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A Periodic Review Base Stock Inventory System with Sales Rejection

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We consider a system in which an order is placed every T periods to bring the inventory position up to the base stock S. We accept demand until the inventory position reaches a sales rejection threshold M. Our objective is to find the optimal values of S and M which minimize the long-run average cost per period. We establish the stationary distribution of our system and develop structural properties of the optimal solution that facilitate computation. In particular, we show that in an optimal solution, the optimal value of M is non-negative under some reasonable conditions. Hence, in our model a mixture of backorders and lost sales may occur. Additionally, we compare our system against traditional systems in which demand during stockouts is either fully backordered or lost.

Key words: stochastic inventory models; (R, T) systems; base stock systems; backorders; lost sales

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1. Introduction

A cornerstone of stochastic inventory theory is the paper by Scarf (1960) in which he established the optimality of the (s, S) policies under rather general conditions. In an (s, S) inventory system, the state of the inventory is reviewed every T periods and if, since the last review, the stochastic demand is sufficiently high so that the inventory has fallen below s, an order is placed to bring the inventory level up to S. Despite its theoretical value, the practical use of (s, S) policies is inhibited by computational difficulties, especially when the delivery leadtime is positive. Recognizing such impracticalities, Hadley and Whitin (1963) make a pragmatic argument to justify the use and analysis of (R, T) systems.

In an (R, T) policy, the inventory position is reviewed every T periods and an order is placed to bring it up to R. It is easily seen that this base stock policy is a restricted (s, S) policy with $s = S$, but easier to analyze since the decision maker only has to choose one variable, R. Nevertheless, to proceed, Hadley and Whitin (1963) make the key assumption that the base stock is set so high that it is always sufficient to clear all backordered demands when the latest order is received. Under this assumption, the problem of choosing R to minimize the average cost per unit time, including that of inventory holding and backlogging, reduces to myopically considering the next $T + L$ periods, where L is the constant delivery lead time.

While base stock models with backorders remain pervasive in the literature, one disadvantage is that order fulfillment can be delayed by up to $T + L$ periods, that is, until the next order is placed and received. Consequently, delayed order fulfillment has explicit costs that include discounts under rain check policies in the retail industry, and automatically triggered penalties in many field service contracts. Montgomery et al. (1973) were the first to recognize the economic benefits of using a hybrid system in which some demand was rejected or "lost" so that the average delay for backordered demand could be reduced. In their model as much as possible of the demand is filled from inventory on-hand and a pre-determined fraction of demand is rejected when there is no inventory on-hand, while the rest is backordered. Using the deterministic model as a building block, Montgomery et al. (1973) provide an approximate treatment of the problem. Notably, DeCroix and Arreola-Risa (1998) in a periodic review setting, with negligible delivery leadtimes, recognized that this pre-determined fraction of rejected demand can be controlled by offering economic incentives to customers who face backorders. And, recently, Bhargava et al. (2006) provide a complementary utility-theoretic treatment of the consumer's response to delayed fulfillment. Some important extensions are due to Cheung (1998) and Zhang et al. (2003), who consider continuous review models adapting the approximate treatment due to Montgomery et al. (1973). In particular, in Zhang et al. (2003), the decision of whether or not to backorder is made by the retailer and not the consumer. Explicitly recognizing that some retailers impose time limits on backlog duration, demand

during a stockout during the early part of the restock leadtime is rejected or lost; in contrast, demand close to the delivery leadtime is backordered.

One feature of such systems is that during stockouts, some demand is rejected while the rest is backordered. In contrast, in the two priority class model of Deshpande et al. (2003), Poisson demand from the lower priority class is backordered when the inventory on-hand is below a holdback level, which leads to easily implementable systems as has been reported by them. In our variation of the partial sales rejection system of Montgomery et al. (1973), we reject all demand when the inventory position falls to a threshold. In that sense, our threshold is analogous to the holdback level of Deshpande et al. (2003). Moreover, as will be seen it leads to an exact representation that is analytically tractable and is applicable to a wide class of demand distributions.

As in traditional base stock systems, we assume that the demand in each period is represented by an independently and identically distributed (*iid*) random variable. The state of the system is reviewed every T periods, and if IP, the inventory position (the net inventory after accounting for all pipeline stock, inventory on-hand and backorders) falls below S an order is placed, with delivery leadtime L, to bring the inventory position up to the base stock S. Demand is accepted until the inventory position falls to a sales rejection threshold $M < S$. All demand that is not accepted is lost. The objective is to choose that pair of M and S that minimizes the long-run average cost, after taking into account the cost of purchasing, inventory carrying, backordering and lost sales. The optimal choice of M trades off the incremental cost from lost sales in one order cycle against the future cost of backorders.

Since our order fulfillment policy makes a rather novel use of partial backorders and lost sales, we discuss the dynamics of the system which are fully captured by the evolution of IP. Notice that IP takes on values in the interval (M, S). Hence, $M = -\infty$ yields the full backorder case as a special case of our model. If M is negative, some demand that is accepted may be filled from the order that will be placed at the next review epoch, leading to considerable delay in fulfilling such demand. On the other hand, if M is non-negative, all accepted demand is either satisfied from the inventory on-hand or from the pipeline inventory, assuring that it is filled within time L of acceptance.

In Section 2, we present our base stock system with a mixture of backorders and lost sales and develop some structural properties. In particular, we prove that M cannot take a finite negative value under some reasonable conditions, ruling out all negative values for M except the full backorder policy ($M = -\infty$) as a candidate optimal policy. In addition, we develop structural properties to analyze the full backorder system for the case when T and L are arbitrary. Consequently, our analysis of the full backorder case may be viewed as a generalization of the analyses of Chen and Zheng (1993) and Zipkin (2000) who consider the case of $T = 1$. Subsequently, in Section 3, we examine the structure of the optimal policy for non-negative thresholds or $M \geq 0$, and limit attention to the benchmark case $T = L = 1$. In particular, we show that the optimal M is independent of S, guaranteeing that the optimal solution can be found easily. Then, in Section 4, using this benchmark system, we examine the value of sales rejection by comparing our system against the full backorder and full lost sales system. Section 5 summarizes our results for more general systems that are discussed in detail in the Addendum to this manuscript. Concluding comments are in Section 6. All proofs are provided in the Appendix or the Addendum.

2. Model Formulation

In this section, we will formulate our model and develop some of its basic properties. In our system, an order is placed at the start of every T periods that brings the inventory position up to S. During the subsequent T periods, demand is accepted until the inventory position reaches a threshold $M < S$. All demand that is not accepted is lost. Thus the system is conveniently represented as a (T, M, S) system. We will begin by defining the evolution of our (T, M, S) system by focusing on events starting at time t, when a scheduled review of the system is about to occur.

At time t, before the review, the inventory position is IP_t, where $M \leq IP_t \leq S$. After the review, the inventory position, $IP_t^+ = S$. If $IP_t = S$, no order is placed; otherwise, an order of $Q_t = S - IP_t$ is placed. If an order is placed at the beginning of period t, it will be received at the beginning of period $t + L$. Subsequently, D_t, the random demand in period t is realized. If $IP_t > M$, the demand is accepted;

otherwise the demand is lost or rejected. If there is inventory on-hand, accepted demand is filled immediately; otherwise it is backordered until the start of the next review period. Accepted demand not satisfied from inventory on-hand is backordered incurring a fixed unit penalty π . Additionally, all backordered demand, including that which was not cleared at the beginning of the period, is charged a unit cost of b per period. Unaccepted demand is lost at a unit cost l. In the event that accepted demand is sufficiently low so that there is inventory on-hand, the leftover inventory is charged a unit cost of h per period. In the following periods, $t + 1$ to $t + T - 1$, no order is placed. Hence, both the starting inventory position and the inventory on-hand are depleted at a random rate. And, in each of these periods demand is accepted as long as the total sales in that period do not exceed the starting inventory position minus M. Furthermore, as in period t, as much as possible of accepted demand is filled from the inventory on-hand. At the beginning of period $t + T$, another order is placed to bring the inventory position back to S, and this process repeats.

Since the sales in each period do not exceed that period's starting inventory position minus M, it is assured that there is sufficient inventory in the system to clear all backlogged orders if $M > 0$, whereas there could be a shortage of $-M$ units to clear all backlogged orders if $M < 0$. Moreover, during the review period T, statistically there will be periods in which the inventory position equals M and the inventory on-hand is strictly positive. Hence, there may be intervals in which we will have lost sales and inventory on-hand concurrently. Nevertheless, since we order every T periods to bring the inventory position back to the order-up-to level S, and, whether or not demand is accepted only depends on the value of the starting inventory position minus M in each period, it is possible to describe the dynamics of the system by a one-dimensional irreducible Markov process, which guarantees a stationary distribution. This stationary distribution has a natural solution which is then used to compute the average expected cost per period. We focus our analysis on continuous demand whereas, as described by Xu (2006), the results can be easily extended to the case of discrete demand. In developing the model, we will use the following notation:

<u>Cost Parameters</u> c = unit purchase cost. b = unit backorder cost per period. π = fixed penalty cost for each unit of backorder. l = unit lost sales cost. h = unit holding cost.
<u>Demand and Supply Information</u> L = delivery lead time (non-negative integer). D_t = the demand in period t. $F(x)$ = cumulative distribution function of demand. $f(x)$ = probability density function of demand. u = mean demand in each period. ξ_t = the random observation of demand in period t. $\mathbb{S}_t$ = sales in period t. D^n = aggregate demand during n consecutive periods (n is an integer, $D^0 = 0$). $f_n(.)$ = probability density function for the total demand during n consecutive periods. $F_n(.)$ = cumulative distribution function for the total demand during n consecutive periods. $\overline{F}_n(.) = 1 - F_n(.)$ = complementary cumulative distribution function.
<u>Performance-Related Variables</u> T = order cycle. S = target inventory position. M = sales rejection threshold. $y = S - M$, the sales limit. IP_t = the starting inventory position in period t before order (if an order is placed, we have $IP_t^+ = S$) Q_t = order quantity in period t ($Q_t = S - IP_t$). IO_t = inventory-on-hand at the beginning of period t.
<u>Cost Functions</u> $TC(T, L, M, S)$ = the expected total cost per cycle after the system becomes stable. $PC(T, L, M, S)$ = the expected order quantity per cycle after the system becomes stable. $LS(T, L, M, S)$ = the expected lost sales per cycle after the system becomes stable. $HC(T, L, M, S)$ = the expected leftovers per cycle after the system becomes stable. $BO_1(T, L, M, S)$ = the total expected backorders per cycle after the system becomes stable. $BO_2(T, L, M, S)$ = the expected incremental backorders per cycle after system becomes stable.

2.1. Operating Characteristics

Since we have a discrete time model, with positive T and L, it is convenient to define $L = mT + n$. Here $m \geq 0$, $0 \leq n \leq T - 1$, m and n are integers. Notice that if $m = 0$, we have $L < T$; if $m = 1$ and $n = 0$, $L =$

T; and, if $m \geq 1$, $L \geq T$; and, if $m \geq 1$ and $n = 0$, then L is a multiple of T. Moreover, since $L > 0$, we do not allow $m = n = 0$. Also, it is useful to notice that since an order is placed at the beginning of some period t, and the review cycle is T, orders are placed at the beginning of periods $t, t \pm T, \ldots, t \pm kT, \ldots$ to bring the inventory position back to S. Specifically, if $T = 1$, then an order is placed every period.

Before proceeding with the formulation of the cost function, it is necessary to identify how the operating characteristics of this system evolve over time. Then, taking the appropriate limits yields the stationary distributions of the key random variables. We begin with $\mathbb{S}_t$, the sales in period t, and its cumulative distribution function. Notice that at the beginning of some period t, the inventory position after an order is placed is $IP_t^+ = S$, and then demand D_t occurs. Hence, under our policy, we have,

$$\begin{cases} \mathbb{S}_t = \min\{IP_t^+ - M, D_t\} = \min\{S - M, D_t\}, \\ \mathbb{S}_{t+i} = \min\{IP_{t+i} - M, D_{t+i}\} \quad \text{if} \quad i \in \{1, \ldots, T-1\}. \end{cases} \tag{1}$$

In each of periods $t + 1, t + 2, \ldots, t + T - 1$, no order is placed; hence,

$$IP_{t+i+1} = IP_{t+i} - \mathbb{S}_{t+i} = IP_t^+ - \mathbb{S}_t - \mathbb{S}_{t+1} - \ldots - \mathbb{S}_{t+i} = S - \sum_{j=0}^{i} \mathbb{S}_{t+j} \quad \text{if} \quad i \in \{0, 1, \ldots, T-1\}. \tag{2}$$

Finally, at the beginning of period $t + T$, a new order is placed to bring the inventory position back to the target $IP_{t+T}^+ = S$, initiating another cycle. Clearly, this sales process is a renewal process with deterministic renewal cycle of T. And, from classical renewal theory (Hoel, Port and Stone (1972) or Ross (1996)), we have the following lemma, whose proof is given in the Appendix.

Lemma 1

Given that orders are placed at the beginning of period $t + kT$ ($k \in Z$, the set of integers), we have

1.1) $\mathbb{S}_{t+i}$ *and* $\mathbb{S}_{t+kT+i}$ ($i \in \{0, 1, \ldots, T-1\}$) *follow independent and identical distributions;*

1.2) $\begin{cases} P\{\mathbb{S}_{t+i} = 0\} = \bar{F}_i(S - M) \\ P\{\mathbb{S}_{t+i} \leq r\} = F(r) + \bar{F}_i(S - M - r)\bar{F}(r) \quad \text{if } 0 < r \leq S - M; \end{cases}$ ($i \in \{1, \ldots, T-1\}$)

1.3) $Q_{t+kT} = \sum_{i=0}^{T-1} \mathbb{S}_{t+(k-1)T+i}$; *and,*

1.4) *If we define* $A_t^i = \left\{ \sum_{j=0}^{i} \mathbb{S}_{t+j} < S - M \right\}$ *and* $B_t^i = \left\{ \sum_{j=0}^{i} D_{t+j} < S - M \right\}$ *, for* $i \in \{1, \ldots, T-1\}$ *, then* $A_t^i = B_t^i$; $\overline{A}_t^i = \overline{B}_t^i$.

It follows from Lemma 1.1 that the sales in period $t + i$ of one review (order) cycle are independent of sales in the corresponding period in another review cycle, whereas Lemma 1.2 gives the stationary distribution of sales in any period of the review cycle. Lemma 1.3 formalizes the intuitive result that when a (T, M, S) policy is used, the order quantity is exactly the total sales of the previous review cycle. Finally, from Lemma 1.4 it follows that when total demand in any review cycle is less than $S - M$, then it exactly equals the total sales of that review cycle. In this case, there may be some backorders but no lost sales. Using the above lemma, we can easily derive that

$$IP_{t+kT+i} = IP_{t+kT}^{+} - \mathbb{S}_{t+kT} - \mathbb{S}_{t+kT+1} - \ldots - \mathbb{S}_{t+kT+i-1} = S - \sum_{j=0}^{i-1} \mathbb{S}_{t+kT+j} \quad (1 \le i \le T).$$

Since $IP_t^+ = S$ after an order is placed at the beginning of period t, which will reach the retailer at the beginning of period $t + L$, the inventory-on-hand at the beginning of period $t + L$ will be

$$IO_{t+L} = S - \sum_{j=0}^{L-1} \mathbb{S}_{t+j} = S - \sum_{j=0}^{mT+n-1} \mathbb{S}_{t+j} = S - \sum_{k=0}^{m-1} \left(\sum_{i=0}^{T-1} \mathbb{S}_{t+kT+i} \right) - \sum_{j=0}^{n-1} \mathbb{S}_{t+mT+j} \ .$$

(If $n = 0$, then $\sum_{j=0}^{n-1} \mathbb{S}_{t+mT+j}$ is omitted; if $m = 0$, then $\sum_{k=0}^{m-1} \left(\sum_{i=0}^{T-1} \mathbb{S}_{t+kT+i} \right)$ is omitted.)

For the inventory-on-hand of period $t + L + i$, we need to consider two cases:

$$IO_{t+L+i} = IO_{t+L+i-1} - \mathbb{S}_{t+L+i-1} = S - \sum_{k=0}^{m-1} \left(\sum_{i=0}^{T-1} \mathbb{S}_{t+kT+i} \right) - \sum_{j=0}^{i-1} \mathbb{S}_{t+mT+j} \quad (1 \le i \le T-1) \text{ if } n = 0,\ m > 0; \text{ and,}$$

$$\begin{cases} IO_{t+L+i} = S - \sum_{k=0}^{m-1} \left(\sum_{i=0}^{T-1} \mathbb{S}_{t+kT+i} \right) - \sum_{j=0}^{n+i-1} \mathbb{S}_{t+mT+j} & (1 \le i \le T-n) \\ IO_{t+L+i} = S - \sum_{k=0}^{m} \left(\sum_{i=0}^{T-1} \mathbb{S}_{t+kT+i} \right) - \sum_{j=0}^{i+n-T-1} \mathbb{S}_{t+(m+1)T+j} & (T-n < i \le T-1) \end{cases} \quad \text{if } n > 0\ ;\ m \ge 0.$$

The difference lies in that if $n > 0$, then the order will arrive at the beginning of $t + L$ which is between $t + mT$ and $t + (m + 1)T$, two consecutive order epochs, given $mT < L = mT + n < (m + 1)T$; if $n = 0$, then the order will arrive at the beginning of period $t + mT$ (and the next one will arrive at $t + (m + 1)T$).

