_1 \). By (b,i) we have \( \overline{P} \cap P_e \subseteq \overline{G}^Z \), and therefore there is a \( y \in \overline{P} \cap P_e \). By (c,f), \( \{zt; u \leq t < y\} = C \), and by (d), we also have \( Z y = e \). Hence, as \( Z[u,z] \) is a run of \( H, \kappa_0, \kappa_1 \) and \( Z u = c \), we have by definition (2), \( e \in s^C_C y \). As \( y \in \overline{P} \) we have by (k) that \( S_u y = s_0 \), and so \( e \in s^C_C \). This argument proves (t)

\[ D_1 \subseteq s^C_C. \]

Now define \( <e,w> \in Q \iff e \in D_1 \wedge w \in W \wedge P_e \cap P_w \neq 0 \). Clearly \( Q \subseteq D_1 x W \). Suppose \( e \in D_1 \). By (b,i) we have a \( w \in W \) such that \( P_e \cap P_w \subseteq \overline{G}^Z \). Hence \( <e,w> \in Q \). This argument shows that \( Q \) projects onto \( D_1 \). Similarly, (c,h) shows that \( Q \) projects onto \( W \). Thus, (m) \( Q \subseteq D_1 x W, Q \) onto \( D_1, Q \) onto \( W \).

Assume \( <e,w> \in Q \) and \( d \in D_1 \). Then \( e \in D_1, w \in W \), and there is a \( v \in P_e \cap P_w \). By (b,i) \( \overline{P} \cap P_d \subseteq \overline{G}^Z \), and hence there
is a $y \in \bar{P}_d \cap P_d$ with $v < y$. Using (c,e), this yields 
$\{Zt; v \leq t < y\} = D_o$. By (d) we have $Zv = e$ and $Zy = d$.

Hence, as $Z[u,z]$ is a run of $H_0, H_o, H_o$, it follows 
by (2), that $d \in S_v e, D_o y$. As $v \in P_w, y \in \bar{P}$, and $v < y$ we 
get by (j) that $v \approx w(y)$. Therefore $S_y v = S_y w$. But by (k) 
$S_w y = s_w$, and so $d \in S_w e, D_o y = s_w e, D_o$. We have thus shown that 
ed, D_o for any $<e, w> \in Q$. I.e., (n) $<e, w> \in Q \Rightarrow D_1 \subseteq s_w e, D_o$.

By (a,g,l,m,n) it follows that $b \in B^c, C[z_o, s_w^{SW}, w \in W]$. 
Thus with (i,k) we obtain (o) $(\forall 1^y) b \in B^c, C[z_o y, s_w y w \in W]$.

From $b \in S^c, C z$ we have now found $D_o, D_1, Q, W$ such that (h,i,o) 
and therefore the right side of (5) holds.

We now suppose the right side of (5) holds and show that 
b \in S_u^c, C z. First note that as $O_1^x$ is an ideal $(\forall 1^y) \bigvee_{D_0, D_1, Q} \delta = 
\bigvee_{D_0, D_1, Q} (\forall 1^y) \delta$. From this and the assumption that the right side 
of (5) holds, there is a finite $W$ and $P, D_0, D_1, Q$ such that (p) 
$\bigvee_{w \in W} (\forall 1^y) w(z) \land (w_1 z) \lor y \approx w(z)$, (q) $P \subseteq O_1^x$, $D_0 \subseteq C$, (r) 
ed, $D_1 \subseteq K_0 [D_o, e], K_1 [D_1, b]$, (s) $y \in \bar{P} \Rightarrow D_1 \subseteq S_u^c, C y$, 
$Q \subseteq D_1 \times W, Q$ onto $D_1, Q$ onto $W$, and (t) 
$<e, w> \in Q \land y \in \bar{P} \Rightarrow D_1 \subseteq S_w e, D_o y$. Let $\bar{W} = \{w; w \in W \land 
(wv)^w \}$. $W \supset v \approx w(z)$ and let $\bar{Q} = \{<e, w>; w \in \bar{W} \land (qv). v \in W \land 
\langle e, v \rangle \in Q \land v \approx w(z)\}$. Clearly, $\bar{W}$ and $\bar{Q}$ satisfy all the 
above properties of $W$ and $Q$ respectively. We therefore
may further assume (u) \( v, w \in W \land v \neq w \Rightarrow v \neq w(z) \). From 
(p,u) and the second part of Remark 2.2, there is an \( w_1 \)-
sequence \( P \) of \( w_0 \)-limits and sets \( P_w \) such that (v)
\( w \in W \Rightarrow P_w \notin \mathcal{O}_1 \). (w) \( P = \bigcup_{w \in W} P_w \) is cofinal closed in \( z \), (x)
\( v < y \land P_y \land \mu_y \Rightarrow v \approx w(y) \), and (y) \( v, w \in W \land v \neq w \Rightarrow P_v \cap P_w = 0 \).
By (q,w) we may also assume (z) \( \bar{P} \subseteq P \).

Now we use the fact that \( \mathcal{O}_1/\mathcal{O}_1 \) has no atoms. Because of (v)
we can split each \( P_w \) into an arbitrary finite number of disjoint
non-null parts. So for each \( w \in W \) there are sets \( P_{e,w} \) such
that \( P_w = \bigcup_{e,w} P_{e,w} \) (a non-empty union since \( Q \) projects onto
\( W \)), (v') \( <e,w> \in Q \Rightarrow P_{e,w} \notin \mathcal{O}_1 \) and \( <e,w> \neq <d,w> \in Q \Rightarrow P_{e,w} \cap
P_{d,w} = 0 \). By (w,x,y) we now have (y') \( <e,w> \neq <d,v> \in Q \Rightarrow
P_{e,w} \cap P_{d,v} = 0 \), (w') \( P = \bigcup_{e,w} P_{e,w} \) is cofinal closed, and
(x') \( <e,w> \in Q \land v < y \land P_{e,w} \land P_{y} \Rightarrow v \approx w(y) \).

By induction on \( i < w_1 \), we now define a subsequence of the
\( w_1 \)-sequence \( P \) which is cofinal closed in \( z \).

\[
Y_i = (uY)\bar{P}_Y \quad \text{exists by (q)} ; \quad \bar{P}_Y \Rightarrow \text{hence by (z)} \ Y_i
\]

\[
Y_{i+1} = (uY)\bar{P}_Y \land Y > Y_i \quad \text{exists by (q)} ; \quad \bar{Y}_i < Y_{i+1} \quad \bar{P}_{Y_{i+1}}
\]

\[
Y_\lambda = \lim_{i<\lambda} Y_i \quad \text{exists as} \quad w_0 \neq z \quad \text{and} \quad \bar{P}_Y \text{ as } \bar{P}_{Y_i}
\]
By \((w',y')\) each element of \(P\) belongs to exactly one of the \(P_{e,w}\). So for each \(i < w_1\) there are \(e_i, w_i\) such that

\[<e_i, w_i> \in Q \cdot P_{e_i, w_i} Y_i \quad \text{and} \quad <e, w> \in Q \land P_{e, w} Y_i \Rightarrow e = e_i.\]

As \(Q \subseteq D_1 \times W\), by (aa) we also have (cc) \(e_i \in D_1\).

Since \(y_0 \in \bar{P}\), by (x,cc) \(e_0 \in S_{u}^{c_1, c} Y_0\). Thus, there is a \(C\)-exact run \(Z[u, y_0]\) of \(\Gamma\) from \(c\) to \(e_0\). By (aa), \(<e_i, w_i> \in Q\) and \(P_{e_i, w_i} Y_i\). As \(Y_{i+1} \in \bar{P}\), \(y_i < Y_{i+1}\), from (x') it follows that

\(y_i \approx w_i (y_{i+1})\). Hence, \(S_{Y_i} Y_{i+1} = S_{Y_i} Y_{i+1}\). Now \(Y_{i+1} \in \bar{P}\), and therefore by (t,cc), \(e_{i+1} \in S_{e_i} D_0\). Thus \(e_{i+1} \in S_{Y_i} Y_{i+1}\) and consequently there is a \(D_0\)-exact run \(Z[y_i, y_{i+1}]\) from \(e_i\) to \(e_{i+1}\). Note that the runs \(Z[y_i, y_{i+1}]\) and \(Z[y_{i+1}, y_{i+2}]\) have equal states, \(e_{i+1}\) at \(y_{i+1}\), and similarly \(e_0\) occurs as state \(Z y_0\) in both \(Z[u, y_0]\) and \(Z[y_o, y_1]\). As \(D_0 \subseteq C\), these partial runs can be spliced together to form a \(C\)-exact run \(Z[u, z]\). Since the splicing places form an \(\omega_1\)-sequence which is cofinal-closed in \(z\) \(\{y_i\}_{i < \omega_1} \rightarrow y_\lambda = \lim_{i < \lambda} y_i\), this run has no gaps.

We still must show that \(Z[u, z]\) is a run of \(\Gamma\).

Clearly, \(Z\) satisfies the transition-conditions \(K, \kappa_0, \kappa_1\) at every place other than \(y_\lambda\). As \(\sup_{\rho} Y Z = D_0\) and \(e_\lambda \in D_1\) by (cc), it follows from (r) that \(\kappa_0[D_0, e_\lambda]\) and therefore \(\kappa_0[\sup_{\rho} Y Z, Zy_\lambda]\). Since \(\omega_0 = y_\lambda\), \(Z\) satisfies the transition conditions at \(y_\lambda\). Thus, \(Z[u, z]\) is an \(X[u, z]\)-run of \(\Gamma\). It remains to show that the run can be completed to
$Z[u,z]$. We define $Zz = b$ and show $\chi_1[\sup^Z_1Z, b]$. Since we already know $\chi_1[D_1, b]$, it suffices to prove $\sup^Z_1Z = D_1$.

This will be done in the remaining two paragraphs.

Suppose first that $e \in D_1$. Because $Q$ projects onto $D_1$, there is a $w \in W$ such that $<e, w> \in Q$. As $\{y_i; i < \omega_1\}$ is cofinal-closed in $z$, by $(v')$ we have $\{y_i; i < \omega_1\} \cap P_{e, w} \in \mathcal{O}_1^z$.

Let $y_i \in P_{e, w}$. From $(bb)$ it follows that $e = e_i$, and since $Zy_i = e_i$ we have $\{y_i; i < \omega_1\} \cap P_{e, w} \subseteq [t; Zt = e]$. Hence $[t; Zt = e] \in \mathcal{O}_1^z$, i.e. $e \in \sup^Z_1Z$. This argument shows $D_1 \subseteq \sup^Z_1Z$.

Now suppose $e \in \sup^Z_1Z$: There is a $U \in \mathcal{O}_1^z$ such that $U \supset Zt = e$. As $\{y_i; i < \omega_1\}$ is cofinal closed in $z$, we have $U \cap \{y_i; i < \omega_1\} \in \mathcal{O}_1^z$. Hence, there is a $y_i \in U$. So $Zy_i = e_i$. But we already know that $Zy_i = e_i$. Thus $e = e_i \in D_1$ by $(cc)$.

This argument shows $\sup^Z_1Z \subseteq D_1$.

Q.E.D.

At first glance formula (5) is a bit nicer than the corresponding formulas in Sections 3 and 4 of [Bü]. This is because the reference to the propositional expression $B$ is isolated, and therefore the piggyback expressions $Q_{u, w}$ aren't needed. But, unfortunately a new problem arises. It is not hard to see that $(\exists^Z_1 y)yzw(...)$.

$(\exists^Z_1 t)(\exists y)yzw(t)$. Thus, the formula which corresponds to (5) of [Bü]. Section 4 does not holds. We will see that in our
effort to remedy this situation, it is once again necessary to
use the piggyback device.