Since we have derived expressions for the state variables, we will now use them to study system performance.

2.2. The Model with a Negative Threshold

Since it follows from Lemma 1 that system performance only depends on the operating characteristics during a review cycle, for the purpose of calculating the expected cost per period it is sufficient to focus on minimizing the expected cost per cycle. To proceed, we conveniently look at the cycle which consists of period $t + L$ to period $t + L + T - 1$. Since we place exactly one order every T periods, we will incur exactly one purchase cost every cycle and receive exactly one order. Similarly, we will incur inventory holding cost or backorder cost and lost sales cost in each period within the cycle. In the following analysis, we assume $n > 0$ and that we will place an order at the beginning of period $t + L + (T - n) = t + (m + 1)\,T$. (If $n = 0$, the order is placed at the beginning of period $t + L = t + mT$, making the analysis for the case $n = 0$ virtually identical). In general, the long-run expected cost per cycle can be written as:

$$\begin{aligned} TC(T,L,M,S) &= c \cdot PC(T,L,M,S) + l \cdot LS(T,L,M,S) + h \cdot HC(T,L,M,S) \\ &\quad + b \cdot BO_1(T,L,M,S) + \pi \cdot BO_2(T,L,M,S). \end{aligned} \tag{3}$$

In (3), $PC(T,L,M,S) = E(Q_{t+(m+1)T}) = E\left(\sum_{i=0}^{T-1} \mathbb{S}_{t+mT+i}\right) = \int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi.$

Then, using the equality, *lost sales* = *demand* − *sales*, we can write

$$LS(T,L,M,S) = E(T \cdot D) - E\left(\sum_{i=0}^{T-1} \mathbb{S}_{t+mT+i}\right) = Tu - \left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right].$$

While these two expectations depend only on the cumulative demand, the expected leftovers and backorders have to be assessed and summed over each period of the cycle. When the acceptance limit M is greater than zero, the difficulty arises that there are instances in which there may be lost sales when there is inventory on-hand. Accounting for such contingencies contributes significantly to the ensuing complexity of the problem. Fortunately, when $M < 0$, such contingencies cannot arise because a negative inventory position assures that there is no uncommitted pipeline stock and therefore, inventory on-hand.

In the following, we first calculate the expected leftovers and backorders for $M < 0$, while the complementary case $M \geq 0$ is presented later.

When $M < 0$, the expected leftovers at the end of period $t + L + i$ ($0 \leq i \leq T-1$) is given by $HC_{t+L+i} = \int_0^S (S-\xi) f_{L+i+1}(\xi) d\xi$. The intuitive explanation, again from Lemma 1, is that if we do have any leftovers at the end of some period $t + L + i$, then it means that the total demand from period t (when an order is placed) to period $t + L + i$ (over $L + i + 1$ periods) is less than S.

Thus, the sum of expected leftovers for the focal cycle is given by

$$HC(T, L, M, S) = \sum_{i=0}^{T-1} HC_{t+L+i} = \sum_{i=0}^{T-1} \left[\int_0^S (S-\xi) f_{L+i+1}(\xi) d\xi \right].$$

Finally, we can relate BO_{t+L+i} (the expected backorders in period $t + L + i$) with HC_{t+L+i} (the expected leftovers) to write BO_{t+L+i} as: BO_{t+L+i} = {Total expected sales from period t to period $t + L + i$} − {S − Expected Leftovers at the end of period $t + L + i$}, so that for $0 \leq i < T-n$,

$$\begin{aligned} BO_{t+L+i} &= E\left(\sum_{j=t}^{t+L+i} \mathbb{S}_j \right) - \left(S - HC_{t+L+i} \right) \\ &= m\left[\int_0^{S-M} \xi f_T(\xi) d\xi + (S-M) \int_{S-M}^{\infty} f_T(\xi) d\xi \right] + \left[\int_0^{S-M} \xi f_{n+i+1}(\xi) d\xi + (S-M) \int_{S-M}^{\infty} f_{n+i+1}(\xi) d\xi \right] \\ &\quad - \left(S - HC_{t+L+i} \right); \end{aligned}$$

and for $T-n \leq i \leq T-1$,

$$\begin{aligned} BO_{t+L+i} &= (m+1)\left[\int_0^{S-M} \xi f_T(\xi) d\xi + (S-M) \int_{S-M}^{\infty} f_T(\xi) d\xi \right] + \left[\int_0^{S-M} \xi f_{n+i+1-T}(\xi) d\xi + (S-M) \int_{S-M}^{\infty} f_{n+i+1-T}(\xi) d\xi \right] \\ &\quad - \left(S - HC_{t+L+i} \right). \end{aligned}$$

The explanation is that from the beginning of period t to the end of period $t + L + i$, only S units (the inventory position in period t) are physically available to meet demand. Hence, as in the classical newsvendor model, given the leftover inventory at the end of period $t + L + i$ is HC_{t+L+i}, $\left[S - HC_{t+L+i} \right]$ represents the expected sales (from period t to period $t + L + i$, over $L + i + 1$ periods) that are satisfied from S units of available physical inventory. Therefore, the difference between the total expected sales from period t to period $t + L + i$, and the sales that are satisfied from physically available S units of

inventory during period t to period $t + L + i$, represents that part of sales that are not satisfied from the inventory-on-hand. In other words, these are backlogged cumulatively until the end of period $t + L + i$. Aggregating over the cycle, we write:

$$\begin{aligned} BO_1(T,L,M,S) &= \sum_{i=0}^{T-1} BO_{t+L+i} \\ &= \sum_{i=0}^{T-n-1} E\left(\sum_{j=t}^{t+L+i} \mathbb{S}_j\right) + \sum_{i=T-n}^{T-1} E\left(\sum_{j=t}^{t+L+i} \mathbb{S}_j\right) - T\cdot S + HC(T,L,M,S) \\ &= (L+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\ &\quad + \sum_{j=1}^{T-1}\left[\int_0^{S-M} \xi f_j(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_j(\xi)d\xi\right] - TS + \sum_{i=0}^{T-1}\int_0^S (S-\xi) f_{L+i+1}(\xi)d\xi . \end{aligned} \tag{4}$$

As for the expected incremental backorders incurred in the focal cycle, proceeding as in Chen and Zheng (1993) we get:

$$\begin{aligned} BO_2(T,L,M,S) &= BO_{t+L+T-1} - BO_{t+L}^{-} \\ &= (m+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] + \left[\int_0^{S-M} \xi f_n(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_n(\xi)d\xi\right] \\ &\quad - \left(S - HC_{t+L+T-1}\right) - \left[E\left(\sum_{j=t}^{t+L-1} \mathbb{S}_j\right) - \left(S - HC_{t+L}^{-}\right)\right] \\ &= (m+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] + \left[\int_0^{S-M} \xi f_n(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_n(\xi)d\xi\right] \\ &\quad - \left(S - \int_0^S (S-\xi) f_{L+T}(\xi)d\xi\right) - m\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\ &\quad - \left[\int_0^{S-M} \xi f_n(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_n(\xi)d\xi\right] + \left(S - \int_0^S (S-\xi) f_L(\xi)d\xi\right) \\ &= \left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] + \left[\int_0^S (S-\xi) f_{L+T}(\xi)d\xi - \int_0^S (S-\xi) f_L(\xi)d\xi\right], \end{aligned} \tag{5}$$

where BO_{t+L}^{-} and HC_{t+L}^{-} represent the starting expected backorders (not cleared by the order received at the beginning of period $t + L$) and starting expected leftovers (after backorders are cleared). Hence, when $M < 0$, the total expected cost for the focal cycle simplifies to:

$$\begin{aligned} TC(T,L,M,S) &= [c - l + (L+1)b + \pi]\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\ &\quad + (h+b)\sum_{i=0}^{T-1}\int_0^S (S-\xi) f_{L+i+1}(\xi)d\xi + b\sum_{j=1}^{T-1}\left[\int_0^{S-M} \xi f_j(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_j(\xi)d\xi\right] \\ &\quad + (lTu - bTS) + \pi\int_0^S (S-\xi) f_{L+T}(\xi)d\xi - \pi\int_0^S (S-\xi) f_L(\xi)d\xi. \end{aligned} \tag{6}$$

With the formulation of the expected total cost for the cycle, we will now prove that if the demand distribution is from the monotone likelihood ratio (MLR) family, then it is not optimal to have a finite

negative M. Following Ross (1996) and Rosling (2002), we define the MLR family of distributions as follows.

Monotone Likelihood Ratio (MLR) Family: If the set $\{f_n(x)\}$ of density functions of all n-fold convolutions of $f(x)$ is such that for all $n \geq 1$, $f_{n+1}(x)/f_n(x)$ is non-decreasing in x (assume $f_{n+1}(0)/f_n(0) = 0$ when $f_n(0) = 0$), then we say that $f(x)$ is MLR (or MCR, monotone convolution ratios).

Proposition 1-2 of Rosling (2002) shows that if $f(x)$ is logconcave, then $f(x)$ is MLR , so that MLR is a generalization of logconcavity. Most commonly used discrete distributions, e.g., Poisson, uniform, binomial, negative binomial, geometric, logarithmic as well as continuous distributions, e.g., normal, exponential, gamma with shape parameter $m \geq 1$, uniform and their translation, convolution or truncation belong to the MLR family.

To proceed, we now define $y = S - M$. It is clear that $y \in (S, +\infty)$. Then, (6) can be rewritten as

$$\begin{aligned} TC(T,L,y,S) = {} & [c - l + (L+1)b + \pi]\left[\int_0^y \xi f_T(\xi)d\xi + y\int_y^\infty f_T(\xi)d\xi\right] \\ & +(h+b)\sum_{i=0}^{T-1}\int_0^S (S-\xi)f_{L+i+1}(\xi)d\xi + b\sum_{j=1}^{T-1}\left[\int_0^y \xi f_j(\xi)d\xi + y\int_y^\infty f_j(\xi)d\xi\right] \\ & +(lTu - bTS) + \pi\int_0^S (S-\xi)f_{L+T}(\xi)d\xi - \pi\int_0^S (S-\xi)f_L(\xi)d\xi. \end{aligned} \tag{7}$$

Therefore,

$$\frac{\partial TC}{\partial y} = [c - l + (L+1)b + \pi]\overline{F}_T(y) + b\sum_{j=1}^{T-1}\overline{F}_j(y).$$

Notice that this partial derivative does not depend on S so that we are able to assert that:

Theorem 1

1.1) *If* $l - c \leq (L+1)b + \pi$ *, then the optimal* M^* *cannot be negative;*

1.2) *If* $l - c \geq (L+T)b + \pi$ *, then the optimal* $M^* = -\infty$ *;*

1.3) *If* $(L+1)b + \pi < l - c < (L+T)b + \pi$ *, and the demand has the MLR property, then the optimal* M^* *cannot be a finite negative value.*

An important consequence of Theorems 1.1 and 1.2 is that in these parameter sets, it is never optimal to have a finite negative sales rejection threshold *M*. For intermediate parameter ranges, this observation is also true but demand must be from the MLR family. An intuitive explanation for Theorem 1 follows by first observing if the demand is rejected or lost, as is the case in *lost sales* models, then the marginal cost of an understock is $l-c$. And then noticing that if the demand is backordered, as in *backorder* models, we consider the unit cost of understock to be $\pi+bt$. Hence, given that there is a backorder, the understocking cost would vary between $\pi+b(L+1)$ (fulfillment is delayed by one period of the cycle) and $\pi+b(L+T)$ (fulfillment is delayed by one full cycle). Thus, it follows that if the unit cost of lost sales $l-c>\pi+b(L+T)$, a system with **only** backorders dominates systems with lost sales. And, it also follows that if $l-c\le\pi+b(L+1)$, an optimal (*T, M, S*) system with a non-negative sales rejection threshold dominates systems with **only** backorders. Hence, in developing structural properties, we can conclude that in a (*T, M, S*) model, only non-negative *M* must be considered; otherwise the full backorder policy will emerge, which is the case we now analyze.

2.3. The Full Backorder Case

Since our modeling framework includes the full backorder system as a special case with $M=-\infty$, it is sufficient to proceed with its analysis by focusing on calculating the cost incurred in the *T* periods from *t* + *L* to *t* + *L* + *T* −1. Clearly, in this case, the expected cost per cycle consists of purchase, holding and backorder costs only, since no demand is lost. Since unmet demand is backlogged in each period, the expected purchase cost per cycle is exactly equal to cTu. We calculate the expected holding and backorder costs period by period and then aggregate them to get the total. Recall that after a review at time *t*, $IP_t^+=S$, and the inventory cost at the end of period *t* + *L*, $C_{t+L}(.)$ is:

$$\begin{aligned} C_{t+L}(S-\textstyle\sum_{i=t}^{t+L}D_i) &= h\cdot E[S-\textstyle\sum_{i=t}^{t+L}D_i]^+ + b\cdot E[S-\textstyle\sum_{i=t}^{t+L}D_i]^- \\ &= h\cdot\int_0^S (S-\xi)f_{L+1}(\xi)d\xi + b\cdot\int_S^{+\infty}(\xi-S)f_{L+1}(\xi)d\xi. \end{aligned}$$

Similarly, for $1\le k\le T-1$,

$$C_{t+L+k}(S-\sum_{i=t}^{t+L+k}D_i)=h\cdot E[S-\sum_{i=t}^{t+L+k}D_i]^+ + b\cdot E[S-\sum_{i=t}^{t+L+k}D_i]^-$$
$$=h\cdot\int_0^S(S-\xi)f_{L+k+1}(\xi)d\xi+b\cdot\int_S^{+\infty}(\xi-S)f_{L+k+1}(\xi)d\xi.$$

In the calculation above, we did not include the expected incremental backorders incurred in the focal cycle, which can be written as

$$BO_{t+L+T-1}^{t+L}=E[S-\sum_{i=t}^{t+L+T-1}D_i]^- - E[S-\sum_{i=t}^{t+L-1}D_i]^- \ .$$

Hence, the total cost incurred in the focal cycle is

$$TC_{FB}(T,L,S)=c\cdot T\cdot u+\sum_{j=t+L}^{t+L+T-1}C_j(.)+\pi BO_{t+L+T-1}^{t+L}. \tag{8}$$

Then

$$\begin{cases}\dfrac{dTC_{FB}(T,L,S)}{dS}=(h+b)\sum_{k=1}^{T}F_{L+k}(S)-T\cdot b+\pi F_{L+T}(S)-\pi F_L(S),\\ \dfrac{d^2TC_{FB}(T,L,S)}{dS^2}=(h+b)\sum_{k=1}^{T}f_{L+k}(S)+\pi f_{L+T}(S)-\pi f_L(S).\end{cases} \tag{9}$$

Equations (9) generalize the optimality condition due to Chen and Zheng (1993), who only consider the case $T = 1$. They also generalize the optimality condition in Zipkin (2000) who considers the case $T = 1$ and $\pi = 0$. Adapting an argument in Chen and Zheng (1993), it easily follows that with MLR demand, there exists a unique S_{FB}, which minimizes $TC_{FB}(T,L,S)$, i.e., $(dTC_{FB}(T,L,S)/dS)|_{S=S_{FB}}=0$.

Now that we have completed the analysis of models with negative *M*, we consider the complementary case of non-negative thresholds.

3. The Model with a Non-Negative Threshold

In this and the next section we consider the case when $M \geq 0$ and $T = L = 1$. We will discuss the more general cases with $T > 1$ and/or $L > 1$ in Section 5.

When considering the system with $T = L = 1$, it is appropriate to limit attention to the parameters of Theorem 1.1, namely, $l-c\leq 2b+\pi$. Since Theorem 1.3 does not apply, we can make significant progress in this case without invoking the MLR property. Moreover, when $T = L = 1$, there is no distinction between b and π, hence, without loss of generality, we simply drop π for this situation.

As we discussed earlier, when $M > 0$, there can be periods in which we may have leftover inventory and yet have lost sales. In such situations there will be no backorders as there is inventory on-hand. To see this, let $y = S - M$. Since we start each period with inventory position S, this implies that sales in each period cannot exceed y. So the sales over the next two periods cannot exceed $2y$. Therefore, if $2y = 2(S - M) < S$, or equivalently, $M > S/2$, then there can be no backorders and there will be inventory on-hand at the end of each period.

For such values of M,

$$PC(1, M, S) = [\int_0^{S-M} \xi f(\xi) d\xi + (S - M)\bar{F}(S - M)],$$
$$LS(1, M, S) = u - [\int_0^{S-M} \xi f(\xi) d\xi + (S - M)\bar{F}(S - M)],$$
$$HC(1, M, S) = S - 2[\int_0^{S-M} \xi f(\xi) d\xi + (S - M)\bar{F}(S - M)].$$

Therefore,

$$TC(1, M, S) = [c - l - 2h][\int_0^{S-M} \xi f(\xi) d\xi + (S - M)\bar{F}(S - M)] + h \cdot S + l \cdot u,$$

Letting $y = S - M$, so that $y \in [0, S)$, the above equation can be rewritten as

$$TC(1, y, S) = [c - l - 2h][\int_0^{y} \xi f(\xi) d\xi + y\bar{F}(y)] + h \cdot S + l \cdot u.$$

And $\frac{\partial TC}{\partial y} = [c - l - 2h]\bar{F}(y) < 0$, hence, TC decreases in y and increases in M.