Let \( z = \omega_1 \) and \( V \) be a (finite) sequence of representatives
of all the equivalence classes of \( \equiv(-z) \) in their natural order.
Clearly,

\[
(6) \quad \forall y < z. v \in V. y \equiv v(-z)
\]

\[
\forall v, w \in V. v \neq w \rightarrow v \not\equiv w(-z)
\]

For any such sequence \( V \) we may define the following sequences
by induction:

\[
(7) \quad y_0 = (\mu y). \forall v \in V. y > v
\]

\[
t_i = (\mu t). \forall v \in V. y_i \sim v(t) \quad \exists \text{ exists by (6)}
\]

\[
y_{i+1} = t_i
\]

\[
y_\lambda = \lim_{i < \lambda} t_i \quad ; \quad y_\lambda = \lim_{i < \lambda} y_i
\]

Assume this induction halts at the limit ordinal \( \beta \); i.e.
\( z = \lim t_i \). Define \( Y_V = \{ y_i; i < \beta \} \) and \( T_V = \{ t_i; i < \beta \} \).
The following lemma is just what we need to get rid of the \( (\exists^2 y) \)
quantifier from formula (5).
Lemma 2.3: Let $z = w_1$ and $V$ be a sequence of representatives of all the equivalence classes of $\approx (-z)$ in their natural order. Then for any $w$ in $V$,

$$(\exists_1^2 y) y \approx w(-z) \iff (\exists_1^2 t) (\exists y). y \approx w(t) \land y \in T_v^i$$

Proof: As $Y_V$ is clearly cofinal-closed in $z$, we have $Y_V \in \mathcal{Z}$. Therefore, by Corollary 1.8 it follows that

$$[y; y \approx w(-z)] \in \mathcal{O}_1^Z = [y; y \in Y_V \land y \approx w(-z)] \in \mathcal{O}_1^{Y_V \cdot z}$$

Now, $i - y_i$ is an isomorphism $<\mathcal{B}, < > \equiv <Y_V, < >$; consequently,

$$[y; y \in Y_V \land y \approx w(-z)] \in \mathcal{O}_1^{Y_V \cdot z} = [i; y_i \approx w(-z)] \in \mathcal{O}_1^\emptyset$$

Assume $y_i \approx w(-z)$. By (7), there is a $v$ in $V$ such that $y_i \approx v(t_i)$, and so $w \approx v(-z)$. As $w$ is in $V$, from (6) it follows that $v = w$. Hence $y_i \approx w(t_i)$. We have thus shown $y_i \approx w(-z) \supset y_i \sim w(t_i)$. Since the other implication is trivial, we have $y_i \approx w(-z) = y_i \sim w(t_i)$. From this and the general fact that $[i; i < \mathcal{B} \land i \text{ is a limit}] \in \mathcal{O}_1^\emptyset$ we have the equivalence,

$$[i; y_i \sim w(-z)] \in \mathcal{O}_1^\emptyset = [\lambda; y_\lambda \sim w(t_\lambda)] \in \mathcal{O}_1^\emptyset$$
Assume \( y \sim w(t_i) \) and \( y \in T_V' \). Since \( y \in T_V' \), there is a \( \lambda \) such that \( y = \lim_{j<\lambda} t_j \). Thus \( y = y_\lambda \). Using (7), there is a \( v \) in \( V \) such that \( y \sim v(t_\lambda) \). Since \( y \sim w(t_i) \) and \( y \sim v(t_\lambda) \), by (6), we get \( w = v \). Now by definition of the exact merging relation \( \sim \), we have \( t_i = t_\lambda \), and so \( i = \lambda \) by (7). Hence \( y = y_i = y_\lambda \) and so \( y_\lambda \sim w(t_\lambda) \). We have thus shown \( \{i; (sy) \cdot y \sim w(t_i) \wedge y \in T_V' \} \subseteq \{\lambda; y_\lambda \sim w(t_\lambda) \} \). Since the other inclusion is trivial we have,

\[
\{\lambda; y_\lambda \sim w(t_\lambda) \} \in \mathcal{O}_1^\beta = \{i; (sy) \cdot y \sim w(t_i) \wedge y \in T_V' \} \in \mathcal{O}_1^\beta
\]

Now \( i = t_i \) is an isomorphism of \( \prec \beta, \succ \succ \prec T_V, \succ \succ \), and therefore

\[
\{i; (sy) \cdot y \sim w(t_i) \wedge y \in T_V' \} \in \mathcal{O}_1^\beta = \{t \in T_V \wedge (sy) \cdot y \sim w(t) \wedge y \in T_V' \} \in \mathcal{T}_V, \mathcal{Z}
\]

Following the string of equivalences we get the desired result,

\[
\{y; y \approx w(-z) \} \in \mathcal{O}_1^\mathcal{Z} = \{t; t \in T_V \wedge (sy) \cdot y \sim w(t) \wedge y \in T_V' \} \in \mathcal{T}_V, \mathcal{Z}
\]

Q.E.D.

Corollary 2.4: Let \( z = w_1 \) and \( V \) be a sequence satisfying (6). If \( W \) is a subsequence of \( V \), denoted \( W \subseteq V \), then
Proof: Let $V$ satisfy (6) and $W \subseteq V$. For each $v$ in $V$ let

$$P_v = \{y; y \approx v(-z)\}.$$ 

Then by (6), $z = \bigcup_{v \in V} P_v$ and

$$v, w \in V \land v \neq w \Rightarrow P_v \cap P_w = 0.$$ 

Consequently, $\bigcup_{v \in W} P_v \in \mathcal{O}_1$ and so (a) $(\forall^2_1 y) \forall_{w \in W} \forall_{v \in V} w \approx w(-z) = \sim \forall_{v \in V} w \in W \land v \not \in W (\forall^2_1 y) w \approx w(-z)$. For each $v$ in $V$, let $Q_v = \{t; t \in T_v \land (\forall y). y \approx v(t) \land y \in T_v \}$. From the proof of Lemma 8.3 we know that $Q_v = \{t; y \approx v(t)\}$. Then by (6) and (7), $T'_V = \bigcup_{v \in V} Q_v$ and

$$v, w \in V \land v \neq w \Rightarrow Q_v \cap Q_w = 0.$$ 

So once again, $\bigcup_{v \in W} Q_v \in \mathcal{O}_1$ and therefore (b) $(\forall^2_1 y) \forall_{w \in W} \forall_{v \in V} w \approx w(-z) = \sim \forall_{v \in V} w \in W \land v \not \in W (\forall^2_1 y) w \approx w(-z)$. The desired result now follows from (a) and (b) by using Lemma 2.3.

Q.E.D.

Our next step is to use Lemma 2.3 and its corollary to put formula (5) into a form in which the piggyback device is applicable. Let $\alpha < \omega_2$ satisfy the induction hypothesis, $\omega_1 = z < \alpha$, $u < z$, $X$ be any input, and $V$ satisfy (6). Then,
Looking at (8) it is clear that in order to get a deterministic expression for \( b \in S^c, C \) it will be necessary to find recursions which define the \( Y_v \) and \( T_v \) for \( V \) and which are easily simulated by ones which are finite-state. Using (6) and (7), it is easy to see that if \( z = w_1 \) and \( V \) is a sequence for which (6) holds, then \( Y_v \) and \( T_v \) satisfy the recursion:

\[
Y_v = Y_v \rightarrow T_v \text{ where } \rightarrow \text{ is the last member of } V
\]

(9) \( T_v t =. \bigvee_{v \in V} (sy) \cdot Y_v y \wedge y \sim v(t) \)

\( Y_v t' =. T_v t \)

\( Y_v x =. (x^t) T_v t \) for any limit \( x \)

We now reintroduce the piggyback device in order to fix up formula (8) and recursions (9). As in Section 3 and 4 of [Bu] we define,

\( r_y t =. (u^v) y \simeq y(-t) \)

\( U_t = \text{ sequence of all } r_y t \), \( y < t \)

(10) \( u \sim v(t) = v \text{ in } U_t, u \text{ the first in } U_t \text{ such that } u \simeq v(t) \)

\( P_v t = [r_y t; y < t \wedge y \in Y_v] \)

\( R_v t = [r_y t; y < t \wedge y \in T_v] \)
Lemma 2.5: Let $z = w_1$. Then $U_z$ satisfies (6) and $Y_{U_z}, T_{U_z}$ satisfy the following recursion $t < z$:

$$
y_{U_z} \bar{v}' \quad \text{where } \bar{v} = \text{last member of } U_z$$

$$T_{U_z} t = \bigvee_{v \in U_z} (gr)(r \neq v \dashv r(t) \land r \in P_{U_z} t)$$

(11)

$$Y_{U_z} t' = T_{U_z} t$$

$$Y_{U_z} x = \left( g_{x}^{0} t \right) T_{U_z} t \quad \text{for any limit } x$$

Proof: Using (10) it is obvious that $U_z$ satisfies the properties of $V$ in (6). As the rest follows immediately from (9), all we need to show is (a) $T_{U_z} t = \bigvee_{v \in U_z} (gr)[r \neq v \dashv r(t) \land r \in P_{U_z} t]$. The basic properties listed after (13) in [Bü] Section 3 still apply and will be used in the proof.

Assume $T_{U_z} t$. From (9), there are $v \in U_z$ and $y \in Y_{U_z}$ such that $y \sim v(t)$. By (10) $v = r_y z$ is the first in its class modulo $\approx (-z)$. Thus, by (iii) we have $v \dashv v(t)$. As $\bar{v}$ is the last member of $U_z$ and $\bar{v}'$ is the first member $Y_{U_z}$, we have
\[ v \leq \tilde{v} < y \leq t. \] Consequently, from (jw) and \( \nu < t \) we get that \( t \not\ni \nu(t) \). From \( t \not\ni \nu(t) \) and \( y \sim \nu(t) \) it follows that \( y < t \), and therefore \( r = r_y t \) exists. By (i), \( r \leq y < t \) and \( y \cong r(-t) \).

Since \( y \sim \nu(t) \), this yields \( \nu \not\ni r \) and \( r \cong \nu(t) \). Hence, by (ii) and \( \nu \sim \nu(t) \), we now have \( \nu \sim r(t(t) \). As \( y < t \) and \( y \in Y_Uz \), by (10) \( r \in P_{Uz} t \). This shows that \( T_{Uz} t \) implies the right side of (a).

Now we assume the right side of (a) and show that \( T_{Uz} t \). Thus, there are \( v \) and \( r \) such that \( v \) in \( Uz, r \not\ni v, \nu \sim r(t(t) \), and \( r \in P_{Uz} t \). By (10), there is a \( y \) such that \( y < t, r = r_y t \), and \( y \in Y_Uz \). Hence \( r \cong y(t) \) and \( v \cong r(t) \), so \( y \cong \nu(t) \). We have already seen that \( v \) in \( Uz \ni y \sim \nu(t) \). Thus, using (j) we have \( r_y t = v \not\ni r = r_y t \). Hence, by (iv), \( y \not\ni \nu(-t) \). So \( y \sim \nu(t) \).

By (9) we now have \( T_{Uz} t \). This completes the proof of (a).

\text{Q.E.D.}

We now give recursions for the piggyback components. That is, the \( \omega_2 \)-sequence \( U \) and for each finite sequence \( V \), the \( \omega_2 \)-sequences \( P_V \) and \( R_V \) defined by (10) satisfy the following recursion (\( x \) is any limit):

\[ U0 = \text{empty} \quad R_V 0 = 0 \quad P_V 0 = 0 \]

\[ Ut' = \text{remove from } Ut \text{ all } u \uparrow u(t), \text{ and add } t \text{ at the end.} \]

\[ Ux = \text{all } u \text{ such that } (v^x t)u \ni u(t), \text{ in natural order.} \]

(12) \[ P_V t' = [r: (\exists p)[r \sim p(t) \land p \in P_V t] \lor [r = t \in Y_V]] \]

\[ P_V x = \{r: (\nu^x t)r \in P_V t\} \]
\[ R_{V}t' = \{ r; (\exists p)[r \not\rightarrow p(t) \land p \in R_{V}t] \lor [r = t \in T'_{V}] \} \]

\[ R_{V}x = \{ r; (x^{x}_{0}t)r \in R_{V}t \} \]

The proof of the above is almost identical to that of [Bü] Lemma 4.3 and will be omitted. We next use the piggyback device to put \( S_{u}z \), for \( \omega_{1} \)-limits \( z \), in a form which is more useful for our purposes. Let \( \alpha < \omega_{2} \) satisfy the induction hypothesis, \( \omega_{1} = z < \alpha \), \( u < z \), and \( X \) be any input. As \( Uz \) satisfies (6), by (8) and (10) we have,

\begin{align*}
&b \in S_{u}^{C,C}z \iff \bigvee_{w \in Uz} \bigwedge_{w \in W} (q_{1}^{T_{Uz}z} t). (\exists r)[r \not\rightarrow w - r(t) \land r \in R_{Uz}t] \\
&\quad \land (q_{1}^{T_{Uz}z} t) b \in B_{C,C}^{C,S_{u}t,\{S_{w}t, w \in W\}}. \tag{13}
\end{align*}

We are now ready to set up a finite-state recursion for the \( \omega_{2} \)-sequences \( S_{1}, V, Y, T, P, R \), and \( Q_{i,k} \) which simulate the sequences \( S_{u}, U, Y_{V}, T_{V}, P_{V} \), and \( Q_{u,w} \) where \( Q_{u,w} \) (and also \( B[s_{0}, s] \)) are defined as in [Bü] Section 4 (\( x \) here ranges over all limits, except in the statement of \( s_{i}^{C,C} \)).
28.

\[ S_i(t) = \begin{cases} 
\emptyset & \text{if } i \text{ not in } V_t \\
\sigma_i(t) & \text{if } i \text{ in } V_t 
\end{cases} \]

\[ V_0 = \emptyset \]

\[ V_t' = \text{remove from } V_t \text{ all } i \neq i(t), \text{ add } p(t) \text{ at end} \]

\[ V_x = \text{all } i \text{ such that } (\varphi^x t)i \neq i(t), \text{ i before } j \text{ in case } (\varphi^x t)[i \text{ before } j] \]

\[ Q_{ik}(t) = \begin{cases} 
\emptyset & \text{if } i \text{ or } k \text{ not in } V_t \\
\varphi_{ik}(t) & \text{if } i, k \text{ in } V_t
\end{cases} \]

\[ \tilde{T}_j = Y_j t' . = . T_j t \vee p(t) \text{ in } J \quad Y_j x = T_j x \]

(14)

\[ \tilde{T}_j' = \tilde{T}_j x'. =. (\varphi^x t)T_j t \]

\[ P_{j0} = 0 \quad \text{and } \quad R_{j0} = 0 \]

\[ P_{jt'} = \begin{cases} 
0 & \text{if } p(t) \text{ in } J \\
[\ell; \bigvee_{h} [\ell - h(t) \land h \in P_{jt'}] \bigvee [\ell = p(t) \land Y_j t]] & \text{if } p(t) \text{ not in } J
\end{cases} \]

\[ R_{jt'} = \begin{cases} 
0 & \text{if } p(t) \text{ in } J \\
[\ell; \bigvee_{h} [\ell - h(t) \land h \in R_{jt'}] \bigvee [\ell = p(t) \land T_j t']] & \text{if } p(t) \text{ not in } J
\end{cases} \]

\[ P_{jx} = [\ell; (\varphi^x t) \ell \in P_{jt}] \]

\[ R_{jx} = [\ell; (\varphi^x t) \ell \in R_{jt}] \]
Here the indices $h, i, j, k, l$ range over $\{0, \ldots, g\}$ and $J, K$ range over finite sequences with distinct components in $\{0, \ldots, g\}$ ($g$ = the number of states of the $S_u$'s), $e^C, C = \{c\}$ and $e^C, C = 0$ if $C \neq 0$, $\varnothing$ is the empty sequence, and the following abbreviations are used:

$S_{i t} : <S^C_i, C_t; c \text{ a state of } \Gamma, C \text{ a set of states of } \Gamma>$

$Q_{i, k t} : <Q^C_k, C_t; c \text{ a state of } \Gamma, C \text{ a set of states of } \Gamma>$

$i \sim j(t) : j \text{ in } V_t, \text{ and } i \text{ is the first in } V_t \text{ such that } S_{i t} = S_{j t}$

$o(t) : \text{ the first } i \leq g, \text{ such that } i \text{ not in } V_t$

$S^C_i, C(t) : \{b; \bigvee_a a \in C \land E[\mathcal{X}_t, a, b] \land a \in S^C_i, C_t \lor S^C_i, C - \{a\}_t\}$

$S^C_i, C(x) : \{b; \bigvee_k \{v^x_0, t\} k \sim k(t) \land (a^x_0) \bigvee_j [j \neq k \sim j(t)] \land \langle \emptyset, j \rangle \in Q^C_i, C_k\}$

$S^C_i, C(z) : \{b; \bigwedge_J J = V_z \land \bigvee_k \bigwedge_{K \subseteq J} (s^x_{i, t} \bigwedge_{K} (s^z_{k, t}) V[l \neq k \sim l(t) \land l \in R_{Jt}] \land (v^y_{Jz}, t) \bigwedge_k \bigvee_{l} V[l \neq k \sim l(t) \land l \in R_{Jt}] \land (a^z_1) \in B^C, C[S_{i t}, \{S_{k t}; k \in K\}]$
\[ 2_{i,k}^{C,c}(t') : [\Phi,j] : \forall h [j \rightarrow h(t) \land \Phi,h] \in Q_{i,k}^{C,c} \]

\[ \lor [j = \rho(t) \land b \in B_{i,k}^{C,c}\{s_i,t,s_k t\}] \]

\[ 2_{i,k}^{C,c}(x) : [\Phi,j] : (\forall x_t) [\Phi,j] \in Q_{i,k}^{C,c} \]

\[ T_{j,t} : \forall j \in J \forall l [l \neq j \rightarrow t(t) \land l \in P_j t] \]

Note that the recursion (14) has finitely many components \( S_i.V_j, T_j, P_j, R_j \) each taking on finitely many states. Thus, by the remark on sup-conditions [Bu] Lemma 5.8, the recursion (14) clearly is finite-state of the form (3).

**Lemma 2.6:** (the subset-construction for ordinals \(< \omega_2\)): To any transition-system \( \Gamma \) of the form (1) with initial condition \( E[\cdot], L[\cdot] \) and terminal condition \( L'[\cdot] \), one can construct a finite-state recursion \( A \) of form (3) with terminal condition \( L'[\cdot] \) such that, for any ordinal \( \alpha < \omega_2 \) and any input \( X[0,\alpha] \), \( [E, \Gamma, L] \) accepts \( X \) if and only if \( [A, L'] \) accepts \( X \). That is \( (\xi 2). E[\xi 0] \land \Gamma_0(x, z) \land L[\xi z] \) holds just in case the recursion \( A \) applied to \( X \) yields a final state \( Y_\alpha \) which satisfies \( L'[Y_\alpha] \). In fact, the recursion (14) is such an \( A \), the terminal condition \( L' \) being

\[ \forall c. E[c] \land b \in L \land S_{0,c}^{C,c} \land c_{C,b} \]
Proof: As we have remarked earlier the proof is by induction on $\alpha$. So we assume $\alpha < \omega_2$ and for all $t < \alpha$ that the $S_0^t$ defined by (2) satisfies the recursion, (14). Let $X[0, \alpha]$ be an arbitrary $\alpha$-sequence. We ask the reader to make up the formulas (14), obtained from (14) by making the following replacements: $h, i, j, k, t, J, K$ by $p, u, v, w, r, V, W$; $S, \mathcal{S}, Q, 2, P, R, Y, T, T'$ by $\mathcal{S}, \mathcal{S}', Q, 2, P, R, Y, T, T'$; $V$ by $U$; $\tau$ by $\varepsilon$; $\rho(t)$ by $t$.

Note that (14) contains the recursions which occur in Lemmas 4.2 and 4.3, and in (7) of Section 4, as well as those of (11), (12), and (13) of this section. By the results stated in these (using the induction assumption for Remark 2.1) it follows that $S_0^C, C^\alpha$ is just the set introduced in (2) (note that $0 \in U_\alpha$).

So we have

\[(a) \quad b \in S_0^C, C^\alpha \iff (x^2, \omega) \cdot \pi_0 = c \land \bar{\Gamma}_0(x, z) \land z_\alpha = b\]

The recursion (14) together with the terminal condition $\forall \mathcal{C} \in [c] \land b \in C \cap S_0^C, C^\alpha$ would therefore answer the requirement in our lemma, except (14) is not finite-state. The lemma will be proved if we can show $S_0^C = S_0^\alpha$. To see this we once again use the function $u_i t$ which is defined:

\[(c) \quad i = \rho(t) \Rightarrow u_i t' = t \quad i = i(t) \Rightarrow u_i t' = u_i t \]

$\mathcal{V}_0^x i = i(t)$ $\Rightarrow u_i x = \text{that } u, \mathcal{V}_0^x u_i t = u$
This function shows in just which way the finite-state recursion (14) simulates the "quasi finite-state" recursion (14).

Namely, we will prove:

(b_1) i in Vt \supseteq u_i t \text{ defined and in Ut}
(b_2) u in Ut \supseteq (\forall i)[i in Vt \land u_i t = u]
(b_3) i before j in Vt \supseteq u_i t < u_j t
(b_4) i in Vt \land u = u_i t \supseteq S_i t = S u t
(b_5) j in Vt \supseteq i \rightarrow j(t) = u_i t = u_j t(t)
(b_6) i, k in Vt \land u = u_i t \land w = u_k t \supseteq
\forall j > \in \Omega_{i, k} \cap \exists u_j t > \in \Omega_{V, W}
(b_7) (v^z_t) Y_{V Z} t = \tilde{Y}_{U Z} t
(b_8) (v^z_t) T_{V Z} t = \tilde{T}_{U Z} t
(b_9) (v^z_t)(l in Vt \land u = u_l t \supseteq l \in P_{V Z} t = u \in \tilde{P}_{U Z} t]
(b_10) (v^z_t)(l in Vt \land u = u_l t \supseteq l \in R_{V Z} t = u \in \tilde{R}_{U Z} t]
(b_11) (v^z_t) T_{V Z} t = \tilde{T}_{U Z} t

The proof of (b) is by induction on \( t \leq a \). The inductive steps for \( b_1, b_2, b_3, b_5, b_6 \) follow exactly as in [Bül] Lemma 4.4. From the proof of (b_2) we also have for any limit \( x \leq a \), \( d \) \( u = u_i y \land (\forall t)^x u = u(t) \supseteq (\forall t)^x u = u_i t \land (\forall t)^x i - i(t) \). We now give a few consequences of (d). Suppose \( x \) is a limit \( \leq a \), \( \forall \) is
the last member of $U_x$, and $u$ is in $U_x$. Then $u$ is in $U_{v'}$, and by the induction hypothesis, $u = u_i v'$ for some $i$ in $V_{v'}$. As $u$ is in $U_x$ we have $(\forall t)^{U_x} u = u_i v'$, and by (d) and (14), $i$ is in $V_x$ with $(\forall t)^{x'} u = u_i t$. This argument shows that (e) $v$ is the last member of $U_x \land u$ in $U_x$. $\Rightarrow \forall i$ in $V_x$ $(\forall t)^{x'} u = u_i t$. A slightly more complicated argument shows that (f) $v$ is the last member of $U_x \land i$ in $V_x$. $\Rightarrow (\forall t)^{x}[i = i(t) \land u_i x = u_i t]$

Now suppose $z = u_1$, $z \leq a$, and $v$ is the last member of $U_z$. We show by induction on $t$, $v < t < z$, that $t_{U_z} = t_{U_z}$, $Y_{U_z} = Y_{U_z}$, $T_{U_z} = T_{U_z}$, and $t$ in $V_t \land u = u_t$. $\Rightarrow [t \in P_{U_z} \land u \in P_{U_z}] \land [t \in R_{U_z} \land u \in R_{U_z}]$. This then will yield the inductive step for $(b_7, b_8, b_9, b_{10}, b_{11})$.

(b_7): As $v$ is in $U_z$ we know that $(\forall t)^{z} v = v(t)$. From (c), $v = u_1 v'$, and thus by (d) and (14) we have that $\rho(v)$ is in $V_z$. Hence, by (14) $Y_{U_z} v' = T$. As $v$ is in $U_z$, by (14) $Y_{U_z} v' = T$, and so $Y_{U_z} v' = Y_{U_z} v'$.