Now, to attain the optimal solution, if M is decreased, then $2(S - M) < S$ would not be satisfied, which means that this case does not yield the optimal solution, and we only need to consider the case of $0 \le M \le S/2$. And in this case, the formulation of expected sales and expected lost sales remain unchanged, while the formulation of expected leftovers now becomes

$$\begin{aligned} HC(1, M, S) &= E\left[\left(S - \min\{D_t, S - M\} - \min\{D_{t+1}, S - M\}\right)^+\right] \\ &= \int_0^S (S - \xi) f_2(\xi) d\xi - 2\int_{S-M}^S f(\xi_1)\left[\int_0^{S-\xi_1} (S - \xi_1 - \xi_2) f(\xi_2) d\xi_2\right] d\xi_1 \\ &\quad + 2\bar{F}(S - M)\int_0^M (M - \xi) f(\xi) d\xi, \end{aligned} \tag{10}$$

And we can write the total cost as

$$TC(M,S)=[c-l+2b]\left[\int_0^{S-M}\xi f(\xi)d\xi+(S-M)\bar{F}(S-M)\right]+lu-bS$$
$$+(h+b)\left\{\int_0^S(S-\xi)f_2(\xi)d\xi-2\int_{S-M}^S f(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\xi_1\right. \tag{11}$$
$$\left.+2\bar{F}(S-M)\int_0^M(M-\xi)f(\xi)d\xi\right\}.$$

Taking the partial derivative, we get

$$\frac{\partial TC(M,S)}{\partial M}=[l-c-2b]\bar{F}(S-M)+2(h+b)\bar{F}(S-M)F(M). \tag{11a}$$

Notice that in (11a), if $l-c-2b\geq 0$, then $\frac{\partial TC(M,S)}{\partial M}\geq 0$, which implies that the optimal M should be as small as possible. Hence, we have $M\leq 0$, and this is consistent with Theorem 1.2, that is, the optimal $M^*=-\infty$.

Next, we consider the case $l-c-2b<0$.

Since $0\leq M\leq S/2$ in this case, $0\leq S-M<+\infty$ and $\bar{F}(S-M)>0$. Therefore, from the first order optimality condition $\frac{\partial TC(M,S)}{\partial M}=0$, we obtain $F(M^*)=\frac{c+2b-l}{2(h+b)}$. Moreover, the first and second partial derivatives over S are

$$\frac{\partial TC(M,S)}{\partial S}=[c-l+2b]\bar{F}(S-M)-b+(h+b)\left\{F_2(S)-2\int_{S-M}^S f(\xi)F(S-\xi)d\xi\right\}, \tag{11b}$$

$$\frac{\partial TC^2(M,S)}{\partial S^2}=-[c-l+2b]f(S-M)+(h+b)\left\{f_2(S)-2\int_{S-M}^S f(\xi)f(S-\xi)d\xi\right.$$
$$\left.+2f(S-M)F(M)\right\}. \tag{12}$$

For the optimal M, we can rewrite (12) as

$$\frac{\partial TC^2(M^*,S)}{\partial S^2}=[2(h+b)F(M^*)-(c-l+2b)]f(S-M^*)+(h+b)\left\{f_2(S)-2\int_{S-M}^S f(\xi)f(S-\xi)d\xi\right\}$$
$$=(h+b)\left\{f_2(S)-2\int_{S-M}^S f(\xi)f(S-\xi)d\xi\right\}$$
$$>0,$$

namely, for optimal M, $TC(M^*,S)$ is convex in S; moreover, for $0 \le M \le S/2$,

$$\frac{\partial TC(M^*,S)}{\partial S}|_{S\to+\infty} = -b+(h+b) = h > 0, \quad \text{and}$$

$$\begin{aligned}\frac{\partial TC(M^*,S)}{\partial S}|_{S=2M^*} &= [c-l+2b]\bar{F}(M^*)-b+(h+b)\left\{F_2(2M^*)-2\int_{M^*}^{2M^*} f(\xi)F(2M^*-\xi)d\xi\right\} \\ &= [c-l+2b]\bar{F}(M^*)-b+(h+b)F^2(M^*) \\ &= (h+b)F^2(M^*)-[c-l+2b]F(M^*)+[c-l+b] \\ &= -\frac{(c-l+2b)^2}{4(h+b)}+[c-l+b] < 0.\end{aligned} \tag{13}$$

In (13), we necessarily assume that $l-c \ge b$, namely, the unit backorder cost is cheaper than the unit lost sales cost. From the above analysis, we conclude that:

Theorem 2

When $T = L = 1$,

2.1) *If $b \le l-c < 2b$, then the optimal M^* is independent of S^* and satisfies* $F(M^*) = \dfrac{c+2b-l}{2(h+b)}$*; the optimal S^* is unique and obtained by solving (11b) set equal to zero;*

2.2) *If $l-c \ge 2b$, then the optimal $M^* = -\infty$, and the optimal S^* satisfies* $F_2(S^*) = \dfrac{b}{h+b}$.

Furthermore, using the expression of the total cost function $TC(M, S)$ given in (11), we can obtain the following comparative statics for the optimal M and S:

(i) The optimal M decreases in l and h, and increases in b; and

(ii) The optimal S increases in l, decreases in h, and may either increase or decrease in b depending on the demand distribution F and the relative values of M and S.

While the above results are intuitive, we include their derivations in Addendum A1.

Now that we have analyzed the case of full backorders and optimal non-negative thresholds, we want to compare them against the case of full lost sales to examine the value of sales rejection.

4. The Value of Sales Rejection

In the analysis so far, we have considered systems with backorders. For the case when $T = L = 1$, we have established that systems with non-negative M can exist, and they are optimal when $l - c \le 2b$. The standard treatment in which all demand that occurs during a stockout is backordered arises as the special case with $M = -\infty$ in our model. In this section we focus on comparing these backorder based systems with a system of lost sales only where all demand not filled from inventory on-hand is lost. This allows us to provide very sharp results on the ranking between the full backorder, full lost sales and the (*T, M, S*) systems.

Although significant progress has been made on (*R, T*) systems with full backorders, there has been limited analysis of such systems in the case of full lost sales. An important development is due to Karlin and Scarf (1958) who have provided a complete treatment of the lost sales case for $T = L = 1$ for arbitrary demand distributions. Since we wish to exploit an analytically tractable treatment of the lost sales case as a benchmark for comparison, in this section we will analyze our inventory system for the case of $T = L = 1$, and any demand distribution.

When demand in each period is drawn from an arbitrary distribution, the average cost function for the lost sales model of Karlin and Scarf (1958) can be written as:

$$\begin{aligned} TC_{FL}(S) &= cu + (l - c + 2h)\int_0^S \frac{F(x)}{1 - \bar{F}(x)\bar{F}(S - x)}dx - hS + (l - c)\left[\int_S^{+\infty}(\xi - S)f(\xi)d\xi - \int_0^S F(x)dx\right] \\ &= cu + (l - c + 2h)\int_0^S \frac{F(x)}{1 - \bar{F}(x)\bar{F}(S - x)}dx - hS + (l - c)(u - S) \qquad (14) \\ &= lu - (l - c + h)S + (l - c + 2h)\int_0^S \frac{F(x)}{1 - \bar{F}(x)\bar{F}(S - x)}dx \end{aligned}$$

Also, from (8), the average cost function for the full backorder case ($M = -\infty$) can be written as:

$$TC_{FB}(S) = cu + h\int_0^S (S - \xi)f_2(\xi)d\xi + b\int_S^{+\infty}(\xi - S)f_2(\xi)d\xi \qquad (15)$$

Now that we have presented the cost functions for these two cases, we may compare them to show the following:

Theorem 3

3.1) *If* $l - c \le b - h$, $TC_{FB}(S) \ge TC_{FL}(S)$;

3.2) *If* $l - c \ge 2b$, $TC_{FL}(S) \ge TC_{FB}(S)$.

Theorem 3 along with Theorem 1 allow us to conclude that when $l - c \ge 2b$, the full backorder policy is optimal. Thus the choice narrows to choosing between the optimal (*T, M, S*) policies with $0 \le M \le S/2$ and the optimal policy with full lost sales when $l - c \le 2b$.

Since we have compared the full backorder policy and the full lost sales policy in Theorem 3 for arbitrary base stock *S* under certain regions of the cost parameters, it is natural to ask whether a similar comparison can be made between the (*T*, *M*, *S*) policy and the full backorder or the full lost sales policy. This is formalized as:

Theorem 4

4.1) *If* $l - c \le b - h$, $TC(M, S) \ge TC_{FL}(S)$;

4.2) *If* $l - c \ge 2b$, $TC(M, S) \ge TC_{FB}(S)$.

An implication of Theorem 4.1 is that a (*T, M, S*) policy is dominated by a full lost sales policy when unit lost sales cost $l - c$ and unit inventory holding cost *h* are relatively low compared to the unit backorder cost *b*; otherwise, incurring a backorder cost may be preferable as it potentially saves inventory carrying cost *h* and unit lost sale cost $l - c$. And, Theorem 4.2 is consistent with Theorem 1.2. Now that we have provided a partial ranking of various inventory systems, to provide additional insights, we will show via numerical examples that in the region $b - h < l - c < 2b$, the (*T, M, S*) policy may yield lower expected costs compared to the full lost sales or the full backordering policy.

4.1. Some Numerical Results

To complement the analytical results above, we conducted numerical studies. Specifically, for the full lost sales, and the (*T, M, S*) models, we have found the optimal values of *S* and the optimal costs for each set of the parameter values. A representative set of results are reported in Tables 1a, 1b, and 1c,

respectively for exponential (Erlang-1), Erlang-2, and Erlang-3 demand distributions with a focus on varying l. To compare different costs we compute the relative cost error $\Delta_1 = \frac{TC_{FL}(S_{FL}) - TC(S_M^*, M^*)}{TC_{FL}(S_{FL})}$.

Also, for convenience of comparisons, we choose $E(X) = 1$ for all three distributions. In all the examples, the parameter values are chosen as follows: $c = 1$; $h = 0.1$; $b = 0.6$; $\pi = 0$; $l = \{1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2\}$. Note that since the full backorder policy yields the lowest cost when $l - c > 2b$, this case need not be considered.

We can make the following observations for a given demand distribution: 1) As l increases, the optimal S increases and M decreases (as shown in the previous section) and the expected cost increases. 2) As M^* decreases, the optimal S increases so there would be more backorders. And, 3) when l is low, the full lost sales policy dominates the optimal (T, M, S) policy; but for high values of l, the ranking is reversed.

l	S_{FL}	TC_{FL}	S_M^*	M^*	$TC(S_M^*)$	Δ_1
1.5	2.62570	1.20600	2.84592	0.693	1.22525	-0.0160
1.6	2.84139	1.22498	3.00607	0.560	1.23525	-0.0084
1.7	3.02662	1.24145	3.00761	0.442	1.24340	-0.0016
1.8	3.18878	1.25598	3.20889	0.336	1.24917	0.0050
1.9	3.33286	1.26898	3.28040	0.241	1.25437	0.0115
2.0	3.46241	1.28073	3.34013	0.154	1.25882	0.0171
2.1	3.58004	1.29145	3.39117	0.074	1.26269	0.0223
2.2	3.68771	1.30131	3.43557	0.000	1.26610	0.0271

Table 1a. Policy and cost comparisons for exponential demand with density $f(x) = e^{-x}$

l	S_{FL}	TC_{FL}	S_M^*	M^*	$TC(S_M^*)$	Δ_1
1.5	3.00378	1.155552	2.72544	0.839174	1.16152	-0.00516
1.6	3.11273	1.167313	2.82410	0.729813	1.16709	0.00019
1.7	3.20792	1.176421	2.89115	0.627485	1.17116	0.00450
1.8	3.29234	1.185004	2.94032	0.529202	1.17427	0.00910
1.9	3.36811	1.192702	2.97792	0.431878	1.17672	0.01340
2.0	3.43679	1.199659	3.00732	0.331217	1.17867	0.01750
2.1	3.49956	1.206012	3.03042	0.217950	1.18023	0.02120
2.2	3.55734	1.211846	3.04790	0.000000	1.18141	0.02510

Table 1b. Policy and cost comparisons for Erlang-2 demand with density $f(x) = 4xe^{-2x}$

l	S_{FL}	TC_{FL}	S_M^*	M^*	$TC(S_M^*)$	Δ_1
1.5	2.858893	1.127208	2.41030	0.891355	1.13472	-0.00666
1.6	2.948172	1.135944	2.64476	0.797808	1.13613	-0.00016
1.7	3.020840	1.143340	2.72894	0.708369	1.13878	0.00399
1.8	3.085160	1.149866	2.77920	0.620274	1.14085	0.00784
1.9	3.142800	1.155706	2.81288	0.530293	1.14244	0.01150
2.0	3.194970	1.160981	2.83650	0.433297	1.14367	0.01490
2.1	3.242600	1.165795	2.85310	0.316840	1.14458	0.01820
2.2	3.286400	1.170206	2.86377	0.000000	1.14521	0.02140

Table 1c. Policy and cost comparisons for Erlang-3 demand with density $f(x) = \frac{3^3}{2!} x^2 e^{-3x}$

Keeping the parameter set fixed, we also notice that as we move from exponential to Erlang-3, the optimal value of S_M^* decreases. This may be explained by recognizing that given a fixed mean, the variability decreases. Thus less (positive) safety stock is needed. In contrast, the optimal value of M increases because in these cases the value of M is set in the range where it may be interpreted as having negative safety stock. Thus, as variability decreases, this negative safety stock is of lower magnitude, resulting in a higher value for M. And, the variability impact also has the predictable result that expected costs decline.

5. Models with Arbitrary Lead Time and Review Period

In addition to the analysis of the case with $T = L = 1$, we also considered the three more general cases that arise: $L < T$, $1 = T < L$ and $1 < L < T$. It is not surprising that the analysis of these cases is far more complex and therefore it is presented in the Addendum in Theorems 5, 6, 7 and 8. The structural results are summarized in Table 2 below. The notation $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

	$L = T = 1$	$L > T = 1$	$L \le T,\ T > 1$	$L > T > 1$
Upper bound on M	$M \le S/2$, see Section 3	$M \le SL/(L+1)$, see Theorem 5	$M \le S/2$, see the proof of Theorem 7	$M \le \frac{(\lceil L/T \rceil + 1)}{(\lceil L/T \rceil + 2)} S$, see the proof of Theorem 8
Uniqueness of S	Yes	Yes, Theorem 6.1, under: $\pi = 0$; no requirements on cost parameters; no requirements on demand	Yes, Theorem 7.1, under: $\pi > 0$; $h(T-L) \le Lb + \pi$; MLR demand	Yes, Theorem 8.1, under: $\pi = 0$; no requirements on cost parameters; MLR demand
Uniqueness of M	Yes	Yes, Theorem 6.2, under: $\pi = 0$; no requirements on cost parameters; no requirements on demand	Yes, Theorem 7.2, under: $\pi > 0$; $h(T-L) \le Lb + \pi$, $Lb + \pi \le l - c \le 2(Lb + \pi) - h(T-L)$ MLR demand	Yes, Theorem 8.2, under: $\pi = 0$; $Lb \le l - c < (L+T)b$; MLR demand
Uniqueness of S and M	Yes, Theorem 2	No	No	No

Table 2. Structural results for the general cases

As for the case $T = L = 1$, we find that there is an upper bound on the optimal M for any given S. And, consistent with Theorem 2, we find that the unimodality of M in S and S in M can be established but that this requires additional conditions on the parameters. However, we are not able to establish that the optimal pair M and S is unique; thus a line search in S may be required.

6. Conclusion

In this paper we have developed a periodic review base stock system with sales rejection that results in a mixture of lost sales and backorders. The inventory position in our (T, M, S) system is reviewed every T periods and an order is placed to bring the inventory position up to S. Between reviews demand is

accepted as long as the inventory position does not fall below the sales rejection threshold M. We have established that in an optimal solution the threshold M cannot take finite negative values. In addition to establishing structural results on the form of the objective function, we have provided some comparative results. Analytical and computational results are used to demonstrate that when the unit cost of lost sales has intermediate values, there may be economic benefit of using the (T, M, S) system, rather than using a system in which demand is never backordered (the lost sales case) or always backordered.

One important aspect of our model is that since demand is backordered as long as the inventory position is above the threshold M, our system includes the full backorder system ($M = -\infty$) as a special case. Hence, as in Section 2.3, one contribution of our work is that we have shown that under MLR demand, the optimal target inventory level S for the full backorder model is easily determined from the first-order optimality conditions. In this sense we have generalized the work of Chen and Zheng (1993) and Zipkin (2000).

More generally, our (T, M, S) system may also be viewed as contributing to the literature by providing a complementary and rigorous alternative to that of Montgomery et al. (1973) who adapted the heuristic single-cycle analysis of Hadley and Whitin (1963) to consider a model with a mixture of partial backorders and partial lost sales. In their model, a preset fraction of customers is turned away in each cycle; the remaining demand is either satisfied from inventory on-hand or is backordered. In contrast, in an analogous way to Zhang et al. (2003), the fraction of customers lost in our model is selected optimally; it may vary across cycles but attains a constant fraction as a long-term average. Interestingly, our numerical results show that even though it is possible to guarantee that there are no lost sales when there is inventory on-hand (by setting $M = 0$), it is optimal to tolerate some lost sales when there is inventory on-hand.