We now give the inductive step from $t$ to $t'$. Let $v < t < z$ and suppose $\rho(t)$ is in $V_z$. Then by (f), $\rho(t) = \rho(t)(t)$, and thus by the induction hypothesis $\rho(t)$ is in $V_t$. But this is a contradiction as $\rho(t)$ is not in $V_t$, and so $v < t < z \Rightarrow \rho(t)$
is not in \( V_z \). As clearly \( \bar{v} < t < z \Rightarrow t \) is not in \( U_z \), we have by (14), (14), and the inductive assumption on \( t \) that,

\[
V_{V_z} t' = T_{V_z} t = T_{U_z} t = \bar{V}_{U_z} t'.
\]

The inductive step at limits follows directly from (14), (14), and the inductive assumption on \( t \).

\( (b_g): \) The inductive steps at non-limits is trivial. The result for limits follows easily from the induction assumption on \( t \).

\( (b_g): \) We have already seen in the proof of \( (b_7) \) that \( p(\bar{v}) \) is in \( V_z \) and \( \bar{v} \) is in \( U_z \). Thus by (14), (14), \( P_{V_z} \bar{v}' = 0 = P_{U_z} \bar{v}' \).

We next sketch the proof of the inductive step from \( t \) to \( t' \).

Suppose \( \bar{v} < t < z, t \) is in \( V_t' \), and \( r = u_t t' \). From the proof of \( (b_7) \) we have that \( p(t) \) is not in \( V_t \), and therefore by (14), the induction assumption, and (c) \( (t \neq p(t) \Rightarrow v = u_t t) \), it follows that

\[
\ell \in P_{V_z} t' = \bigvee_{h} [r = u_t(t) \land u_t(t) \in P_{U_z} t] \lor [r = t \land \bar{V}_{U_z} t] = u_t(t)
\]

From this equivalence we have by (14) that \( \ell \) in \( P_{V_z} t' \Rightarrow u \) in \( P_{U_z} t' \). For the implication in the other direction we note that from \( r = p(t) \) we get, by (14), that \( p \) is in \( U_t \) and therefore by the inductive assumption, \( p = u_{t'} \) for some \( h \) (in \( V_t \)). As \( t \) is \( \neq t \) in \( U_z \), the above remarks and equivalence yield \( \ell \) in \( P_{V_z} t' \Rightarrow u \) in \( P_{U_z} t' \). The proof of the inductive step at limits is similar and will be omitted (see the proof of \( (b_6) \) in [Bü] Lemma 4.4). As the proof of \( (b_{10}) \) is almost identical to that of \( (b_g) \) it too will be omitted.
(b_{11}): By (14), (b_9), and the inductive assumption we have,

\[ T_{Vz} t = \bigvee_{j \in Vz} \bigvee_{l} [u_{l} t \neq u_{j} t = u_{l} t(t) \land u_{l} t \in \overline{P}_{Uz} t] \]

By (f), \( j \in Vz \supset u_{j} t = u_{j} z \) in \( Uz \), and by (e) \( v \) in \( Uz \supset \bigvee_{j \in Vz} v = u_{j} t \). Thus it follows from the above equivalence that,

\[ T_{Vz} t = \bigvee_{v \in Uz} \bigvee_{l} [u_{l} t \neq v = u_{l} t(t) \land u_{l} t \in \overline{P}_{Uz} t] \]

From this equivalence we get by (14) (see Lemma 8.5) that \( T_{Vz} t \supset \overline{T}_{Uz} t \). The other implication follows from the above equivalence as in the proof of (b_9).

Lastly, we prove (b_4). The inductive steps at 0, successors, and \( \omega \)-limits is given in [Bru] Lemma 4.4. We now give the inductive step at \( \omega_1 \)-limits \( z \).

(b_4): Suppose \( i \) is in \( Vz \) and \( u = u_{i} z \). By (14), (b_{10}), (b_{11}) and the inductive assumption we have,

\[ b \in S^{C}_{1} C_{z} = \bigvee_{K \subseteq Vz} \bigwedge_{k \in K} \bigvee_{l} (\overline{T}_{Uz} t) \bigwedge_{l} [u_{l} t \neq u_{k} t = u_{l} t \land u_{l} t \in \overline{P}_{Uz} t] \]

\[ \land (\overline{T}_{Uz} t) \bigvee_{k \in K} \bigvee_{l} [u_{l} t \neq u_{k} t = u_{l} t \land u_{l} t \in \overline{P}_{Uz} t] \]

\[ \land (3^{z} t)b \in B^{C}_{1} C_{z} [S_{l} t, S_{k} t; k \in K] \]
Exactly as in the proof of \((b_{11})\), from this equivalence and 
\((e,f)\) we have,

\[
\begin{align*}
b \in S_i^C \cdot C_z := \forall_{w \subseteq Uz} \forall_{w \in W} (G_1^{T_{uz}^{2}}t) \forall_{(u_k \neq w)} \left( u_k \land u_k \in \bar{R}_{uz}t \right) \\
\land (G_1^{T_{uz}^{2}}t) \forall_{w \in W} \forall_{(u_k \neq w)} \left( u_k \land u_k \in \bar{R}_{uz}t \right) \\
\land (G_1^{T_{uz}^{2}}t) b \in B^C \cdot C \left[ \subseteq t, \subseteq w \in W \right]
\end{align*}
\]

That \(b \in S_i^C \cdot C_z \equiv b \in S_u^C \cdot C_z\), follows as in the proof of \((b_9)\) from the above equivalence.

We now have completed the proof of \((b)\). It remains to note that 0 is in all \(V_t\), from \(t=1\) on, and \(u_0t = 0\). By \((b_4)\) this yields the required \(S_0\alpha = \bar{S}_0\alpha\). Q.E.D.

Now all that is needed for our complementation lemma is a method for eliminating relativized quantifiers. This is done in the next lemma.

**Lemma 2.7** Let \(z \vdash t_0\) and let

\[
(*) \quad (\forall t) \left[ \forall t \land \forall t \land \forall t' \right] \land \left[ \forall t \land \forall t \land \forall t \right] \\
\land (\forall x) \left( \forall x^t \right) \forall x \quad (x \text{ ranges over all limits})
\]

Then,

\[
(a) \quad (G_1^{T_{uz}^{2}}t)U_t \equiv (G \forall \left[ \forall \text{ satisfies } (*) \right] \land (G_0^t)t \land (G_1^t)t)
\]
(b) \((\exists W' \exists z) Wt = (\exists V)[V \text{ satisfies (*)} \land (\exists W) Wt \land (\exists V) Wt]\)

**Proof:** Let \(W\) be cofinal \(z\). For simplicity we abbreviate \(W'z\) as \(W'\), and \(W'W, z\) as \(W^+\). Then we may define an isomorphism \(f\): \(<W', < > \simeq <W^+, < >\) by \(f(x) = (\mu y). Wx \land (\forall t) \gamma_t \sim Wt\).

Now suppose \(U \subseteq W'\). From the above isomorphism clearly \(U \in f[U] \in f[W^+].\) But as \(W' \in f_1^W.\) But as \(W' \in f_1^Z,\) by lemma 1.7, \(U \in f_1^W = U \in f_1^Z,\) and similarly as \(W^+ \in f_1^W, z\) by a relativized version of the same lemma we have \(f[U] \in f_1^W, z.\)

Combining all these equivalences we have shown that \(U \subseteq W' \Rightarrow U \in f_1^W = f[U] \in f_1^W, z.\) Therefore taking \(U\) to be \(f^{-1}[V]\) yields \((c) V \subseteq W^+ \Rightarrow f^{-1}[V] \in f_1^W, z.\)

As \(W^+ \in f_1^W, z\) we know that \(U \in f_1^W, z = U \cap W^+ \in f_1^W, z.\) Now by \((c), U \cap W^+ \in f_1^W, z = f^{-1}[U \cap W^+] \in f_2^Z.\) Using these two equivalences, the definition of \(f\) and the fact that \(W' \in f_1^Z\) we get that \((d) U \in f_1^W, z = \{v; (\forall y) \gamma_v[Wy \land (\forall t) \gamma_t v \sim Wt. \Rightarrow Uy]\} \in f_1^Z.\)

Not let \(\bar{V} = \{v; (\forall y) \gamma_v[Wy \land (\forall t) \gamma_t v \sim Wt. \Rightarrow Uy]\}.\) Note that \(\bar{V}\) satisfies (*), and in turn any \(V\) which satisfies (*) is a subset of \(\bar{V}.\) Thus, using \((d)\) and our convention that \(f_1^W, z = 0\) if \(W\) isn't cofinal \(z,\) we get \((a). Using the same reasoning we get \((b)\) also.

Q.E.D.
Lemma 2.8 (complementation lemma for ordinals $< \omega_2$): To any transition system $\Gamma$ of the form (1) with initial and terminal conditions $E[\cdot]$ and $L[\cdot]$, one can construct a system $[\tilde{E}, \tilde{\Gamma}, \tilde{L}]$ of the same form, such that for all $\alpha < \omega_2$, $\sim (\exists \tilde{Z}). E[\tilde{Z}] \land \Gamma_{\tilde{0}}(X,Z) \land L[\tilde{Z} \alpha] = (\exists \tilde{Y}). \tilde{E}[\tilde{Y}] \land \tilde{\Gamma}(X,Y) \land \tilde{L}[\tilde{Y} \alpha]$. 

Proof: Given any system $[E, \Gamma, L]$, using the previous lemma we can construct an automata $A$ of form (3) with terminal condition $L'$ of equal behavior. Thus $\sim (\exists \tilde{Z}). E[\tilde{Z}] \land \Gamma_{\tilde{0}}(X,Z) \land L[\tilde{Z} \alpha] = \sup_0^{\alpha}$-conditions by $\sup_1^{\alpha}$-conditions and Lemma 2.7 to replace relativized $\sup_1$-conditions by unrelativized $\sup_1$-conditions in $A(X) = Y$. The result is a non-deterministic system $\tilde{\Gamma} = [\tilde{E}, \tilde{\Gamma}, \tilde{L}]$ of the form (1) with the same behavior as $[A, L']$. Q.E.D.

Theorem 2.9 (definability in $MT[a, <]$, $a < \omega_2$): To every formula $\Sigma(x)$ of $MT[a, <]$, $a < \omega_2$, one can construct a transition system $\Gamma$ of form (1) with initial and terminal conditions $E[\cdot]$ and $L[\cdot]$ (or a finite state recursion $A$ of form (3) with terminal condition $L'[\cdot]$), such that for any $\alpha$-sequence $X[0, \alpha]$, $\Sigma(x)$ holds in $[\alpha, <]$ just in case $\Gamma(A)$ accepts $X[0, \alpha]$, i.e. $\Sigma(x) = (\exists \tilde{Z}). E[\tilde{Z}] \land \Gamma_{\tilde{0}}(X,Z) \land L[\tilde{Z} \alpha] (\Sigma(x) = (\exists \tilde{Y}). Y = A(X) \land L'[\tilde{Y} \alpha])$. 
Proof: The prenex form lemma 1.5 of [Büchi] will at first give a system $\Gamma_0$, which is like (1), except that the transition condition for $\omega_1$-limits is a $\sup_0$-condition, $\kappa_1[\sup_0^Z Z, Z]$. Remark 1.3 shows that a $\sup_1$-condition, $\kappa_1[\sup_1^Z Z, Z]$ can be put in its place. The first part of the theorem now follows by the elimination of quantifiers using the complementation lemma. The part in parentheses follows from this by one further application of Theorem 2.6. Q.E.D.
3. A decision method for $MT[\alpha,<]$, $\alpha < \omega_2$.

The decision method we now present is a non-deterministic version of that found in [Bü] Section 4. We begin by briefly repeating some definitions from [Bü] Section 4.

Let us define the $\omega_2$-behavior, $\text{beh } \omega_2 \Gamma$ of a transition system to be the set consisting of all inputs, $X[0,\alpha)$, accepted by $\Gamma$. Theorem 2.9 then says that questions about definability in the MT of ordinals $< \omega_2$ are really questions about $\omega_2$-regular events. We define the $\omega_2$-spectrum of an MT-sentence $\Sigma$ (in the primitive $<$) to be the set of all ordinals $\alpha$ such that $\Sigma$ holds in $[\alpha,<]$. Since an input for an "input free" transition system is an ordinal $\alpha$, we have from Theorem 2.9,

**Remark 3.1:** To every MT-sentence $\Sigma$ in $<$, we can construct an input-free transition system $\Gamma$ with initial condition $E$ and terminal condition $L$ such that the $\omega_2$-spectrum of $\Sigma$ is equal to $\text{beh } \omega_2 \Gamma$.

In the sequel we will consider a fixed input-free transition-system $\Gamma = [H, X_0, X_1]$ of form

$$(1) \quad \Gamma^V u(Z): (\forall t) H[Zt, Zt'] \land (\forall x) X_0^V [\sup x Z, Zx] \land (\forall z) X_1^V [\sup z Z, Zz]$$

We also assume that $K$ is the set of states of $\Gamma$ (i.e. of $Z$) with $s = |K|$ and $r = |\emptyset K \times K| = 2^S s$. For each $\mu < \omega_2$ and $\nu < \omega_1$ we define the following operator $G_{\mu, \nu}: K \to \emptyset K \times K$. 

(2) \( G_{\mu, \nu}[c] = \{<C, d>; (\exists [0, \omega_{\omega_0}]^\mu_\nu, \lambda_0^\mu_\nu(z) \wedge z_0 = c. \wedge z_0^\mu_\nu = d. \wedge \{zt; 0 \leq t < \omega_{\omega_0}^\mu_\nu\} = c\} \)

That is, \(<C, d>\) is in \( G_{\mu, \nu}[c] \) just in case there is a \( C \)-exact \( \omega_{\omega_0}^\mu_\nu \)-run \( z \) of \( \Gamma \) from \( c \) to \( d \). We define composition for these operators in the following way:

(3) \( G_{\mu, \nu} \circ G_{\gamma, \delta}[c] = \{<C, d>; \bigvee_{D, E, e} <D, e> \in G_{\gamma, \delta}[c] \wedge <E, d> \in G_{\mu, \nu}[e] \wedge C = D \cup E\} \)

From these we have the natural recursive definition for powers:

(4) \( G_{\mu, \nu}^0[c] = \{<0, c>\} \)
\( G_{\mu, \nu}^1[c] = G_{\mu, \nu}[c] \)
\( G_{\mu, \nu}^{h+1}[c] = G_{\mu, \nu} \circ G_{\mu, \nu}^h[c] \)

It is useful to note from (2) and (4) that for \( \mu < \omega_2, \nu < \omega_1 \) and \( h < \omega_0, <C, d> \in G_{\mu, \nu}^h[c] \) just in case there is a \( C \)-exact \( \omega_{\omega_0}^\mu_\nu \)-run \( z \) of \( \Gamma \) from \( c \) to \( d \). The operators of (2) also satisfy the following absorption laws, the proofs of which are as for (12) of [Bü] Section 4.

(5) \( \gamma < \nu < \omega_1 \wedge \mu < \omega_2 \Rightarrow G_{\mu, \nu} \circ G_{\mu, \gamma} = G_{\mu, \nu} \)
\( \delta < \mu < \omega_2 \wedge \gamma < \omega_1 \Rightarrow G_{\mu, \gamma} \circ G_{\delta, \gamma} = G_{\mu, \nu} \)
It should be noted that these operators \( G_{\mu, \nu} \) are taking the place of the simpler \( F_{\mu} \) of [Bü] Section 4. That these \( G_{\mu, \nu} \) are more complicated is understandable as (1) is nondeterministic.

**Lemma 3.2:** For \( \mu, \delta < \omega_2; \nu, \gamma < \omega_1; h < \omega_0 \),

\[
(6) \quad <c,d> \in G_{\mu, \nu}[c] \cdot \exists (3k)^{I+1} <c,d> \in G_{\mu, \nu}[c]
\]

\[
(7) \quad <c,d> \in G_{\mu, \nu+1}[c] \cdot \exists \bigvee_{D \subseteq C \land \in C} \land <c,e> \in G_{\delta, \gamma}[c] \land <d,e> \in G_{\delta, \gamma}[e]
\]

\[
(8) \quad \nu = \lim_{i<\omega} \nu_i \land (\forall i) \! \omega \quad G_{\mu, \nu_i} = G_{\delta, \gamma} \cdot \exists [c,d] \in G_{\mu, \nu}[c] \cdot \exists.
\]

\[
\bigvee_{D \subseteq C \land \in C} \land <c,e> \in G_{\delta, \gamma}[c] \land <d,e> \in G_{\delta, \gamma}[e]
\]

\[
(9) \quad \mu = \lim_{i<\omega} \mu_i \land (\forall i) \! \omega \quad G_{\mu_i, \nu_i} = G_{\delta, \gamma} \cdot \exists [c,d] \in G_{\mu_0, c}[c] \cdot \exists.
\]

\[
\bigvee_{D \subseteq C \land \in C} \land <c,e> \in G_{\delta, \gamma}[c] \land <d,e> \in G_{\delta, \gamma}[e]
\]

\[
(10) \quad \{\nu_i; i < \omega_1\} \text{ cofinal closed } \omega_1 \land (\forall i) \! \omega \quad G_{\mu, \nu_i} = G_{\delta, \gamma} \cdot \exists [c,d] \in G_{\mu+1, 0}[c] \cdot \exists.
\]

\[
\bigvee_{D \subseteq C \land \in C} \land <c,d> \in G_{\mu+1, 0}[c] \cdot \exists.
\]

\[
\bigvee_{D \subseteq C \land \in C} \land <c,e> \in G_{\delta, \gamma}[c] \land <d,e> \in G_{\delta, \gamma}[e]
\]

\[
\bigwedge_{e \in \mathcal{D}_1} <d,e> \in G_{\delta, \gamma}[e]
\]
(11) \{u_i; i < \omega_1\} cofinal closed \(\mu \land (\forall i) u_{i,0} = G_\delta, o \Rightarrow\)

\[
\begin{align*}
<&c,d> \in G_{\mu, o}[c]. \exists \\lambda \left( D_o, D_1 \subseteq C \land \mathcal{F}_1[D_1, d] \land \bigwedge_{e \in D_1} \mathcal{F}_o[D_o, e] \land \bigwedge_{e \in D_1} \langle c, e \rangle \in G_{\delta, o}[c] \land \\
\bigwedge_{e, f \in D_1} \langle D_o, f \rangle \in G_{\delta, o}[e]\right)
\end{align*}
\]

Proof:

(6): Let \( h \) be the least number such that \(<c,d> \in G_{\mu, \upsilon}[c], \) and suppose \( h > r. \) Thus, there is a C-exact run \( Z[0, \omega_1^\mu, \omega_0^\nu, h]\) of \( \Gamma \) from \( c \) to \( d. \) For each \( k < h \) let \( c_k = 2\omega_1^\mu, \omega_0^\nu, k \) and \( C_k = \{zt; 0 \leq t < \omega_1^\mu, \omega_0^\nu, k\}. \) As \( h > r, \) a repetition must occur among \( <0, c> = <c_0, 0>, <c_1, 1>, \ldots, <c_h, h> = <c, d>, \) say,

\(<c_m, c_m'> = <c_{m+n}, c_{m+n}>, n > 0. \) Then

\( Z[0, \omega_1^\mu, \omega_0^\nu, m]Z[\omega_1^\mu, \omega_0^\nu, (m+n), \omega_1^\mu, \omega_0^\nu, h]\) is a C-exact \( \omega_1^\mu, \omega_0^\nu, (h-n)-run\) of \( \Gamma \) from \( c \) to \( d. \) Thus \(<c, d> \in G_{\mu, \nu}[c], \) which contra-
dicts the assumption about \( h. \) This shows that \( h \leq r \) and consequently, (6) is proved.

(7): Suppose \(<c,d> \in G_{\mu, \nu+1}[c]\) and let \( x = \omega_1^\mu, \omega_0^\nu, 1. \) There is a C-exact run \( Z[0, x] \) from \( c \) to \( d. \) Let \( D = \sup_{o,x} Z. \)

Thus \( D \subseteq C \) and \( \mathcal{F}_o[D, d]. \) As \( \{\omega_1^\mu, \omega_0^\nu, n; n < \omega\}\) converges to \( x, \) there is a state \( e \) and a subsequence

\( \{\omega_1^\mu, \omega_0^\nu, m_n; n < \omega\}\) which also converges to \( x \) such that
n < \omega \cdot \omega \cdot Z \omega_1^\mu \omega_0^\nu m_n = e \wedge \{Zt; \omega_1^\mu \omega_0^\nu m_n < t < \omega_1^\mu \omega_0^\nu m_{n+1}\} = D.

Thus, \langle C,e \rangle \in G_{\mu,v}^m[c] \wedge \langle D,e \rangle \in G_{\mu,v}^{m_n - m_0}[e], \text{ and so by (6), there are}

h, k \leq r \text{ such that } \langle C,e \rangle \in G_{\mu,v}^h[c] \wedge \langle D,e \rangle \in G_{\mu,v}^k[e]. \text{ This}

shows that if } \langle C,d \rangle \in G_{\mu,v+1}[c], \text{ then the right side of (7) holds. }

Conversely, suppose } D \subseteq C, \mathcal{K}_0[D,d], \langle C,e \rangle \in G_{\mu,v}^h[c], \text{ and }

\langle D,e \rangle \in G_{\mu,v}^k[c]. \text{ So there is a } C\text{-exact run } Z_1[0,\omega_1^\mu \omega_0^\nu h]\text{ from c to e, and a } D\text{-exact run } Z_2[0,\omega_1^\mu \omega_0^\nu k]\text{ from e to e. As}

D \subseteq C \text{ and } \mathcal{K}_0[D,d], Z_1[0,\omega_1^\mu \omega_0^\nu h] Z_2[0,\omega_1^\mu \omega_0^\nu k] Z_2[0,\omega_1^\mu \omega_0^\nu k] \cdots d

is a } C\text{-exact run of } \mathcal{F} \text{ from } c \text{ to } d \text{ and therefore}

\langle C,d \rangle \in G_{\mu,v+1}[c].

(8): Suppose } v = \lim_{i<\omega} v_i, \text{ i < } \omega \cdot \omega \cdot G_{\mu,v_i} = G_{\delta,\gamma}, \langle C,d \rangle \in G_{\mu,v}[c]

and let } x = \omega_1^\mu \omega_0^\nu . \text{ There is a } C\text{-exact run } Z[0,x]\text{ from c to d. Let } D = \sup_{x Z}. \text{ Thus } D \subseteq C \text{ and } \mathcal{K}_0[D,d]. \text{ As}

v = \lim_{i<\omega} v_i, \{\omega_1^\mu \omega_0^\nu i; i < \omega\} \text{ converges to } x, \text{ and so there is }

a subsequence } \{\omega_1^\mu \omega_0^\nu m_i, i < \omega\} \text{ which also converges to }

x \text{ such that } i < \omega \cdot \omega \cdot Z \omega_1^\mu \omega_0^\nu m_i = e \wedge

\{Zt; \omega_1^\mu \omega_0^\nu m_i < t < \omega_1^\mu \omega_0^\nu m_{i+1}\} = D.
So $Z[0, \omega_1^{\omega_0^{\nu_i}}]$ is a C-exact run from $c$ to $e$, and $Z[\omega_1^{\omega_0^{\nu_i}}, \omega_1^{\omega_0^{\nu_i}+1}]$ is a D-exact $\omega_1^{\nu_i+1}$-run from $e$ to $e$. Since $G_{\nu_i} = G_{\delta, \gamma}$ we have by (2),

\[ <c,e> \in G_{\delta, \gamma}[c] \land <d,e> \in G_{\delta, \gamma}[e]. \]

This proves (8) with "=" replaced by "\in". The converse is shown as for (7). The proof of (9) is like that of (8) and will be omitted.

(10): The proof is basically a miniature version of the proof of formula (5) of section 2. Let (a) $\{\nu_i; i < \omega_1\}$ be cofinal-closed $\omega_1$ and (b) $i < \omega_1 \quad \sup_{G_{\nu_i}, \nu_i} = G_{\delta, \gamma}$. First suppose $<c,d> \in G_{\mu+1, \omega}[c]$ and $z = \omega_1^{\mu+1}$. So there is a C-exact run $Z[0, z]$ of $\Gamma$ from $c$ to $d$.