Some insight into why we have a positive threshold can be gleaned by looking at inventory models when there are multiple classes of customers. For example, Deshpande et al. (2003) consider a system where there are two classes of customers who may use the same product. When inventory on-hand falls below a threshold (holdback level) the demand from the lower priority class is backordered. In other

words, some stock is reserved for the higher priority customers. Analogously, our threshold may also be interpreted as a holdback level that is based on the inventory position where some stock is reserved for customers from the next period.

While our holdback level or threshold is static, there are models in which this threshold is dynamic. This is the case in the model of Archibald et al. (1997) whose application is motivated by an auto dealership network. In their two retailer model with Poisson demands and negligible lead times, each retailer has to dynamically decide whether to accept a demand from the other retailer. Archibald et al. (1997) show that the optimal response is to have a holdback level that is monotonically non-increasing during the review period. A generalization of this model for negligible replenishment lead times but with positive transshipment times between retailers is due to Comez et al. (2006). Hence, as a future research question, it would be interesting to examine, in the instances of our model with positive replenishment lead times and $T \geq 1$, whether a time-dependent threshold policy would emerge as optimal, and whether such a policy would be a monotone function of the inventory on-hand.

Finally, we close by observing that our single-retailer model may be used as a building block to model more complex systems. One such application to a multi-retailer-one-warehouse system is described in detail by Xu (2006) for variants of the cross-docking model of Eppen and Schrage (1981).

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Appendix

Proof of Lemma 1

1.1) Since orders are placed at the beginning of period $t + kT$ ($k \in Z$) to bring the inventory position back to S, and $\mathbb{S}_t = \min\{IP_t^+ - M, D_t\} = \min\{S - M, D_t\}$, it is independent of historical sales $\mathbb{S}_{t-1}, \mathbb{S}_{t-2}, \ldots\ldots$ Also $\mathbb{S}_{t+i} = \min\{IP_{t+i} - M, D_{t+i}\} = \min\{S - \sum_{j=0}^{i-1} \mathbb{S}_{t+j}, D_{t+i}\}$, which depends on the sales from period t to period $t + i - 1$ only through the current inventory position IP_{t+i}, hence $\mathbb{S}_{t+i}$ is also independent of sales in previous cycle. Since D_t in each period t is i.i.d., $\mathbb{S}_t = \min\{S - M, D_t\}$ should have the same distribution as $\mathbb{S}_{t+kT} = \min\{S - M, D_{t+kT}\}$ for any $k \in Z$, hence $\mathbb{S}_t$ and $\mathbb{S}_{t+kT}$ are i.i.d.; then it follows that $\mathbb{S}_{t+1} = \min\{S - M - \mathbb{S}_t, D_{t+1}\}$ and $\mathbb{S}_{t+kT+1} = \min\{S - M - \mathbb{S}_{t+kT}, D_{t+kT+1}\}$ are i.i.d. too. Then arguing similiarly we get $\mathbb{S}_{t+i} = \min\{S - M - \sum_{j=0}^{i-1} \mathbb{S}_{t+j}, D_{t+i}\}$ and $\mathbb{S}_{t+kT+i} = \min\{S - M - \sum_{j=0}^{i-1} \mathbb{S}_{t+kT+j}, D_{t+kT+i}\}$ are i.i.d. for any $i \in \{1, \ldots, T-1\}$.

1.2) $\mathbb{S}_t = \min\{IP_t^+ - M, D_t\} = \min\{S - M, D_t\} \Rightarrow \begin{cases} P\{\mathbb{S}_t \le r\} = P\{D_t \le r\} = F(r) \quad if \quad 0 \le r < S - M, \\ P\{\mathbb{S}_t = S - M\} = P\{D_t \ge S - M\} = \bar{F}(S - M); \end{cases}$

$$
\begin{aligned}
P\{\mathbb{S}_{t+i} \le r\} &= P\left\{\min\{IP_{t+i} - M, D_{t+i}\} \le r\right\} \\
&= P\{IP_{t+i} - M \le r, D_{t+i} \le r\} + P\{IP_{t+i} - M > r, D_{t+i} \le r\} + P\{IP_{t+i} - M \le r, D_{t+i} > r\} \\
&= P\{D_{t+i} \le r\} + P\{IP_{t+i} - M \le r\} \cdot P\{D_{t+i} > r\} \\
&= F(r) + P\left\{\sum_{j=0}^{i-1} \mathbb{S}_{t+j} \ge S - M - r\right\} \bar{F}(r) \\
&= F(r) + \bar{F}_i(S - M - r)\bar{F}(r) \quad \text{for } 0 \le r \le S - M.
\end{aligned}
$$

1.3) Since we are using an (*T, S, M*) policy, the order quantity will be exactly the consumption (total sales) of the pervious cycle.

1.4) The result is obvious. Q.E.D.

Proof of Theorem 1

1.1) If $l - c \le (L+1)b + \pi$, then $\frac{\partial TC}{\partial y} = [c - l + (L+1)b + \pi]\bar{F}_T(y) + b\sum_{j=1}^{T-1}\bar{F}_j(y) > 0.$ So, *TC* increases in *y*. Hence, optimal *y* should be as small as possible, that is, $y^* \le S$. Therefore, the optimal M^* cannot be negative.

1.2) If $l - c \ge (L+T)b + \pi$, then

$$\begin{aligned}\frac{\partial TC}{\partial y} &= [c - l + (L+1)b + \pi]\bar{F}_T(y) + b\sum_{j=1}^{T-1}\bar{F}_j(y) \\ &= [c - l + (L+T)b + \pi]\bar{F}_T(y) + b\sum_{j=1}^{T-1}[\bar{F}_j(y) - \bar{F}_T(y)] < 0.\end{aligned}$$

So, *TC* decreases with *y*, which implies that optimal *y* should be as large as possible, that is, $y^* = +\infty$.

Hence, $M^* = -\infty$.

1.3) If $(L+1)b + \pi < l - c < (L+T)b + \pi$, then we write

$$\begin{aligned}\frac{\partial TC}{\partial y} &= [c - l + (L+1)b + \pi]\bar{F}_T(y) + b\sum_{j=1}^{T-1}\bar{F}_j(y) \\ &= [c - l + (L+1)b + \pi] - [c - l + (L+1)b + \pi]F_T(y) + b(T-1) - b\sum_{j=1}^{T-1}F_j(y) \\ &= [c - l + (L+T)b + \pi] + [l - c - (L+1)b - \pi]F_T(y) - b\sum_{j=1}^{T-1}F_j(y).\end{aligned}$$

Notice that $\frac{\partial TC}{\partial y}|_{y=0} = c - l + (L+T)b + \pi > 0$, $\frac{\partial TC}{\partial y}|_{y=+\infty} = 0$.

With MLR demand, we can verify that either $\frac{\partial TC}{\partial y} > 0$ for all $y \ge 0$, or there exists a unique $\tilde{y}$, such that $\frac{\partial TC}{\partial y}|_{y=\tilde{y}} = 0$. In the first case, we are always better-off to have smaller *y*, hence the optimal $y^* \le S$, that is, the optimal M^* cannot be a negative value. In the second case, there are following two sub-cases: a) If $y < \tilde{y}$, then $\frac{\partial TC}{\partial y} > 0$, so *TC* increases in *y*; and b) If $y > \tilde{y}$, then $\frac{\partial TC}{\partial y} < 0$, so *TC* decreases in *y*.

Now, we examine two situations: i) $\tilde{y} \le S$, and ii) $\tilde{y} > S$. When $\tilde{y} \le S$, we don't have to consider sub-

case a); but in sub-case b), *TC* decreases for $y > S$ which implies that the optimal *y* should be as large as possible, hence, $y^* = +\infty$, that is, $M^* = -\infty$. On the other hand, when $\tilde{y} > S$, then in sub-case a), the optimal *y* should be as small as possible, that is, $y^* \le S$, hence, the optimal M^* cannot be negative; and in sub-case b), the optimal *y* should be as large as possible, hence, $y^* = +\infty$, that is, $M^* = -\infty$. This completes the proof. Q.E.D.

Proof of Theorem 3

3.1) To prove Theorem 3.1, it is sufficient to show that the total demand satisfied by the on-hand inventory in a full lost sales model exceeds the total demand satisfied by the on-hand inventory in a full backorder model. This is because each unit satisfied by the on-hand inventory is cost neutral, whereas each unit satisfied via a backorder actually increases marginal costs relative to a lost sales when $l - c \le b - h$.

Note that, in a full backorder model, the total demand satisfied by on-hand inventory equals expected sales (demand) per period minus expected backorders per period, namely,

$$PC_{FB}(S) = u - \int_S^{+\infty} (\xi - S) f_2(\xi) d\xi .$$

In the full lost sales model, lost sales can be written as

$$LS_{FL}(S) = \int_S^{+\infty} (\xi - S) f(\xi) d\xi - \int_0^S F(x) dx + \int_0^S \frac{F(x)}{1 - \bar{F}(x)\bar{F}(S-x)} dx,$$

so that the total demand satisfied by on-hand inventory equals

$$PC_{FL}(S) = u - LS_{FL}(S) = S - \int_0^S \frac{F(x)}{1 - \bar{F}(x)\bar{F}(S-x)} dx .$$

Therefore,

$$
\begin{aligned}
&PC_{FL}(S)-PC_{FB}(S)\\
&=S-\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\left[u-\int_S^{+\infty}(\xi-S)f_2(\xi)d\xi\right]\\
&=S-u-\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx+\int_S^{+\infty}(\xi-S)f_2(\xi)d\xi\\
&=\int_0^S (S-\xi)f(\xi)d\xi-\int_S^{+\infty}(\xi-S)f(\xi)d\xi-\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx+\int_S^{+\infty}(\xi-S)f_2(\xi)d\xi\\
&=\int_0^S F(\xi)d\xi-\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\int_S^{+\infty}(\xi-S)f(\xi)d\xi+\int_S^{+\infty}(\xi-S)f_2(\xi)d\xi\\
&=-\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\left(u-\int_0^S \bar{F}(\xi)d\xi\right)+\left(2u-\int_0^S \bar{F}_2(\xi)d\xi\right)\\
&=-\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\left(S+u-\int_0^S \bar{F}(\xi)d\xi\right)+\left(S+2u-\int_0^S \bar{F}_2(\xi)d\xi\right)\\
&=-\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\left(S+u-\int_0^S \bar{F}(\xi)d\xi\right)+\left(2u+\int_0^S F_2(\xi)d\xi\right)\\
&=-\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx-\left(S+u-\int_0^S \bar{F}(\xi)d\xi\right)+2u+\int_0^S (S-\xi)f_2(\xi)d\xi,
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
&\int_0^S (S-\xi)f_2(\xi)d\xi\\
&=\int_0^S f(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\xi_1\\
&=-\int_0^S\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\bar{F}(\xi_1)\\
&=-\left(\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right)\bar{F}(\xi_1)\Big|_0^S\\
&\quad+\int_0^S \bar{F}(\xi_1)\,d\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right] \qquad \text{(integration by parts)}\\
&=\int_0^S (S-\xi)f(\xi)d\xi-\int_0^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1\\
&=\int_0^S F(\xi)d\xi-\int_0^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1\\
&=\int_0^S [1-\bar{F}(\xi)]d\xi-\int_0^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1\\
&=S-\int_0^S \bar{F}(\xi)d\xi-\int_0^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1\\
&=S-2\int_0^S \bar{F}(\xi)d\xi+\int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1.
\end{aligned}
\tag{17}
$$

Hence,

$$
\begin{aligned}
& PC_{FL}(S) - PC_{FB}(S) \\
&= -\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - \left(S + u - \int_0^S \bar{F}(\xi)d\xi\right) + 2u + \int_0^S (S-\xi) f_2(\xi)d\xi \\
&= -\int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - \left(S + u - \int_0^S \bar{F}(\xi)d\xi\right) + 2u \\
&\quad + S - 2\int_0^S \bar{F}(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1 \\
&= u - \int_0^S \bar{F}(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1 - \int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)[1-F(S-x)]}dx \\
&= \int_S^{+\infty} (\xi - S) f(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1 - \int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{F(x)+\bar{F}(x)F(S-x)}dx \\
&\geq \int_S^{+\infty} (\xi - S) f(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1 - \int_0^S \frac{F(x)\bar{F}(x)\bar{F}(S-x)}{F(x)+\bar{F}(x)F(S-x)}dx \\
&\geq \int_S^{+\infty} (\xi - S) f(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1 \\
&\quad - \int_0^S \bar{F}(x)\bar{F}(S-x)dx \qquad \left(\text{since } \frac{F(x)}{F(x)+\bar{F}(x)F(S-x)} < 1\right) \\
&= \int_S^{+\infty} (\xi - S) f(\xi)d\xi \\
&\geq 0.
\end{aligned}
$$

3.2) From (14) we can write

$$
TC_{FL}(S) = (c-2h)u + hS + (l-c+2h)\left[\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - (S-u)\right]. \tag{18}
$$

From (15), after some simplification, we write

$$
TC_{FB}(S) = (c-2h)u + hS + (h+b)\left\{2u - \int_0^S \bar{F}_2(x)dx\right\}. \tag{19}
$$

From (18) and (19) we get

$$
TC_{FL}(S) - TC_{FB}(S) = (l-c+2h)\left[\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - (S-u)\right] + (h+b)\left\{2u - \int_0^S \bar{F}_2(x)dx\right\}.
$$

Now, we can write

$$
2\left[\int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - (S-u)\right] - \left[2u - \int_0^S \bar{F}_2(x)dx\right] = \int_0^S \bar{F}_2(x)dx - 2\int_0^S \frac{\bar{F}(x)F(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx.
$$

From (16) and (17), we know that $2\int_0^S \bar{F}(x)dx - \int_0^S \bar{F}_2(x)dx - \int_0^S \bar{F}(x)\bar{F}(S-x)dx = 0,$ so that after some simplification we obtain

$$\begin{aligned}
&\int_0^S \bar{F}_2(x)dx - 2\int_0^S \frac{\bar{F}(x)F(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx \\
&= 2\int_0^S \bar{F}(x)dx - \int_0^S \bar{F}(x)\bar{F}(S-x)dx - 2\int_0^S \frac{\bar{F}(x)F(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx \\
&= \int_0^S \bar{F}(x)dx + \int_0^S \bar{F}(x)F(S-x)dx - \int_0^S \frac{\bar{F}(x)F(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx - \int_0^S \frac{\bar{F}(x)F(S-x)}{F(S-x)+F(x)\bar{F}(S-x)}dx \\
&= \int_0^S \frac{F(x)F(S-x)\bar{F}(x)\bar{F}(S-x)}{1-\bar{F}(x)\bar{F}(S-x)}dx \\
&\geq 0.
\end{aligned}$$

This implies that if $l - c + 2h \geq 2(h+b)$, i.e., $l - c \geq 2b$, then $TC_{FL}(S) \geq TC_{FB}(S)$. Q.E.D.

Proof of Theorem 4

4.1) As in the proof of Theorem 3.1, here also it is sufficient to show that the total demand satisfied by on-hand inventory in a full lost sales model exceeds the total demand satisfied by on-hand inventory in a (*T*, *M*, *S*) model when $l - c \leq b - h$.