Let $D_0 = \sup_0^Z$ and $D_1 = \sup_1^Z$. Thus, (C) $D_0 \subseteq C$, $K_1[D_1,d]$. As we saw in the proof of formula (5) of Section 2, there is a set $P$ and for each $e \in D_1$, sets $P_e$ such that (d) $e \in D_1 \supset P_e \subseteq \sup_{D_1}^Z$, (e)

\[ P = \bigcup_{e \in D_1} P_e \text{ is cofinal-closed } Z, \quad (f) \quad e \in D_1 \supset P_e \ni \sup_{D_1}^Z. \]

(g) $v < y \land P_v \land P_y \ni \mathcal{Z}(t; v \leq t < y) = D_0$, (h)

\[ P_y \ni \{Zt; 0 \leq t < y\} = C, \quad \text{and (i) } P_x \ni \sup_{D_0}^Z = D_0. \]

By (a), we may further assume (j) $P \ni \{\omega_1^{\mu \nu_i}; i < \omega\}$. As $Z$ is a run of $K_0$, by (d,e,f,i) it follows that
(k) \( e \in D_1 \supset \mathcal{S}_o[D_o,e] \).

Let \( e \in D_1 \). By (d) there is a \( y \in \mathcal{P}_e \) and by \((e,f,h)\), \( Z_y = e \) and \( \{Z_t; 0 \leq t < y\} = C \). Using \((e,j)\) there is an \( i < \omega_1 \) such that \( y = \omega_1 \omega_i \), and so \( Z[o,y] \) is a C-exact \( \omega_1^{\omega_1} \omega_i \)-run of \( \Gamma \) from \( c \) to \( e \). Thus by (2), \( <C,e> \in G_\mu,\nu_i [c] \). By (b), \( G_\mu,\nu_i = G_\delta,\gamma \), and hence we have shown, \((1) \bigwedge_{e \in D_1} <C,e> \in G_\delta,\gamma [c] \).

Now let \( e, f \in D_1 \). By (d) there are \( v, y \) such that \( v < y \), \( \mathcal{P}_e v \), and \( \mathcal{P}_f y \). By \((e,f,g)\), \( Z_v = e \), \( Z_y = f \), and \( \{Z_t; v \leq t < y\} = D_o \). Using \((e,j)\) there are \( i < j < \omega_1 \) such that \( v = \omega_1^{\omega_i} \), \( y = \omega_1^{\omega_j} \), and so \( Z[v,y] \) is a \( D_o \)-exact \( \omega_1^{\omega_i} \omega_o^{\omega_j} \)-run of \( \Gamma \) from \( e \) to \( f \). Thus by (2), \( <D_o,f> \in G_\mu,\nu_j [e] \) and so by (b), \( <D_o,f> \in G_\delta,\gamma [e] \).

This shows that \((m) \bigwedge_{e,f \in D_1} <D_o,f> \in G_\delta,\gamma [e] \). Thus from \((a,b)\) and \( <C,d> \in G_{\mu+1,\nu_0} [c] \), we have shown \((c,k,\lambda,m)\) and consequently (10) holds with "\( = \)" replaced by "\( \Rightarrow \)".

In the next two paragraphs we complete the proof by showing the converse.

Suppose \((a,b)\) and \((n) \ D_o \subseteq C, \mathcal{S}_1[D_1,d], \ (o)\n\ e \in D_1 \supset \mathcal{S}_o[D_o,e] \), \((p)\ e \in D_1 \supset <C,e> \in G_\delta,\gamma [c] \), \((q)\ e, f \in D_1 \supset <D_o,f> \in G_\delta,\gamma [e] \). Let
\[ P = \{ \omega^1_i \omega^0_o ; i < \omega_1 \} \text{ and } z = \omega^1_i + 1. \] By (a), \( P \) is cofinal closed \( z \) and so \( P \models \exists^z \) has no atoms. Thus there are sets \( P_e \) such that (r) \( P = \cup P_e \), (s) \( e \in D \supset P_e \& Z \), and (e)\[ D \]

\begin{align*}
\text{(t)} \quad & e, f \in D \land e \neq f \Rightarrow P_e \cap P_f = 0. \text{ By (r,t) each } \\
& \omega^1_i \omega^0_o \text{ is in exactly one of the } P_e. \text{ So there are } e_i \text{ such that } \\
& \text{(u)} \ P_e \omega^1_i \omega^0_o, (v) \ P_e \omega^1_i \omega^0_o \Rightarrow e = e_i, \text{ and} \\
& \text{(w)} \ e_i \in D.
\end{align*}

By (b,p,w) and (2), there is a C-exact run \( Z[0, \omega^1_i \omega^0_o] \) of \( \Gamma \) from \( c \) to \( e_o \). Similarly, by (b,q,w), there are \( D_o \)-exact \( \omega^1_i \omega^0_o + 1 \)-runs \( Z[\omega^1_i \omega^0_o ; \omega^1_i \omega^0_o + 1] \) of \( \Gamma \) from \( e_i \) to \( e_{i+1} \). As \( P \) is cofinal closed \( z \) and the runs have equal states where they overlap, we may splice all these runs together to form a run \( Z[0, z] \). We prove that \( Z[0, z] \) is a run of \( \Gamma \) as in the proof of (5) of section 2, using (o,w).

Now complete this run to \( z \) by defining \( Zz = d. \) Using (s,u,v,w) it is easy to prove \( D_1 = \sup Z \) and so by (n), \( Z[0, z] \) is a C-exact run of \( \Gamma \) from \( c \) to \( d. \) Hence \( <C, d> \supset G_{\mu+1, o}[c] \). This completes the proof of (10). The proof of (11) is similar and will be omitted.

Q. E. D.

We now show that for each \( \mu < \omega_2 \), the \( G_{\mu, v} \) become equal from some \( q < \omega \) on, thus extending Lemma 4.7 of \[ Bû].
Lemma 3.3: For each $\mu < \omega_2$, there is a finite $q_\mu$ such that $G_{\mu,q_\mu} = G_{\mu,q_\mu+1}$. Furthermore for such $q_\mu$,

$$(\forall \nu) \forall_1 (\forall \pi) G_{\mu,\nu}^\omega \subseteq G_{\mu,\nu}^\omega$$

$$(\forall \nu) \forall_1 (\forall \pi) G_{\mu,\nu}^\omega \subseteq G_{\mu,\nu}^\omega$$

Proof: Fix $\mu < \omega_2$. Note that the $G_{\mu,\nu}$ are finite operators, i.e. $\langle G_{\mu,\nu}[c] ; c \in K \rangle$ takes on only finitely many values.

Thus there are finite $p,m$; $m > 0$, such that (a) $G_{\mu,p} = G_{\mu,p+m}$.

Now by induction on $n,j,h < \omega$ using (a), (4), and (7), it is easy to show (b) $(\forall n,j,h) G_{\mu,p+n+m}^h = G_{\mu,p+n}^h$. We now prove the following claim by induction on $h$:

(c) $(\forall n)^m (\forall j)^\omega (\forall h)^\omega$ $G_{\mu,p+n+m}^h = G_{\mu,p+n}$

If $h = 1$, then (c) follows by (b). Let $h > 1$. We now have the following sequence of equalities:

$$G_{\mu,p+n+m}^h = G_{\mu,p+n}^h$$

(4) $G_{\mu,p+n} = G_{\mu,p+n+1}$

(ind hyp) $G_{\mu,p+n} = G_{\mu,p+n}$

(b) $G_{\mu,p+n+m} = G_{\mu,p+n+m}$

(5) $G_{\mu,p+n+m} = G_{\mu,p+n+m}$

(b) $G_{\mu,p+n} = G_{\mu,p+n}$
Thus, $G_{\mu,p+n+m_j}^h = G_{\mu,p+n}$, proving (c). Note that $\{p + m_j; j < \omega\}$ converges to $\omega$. By (b), $j < \omega \Rightarrow G_{\mu,p+m_j}^h = G_{\mu,p}^h$, and so by (8), $<c,d> \in G_{\mu,\omega}[c]$ if $\forall D \subseteq c \land \mathcal{K}_c[D,d] \land <c,e> \in G_{\mu,p}[c] \land <D,e> \in G_{\mu,p}[e]$. So by (7) and (c) we have $<c,d> \in G_{\mu,\omega}[c]$ if $<c,d> \in G_{\mu,p+1}$. This proves $G_{i,\omega} = G_{i,p+1}$. Now $\{p + 1 + m_j; j < \omega\}$ also converges to $\omega$, and so by (b), $j < \omega \Rightarrow G_{i,p+1+m_j} = G_{i,p+1}$. Using (7), (8), and (c) by a similar argument we have $<c,d> \in G_{\mu,\omega}[c]$ if $\forall D \subseteq c \land \mathcal{K}_c[D,d] \land <c,e> \in G_{\mu,p+1}[c] \land <c,e> \in G_{\mu,p+2}[c]$. Consequently, $G_{\mu,p+1} = G_{\mu,\omega} = G_{\mu,p+2}$. Letting $q_{\mu} = p+1$, we have (d) $G_{\mu,q_{\mu}} = G_{\mu,q_{\mu}+1}$. Now by (c) it follows that (e) $G_{\mu,q_{\mu}}^2 = G_{\mu,q_{\mu}}$. Using (d,e), (7), (8), it is easy to show by induction on $h,v$ that $(\forall v) \omega^1(\forall h)\omega^h_c = G_{\mu,q_{\mu}}$, thus completing the proof of the lemma. Q. E. D.

The above lemma shows that for any $\mu$ there is a $q_{\mu}$ such that $G_{\mu,q_{\mu}} = G_{\mu,q_{\mu}+1}$. As a matter of fact we have from the proof that $q < 2^s s^2 (s = |k|)$. The next lemma shows that a similar fact is true of the $\mu$ part of the operator $G_{\mu,v}$.

Lemma 3.4: There is a finite $p$ such that $G_{p+1,0} = G_{p,0}$.
Furthermore for such \( p \), we have \( G_{p,1} = G_{p,2} \).

**Proof:** \( <G_{i,0}[c] ; c \in K> \) takes on finitely many values. Thus there are \( p, m, m > 0 \), such that (a) \( G_{p,0} = G_{p+m,0} \). By induction on \( n, h, i, j \), using (a), (4), (7), (10), it is easy to show (b) \( (\forall n)^{m}(\forall h, i, j)\omega. G_{p+n+i,j}^{h} = G_{p+n+m+i,j}^{h} \). Using the second part of (5), (4), (b), and a proof similar to that used in the previously lemma, we have (c) \( (\forall n)^{m}(\forall i, j)\omega (\forall h)_{1}^{\omega} G_{p+n+i,j}^{h} = G_{p+n},j \). Now, \( \{p + mi; i < \omega\} \) converges to \( \omega \). By (b), \( G_{p+mi,0} = G_{p,0} \) and so by (7), (9), (c) we have \( \langle C, d \rangle \in G_{\omega,0}[c] \). .\( \forall D \subseteq C \wedge \forall [D,d] \wedge <C,e> \in G_{p,0}[c] \wedge <D,e> \in G_{p,0}[e] \)

. .\( \forall <C,d> \in G_{p,1}[c] \). Thus, \( G_{\omega,0} = G_{p,1} \). As \( \{p + 1 + mi; i < \omega\} \) converges to \( \omega \). using (b), (7), (9), (c) and a similar argument we have \( G_{\omega,0} = G_{p+1,1} \). Thus, \( G_{p,1} = G_{p+1,1} \). From this we can show by an inductive argument that \( (\forall h)^{\omega} (\forall j)^{\omega} G_{p,j}^{h} = G_{p+1,j}^{h} \). Hence \( G_{p+1,q_{p}} = G_{p+1,q_{p+1}} \), and so by (10) and lemma 3.3 it follows that \( G_{p+1,0} = G_{p+2,0} \). This proves the first part of the lemma.