In the full lost sales model, as in the proof of Theorem 3.1, the total demand satisfied by on-hand inventory equals

$$PC_{FL}(S) = S - \int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx. \tag{20}$$

The total demand satisfied by on-hand inventory in a (*T*, *M, S*) model equals expected sales per period minus expected backorders per period, namely,

$$\begin{aligned}
P\bar{C}(M,S) &= \left[\int_0^{S-M} \xi f(\xi)d\xi + (S-M)\bar{F}(S-M)\right] \\
&\quad - \left\{2\left[\int_0^{S-M} \xi f(\xi)d\xi + (S-M)\bar{F}(S-M)\right] - \left[S - HC(1,M,S)\right]\right\} \\
&= S - HC(1,M,S) - \left[\int_0^{S-M} \xi f(\xi)d\xi + (S-M)\bar{F}(S-M)\right].
\end{aligned} \tag{21}$$

Hence,

$$
\begin{aligned}
&PC_{FL}(S) - P\bar{C}(M,S) \\
&= HC(1,M,S) + \left[\int_0^{S-M} \xi f(\xi)d\xi + (S-M)\bar{F}(S-M)\right] - \int_0^S \frac{F(x)}{1-\bar{F}(x)\bar{F}(S-x)}dx,
\end{aligned} \tag{22}
$$

with

$$
\begin{aligned}
HC(1,M,S) = &\int_0^S (S-\xi)f_2(\xi)d\xi - 2\int_{S-M}^S f(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\xi_1 \\
&+ 2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi,
\end{aligned} \tag{23}
$$

where the sum of the second and third terms in (23) equals

$$
\begin{aligned}
&-2\int_{S-M}^S f(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\xi_1 + 2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi \\
&= 2\int_{S-M}^S \left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right]d\bar{F}(\xi_1) + 2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi \\
&= 2\left(\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right)\bar{F}(\xi_1)\Big|_{S-M}^S - 2\int_{S-M}^S \bar{F}(\xi_1)d\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f(\xi_2)d\xi_2\right] \\
&+ 2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi \qquad \text{(using integration by parts)} \\
&= -2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi + 2\int_{S-M}^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1 + 2\bar{F}(S-M)\int_0^M (M-\xi)f(\xi)d\xi \\
&= 2\int_{S-M}^S \bar{F}(\xi_1)F(S-\xi_1)d\xi_1,
\end{aligned} \tag{24}
$$

and by (17), the first term in (23) equals

$$
\int_0^S (S-\xi)f_2(\xi)d\xi = S - 2\int_0^S \bar{F}(\xi)d\xi + \int_0^S \bar{F}(\xi_1)\bar{F}(S-\xi_1)d\xi_1.
$$

Therefore, (22) can be written as

$$
\begin{aligned}
&PC_{FL}(S)-P\overline{C}(M,S)\\
&=HC(1,M,S)+\left[\int_0^{S-M}\xi f(\xi)d\xi+(S-M)\overline{F}(S-M)\right]-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\\
&=S-2\int_0^S\overline{F}(\xi)d\xi+\int_0^S\overline{F}(\xi_1)\overline{F}(S-\xi_1)d\xi_1+2\int_{S-M}^S\overline{F}(x)F(S-x)dx\\
&\quad+\left[\int_0^{S-M}\xi f(\xi)d\xi+(S-M)\overline{F}(S-M)\right]-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\\
&=S-2\int_0^S\overline{F}(\xi)d\xi+\int_0^S\overline{F}(\xi_1)\overline{F}(S-\xi_1)d\xi_1+2\int_{S-M}^S\overline{F}(x)F(S-x)dx\\
&\quad+2\int_0^{S-M}\overline{F}(\xi)d\xi-\int_0^{S-M}\overline{F}(\xi)d\xi-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\\
&=\left[S-\int_0^{S-M}\overline{F}(\xi)d\xi-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\right]-2\int_{S-M}^S\overline{F}(\xi)d\xi+\int_0^S\overline{F}(\xi_1)\overline{F}(S-\xi_1)d\xi_1+2\int_{S-M}^S\overline{F}(x)F(S-x)dx\\
&=\left[S-\int_0^{S-M}\overline{F}(\xi)d\xi-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\right]+\int_0^S\overline{F}(\xi_1)\overline{F}(S-\xi_1)d\xi_1-2\int_{S-M}^S\overline{F}(x)\overline{F}(S-x)dx\\
&=\left[M+\int_0^{S-M}F(\xi)d\xi-\int_0^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\right]\\
&\quad+\int_0^S\overline{F}(\xi_1)\overline{F}(S-\xi_1)d\xi_1-\int_{S-M}^S\overline{F}(x)\overline{F}(S-x)dx-\int_0^M\overline{F}(x)\overline{F}(S-x)dx\\
&=\left[M-\int_0^{S-M}\frac{F(x)\overline{F}(x)\overline{F}(S-x)}{1-\overline{F}(x)\overline{F}(S-x)}dx-\int_{S-M}^S\frac{F(x)}{1-\overline{F}(x)\overline{F}(S-x)}dx\right]+\int_M^{S-M}\overline{F}(x)\overline{F}(S-x)dx\\
&=\left[M-\int_0^{S-M}\frac{F(x)\overline{F}(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx-\int_0^M\frac{F(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx\right]+\int_M^{S-M}\overline{F}(x)\overline{F}(S-x)dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&M-\int_0^{S-M}\frac{F(x)\overline{F}(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx-\int_0^M\frac{F(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx\\
&=\int_0^M\frac{F(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx-\int_0^{S-M}\frac{F(x)\overline{F}(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx\\
&=\int_0^M\frac{F(x)F(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx-\int_M^{S-M}\frac{F(x)\overline{F}(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx\\
&\geq\int_0^M\frac{F(x)F(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx-\int_M^{S-M}\overline{F}(x)\overline{F}(S-x)dx\quad\left(\text{since }\frac{F(x)}{F(S-x)+F(x)\overline{F}(S-x)}<1\right),
\end{aligned}
$$

where $M\leq S-M$, since $0\leq M\leq S/2$.

Therefore,

$$
\begin{aligned}
& PC_{FL}(S) - P\overline{C}(M,S) \\
& \geq \int_0^M \frac{F(x)F(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx - \int_M^{S-M} \overline{F}(x)\overline{F}(S-x)dx + \int_M^{S-M} \overline{F}(x)\overline{F}(S-x)dx \\
& = \int_0^M \frac{F(x)F(x)\overline{F}(S-x)}{F(S-x)+F(x)\overline{F}(S-x)}dx \\
& \geq 0.
\end{aligned}
$$

Until now, we have proven Theorem 4.1 for $0 \leq M \leq S/2$. Next, we will prove it for (i) $S/2 < M \leq S$ and (ii) $M < 0$, respectively.

i) For $S/2 < M \leq S$, from the fourth paragraph of Section 3, we know $\frac{\partial TC}{\partial y} = [c - l - 2h]\overline{F}(y) < 0$,

that is, $TC(M, S)$ decreases in y and increases in M. Therefore, we conclude that

$TC(M,S) \geq TC(S/2,S) \geq TC_{FL}(S)$ for $S/2 < M \leq S$.

ii) For $M < 0$, in Theorem 1.1, we have shown that when $l - c \leq (L+1)b + \pi$, the optimal M^* cannot be negative. For $L = 1$, the condition becomes $l - c \leq 2b + \pi$, and we have (see the proof of Theorem 1.1) $TC(M,S) \geq TC(0,S) \geq TC_{FL}(S)$ for $M < 0$. Note that, the condition $l - c \leq b - h$ of Theorem 4.1 satisfies $l - c \leq 2b + \pi$.

Hence, we have $TC(M,S) \geq TC_{FL}(S)$ for any $M \leq S$.

4.2) Proof follows immediately from Theorem 1.2. Q.E.D.

Addendum

A1. Some Comparative Statics

From $\frac{\partial TC(M,S)}{\partial M} = [l-c-2b]\bar{F}(S-M) + 2(h+b)\bar{F}(S-M)F(M) = 0$, we define

$$\Gamma(S,M,h,b,l) = F(M) - \frac{c+2b-l}{2(h+b)} = 0.$$

Similarly, we define

$$\begin{aligned}\Omega(S,M,h,b,l) &= \frac{\partial TC(M,S)}{\partial S} \\ &= [c-l+2b]\bar{F}(S-M) - b + (h+b)\left\{F_2(S) - 2\int_{S-M}^{S} f(\xi)F(S-\xi)d\xi\right\} = 0.\end{aligned}$$

Notice that the above equations in Γ and Ω express S and M as implicit functions of three parameters l, h and b. In the following, we analyze the behavior of S and M in these parameters.

<u>(1) For the parameter l</u>

By the chain rule of partial derivatives for implicit functions, we have

$$\begin{cases}\Gamma_l + \Gamma_S \cdot \frac{\partial S}{\partial l} + \Gamma_M \cdot \frac{\partial M}{\partial l} = 0, \\ \Omega_l + \Omega_S \cdot \frac{\partial S}{\partial l} + \Omega_M \cdot \frac{\partial M}{\partial l} = 0,\end{cases}$$

where

$$\begin{cases}\Gamma_l = \frac{\partial \Gamma}{\partial l} = \frac{1}{2(h+b)}, \qquad \Gamma_S = \frac{\partial \Gamma}{\partial S} = 0, \qquad \Gamma_M = \frac{\partial \Gamma}{\partial M} = f(M), \\ \Omega_l = \frac{\partial \Omega}{\partial l} = -\bar{F}(S-M) \\ \Omega_S = \frac{\partial \Omega}{\partial S} = -[c-l+2b]f(S-M) + (h+b)\left\{f_2(S) - 2\int_{S-M}^{S} f(\xi)f(S-\xi)d\xi + 2f(S-M)F(M)\right\} \\ \qquad = (h+b)\left[f_2(S) - 2\int_{S-M}^{S} f(\xi)f(S-\xi)d\xi\right], \\ \Omega_M = \frac{\partial \Omega}{\partial M} = [c-l+2b]f(S-M) - 2(h+b)f(S-M)F(M) = 0.\end{cases}$$

Hence,

$$\frac{\partial M}{\partial l}=-\frac{\begin{vmatrix}\Gamma_S & \Gamma_l\\ \Omega_S & \Omega_l\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0-\frac{1}{2(h+b)}\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}<0,$$

and

$$\frac{\partial S}{\partial l}=-\frac{\begin{vmatrix}\Gamma_l & \Gamma_M\\ \Omega_l & \Omega_M\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0+f(M)\cdot\bar{F}(S-M)}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}>0.$$

<u>(2) For the parameter *h*</u>

Again, by the chain rule of partial derivatives for implicit functions, we have

$$\begin{cases}\Gamma_h+\Gamma_S\cdot\frac{\partial S}{\partial h}+\Gamma_M\cdot\frac{\partial M}{\partial h}=0,\\ \Omega_h+\Omega_S\cdot\frac{\partial S}{\partial h}+\Omega_M\cdot\frac{\partial M}{\partial h}=0,\end{cases}$$

where

$$\begin{cases}\Gamma_h=\frac{\partial\Gamma}{\partial h}=\frac{c+2b-l}{2(h+b)^2}, \qquad \Gamma_S=\frac{\partial\Gamma}{\partial S}=0, \qquad \Gamma_M=\frac{\partial\Gamma}{\partial M}=f(M),\\ \Omega_h=\frac{\partial\Omega}{\partial h}=F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi,\\ \Omega_S=\frac{\partial\Omega}{\partial S}=-[c-l+2b]f(S-M)+(h+b)\left\{f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi+2f(S-M)F(M)\right\}\\ \qquad=(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right],\\ \Omega_M=\frac{\partial\Omega}{\partial M}=[c-l+2b]f(S-M)-2(h+b)f(S-M)F(M)=0.\end{cases}$$

Hence,

$$\frac{\partial M}{\partial h}=-\frac{\begin{vmatrix}\Gamma_S & \Gamma_h\\ \Omega_S & \Omega_h\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0-\frac{c+2b-l}{2(h+b)^2}\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}<0,$$

and

$$\frac{\partial S}{\partial h}=-\frac{\begin{vmatrix}\Gamma_h & \Gamma_M\\ \Omega_h & \Omega_M\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0-f(M)\cdot\left[F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi\right]}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}<0.$$

<u>(3) For the parameter *b*</u>

Once again, by the chain rule of partial derivatives for implicit functions, we have

$$\begin{cases}\Gamma_b+\Gamma_S\cdot\dfrac{\partial S}{\partial b}+\Gamma_M\cdot\dfrac{\partial M}{\partial b}=0,\\ \Omega_b+\Omega_S\cdot\dfrac{\partial S}{\partial b}+\Omega_M\cdot\dfrac{\partial M}{\partial b}=0,\end{cases}$$

where

$$\begin{cases}\Gamma_b=\dfrac{\partial\Gamma}{\partial b}=\dfrac{c-2h-l}{2(h+b)^2}, \qquad \Gamma_S=\dfrac{\partial\Gamma}{\partial S}=0, \qquad \Gamma_M=\dfrac{\partial\Gamma}{\partial M}=f(M),\\ \Omega_b=\dfrac{\partial\Omega}{\partial b}=2\overline{F}(S-M)-1+F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi,\\ \Omega_S=\dfrac{\partial\Omega}{\partial S}=-[c-l+2b]f(S-M)+(h+b)\left\{f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi+2f(S-M)F(M)\right\}\\ \qquad=(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right],\\ \Omega_M=\dfrac{\partial\Omega}{\partial M}=[c-l+2b]f(S-M)-2(h+b)f(S-M)F(M)=0.\end{cases}$$

Hence,

$$\frac{\partial M}{\partial b}=-\frac{\begin{vmatrix}\Gamma_S & \Gamma_b\\ \Omega_S & \Omega_b\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0-\dfrac{c-2h-l}{2(h+b)^2}\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}>0,$$

and

$$\frac{\partial S}{\partial b}=-\frac{\begin{vmatrix}\Gamma_b & \Gamma_M\\ \Omega_b & \Omega_M\end{vmatrix}}{\begin{vmatrix}\Gamma_S & \Gamma_M\\ \Omega_S & \Omega_M\end{vmatrix}}=-\frac{0-f(M)\cdot\left[2\bar{F}(S-M)-1+F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi\right]}{0-f(M)\cdot(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]}$$

$$=-\frac{\left[2\bar{F}(S-M)-1+F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi\right]}{(h+b)\left[f_2(S)-2\int_{S-M}^{S}f(\xi)f(S-\xi)d\xi\right]},$$

Since the sign of $2\bar{F}(S-M)-1+F_2(S)-2\int_{S-M}^{S}f(\xi)F(S-\xi)d\xi$ is uncertain, $\frac{\partial S}{\partial b}$ may be positive or negative.

A2. Detailed Analysis for Column 3 in Table 2

Now that we have made a complete analysis with the case of $T = L = 1$, we want to analyze a more general setting, and we start with the case of $T = 1$, $L > 1$, we will analyze the case of $T > 1$ later. As in Section 3, if $(L+1)y=(L+1)(S-M)<S$,

or equivalently, $M > SL/(L+1)$, still there can be no backorders and there will be inventory on-hand at the end of each period, and the cost formulation is also similar:

$$TC(L,M,S)=[c-l-(L+1)h][\int_0^{S-M}\xi f(\xi)d\xi+(S-M)\bar{F}(S-M)]+h\cdot S+l\cdot u\,, \tag{25}$$

and we assert that:

Theorem 5 *In an optimal solution,* $M \le SL/(L+1)$ *or* $S \ge M(L+1)/L$.

Proof.

From (25) we have $\frac{\partial TC}{\partial y}=[c-l-(L+1)h]\bar{F}(y)<0$. Hence, *TC* decreases in *y*, or increases in *M*. Now, to attain the optimal solution, if *M* is decreased, then $(L+1)(S-M)<S$ would not be satisfied, which means that this case does not yield the optimal solution. Q.E.D.

Remark 1: Theorem 5 together with Theorem 1, implies that when $T = 1$, to attain the optimal solution, M must lie between 0 and $SL/(L+1)$. We now examine the optimal structure of the cost function when M is in this range. As in Section 2, we can write the total cost function as

$$\begin{aligned} TC(L,y,S) &= c\left[\int_0^y \xi f(\xi)d\xi + y\bar{F}(y)\right] + l\left\{u - \left[\int_0^y \xi f(\xi)d\xi + y\bar{F}(y)\right]\right\} + h\cdot HC_{t+L} \\ &\quad + b\left\{(L+1)[\int_0^y \xi f(\xi)d\xi + y\bar{F}(y)] - (S - HC_{t+L})\right\} \\ &= \left[c - l + (L+1)b\right]\left[\int_0^y \xi f(\xi)d\xi + y\bar{F}(y)\right] + (h+b)HC_{t+L} + lu - bS. \end{aligned} \tag{26}$$

The formulation of HC_{t+L} is rather complex in this case, and we need to consider several cases:

If $(L+1)y \ge S$, then there exists a positive integer $m \le L$ such that $my < S$, $(m+1)y \ge S$, that is, $\frac{S}{m+1} \le y < \frac{S}{m}$. We will now calculate the expected leftovers at the end of each period using a combinatorial argument. For demands from D_t to D_{t+L} (totally $L+1$ periods' demand), leftovers can occur if at most m of these demands exceed y, and we can decompose this event into the following m +1 cases:

1) $D_i < y \qquad (t \le i \le t+L)$;

2) $D_j \ge y$, $D_{i\ne j} < y \qquad (t \le j \ne i \le t+L)$;

3) $D_{j_1} \ge y$, $D_{j_2} \ge y$, $D_{i\ne j_1 \ne j_2} < y \qquad (t \le j_1 \ne j_2 \ne i \le t+L)$;

……

m) $D_{j_1} \ge y$, …, $D_{j_{m-1}} \ge y$, $D_{i\ne j_1\ne j_2\ne\ldots j_{m-1}} < y \qquad (t \le j_1 \ne j_2 \ne \ldots \ne j_{m-1} \ne i \le t+L)$;

m+1) $D_{j_1} \ge y$, …, $D_{j_{m-1}} \ge y$, $D_{j_m} \ge y$, $D_{i\ne j_1\ne j_2\ne\ldots j_m} < y \qquad (t \le j_1 \ne j_2 \ne \ldots \ne j_m \ne i \le t+L)$;

In case 1),

$$\begin{aligned} E_1[S - \textstyle\sum_{i=t}^{t+L} \mathbb{S}_i]^+ &= E[S - \textstyle\sum_{i=t}^{t+L} D_i;\, D_i < y; t \le i \le t+L; \textstyle\sum_{i=t}^{t+L} D_i \le S] \\ &= E[S - \textstyle\sum_{i=t}^{t+L} D_i;\, \textstyle\sum_{i=t}^{t+L} D_i \le S] \\ &\quad - E[S - \textstyle\sum_{i=t}^{t+L} D_i;\, \textstyle\sum_{i=t}^{t+L} D_i \le S; \text{at least one } D_i \ge y; t \le i \le t+L]. \end{aligned}$$