Now assume that \( p \) is any finite ordinal such that \( G_{p,0} = G_{p+1,0} \). By familiar argument, from this it follows that

(d) \( (\forall h, j)^{\omega} (\forall i)^{\omega} G_{p,i,j}^{h} = G_{i+1,j}^{h} \wedge q_{i} = q_{i+1} \) and (e)

(\( \forall h \))_{1}^{\omega} (\forall j)^{\omega} G_{p,j}^{h} = G_{p,j}^{h} \). Now \( \{\omega_{i}; p < i < \omega\} \) converges to
\( \omega_1 \) and so by (d), (7), (9), (e), \( \langle c,d \rangle \in G_{\omega,0} \Rightarrow \bigvee_{D,e} c \wedge \mathcal{H}[D,d] \wedge \langle c,e \rangle \in G_{p,0}[c] \wedge \langle d,e \rangle \in G_{p,0}[e] \Rightarrow \langle c,d \rangle \in G_{p,1}[c] \).

Thus \( G_{\omega,0} = G_{p,1} \). As \( \{ \omega^i_1 \omega_0 : p \leq i < \omega \} \) also converges to \( \omega \), a similar argument shows that \( G_{\omega,0} = G_{p,2} \) and so \( G_{p,1} = G_{p,2} \), which completes the proof of the lemma.

Q. E. D.

**Lemma 3.5:** Let \( G_{p,0} = G_{p+1,0} \) and for each \( i < p \),

\[ G_i,q_i = G_i,q_i + 1. \]

Then for \( 1 \leq h < \omega_0, v < \omega_1 \), we have:

\[ i < p \land q_i \leq v \Rightarrow G^h_{i,v} = G^h_{i,q_i} \]

(12) \[ p \leq \mu < \omega_2 \Rightarrow G^h_{\mu,v} = \begin{cases} G_{p,1} & \text{if } \mu = \omega_0 \lor v \neq 0 \\ G_{p,0} & \text{otherwise} \end{cases} \]

**Proof:** The first implication is a restatement of Lemma 3.3.

The second is proved by induction on \( h, \mu, v \). There are two cases: \( \omega^h \omega_0^v \omega_1 = \omega_0 \), in which case \( G^h_{\mu,v} = G_{p,1} \), and \( \omega^h \omega_0^v \omega_1 = \omega_1 \), in which case \( G^h_{\mu,v} = G_{p,0} \). These yield many subcases: \( v \) as successor or \( v = \omega_0 \) (in which case \( \omega^v_0 \) is the limit of an \( \omega_0 \)-sequence of \( \omega^v_0 \)-limits or of an \( \omega_0 \)-sequence of \( \omega^v_1 \)-limits); \( v = 0 \) and \( \mu = \omega_0 \) (in which case \( \omega^v_1 \) is the limit of an \( \omega^v_0 \)-sequence of \( \omega^v_0 \)-limits or of an \( \omega^v_0 \)-sequence
of \( \omega_1 \)-limits) or \( \nu = 0 \) and \( \mu = \omega_1 \) (in which case \( \omega_1^\mu \) is the limit of a cofinal-closed \( \omega_1 \)-sequence of \( \omega_0 \)-limits). The arguments for each are familiar and are the various parts of Lemma 3.2 and both parts of Lemma 3.4. Q. E. D.

For any \( p \), every ordinal, \( \alpha \) has a unique representation 
\[
\alpha = \omega_1^p \mu + \omega_1^{p-1} \nu_{p-1} + \ldots + \omega_1^0 \nu_0
\]
where each \( \nu_i < \omega_1 \); we call this the \( p \)-expansion (base \( \omega_1 \)) of \( \alpha \). \( \omega_1^p \mu \) is the \( p \)-head (base \( \omega_1 \)) of \( \alpha \) and the rest of the expansion is the \( p \)-tail (base \( \omega_1 \)) of \( \alpha \). Note that for each \( i < p \) and for any \( q_i \), \( \nu_i \) has a unique representation 
\[
\nu_i = \omega_0^q_i \gamma_i + \omega_0^{q_i-1} k_{i,q_i-1} + \ldots + \omega_0^0 k_{i,0}
\]
This is the \( q_i \)-expansion (base \( \omega_0 \)) of \( \nu_i \) that was discussed in [Bü] Section 4. We will now refer to it as the \( i,q_i \)-expansion of \( \alpha \). \( \omega_0^q_i \gamma_i \) is the \( i,q_i \)-head of \( \alpha \) and the rest is the \( i,q_i \)-tail of \( \alpha \). The sequence \( \langle (\gamma_i), k_{i,q_i-1}, \ldots, k_{i,0} \rangle \), where \( (\gamma_i) \) is \( 0 \) if \( \gamma_i = 0 \) and \( 1 \) otherwise, is called the \( i,q_i \)-character of \( \alpha \), denoted \( \text{char}_{i,q_i} \alpha \). Next define

\[
\text{index}_p \alpha = \begin{cases} 
<0,0> & \text{if } \mu = 0 \\
<1,0> & \text{if } \mu \neq 0 \land \mu = \omega_0 \ (\text{i.e. if } \omega_1^p \mu + \omega_0) \\
<0,1> & \text{if } \mu \neq 0 \land [\mu \text{ is a successor or } \mu = \omega_1] \\
& \text{(i.e. if } \omega_1^p \mu = \omega_1) 
\end{cases}
\]

We now define the \( p; q_{p-1}, \ldots, q_0 \)-character (base \( \omega_1 \)) of \( \alpha \)
denoted char \( p_1, q_{p-1}, \ldots, q_0 \) to be \( \langle \text{index}_{p_1}, \text{char}_{p_1, q_{p-1}}, \ldots, \text{char}_{q_0, q_0} \rangle \). Note that associated with each \( \alpha < \omega_2 \) is a unique \( p_1, q_{p-1}, \ldots, q_0 \)-character.

We also have a unique representation \( \alpha = \omega_1^m + 0 \omega_1^{n-1} \nu_{n-1} + \ldots + \omega_1 \nu_0 \), whereby \( n = 0 \lor \nu_{n-1} \neq 0 \) and each \( \nu_i < \omega_1 \); we call this the \( \omega \)-expansion (base \( \omega_1 \)) of \( \alpha \). \( \omega_1^m \) is the \( \omega \)-head (base \( \omega_1 \)) and the rest is the \( \omega \)-tail (base \( \omega_1 \)). Note that each \( \nu_i \) has in turn a unique representation \( \omega_1^{m_i-1} + \omega_0 k_i + \ldots + \omega_0 k_0 \) called the \( i, \omega \)-expansion of \( \alpha \). \( \omega_0 k_0 \) is the \( i, \omega \)-head of \( \alpha \), the rest the \( i, \omega \)-tail of \( \alpha \); \( \langle \gamma_i, k_i, m_i-1, \ldots, k_0, 0 \rangle \) is the \( i, \omega \)-character of \( \alpha \), denoted char\( _{i, \omega} \alpha \). The \( \omega \)-index of \( \alpha \), denoted index\( _{\omega} \alpha \) is defined as follows:

\[
\text{index}_{\omega} \alpha = \begin{cases} 
<0,0> & \text{if } \mu = 0 \\
<1,0> & \text{if } \mu \neq 0 \land [\mu \text{ successor } \lor \mu = \omega_0] \\
<0,1> & \text{if } \mu \neq 0 \land \mu < \omega_1 
\end{cases}
\]

The \( \omega \)-character (base \( \omega_1 \)) of \( \alpha \) denoted char\( _{\omega} \alpha \), is defined to be \( \langle \text{index}_\omega \alpha, \text{char}_{n-1, \omega}^\alpha, \ldots, \text{char}_{0, \omega}^\alpha \rangle \).

We now introduce some abbreviations. For each \( \mu < \omega_2 \) and \( \nu < \omega_1 \) we define the operator \( F_{\mu, \nu}: \mathcal{PK} \to \mathcal{PK} \) as follows:
From (2) we see that \( F_{\mu, \nu}[D] \) consists of all states which are attainable via an \( \omega^u \omega_0 \)-run of \( H, \mathcal{X}_0, \mathcal{X}_1 \) starting from some state in \( D \). Let the \( p; q_{p-1}, \ldots, q_0 \)-character of \( \alpha \) be exactly as above: we define

\[
F_{\text{char}, p; q_{p-1}, \ldots, q_0}[D] = F_{\text{char}, q_0}^{\alpha} F_{\text{char}, q_1}^{\alpha} \ldots F_{\text{char}, q_{p-1}}^{\alpha} F_{\text{char}, q_p}^{\alpha}[D]
\]

(14) \( F_{\text{index}, p}[D] = F_{p, 0}^m F_{p, 1}^l[D] \) where \( \text{index}_p \alpha = <n, m> \)

\[
F_{\text{char}, p; q_{p-1}, \ldots, q_0}[D] = F_{\text{char}, q_0}^{\alpha} F_{\text{char}, q_1}^{\alpha} \ldots F_{\text{char}, q_{p-1}}^{\alpha} F_{\text{index}, p}[D]
\]

**Theorem 3.6 (the normal form for sentences):** Let \( \{ \} \) be an MT-sentence in \( < \). There are finite numbers \( p, q_{p-1}, \ldots, q_0 \) operators

\[
G_{0,0}, \ldots, G_{0,q_0}; G_{1,0}, \ldots, G_{1,q_1}, \ldots, G_{p-1,0}, \ldots, G_{p-1,q_{p-1}}; G_{p,0}, G_{p,1}
\]

on a finite set \( K \), and \( E, L \subseteq K \) such that for any \( \alpha < \omega_2, \{ \} \)

holds in \( [\alpha, \omega] \) just in case \( F_{\text{char}, p; q_{p-1}, \ldots, q_0}[E] \cap L \neq \emptyset \)

(\text{using the previous abbreviations}).

**Proof:** Using Remark 3.1 we can construct an input-free transition system \( \Gamma = [H, \mathcal{X}_0, \mathcal{X}_1] \) with initial condition \( E \) and terminal
condition $L$ and $K$ equal the set of states of $\Gamma$ such that $[E,\Gamma, L]$ accepts just those $\alpha$ in the $\omega_2$-spectrum of $\Sigma$. Let the $G_{i,j}$ be defined from (2), and let $p, q_{p-1}, \ldots, q_0$ be defined as in Lemma 3.3 and 3.4. The proof of the theorem is straightforward and follows from Lemma 3.4, definition (2), (13), (14), and noting that by (5), $G_{p,1}^0 G_{p,0} = G_{p,1}$.

Q. E. D.

The system $[\mathcal{K}; E, F, \ldots, F_{p-1}, q_{p-1}, F_p, 0, F_{p,1}]$ is a finite transition algebra working on words over the input states $\{<i,j>; i \leq p, j \leq q_i\}$ in the usual sense of finite automata. We may also (see [Bü] Section 4) interpret it as a free algebra relative to the absorption laws:

(15) $k < j \implies G_{i,j} \circ G_{i,k} = G_{i,j}$ 
$h > i \implies G_{i,j} \circ G_{h,k} = G_{i,j}$

$G_{i,q_i} = G_{i,q_i+1}$ 
$G_{p,0} = G_{p+1,0}$

In this case an input is a $p, q_{p+1}, \ldots, q_0$-character. We call these automata, $[\mathcal{K}; E, F_0, 0, \ldots, F_{p-1}, q_{p-1}, F_p, 0, F_{p,1}; L]$ $p; q_{p-1}, \ldots, q_0$-absorption automata. Their behavior are sets of $p; q_{p-1}, \ldots, q_0$-characters, and Theorem 3.6 take the form:

**Theorem 3.6:** The $\omega_2$-spectrum of any MT-sentence $\Sigma$ consists of all $\alpha < \omega_2$ whose $p; q_{p-1}, \ldots, q_0$-characters are accepted by an appropriately chosen $p; q_{p-1}, \ldots, q_0$-absorption automata.
Let $\Delta$ be an MT-sentence in $\alpha$ and let $p, q_{p-1}, \ldots, q_0$ be as in Theorem 3.6. From Theorem 3.6 we see that if $\Delta$ has a model of cardinality $< \omega_2$, then it has one of cardinality $< \omega_1^{p+1}$. Furthermore for each $i < p$ if $\Delta$ has a model of cardinality $< \omega_1^{i+1}$ then it has a model of cardinality $< \omega_1^{i+1}$. Thus the MT of all well-orders of cardinality $< \omega_2$ is equal to the MT of all $\alpha < \omega_1^\omega$, and the MT of all well-orders of cardinality $< \omega_1^{i+1}$ is equal to the MT of all $\alpha < \omega_1^{i+1} \omega_0$. In [Bü] Section 4 we made these observations in the case where $i = 0$. The following theorems now follow as in [Bü] Section 4.