If we define $A_i = \{\sum_{i=t}^{t+L} D_i \le S;\, D_i \ge y\}$ for $t \le i \le t+L$, then

$$
\begin{aligned}
&E[S-\textstyle\sum_{i=t}^{t+L} D_i;\ \sum_{i=t}^{t+L} D_i \le S; \text{at least one } D_i \ge y; t \le i \le t+L] \\
&=\binom{L+1}{1} E[S-\textstyle\sum_{i=t}^{t+L} D_i; A_1] - \binom{L+1}{2} E[S-\textstyle\sum_{i=t}^{t+L} D_i; A_1 \cap A_2] \\
&+\ldots+(-1)^{m+1}\binom{L+1}{m} E[S-\textstyle\sum_{i=t}^{t+L} D_i; A_1 \cap A_2 \cap \ldots A_m] \\
&=\binom{L+1}{1}\int_y^S f(\xi_1)[\int_0^{S-\xi_1}(S-\xi_1-\xi_L) f_L(\xi_L) d\xi_L] d\xi_1 \\
&-\binom{L+1}{2}\int_y^{S-y} f(\xi_1)\{\int_y^{S-\xi_1} f(\xi_2)[\int_0^{S-\xi_1-\xi_2}(S-\xi_1-\xi_2-\xi_{L-1}) f_{L-1}(\xi_{L-1}) d\xi_{L-1}] d\xi_2\} d\xi_1 \\
&+\ldots\ldots \\
&+(-1)^{m+1}\binom{L+1}{m}\int_y^{S-(m-1)y} f(\xi_1)\{\int_y^{S-(m-2)y-\xi_1} f(\xi_2)\ldots\int_y^{S-\sum_{j=1}^{m-1}\xi_j} f(\xi_m)[\int_0^{S-\sum_{j=1}^{m}\xi_j}(S-\textstyle\sum_{j=1}^{m}\xi_j-\xi_{L-m+1}) f_{L-m+1}(\xi_{L-m+1}) d\xi_{L-m+1}]\ldots\} d\xi_1
\end{aligned}
$$

In case 2),

$$
\begin{aligned}
E_2[S-\textstyle\sum_{i=t}^{t+L}\mathbb{S}_i]^+ &= \binom{L+1}{1} E[S-y-\textstyle\sum_{i=t}^{t+L-1} D_i;\ D_{t+L} \ge y; D_i < y; t \le i \le t+L-1; \sum_{i=t}^{t+L-1} D_i \le S-y] \\
&= \binom{L+1}{1}\bar{F}(y)\cdot E[S-y-\textstyle\sum_{i=t}^{t+L-1} D_i;\ D_i < y; t \le i \le t+L-1; \sum_{i=t}^{t+L-1} D_i \le S-y] \\
&= \binom{L+1}{1}\bar{F}(y)\cdot E[S-y-\textstyle\sum_{i=t}^{t+L-1} D_i;\ \sum_{i=t}^{t+L-1} D_i \le S-y] \\
&\quad -\binom{L+1}{1}\bar{F}(y)\cdot E[S-y-\textstyle\sum_{i=t}^{t+L-1} D_i;\ \sum_{i=t}^{t+L-1} D_i \le S-y; \text{at least one } D_i \ge y; t \le i \le t+L-1],
\end{aligned}
$$

where

$$
\begin{aligned}
&E[S-y-\textstyle\sum_{i=t}^{t+L-1} D_i;\ \sum_{i=t}^{t+L-1} D_i \le S-y; \text{at least one } D_i \ge y; t \le i \le t+L-1] \\
&=\binom{L}{1}\int_y^{S-y} f(\xi_1)[\int_0^{S-y-\xi_1}(S-y-\xi_1-\xi_{L-1}) f_{L-1}(\xi_{L-1}) d\xi_{L-1}] d\xi_1 \\
&-\binom{L}{2}\int_y^{S-2y} f(\xi_1)\{\int_y^{S-y-\xi_1} f(\xi_2)[\int_0^{S-y-\xi_1-\xi_2}(S-y-\xi_1-\xi_2-\xi_{L-2}) f_{L-2}(\xi_{L-2}) d\xi_{L-2}] d\xi_2\} d\xi_1 \\
&+\ldots\ldots \\
&+(-1)^{m}\binom{L}{m-1}\int_y^{S-(m-1)y} f(\xi_1)\{\int_y^{S-(m-2)y-\xi_1} f(\xi_2)\ldots\int_y^{S-y-\sum_{j=1}^{m-2}\xi_j} f(\xi_{m-1})[\int_0^{S-y-\sum_{j=1}^{m-1}\xi_j}(S-y-\textstyle\sum_{j=1}^{m-1}\xi_j-\xi_{L-m+1}) f_{L-m+1}(\xi_{L-m+1}) d\xi_{L-m+1}]\ldots\} d\xi_1
\end{aligned}
$$

……

Similarly, in case m),

$$
\begin{aligned}
&E_m[S-\textstyle\sum_{i=t}^{t+L}\mathbb{S}_i]^+\\
&=\binom{L+1}{m-1}E[S-(m-1)y-\textstyle\sum_{i=t}^{t+L-m+1}D_i;\ D_j\ge y;t+L-m+2\le j\le t+L;D_i<y;t\le i\le t+L-m+1;\sum_{i=t}^{t+L-m+1}D_i\le S-(m-1)y]\\
&=\binom{L+1}{m-1}\bar{F}^{m-1}(y)\cdot E[S-(m-1)y-\textstyle\sum_{i=t}^{t+L-m+1}D_i;\ D_i<y;t\le i\le t+L-m+1;\sum_{i=t}^{t+L-m+1}D_i\le S-(m-1)y]\\
&=\binom{L+1}{m-1}\bar{F}^{m-1}(y)\cdot E[S-(m-1)y-\textstyle\sum_{i=t}^{t+L-m+1}D_i;\ \sum_{i=t}^{t+L-m+1}D_i\le S-(m-1)y]\\
&-\binom{L+1}{m-1}\bar{F}^{m-1}(y)\cdot E[S-(m-1)y-\textstyle\sum_{i=t}^{t+L-m+1}D_i;\ \sum_{i=t}^{t+L-m+1}D_i\le S-(m-1)y;\text{at least one } D_i\ge y;t\le i\le t+L-m+1].
\end{aligned}
$$

where

$$
\begin{aligned}
&E[S-(m-1)y-\textstyle\sum_{i=t}^{t+L-m+1}D_i;\ \sum_{i=t}^{t+L-m+1}D_i\le S-(m-1)y;\text{at least one } D_i\ge y;t\le i\le t+L-m+1]\\
&=\binom{L-m+2}{1}\int_y^{S-(m-1)y}f(\xi_1)\{\int_0^{S-(m-1)y-\xi_1}[S-(m-1)y-\xi_1-\xi_{L-m+1}]f_{L-m+1}(\xi_{L-m+1})d\xi_{L-m+1}\}d\xi_1.
\end{aligned}
$$

Finally, in case m + 1),

$$
\begin{aligned}
&E_{m+1}[S-\textstyle\sum_{i=t}^{t+L}\mathbb{S}_i]^+\\
&=\binom{L+1}{m}E[S-my-\textstyle\sum_{i=t}^{t+L-m}D_i;\ D_j\ge y;t+L-m+1\le j\le t+L;D_i<y;t\le i\le t+L-m;\sum_{i=t}^{t+L-m}D_i\le S-my]\\
&=\binom{L+1}{m}E[S-my-\textstyle\sum_{i=t}^{t+L-m}D_i;\ D_i<y;t\le i\le t+L-m;\sum_{i=t}^{t+L-m}D_i\le S-my]\\
&=\binom{L+1}{m}\bar{F}^m(y)\cdot\int_0^{S-my}[S-my-\xi]f_{L-m+1}(\xi)d\xi.
\end{aligned}
$$

Thus, combining all m + 1 cases we get $HC_{t+L}=\sum_{n=1}^{m+1}E_n[S-\sum_{i=t}^{t+L}\mathbb{S}_i]^+$. And we are ready to establish the following lemma.

Lemma 2

2.1) $\dfrac{\partial HC_{t+L}}{\partial y}=-(L+1)\bar{F}(y)P\left\{\sum_{i=1}^{L}\mathbb{S}_i\le S-y;0\le\mathbb{S}_i\le y;1\le i\le L\right\}\le 0;$

2.2) $\dfrac{\partial HC_{t+L}}{\partial S}=P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i\le S;0\le\mathbb{S}_i\le y;1\le i\le L+1\right\}\ge 0;$

2.3) $\dfrac{\partial^2 HC_{t+L}}{\partial y^2}\ge 0$, $\dfrac{\partial^2 HC_{t+L}}{\partial y\partial S}\le 0;$

2.4) $\dfrac{\partial^2 HC_{t+L}}{\partial S^2}\ge 0.$

Proof.

$$\frac{\partial HC_{t+L}}{\partial y}$$

$$= -\binom{L+1}{1}\bar{F}(y)\left[\begin{array}{l} F_L(S-y) - \binom{L}{1}\int_y^{S-y} f(\xi_1)F_{L+1}(S-y-\xi_1)d\xi_1 + \binom{L}{2}\int_y^{S-2y} f(\xi_1)\{\int_y^{S-y-\xi_1} f(\xi_2)F_{L-2}(S-y-\xi_1-\xi_2)d\xi_2\}d\xi_1 \\ -.....(-1)^m\binom{L}{m-1}\int_y^{S-(m-1)y} f(\xi_1)\{\int_y^{S-(m-2)y-\xi_1} f(\xi_2).....\int_y^{S-y-\sum_{j=1}^{m-2}\xi_j} f(\xi_{m-1})F_{L-m+1}(S-y-\sum_{j=1}^{m-1}\xi_j)d\xi_{m-1}...\}d\xi_1 \end{array}\right]$$

$$- .....$$

$$-(m-1)\binom{L+1}{m-1}\bar{F}^{m-1}(y)\cdot\left[F_{L-m+2}[S-(m-1)y] - \binom{L-m+2}{m-1}\int_y^{S-(m-1)y} f(\xi_1)F_{L-m+1}[S-(m-1)y-\xi_1]d\xi_1\right]$$

$$-m\binom{L+1}{m}\bar{F}^m(y)F_{L-m+1}[S-my]$$

$$= -(L+1)\bar{F}(y)\left[P\left\{\sum_{i=1}^L D_i \le S-y;\ 0\le D_i < y; 1\le i\le L\right\} + ... + \binom{L}{m-1}\bar{F}^{m-1}(y)P\left\{\sum_{i=1}^{L-m+1} D_i \le S-my;\ 0\le D_i < y; 1\le i\le L\right\}\right]$$

$$= -(L+1)\bar{F}(y)P\left\{\sum_{i=1}^L \mathbb{S}_i \le S-y;\ 0\le \mathbb{S}_i \le y; 1\le i\le L\right\} \le 0,$$

and

$$\frac{\partial HC_{t+L}}{\partial S} = F_{L+1}(S)$$

$$-\left[\begin{array}{l} \binom{L+1}{1}\int_y^S f(\xi_1)F_L(S-\xi_1)d\xi_1 - \binom{L+1}{2}\int_y^{S-y} f(\xi_1)\{\int_y^{S-\xi_1} f(\xi_2)F_{L-1}(S-\xi_1-\xi_2)d\xi_2\}d\xi_1 \\ +.....+(-1)^{m+1}\binom{L+1}{m}\int_y^{S-(m-1)y} f(\xi_1)\{\int_y^{S-(m-2)y-\xi_1} f(\xi_2)...\int_y^{S-\sum_{j=1}^{m-1}\xi_j} f(\xi_m)F_{L-m+1}\left[S-\sum_{j=1}^m \xi_j\right].....\}d\xi_1 \end{array}\right]$$

$$+ .....$$

$$+\binom{L+1}{m-1}\bar{F}^{m-1}(y)\left\{F_{L-m+2}[S-(m-1)y] - \binom{L-m+2}{1}\int_y^{S-(m-1)y} f(\xi_1)F_{L-m+1}[S-(m-1)y-\xi_1]d\xi_1\right\}$$

$$+\binom{L+1}{m}\bar{F}^m(y).F_{L-m+1}[S-my]$$

$$= P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S;\ 0\le \mathbb{S}_i \le y; 1\le i\le L+1\right\} \ge 0.$$

This proves Lemmas 2.1 and 2.2.

Lemma 2.3 follows from the observation that $P\left\{\sum_{i=1}^L \mathbb{S}_i \le S-y; 0\le \mathbb{S}_i \le y; 1\le i\le L\right\}$ is monotonically decreasing in *y* and monotonically increasing in *S*. Lemma 2.4 follows from the observation that $P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S; 0\le \mathbb{S}_i \le y; 1\le i\le L+1\right\}$ is monotonically increasing in *S*. Q.E.D.

Applying Lemma 2, it follows that

$$\frac{\partial TC}{\partial y} = [c - l + (L+1)b]\bar{F}(y) + (h+b)\frac{\partial HC_{t+L}}{\partial y}$$
$$= \bar{F}(y)\left[[c - l + (L+1)b] - (L+1)(h+b)P\left\{\sum_{i=1}^{L}\mathbb{S}_i \le S - y; 0 \le \mathbb{S}_i \le y; 1 \le i \le L\right\}\right]. \tag{27}$$

and

$$\frac{\partial TC}{\partial S} = -b + (h+b)P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S; 0 \le \mathbb{S}_i \le y; 1 \le i \le L+1\right\}. \tag{28}$$

Observe that $P\left\{\sum_{i=1}^{L}\mathbb{S}_i \le S - y; 0 \le \mathbb{S}_i \le y; 1 \le i \le L\right\}$ is monotonically decreasing in *y*, and $P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S; 0 \le \mathbb{S}_i \le y; 1 \le i \le L+1\right\}$ is monotonically increasing in *S*. Now, using Lemma 2 to discern the structures of (27) and (28) yields the following:

Theorem 6

When T = 1, $\pi = 0$, *with* $y = S - M$, *we have*

6.1) *Given a y, there is a unique* S^* *such that* $\frac{\partial TC}{\partial S}|_{S=S^*} = 0$*; and,*

6.2) *Given a S, there is a unique* y^* *such that* $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$.

Proof.

6.1) Notice that for any fixed *y*, $P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S; 0 \le \mathbb{S}_i \le y; 1 \le i \le L+1\right\}$ is monotonically increasing from 0 to 1 as *S* increases. Hence, from (28), we observe that there exists a S^* such that $\frac{\partial TC}{\partial S}|_{S=S^*} = 0$. And, such S^* is unique because $\frac{\partial^2 TC}{\partial S^2} = (h+b)\frac{\partial P\left\{\sum_{i=1}^{L+1}\mathbb{S}_i \le S; 0 \le \mathbb{S}_i \le y; 1 \le i \le L+1\right\}}{\partial S} \ge 0$.

6.2) For any given *S*, $P\left\{\sum_{i=1}^{L}\mathbb{S}_i \le S - y; 0 \le \mathbb{S}_i \le y; 1 \le i \le L\right\}$ is monotonically decreasing from 1 to 0 as *y* increases from $S/(L+1)$ to *S*. Now, from (27), we observe that there exists a unique y^* such that $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$. Q.E.D.

Remark 2: Part 1 of Theorem 6 follows directly from (28) since *TC* is convex in *S*. Part 2 of Theorem 6 implies that *TC* is unimodal in *M*. Hence, one way to proceed to find the optimal policy is to first solve for the optimal M^* for a given *S*, and then do an exhaustive line search to find all local minima for *S*.

A3. Detailed Analysis for Column 4 in Table 2

Now that we have studied in considerable detail the important case of $T = 1$, we will discuss the cases when *T* is greater than 1. In this case, the primary difficulty that arises is that we have to track inventory on-hand at points different than the review epochs. However, sufficient structure is retained so that we can still prove the desired unimodality result in each of the variables. Since from Theorem 1 we know that the optimal *M* can be positive only when $l - c < (L+T)b$, we will focus on this region. We first consider the cases when $L \le T$. In such cases, it is assured that all backorders have cleared when an order is placed so that analogously to Theorems 5 and 6, we get the following:

Theorem 7

When $T > 1$, *and* $L \le T$, *if* $h(T-L) > Lb + \pi$, *then optimal* $S = 0$; *otherwise if* $h(T-L) \le Lb + \pi$, *and the demand distribution is from the MLR family, then with* $y = S - M$, *we have*

7.1) *Given a y, there is a unique* S^* *such that* $\frac{\partial TC}{\partial S}|_{S=S^*} = 0$*; and,*

7.2) *Given a S, if* $Lb + \pi \le l - c \le 2(Lb+\pi) - h(T-L)$, *there is a unique* y^* *such that* $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$.

Proof.

Every period, demand is accepted till the inventory position reaches *M*, where $0 \le M \le S$. From Theorem 1 we know that optimal *M* can be positive when $l - c < (L+T)b$. Hence, we will focus on the case $l - c < (L+T)b$.

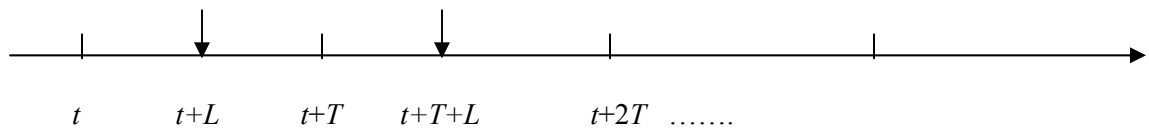

Figure 1: Timeline of events

We are interested in the cycle from period $t+L$ to period $t+L+T-1$. We will calculate the expected order quantity, expected lost sales, expected leftovers and expected backorders. For expositional convenience, we will consider two exclusive cases: 1) $S/2 < M \le S$, and 2) $M \le S/2$.

Case 1) $S/2 < M \le S$:

Similar to the case of $M \le 0$, the calculation of the expected order quantity and expected lost sales are rather straightforward,

$$\begin{cases} PC(T,L,M,S) = \int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi, \\ LS(T,L,M,S) = E(T\cdot D) - PC(M,S) = Tu - \left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right], \end{cases}$$

while the calculation of the expected leftovers and expected backorders require more effort because these have to be assessed and summed over each period of the cycle. To proceed, we start with period $t+L+i$ ($0 \le i \le T-1$). When an order is placed at the beginning of period *t*, it will arrive at the beginning of period $t+L$. If $0 \le i < T-L$, then the time duration from the beginning of period *t* to the end of period $t+L+i$ consists of $L+i+1$ periods; if $T-L \le i \le T-1$, the time duration consists of one regular cycle and a short cycle of $L+i+1-T$ periods. Hence, the total expected sales incurred in this time duration will be

$$\begin{cases} \underset{t\to t+L+i}{\mathbb{S}} = \int_0^{S-M} \xi f_{L+i+1}(\xi)d\xi + (S-M)\bar{F}_{L+i+1}(S-M) \quad \text{if} \quad 0 \le i < T-L, \\ \underset{t\to t+L+i}{\mathbb{S}} = \int_0^{S-M} \xi f_{L+i+1-T}(\xi)d\xi + (S-M)\bar{F}_{L+i+1-T}(S-M) \quad \text{if} \quad T-L \le i \le T-1, \end{cases}$$

where $\underset{t\to t+L+i}{\mathbb{S}}$ denotes the total sales from period t to period $t+L+i$. Since the sales in each cycle cannot exceed $S-M$, we must have

$$\begin{cases} HC_{t+L+i} = S - \left[\int_0^{S-M} \xi f_{L+i+1}(\xi) d\xi + (S-M)\bar{F}_{L+i+1}(S-M) \right] & \text{if} \quad 0 \le i < T-L, \\ HC_{t+L+i} = S - \left[\int_0^{S-M} \xi f_T(\xi) d\xi + (S-M)\bar{F}_T(S-M) \right] & \\ \qquad - \left[\int_0^{S-M} \xi f_{L+i+1-T}(\xi) d\xi + (S-M)\bar{F}_{L+i+1-T}(S-M) \right] & \text{if} \quad T-L \le i \le T-1. \end{cases}$$

(When $HC_{t+L+i} < 0$, it represents backorders.)

If $S/2 < M \le S$, then $2(S-M) \le S$, $HC_{t+L+i} \ge 0$ for $0 \le i \le T-1$, and no backorder will be incurred in this case. Therefore,

$$\begin{aligned} TC(T,L,M,S) &= c \cdot PC(T,L,M,S) + l \cdot LS(T,L,M,S) + h \cdot HC(T,L,M,S) \\ &= [c - l - (L+1)h] \left[\int_0^{S-M} \xi f_T(\xi) d\xi + (S-M)\bar{F}_T(S-M) \right] \\ &\quad + hTS + lTu - h \sum_{w=1}^{T-1} \left[\int_0^{S-M} \xi f_w(\xi) d\xi + (S-M)\bar{F}_w(S-M) \right], \end{aligned}$$

so that $\dfrac{\partial TC}{\partial M} = -[c - l - (L+1)h]\bar{F}_T(S-M) + h\sum_{w=1}^{T-1} \bar{F}_w(S-M) \ge 0$,

which implies that the total cost always decreases as M decreases, till $M \le S/2$. Hence, this case does not yield the optimal solution, which is consistent with the case of T=1. Now we can focus on the case of $M \le S/2$.

Case 2) $M \le S/2$:

In this case, since $2(S-M) > S$, HC_{t+L+i} could be negative for $T-L \le i \le T-1$, and hence, backorders will be incurred; while for $0 \le i < T-L$, $HC_{t+L+i} \ge 0$ still holds. Therefore,

$$HC_{t+L+i} = S - \left[\int_0^{S-M} \xi f_{L+i+1}(\xi) d\xi + (S-M)\bar{F}_{L+i+1}(S-M) \right] \quad \text{if} \quad 0 \le i < T-L,$$

and

$$\begin{aligned} HC_{t+L+i} &= \int_0^S (S-\xi) f_{L+i+1}(\xi) d\xi - \int_{S-M}^S f_T(\xi_1) \left[\int_0^{S-\xi_1} (S-\xi_1-\xi_2) f_{L+i+1-T}(\xi_2) d\xi_2 \right] d\xi_1 \\ &\quad - \int_{S-M}^S f_{L+i+1-T}(\xi_2) \left[\int_0^{S-\xi_2} (S-\xi_2-\xi_1) f_T(\xi_1) d\xi_1 \right] d\xi_2 \\ &\quad + \bar{F}_T(S-M) \int_0^M (M-\xi) f_{L+i+1-T}(\xi) d\xi + \bar{F}_{L+i+1-T}(S-M) \int_0^M (M-\xi) f_T(\xi) d\xi \quad \text{if} \quad T-L \le i \le T-1. \end{aligned}$$

Therefore,

$$BO_{t+L+i} = \left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\bar{F}_T(S-M)\right]$$
$$+\left[\int_0^{S-M} \xi f_{L+i+1-T}(\xi)d\xi + (S-M)\bar{F}_{L+i+1-T}(S-M)\right] - [S - HC_{t+L+i}] \quad \text{if } T-L \le i \le T-1.$$

Hence, $BO_1(T,L,M,S) = \sum_{i=T-L}^{T-1} BO_{t+L+i}$, and

$$\begin{aligned} BO_2(T,L,M,S) &= BO_{t+L+T-1} - BO_{t+L}^{-} \\ &= BO_{t+L+T-1} - 0 \\ &= \left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\bar{F}_T(S-M)\right] + \left[\int_0^{S-M} \xi f_L(\xi)d\xi + (S-M)\bar{F}_L(S-M)\right] \\ &\quad - S + HC_{t+L+T-1}, \end{aligned}$$

where BO_{t+L}^{-} represents the cumulative backorders till the beginning of period $t + L$; since the total sales from period t to period $t+L-1$ is less than the total sales in the cycle, it has to be less than S, thus, any backorder incurred before that time will be fulfilled by the order arriving at time $t + L$, hence no backorders will left unfulfilled by then.

Combining above equations we get

$$\begin{aligned} TC(T,L,M,S) &= c \cdot PC(T,L,M,S) + l \cdot LS(T,L,M,S) + h \cdot HC(T,L,M,S) + b \cdot BO_1(T,L,M,S) + \pi \cdot BO_2(T,L,M,S) \\ &= (c - l + bL + \pi)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\bar{F}_T(S-M)\right] \\ &\quad + h(T-L)S - bLS - \pi S + lTu - h\sum_{w=L+1}^{T}\left[\int_0^{S-M} \xi f_w(\xi)d\xi + (S-M)\bar{F}_w(S-M)\right] \\ &\quad + b\sum_{w=1}^{L}\left[\int_0^{S-M} \xi f_w(\xi)d\xi + (S-M)\bar{F}_w(S-M)\right] + \pi\left[\int_0^{S-M} \xi f_L(\xi)d\xi + (S-M)\bar{F}_L(S-M)\right] \\ &\quad + (h+b)\sum_{i=T-L}^{T-1}\left[\begin{array}{l} \int_0^S (S-\xi)f_{L+i+1}(\xi)d\xi - \int_{S-M}^S f_T(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f_{L+i+1-T}(\xi_2)d\xi_2\right]d\xi_1 \\ -\int_{S-M}^S f_{L+i+1-T}(\xi_2)\left[\int_0^{S-\xi_2}(S-\xi_2-\xi_1)f_T(\xi_1)d\xi_1\right]d\xi_2 \\ +\bar{F}_T(S-M)\int_0^M (M-\xi)f_{L+i+1-T}(\xi)d\xi + \bar{F}_{L+i+1-T}(S-M)\int_0^M (M-\xi)f_T(\xi)d\xi \end{array}\right] \\ &\quad + \pi\left[\begin{array}{l} \int_0^S (S-\xi)f_{L+T}(\xi)d\xi - \int_{S-M}^S f_T(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f_L(\xi_2)d\xi_2\right]d\xi_1 \\ -\int_{S-M}^S f_L(\xi_2)\left[\int_0^{S-\xi_2}(S-\xi_2-\xi_1)f_T(\xi_1)d\xi_1\right]d\xi_2 + \bar{F}_T(S-M)\int_0^M (M-\xi)f_L(\xi)d\xi \\ +\bar{F}_L(S-M)\int_0^M (M-\xi)f_T(\xi)d\xi \end{array}\right]. \end{aligned}$$

Letting $y = S - M$, we rewrite $TC(T, L, M, S)$ as

$$
\begin{aligned}
TC(T,L,y,S) = {} & (c - l + bL + \pi)\left[\int_0^y \xi f_T(\xi)d\xi + y\bar{F}_T(y)\right] \\
& + h(T-L)S - bLS - \pi S + lTu - h\sum_{w=L+1}^{T}\left[\int_0^y \xi f_w(\xi)d\xi + y\bar{F}_w(y)\right] \\
& + b\sum_{w=1}^{L}\left[\int_0^y \xi f_w(\xi)d\xi + y\bar{F}_w(y)\right] + \pi\left[\int_0^y \xi f_L(\xi)d\xi + y\bar{F}_L(y)\right] \\
& + (h+b)\sum_{w=1}^{L}\begin{bmatrix} \int_0^S (S-\xi)f_{T+w}(\xi)d\xi - \int_y^S f_T(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f_w(\xi_2)d\xi_2\right]d\xi_1 \\ -\int_y^S f_w(\xi_2)\left[\int_0^{S-\xi_2}(S-\xi_2-\xi_1)f_T(\xi_1)d\xi_1\right]d\xi_2 \\ +\bar{F}_T(y)\int_0^{S-y}(S-y-\xi)f_w(\xi)d\xi + \bar{F}_w(y)\int_0^{S-y}(S-y-\xi)f_T(\xi)d\xi \end{bmatrix} \\
& + \pi\begin{bmatrix} \int_0^S (S-\xi)f_{T+L}(\xi)d\xi - \int_y^S f_T(\xi_1)\left[\int_0^{S-\xi_1}(S-\xi_1-\xi_2)f_L(\xi_2)d\xi_2\right]d\xi_1 \\ -\int_y^S f_L(\xi_2)\left[\int_0^{S-\xi_2}(S-\xi_2-\xi_1)f_T(\xi_1)d\xi_1\right]d\xi_2 \\ +\bar{F}_T(y)\int_0^{S-y}(S-y-\xi)f_L(\xi)d\xi + \bar{F}_L(y)\int_0^{S-y}(S-y-\xi)f_T(\xi)d\xi \end{bmatrix}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{\partial TC}{\partial S} = {} & h(T-L) - bL - \pi + (h+b)\sum_{w=1}^{L}\begin{bmatrix} F_{T+w}(S) \\ -\int_y^S f_T(\xi)F_w(S-\xi)d\xi - \int_y^S f_w(\xi)F_T(S-\xi)d\xi \\ +\bar{F}_T(y)F_w(S-y) + \bar{F}_w(y)F_T(S-y) \end{bmatrix} \\
& + \pi\begin{bmatrix} F_{L+T}(S) - \int_y^S f_T(\xi)F_L(S-\xi)d\xi - \int_y^S f_L(\xi)F_T(S-\xi)d\xi \\ +\bar{F}_T(y)F_L(S-y) + \bar{F}_L(y)F_T(S-y) \end{bmatrix} \\
= {} & h(T-L) - bL - \pi + (h+b)\sum_{w=1}^{L} P\{\mathbb{S}_T + \mathbb{S}_w \le S;\ \mathbb{S}_T \le y;\ \mathbb{S}_w \le y\} \\
& + \pi P\{\mathbb{S}_T + \mathbb{S}_L \le S;\ \mathbb{S}_T \le y;\ \mathbb{S}_L \le y\},
\end{aligned}
$$

where $\mathbb{S}_T = \min\{D_T, y\}$, $\mathbb{S}_w = \min\{D_w, y\}$. Clearly, if $h(T-L) > bL + \pi$, then $\frac{\partial TC}{\partial S} > 0$, so that the optimal $S = 0$. When $h(T-L) \le bL + \pi$, notice that $P\{\mathbb{S}_T + \mathbb{S}_w \le S;\ \mathbb{S}_T \le y;\ \mathbb{S}_w \le y\}$ $(1 \le w \le L)$ is monotonically increasing in S for any fixed y, so that $\frac{\partial^2 TC}{\partial S^2} \ge 0$, that is, $TC(T,L,y,S)$ is convex in S for any given y. This proves Theorem 7.1.

Now, after differentiating $TC(T,L,y,S)$ in y we can write

$$\begin{aligned}
\frac{\partial TC}{\partial y} &= (c-l+bL+\pi)\bar{F}_T(y) - h\sum_{w=L+1}^{T}\bar{F}_w(y) + b\sum_{w=1}^{L}\bar{F}_w(y) + \pi\bar{F}_L(y) \\
&\quad -(h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] - \pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right] \\
&\le (c-l+bL+\pi)\bar{F}_T(y) - h(T-L)\bar{F}_{L+1}(y) + bL\bar{F}_L(y) + \pi\bar{F}_L(y) \\
&\quad -(h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] - \pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right] \\
&\le [c-l+2(bL+\pi)-h(T-L)]\bar{F}_T(y) - [(bL+\pi)-h(T-L)]\bar{F}_L(y) \\
&\quad -h(T-L)\bar{F}_L(y) + bL\bar{F}_L(y) + \pi\bar{F}_L(y) - (h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] \\
&\quad -\pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right] \\
&= [c-l+2(bL+\pi)-h(T-L)]\bar{F}_T(y) - (h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] \\
&\quad -\pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right].
\end{aligned}$$

If $l-c \ge 2(bL+\pi)-h(T-L)$, then $\frac{\partial TC}{\partial y} \le 0$, that is, the optimal $y^* \ge S$, and from the analysis in Section 2.2, we know that if optimal $M^* \le 0$, then $M^* = -\infty$.

If $l-c \le 2(bL+\pi)-h(T-L)$, we can rewrite $\frac{\partial TC}{\partial y}$ as

$$\begin{aligned}
\frac{\partial TC}{\partial y} &= (c-l+bL+\pi)\bar{F}_T(y) - h\sum_{w=L+1}^{T}\bar{F}_w(y) + b\sum_{w=1}^{L}\bar{F}_w(y) + \pi\bar{F}_L(y) \\
&\quad -(h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] \\
&\quad -\pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right] \\
&= -(l-c-bL-\pi) + (l-c-bL-\pi)F_T(y) - h(T-L) + h\sum_{w=L+1}^{T}F_w(y) \\
&\quad +bL - b\sum_{w=1}^{L}F_w(y) + \pi - \pi F_L(y) - (h+b)\sum_{w=1}^{L}\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right] \\
&\quad -\pi\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right] \\
&= -(l-c) - h(T-L) + 2(bL+\pi) \\
&\quad + F_L(y)\left\{\begin{array}{l}(l-c-bL-\pi)\dfrac{F_T(y)}{F_L(y)} + h\sum_{w=L+1}^{T}\dfrac{F_w(y)}{F_L(y)} - b\sum_{w=1}^{L}\dfrac{F_w(y)}{F_L(y)} - \pi \\ -(h+b)\sum_{w=1}^{L}\dfrac{\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right]}{F_L(y)} - \pi\dfrac{\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right]}{F_L(y)}\end{array}\right\}.
\end{aligned} \tag{29}$$

If the demand distribution is from the MLR family, then both $\frac{F_T(y)}{F_L(y)}$ and $\sum_{w=L+1}^{T}\frac{F_w(y)}{F_L(y)}$ are non-

decreasing in y, $\sum_{w=1}^{L}\frac{F_w(y)}{F_L(y)}$ is non-increasing in y, while $\sum_{w=1}^{L}\frac{\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right]}{F_L(y)}$

is always decreasing in y. Hence, if $l-c\geq bL$, then

$$\left\{\begin{array}{l}(l-c-bL-\pi)\frac{F_T(y)}{F_L(y)}+h\sum_{w=L+1}^{T}\frac{F_w(y)}{F_L(y)}-b\sum_{w=1}^{L}\frac{F_w(y)}{F_L(y)}-\pi\\ -(h+b)\sum_{w=1}^{L}\frac{\left[\bar{F}_T(y)F_w(S-y)+\bar{F}_w(y)F_T(S-y)\right]}{F_L(y)}\\ -\pi\frac{\left[\bar{F}_T(y)F_L(S-y)+\bar{F}_L(y)F_T(S-y)\right]}{F_L(y)}\end{array}\right\}\text{ is non-decreasing } y. \tag{30}$$

And, in this case, from (27) we get

$$\frac{\partial TC}{\partial y}|_{y=0}=-(l-c)-h(T-L)+2(bL+\pi)-(h+b)\sum_{w=1}^{L}\left[F_w(S)+F_T(S)\right]-\pi\left[F_L(S)+F_T(S)\right].$$

If S is quite small, so that $\frac{\partial TC}{\partial y}|_{y=0}\geq 0$, it implies that it is better to sell nothing although the target inventory position is set at S; to avoid this trivial case, we assume that the inventory position S is chosen high enough so that $\frac{\partial TC}{\partial y}|_{y=0}\leq 0$. Under this condition, by (30), there exists a $y^*\leq S$ such that $\frac{\partial TC}{\partial y}|_{y=y^*}=0$, and also $\frac{\partial TC}{\partial y}>0$ for $y>y^*$, so that y^* is unique. This proves Theorem 7.2.

Also notice that when T=L=1, (28) becomes $\frac{\partial TC}{\partial y}|_{y=0}=-(l-c)+2(b+\pi)-2(h+b+\pi)F(S)$, which is similar to the equation in Theorem 2; since $S\geq 2M$, $\frac{\partial TC}{\partial y}|_{y=0}\leq 0$ holds trivially in this special case.