**Theorem 3.6**: The following four Boolean algebras are isomorphic:

- the algebra of $\omega_2$-spectra of MT-sentences,
- the Tarski-Lindenbaum algebra of the MT-theory of all well-orders of cardinality $< \omega_2$,
- the algebra of all sets of ordinals MT-definable in $[\omega_1^\omega, <]$,
- the algebra of all absorption-regular sets.

**Theorem 3.7**: For any ordinals $\alpha, \beta < \omega_2$ the system $[\alpha, <]$ and $[\beta, <]$ are MT-equivalent just in case $\text{char}_\omega \alpha = \text{char}_\omega \beta$.

**Theorem 3.8**: Let $\alpha < \omega_2$ and let $\omega_1^\omega + \omega_1^{n-1} \cdot \omega_1 \cdot \ldots \cdot \omega_1 \cdot \omega_1^0$ be the $\omega$-expansion (base $\omega_1$) of $\alpha$ where each $\omega_1^i$ is of the form $\omega_1^{m_i-1}$ $\omega_1 \cdot \ldots \cdot \omega_1 \cdot \omega_1^0$. Then MT$[\alpha, <]$ is axiomatizable (i.e. standard axiomatizable in the sense of [BS] and
(a) If \( \mu = \gamma_{n-1} = \ldots = \gamma_0 = 0 \), then \([\alpha, \prec]\) is MT-categorical and finitely MT-axiomatizable.

(b) If \( \mu \neq 0 \) or some \( \gamma_i \neq 0 \), then \([\alpha, \prec]\) is neither MT-categorical nor finitely MT-axiomatizable. Furthermore, \([\alpha, \prec]\) is MT-equivalent to \([\beta, \prec]\) where \( \beta = \eta + \omega^{n-1}_{m_i-1} \delta_{n-1} + \ldots + \omega_0 \delta_0 \), each \( \delta_i \) is \( \omega_0 (\gamma_i) + \omega_0 k_i, m_i-1 + \ldots + \omega_0 k_i, 0 \), and \( \eta = 0 \) if \( \mu = 0 \), \( \eta = \omega_1 \) if \( \omega_1 \mu = \omega_0 \), and \( \eta = \omega_1 + 1 \) if \( \omega_1 \mu = \omega_1 \).

Proof: We begin by defining some formulas and stating some properties of these. Further discussion may be found in [BS]. Also, the reader may want to return to [Bu] Section 1 for an explanation of some of the notation.

\[
\begin{align*}
L_{m_0,0} & : T \\
L_{m_i, j+1} & : (\forall x) (\exists y). y > x \land L_{m_i, j}[y] \\
L_{m_{i+1}, 0} & : \neg \text{Acc}_0 \land (\forall x) (\exists y). y > x \land L_{m_i, 0}[y] \\
L_{\text{lim}_{i, j}} & : L_{m_i, j} \lor L_{m_i, j+1} \\
\text{lim}_{i, j} & : L_{\text{lim}_{i, j}} \land \neg L_{m_i, j+1}
\end{align*}
\]

That \( L_{m_i, j+1} \) holds in a structure \([\alpha, \prec]\), i.e. \( \alpha \in L_{m_i, j+1} \), means that \( \alpha = 0 \) or \( \alpha \) is the limit of a sequence of ordinals in \( L_{m_i, j} \). Similarly, \( L_{m_i+1} \models [\alpha, \prec] \) just in case \( \alpha = 0 \) or \( \alpha \) is the limit of an \( \omega_1 \)-sequence of members of
It is desirable to have \( \text{Lim}_{i,j} \subseteq \text{Lim}_{h,k} \) for \(<i,j> < <h,k>\) in the lexicographical order on \( \omega \times \omega \). Now this is true except, unfortunately, in the case where \( j \neq 0 \land k = 0 \). This property is however true of \( \text{Lim}_{i,j} \) (it is for this reason we define \( \text{Lim}_{i,j} \)). Thus if \( \alpha \in \text{Lim}_{i,j} \) we say that \( \alpha \) is a limit of order at least \(<i,j>\), where \( \alpha \in \text{Lim}_{i,j} \) means that \( \alpha \) is a limit of order exactly \(<i,j>\). Note that \( \text{lim}_{i,j} \equiv \text{Lim}_{i,j} \land \neg \text{Lim}_{i,j+1} \) and that \( \text{lim}_{i,j} \succ [\alpha,\lessdot] \) means \( \alpha = \omega^i + \omega^j + \mu \) for some \( \mu \) where \( \mu = 0 \lor \mu \geq \omega^i \omega^j \).

\[
T^n_{i,j} : (\forall x_0, \ldots, x_{n-1}) (x_0 \prec \cdots \prec x_{n-1} \land \bigwedge_{k<n} \text{Lim}_{i,j}[x_k] \land (\forall y) x_0 \prec \text{Lim}_{i,j+1}[y] \land \bigvee [\text{Lim}_{i,j} \land (\forall x_0, \ldots, x_{n-2}) (x_0 \prec \cdots \prec x_{n-1} \land \bigwedge_{k<n-1} \text{Lim}_{i,j}[x_k] \land (\forall y) x_0 \prec \text{Lim}_{i,j+1}[y])] \quad (\forall y) x_0 \prec \text{Lim}_{i,j+1}[y])
\]

For the above we make the convention that \( (\forall x_0, \ldots, x_{m-1}) (x_0 \prec x_1 \prec \cdots \prec x_{m-1}) \Delta \equiv T, \) if \( m \leq 0 \). Note that \( T^n_{i,j} \succ [\alpha,\lessdot] \) if and only if there is a \( \nu < \omega^i \omega^j \) and a \( \mu \) such that \( \mu = 0 \lor \mu \geq \omega^i \omega^j \), with \( \alpha = \mu + \omega^i \omega^j n + \nu \).

\[
U_{i,j}(\Delta) : [\text{Lim}_{i,j} \land \Delta] \lor [\neg \text{Lim}_{i,j} \land (\forall x) [\text{Lim}_{i,j}[x] \land (\forall y) \text{Lim}_{i,j}[y] \Rightarrow y \leq x)] \Rightarrow \Delta [x]]
\]

\( U_{i,j}(\Delta) \succ [\alpha,\lessdot] \) just in case the largest element \( x \leq \alpha \) such
that \( x \in \text{Lim}_{i,j} \), if it exists (i.e. if \( \alpha \geq \omega_1 \omega_i \)), satisfies \( \Delta \).

We also have the following formula \( \sum_{<\omega_2} \), where \( \sum_{<\omega_2} \models [\alpha,<] \) just in case \( \alpha < \omega_2 \).

\[
\sum_{<\omega_2} : W|o \land \text{Acc}_{<2} \land (\forall x) \text{Acc}_{<2}[x]
\]

Now let \( \alpha \) be as in the statement of the theorem and let \( i < n \). If \( m_i \neq 0 \), then \((\forall j)k_{i,j} \neq 0 \) and so let
\( l_i = (\forall j)k_{i,j} \neq 0 \). We define

\[
\Gamma_{\text{char}_i, \alpha} = \begin{cases} 
\bigwedge_{\text{Li} < j < m_i} [T_{i,j} \land \sim T_{i,j}] \land U_i,0(\text{Lim}, l_i) & \text{if } m_i \neq 0 \\
0 & \text{if } m_i = 0
\end{cases}
\]

\[
\sum_{\text{char}_i, \alpha} = \begin{cases} 
\Gamma_{\text{char}_i, \alpha} U U_i,m_i(\text{Lim}, l_i, 0) & \text{if } \gamma_i = 0 \\
\Gamma_{\text{char}_i, \alpha} U U_i,m_i(\text{Lim}, l_i, 0) \cup U_i,m_i(\text{Lim}, k) & \text{if } \gamma_i \neq 0
\end{cases}
\]

\[
\sum_{\text{index}_\omega \alpha} = \begin{cases} 
U_n,0(\text{Acc}_0) \cup \bigcup_{h,k<\omega} \sim U_n,0(\text{Lim}, k) & \text{if } \mu \notin \omega_o \\
U_n,0(\text{Acc}_1) \cup \bigcup_{h,k<\omega} \sim U_n,0(\text{Lim}, k) & \text{if } \mu \in \text{successor}\ \omega_1
\end{cases}
\]

\[
\sum_{\text{char}_\omega \alpha} = \sum_{<\omega_2} \cup \bigcup_{i<n} \sum_{\text{char}_i, \alpha} \cup \sum_{\text{index}_\omega \alpha}
\]
The proof of the theorem now follows from Theorem 3.6, using $\sum_{\text{char}_\omega \alpha} \alpha$ as an axiomatization for $\text{MT}[\alpha, \prec]$. Q. E. D.

**Theorem 3.8:** (the decision method): There is a method which applies to any MT-sentence $\sum$ in $\prec$, and to any $\omega$-character (base $\omega_1$), $\pi$, and which decides whether or not $\sum$ holds in the system $[\alpha, \prec]$, where $\alpha < \omega_2$ and $\text{char}_\omega \alpha = \pi$.

**Proof:** Going over the proofs which lead to Remark 3.1 one can make up a procedure which constructs the input-free system $\langle E, \Gamma, L \rangle$ from $\sum$. By Lemmas 3.2 and 3.3 we may define the corresponding operators $G_{i,j}$ for finite $i, j$ by the following recursion:

$$G_{0,0}[c] = \{\langle c, d \rangle; H[c, d]\}$$

$$G_{i,j+1}[c] = \{\langle c, d \rangle; \bigvee_{D, e} D \subseteq C \land \varphi_0[D, d] \land$$

$$(\exists h, k) <c, e> \in G_{i,j}^h[c] \land <D, e> \in G_{i,j}^k[e]\}$$

$$q_i = (\mu j)G_{i,j} = G_{i,j+1}$$

$$G_{i+1,0}[c] = \{\langle c, d \rangle; \bigvee_{D_0 \in D_1} D_0 \subseteq C \land \varphi_1[D_0, e] \land \bigwedge_{e \in D_1} \varphi_0[D_0, e]$$

$$\land \bigwedge_{e \in D_1} <c, e> \in G_{i,q_i}[c] \land \bigwedge_{e, f \in D_1} <D_0, f> \in G_{i,q_i}[e]\}$$
Now we use (16) to construct $G_0, q_0, G_1, l' \ldots$. This process is stopped when we reach a $q_0$ such that $G_0, q_0 = G_0, q_0 + 1$ (which exists by Lemma 3.3). Now we construct $G_1, q_1, G_1, l', \ldots$, stopping when we reach a $q_1$ such that $G_1, q_1 = G_1, q_1 + 1$. This above procedure is continued producing $G_0, q_0, G_1, q_1, \ldots, G_2, q_2, \ldots$ until we reach a $p$ such that $G_p, 0 = G_p, 1$ (which exist by Lemma 3.4). $G_0, q_0, \ldots, G_p, q_p, q_p - 1, \ldots, q_0$ will then serve for Theorem 3.6. Now rewrite the $\omega$-character $\pi$ as a $p; q_p - 1, \ldots, q_0$ - character, $\tau$. From the $G_i,j$ above we can make the abbreviations of (18) and (19) and thus we can evaluate $F_{\tau}[E]$. By Theorem 3.6 $F_{\tau}[E] \cap K \neq 0$ just in case $\int$ holds in $[\alpha, <]$, where $\alpha < \omega_2$ is such that $\text{char}_\omega \alpha = \pi$.

Q. E. D.

Corollary 3.9: For any $\alpha < \omega_2$, $\text{MT}[\alpha, <]$ is decidable. The decision method depends only on the $\omega$-character (base $\omega_1$) of $\alpha$. The MT of all ordinals $< \omega_2$ is also decidable.
Bibliography