Q.E.D.

A4. Detailed Analysis for Column 5 in Table 2

The problem is considerably more complex when $L>T>1$. Similarly to Theorem 7 we can prove:

Theorem 8

When $L>T>1$, and $\pi=0$, for MLR demand distribution, we have

8.1) *Given a y, there is a unique* S^* *such that* $\frac{\partial TC}{\partial S}|_{S=S^*}=0$*; and,*

8.2) *Given a S, if* $Lb \le l-c<(L+T)b$*, there is a unique* y^* *such that* $\frac{\partial TC}{\partial y}|_{y=y^*}=0$.

Proof.

Every period, demand is accepted till the inventory position reaches M, where $0 \le M \le S$. From Theorem 1 we know that optimal *M* can be positive only when $l-c<(L+T)b$. Hence, we will focus on this case.

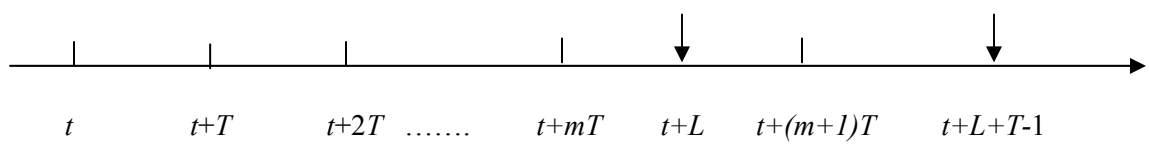

Figure 2: Timeline of events

We are interested in the cycle from period $t+L$ to period $t+L+T-1$. We will calculate the expected order quantity, expected lost sales, expected leftovers and expected backorders.

Similar to the case of $M \le 0$, the calculation of the first two costs are straightforward,

$$\begin{cases} PC(T,L,M,S)=\int_0^{S-M}\xi f_T(\xi)d\xi+(S-M)\int_{S-M}^{\infty}f_T(\xi)d\xi, \\ LS(T,L,M,S)=E(T\cdot D)-PC(T,L,M,S)=Tu-\left[\int_0^{S-M}\xi f_T(\xi)d\xi+(S-M)\int_{S-M}^{\infty}f_T(\xi)d\xi\right], \end{cases}$$

while the calculation of the expected leftovers and expected backorders require more effort. To proceed, we start with period $t+L+i$ ($0 \le i \le T-1$). When an order is placed at the beginning of period *t*, it will arrive at the beginning of period $t+L$. If $0 \le i < T-n$, then the time duration from the beginning of period *t* to the end of period $t+L+i$ consists of m regular cycles of *T* periods and a short cycle of $n+i+1$ periods, specifically, if $i=T-n-1$, then $n+i+1=T$, and the time duration consists of $m+1$ regular cycles. Finally, if $T-n \le i \le T-1$, the time duration consists of $m+1$ regular cycles and a short cycle of $n+i+1-T$ to $n+i+1$ periods. Hence, the total expected sales incurred in this time duration will be

$$
\begin{cases}
\underset{t\to t+L+i}{\mathbb{S}} = \sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} = m\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
+\left[\int_0^{S-M} \xi f_{n+i+1}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1}(\xi)d\xi\right] \quad \text{if } 0 \le i < T-n, \\
\underset{t\to t+L+i}{\mathbb{S}} = \sum_{q=1}^{m+1} \mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} = (m+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
+\left[\int_0^{S-M} \xi f_{n+i+1-T}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1-T}(\xi)d\xi\right] \quad \text{if } T-n \le i \le T-1,
\end{cases}
\tag{31}
$$

where $\underset{t\to t+L+i}{\mathbb{S}}$ denotes the total sales from period t to period $t+L+i$, $\mathbb{S}_T^q$ represents the total sales of the q-th cycle (of T periods) starting from period t, $\mathbb{S}_{n+i+1}^{m+1}$ refers to the total sales of the very last cycle of $n+i+1$ periods when $0 \le i < T-n$, and $\mathbb{S}_{n+i+1-T}^{m+2}$, the total sales of last cycle of $n+i+1-T$ periods when $T-n \le i \le T-1$. Since the sales in each cycle cannot exceed $S-M$, we must have

$$
\begin{cases}
0 \le \underset{t\to t+L+i}{\mathbb{S}} \le (m+1)(S-M) \quad \text{if } 0 \le i < T-n, \\
0 \le \underset{t\to t+L+i}{\mathbb{S}} \le (m+2)(S-M) \quad \text{if } T-n \le i \le T-1.
\end{cases}
$$

Consequently, if M is relatively small, such that $(m+2)(S-M) \le S$, we know that there will be leftovers at the end of each period in the focal cycle, and

$$
\begin{cases}
HC_{t+L+i} = S - m\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
-\left[\int_0^{S-M} \xi f_{n+i+1}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1}(\xi)d\xi\right] \quad \text{if } 0 \le i < T-n, \\
HC_{t+L+i} = S - (m+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
-\left[\int_0^{S-M} \xi f_{n+i+1-T}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1-T}(\xi)d\xi\right] \quad \text{if } T-n \le i \le T-1.
\end{cases}
\tag{32}
$$

In this case, the expected sales per cycle equals to the target inventory position minus the total expected sales that occur during the focal cycle. It is obvious that no backorders will be incurred at the end of each period within the focal cycle, and the total cost in this case is

$$
\begin{aligned}
TC(T,L,M,S) = {} & [c-l-(L+1)h]\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
& + hTS + lTu - h\sum_{w=1}^{T-1}\left[\int_0^{S-M} \xi f_w(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_w(\xi)d\xi\right].
\end{aligned}
$$

If we let $y = S - M$, then

$$TC(T,L,y,S) = [c - l - (L+1)h]\left[\int_0^y \xi f_T(\xi)d\xi + y\int_y^\infty f_T(\xi)d\xi\right]$$
$$+ hTS + lTu - h\sum_{w=1}^{T-1}\left[\int_0^y \xi f_w(\xi)d\xi + y\int_y^\infty f_w(\xi)d\xi\right],$$

and the first partial derivative of $TC(T,L,y,S)$ over y is

$\frac{\partial TC(T,L,M,S)}{\partial y} = [c - l - (L+1)h]\overline{F}(y) - h\sum_{w=1}^{T-1}\overline{F}(y) < 0$, which implies that the total cost always decreases in y, so that it is better to increase y or decrease M, till $(m+2)(S-M) > S$, which implies that this case does not yield the optimal solution. However, when M is relatively large, the calculation is no longer simple, and we need to consider the following two exclusive cases:

I) $(m+1)(S-M) \le S < (m+2)(S-M)$ (or $y \le \frac{S}{m+1}$); and

II) $j(S-M) \le S < (j+1)(S-M)$ (or $\frac{S}{j+1} \le y < \frac{S}{j}$) for $1 \le j \le m$, $m \ge 1$.

Basically, Case I corresponds to smaller value of y, and Case II corresponds to larger value of y given S. In the following, we will start with Case I, and analyze Case II later.

Case I)

Given $(m+1)(S-M) \le M < (m+2)(S-M)$, or $\frac{mS}{m+1} \le M < \frac{(m+1)S}{m+2}$, the retailer will always have positive leftovers (no backorders) in period $t+L+i$ for $0 \le i < T-n$, which follows from the first equation of (31); while in period $t+L+i$ for $T-n \le i \le T-1$, both leftovers and backorders could be incurred depending on the actual sales in each cycle from period $t+L$ to period $t+L+T-1$. Since the distribution of the sales incurred in the first m +1 cycles are identical (each of the cycle consists of T periods) except for the last irregular cycle which only has $n+i+1-T$ periods, it is natural to calculate the expected leftovers by assuming either the total sales of the very last irregular cycle equals $S-M$, or is less than $S-M$. In general, we can write

$$HC_{t+L+i} = S - m\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right]$$
$$-\left[\int_0^{S-M} \xi f_{n+i+1}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1}(\xi)d\xi\right] \quad \text{if } 0 \le i < T-n,$$

and

$$\begin{aligned} HC_{t+L+i} &= \sum_{z=0}^{m+1} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \mathbb{S}_{n+i+1-T}^{m+2}\right]^+ \\ &= \bar{F}_{n+i+1-T}(S-M)\sum_{z=0}^{m} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - (S-M)\right]^+ \\ &+ \int_0^{S-(m+1)(S-M)} f_{n+i+1-T}(\xi)\left[S-(m+1)\left\{\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right\} - \xi\right]d\xi \\ &+ \int_{S-(m+1)(S-M)}^{S-M} f_{n+i+1-T}(\xi)\sum_{z=0}^{m+1} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \xi\right]^+ d\xi \quad \text{if } T-n \le i \le T-1. \end{aligned} \tag{33}$$

For $0 \le i < T-n$, the explanation of HC_{t+L+i} is the same as that of (32); for $T-n \le i \le T-1$, we calculate HC_{t+L+i} in three conditional cases: Ia) the total demand that occurs in the last cycle (of $n+i+1-T$ periods) is greater than or equal to $S-M$, hence the total sales in this cycle will be $S-M$ exactly, which is given in the first term of (33) ; Ib) the total demand that occurs in the last cycle is less than or equal to $S-(m+1)(S-M)$, which corresponds to the second term of (33); Ic) the total demand that occurs in the last cycle is greater than $S-(m+1)(S-M)$, while less than $S-M$, which corresponds to the third term of (33). Here,

$$\begin{aligned} & E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - (S-M)\right]^+ \\ &= \binom{m+1}{z} E\begin{bmatrix} S-(z+1)(S-M) - \sum_{q=z+1}^{m+1} D_T^q; D_T^q \ge (S-M), 0 \le q \le z; \\ D_T^q < (S-M), z+1 \le q \le m+1; \sum_{q=z+1}^{m+1} D_T^q \le S-(z+1)(S-M) \end{bmatrix} \\ &= \binom{m+1}{z}\left[\bar{F}_T(S-M)\right]^z \cdot E\begin{bmatrix} S-(z+1)(S-M) - \sum_{q=z+1}^{m+1} D_T^q; D_T^q < (S-M), z+1 \le q \le m+1; \\ \sum_{q=z+1}^{m+1} D_T^q \le S-(z+1)(S-M) \end{bmatrix} \\ &= \binom{m+1}{z}\left[\bar{F}_T(S-M)\right]^z \cdot E\left[S-(z+1)(S-M) - \sum_{q=z+1}^{m+1} D_T^q; \sum_{q=z+1}^{m+1} D_T^q \le S-(z+1)(S-M)\right] \\ &- \binom{m+1}{z}\left[\bar{F}_T(S-M)\right]^z \cdot E\begin{bmatrix} S-(z+1)(S-M) - \sum_{q=z+1}^{m+1} D_T^q; \sum_{q=z+1}^{m+1} D_T^q \le S-(z+1)(S-M); \\ \text{at least one } D_T^q \ge (S-M); z+1 \le z \le m+1 \end{bmatrix}, \end{aligned} \tag{34}$$

which represents the conditional expected leftovers in period $t+L+i$ when, among the first $m+1$ cycles, z of them have demand that are greater than or equal to $S-M$, given that the total demand of the very last shorter cycle is also larger than $S-M$, where D_T^q denote the total demand of the q-th cycle. Notice that in the first term of (33), z takes value from 0 to m, since the total sales of the last shorter cycle are already $S-M$, if $z=m+1$, then the total sales form period t to period $t+L+i$ will be $(M+2)(S-M)$, and given $(m+1)(S-M)\le S<(m+2)(S-M)$, the retailer will incur backorders but no leftover; while if the total sales in the last shorter cycle are less than $S-M$, then z could equal $m+1$. Moreover,

$$
\begin{aligned}
&E\left[\begin{array}{c} S-(z+1)(S-M)-\sum_{q=z+1}^{m+1}D_T^q;\sum_{q=z+1}^{m+1}D_T^q\le S-(z+1)(S-M) \\ \text{at least one } D_T^q\ge(S-M);z+1\le z\le m+1\end{array}\right] \\
&=\binom{m-z+1}{1}\int_{S-M}^{S-(z+1)(S-M)}f_T(\xi_1)\left[\int_0^{S-(z+1)(S-M)-\xi_1}[S-(z+1)(S-M)-\xi_1-\xi_2]f_{(m-z)T}(\xi_2)d\xi_2\right]d\xi_1 \\
&-\binom{m-z+2}{2}\int_{S-M}^{S-(z+2)(S-M)}f_T(\xi_1)\left[\begin{array}{l}\int_{(S-M)}^{S-(z+1)(S-M)-\xi_1}f_T(\xi_2) \\ \left\{\int_0^{S-(z+1)(S-M)-\xi_1-\xi_2}[S-(z+1)(S-M)-\xi_1-\xi_2-\xi_3]f_{(m-z-1)T}(\xi_3)d\xi_3\right\}d\xi_2\end{array}\right]d\xi_1 \\
&+\ldots\ldots \\
&+(-1)^{m-z+1}\binom{m-z+1}{m-z}\int_{S-M}^{S-m(S-M)}f_T(\xi_1)\left[\begin{array}{l}\int_{(S-M)}^{S-(m-1)(S-M)-\xi_1}f_T(\xi_2)\ldots\int_{(S-M)}^{S-(z+1)(S-M)-\sum_{w=1}^{m-z-1}\xi_w}f_T(\xi_{m-z}) \\ \left\{\int_0^{S-(z+1)(S-M)-\sum_{w=1}^{m-z}\xi_w}[S-(z+1)(S-M)-\sum_{w=1}^{m-z+1}\xi_w]f_T(\xi_{m-z+1})d\xi_{m-z+1}\right\}\ldots\end{array}\right]d\xi_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&E_z\left[S-\sum_{q=1}^{m+1}\mathbb{S}_T^q-\xi\right]^+ \\
&=\binom{m+1}{z}E\left[\begin{array}{c} S-\xi-z(S-M)-\sum_{q=z+1}^{m+1}D_T^q;D_T^q\ge(S-M),0\le q\le z; \\ D_T^q<(S-M),z+1\le q\le m+1;\sum_{q=z+1}^{m+1}D_T^q\le S-\xi-z(S-M)\end{array}\right] \\
&=\binom{m+1}{z}\left[\overline{F}_T(S-M)\right]^z\cdot E\left[\begin{array}{c} S-\xi-z(S-M)-\sum_{q=z+1}^{m+1}D_T^q;D_T^q<(S-M),z+1\le q\le m+1; \\ \sum_{q=z+1}^{m+1}D_T^q\le S-\xi-z(S-M)\end{array}\right] \qquad (35) \\
&=\binom{m+1}{z}\left[\overline{F}_T(S-M)\right]^z\cdot E\left[S-\xi-z(S-M)-\sum_{q=z+1}^{m+1}D_T^q;\sum_{q=z+1}^{m+1}D_T^q\le S-\xi-z(S-M)\right] \\
&-\binom{m+1}{z}\left[\overline{F}_T(S-M)\right]^z\cdot E\left[\begin{array}{c} S-\xi-z(S-M)-\sum_{q=z+1}^{m+1}D_T^q;\sum_{q=z+1}^{m+1}D_T^q\le S-\xi-z(S-M) \\ \text{at least one } D_T^q\ge(S-M);z+1\le z\le m+1\end{array}\right],
\end{aligned}
$$

and for $S-(m+1)(S-M)\le \xi < S-M$,

$$
\begin{aligned}
&E\left[\begin{array}{c} S-\xi-z(S-M)-\sum_{q=z+1}^{m+1} D_T^q ; \sum_{q=z+1}^{m+1} D_T^q \le S-\xi-z(S-M) \\ \text{at least one } D_T^q \ge (S-M); z+1 \le z \le m+1 \end{array}\right] \\
&=\binom{m-z+1}{1}\int_{S-M}^{S-\xi-z(S-M)} f_T(\xi_1)\left[\int_0^{S-\xi-z(S-M)-\xi_1}[S-\xi-z(S-M)-\xi_1-\xi_2] f_{(m-z)T}(\xi_2)d\xi_2\right]d\xi_1 \\
&-\binom{m-z+2}{2}\int_{S-M}^{S-\xi-(z+1)(S-M)} f_T(\xi_1)\left[\begin{array}{l}\int_{(S-M)}^{S-\xi-z(S-M)-\xi_1} f_T(\xi_2) \\ \left\{\int_0^{S-\xi-z(S-M)-\xi_1-\xi_2}[S-\xi-z(S-M)-\xi_1-\xi_2-\xi_3] f_{(m-z-1)T}(\xi_3)d\xi_3\right\}d\xi_2\end{array}\right]d\xi_1 \\
&+\ldots\ldots \\
&+(-1)^{m-z+1}\binom{m-z+1}{m-z}\int_{S-M}^{S-\xi-(m-1)(S-M)} f_T(\xi_1)\left[\begin{array}{l}\int_{(S-M)}^{S-\xi-(m-2)(S-M)-\xi_1} f_T(\xi_2)\ldots\int_{(S-M)}^{S-\xi-z(S-M)-\sum_{w=1}^{m-z-1}\xi_w} f_T(\xi_{m-z}) \\ \left\{\int_0^{S-\xi-z(S-M)-\sum_{w=1}^{m-z}\xi_w}[S-\xi-z(S-M)-\sum_{w=1}^{m-z+1}\xi_w] f_T(\xi_{m-z+1})d\xi_{m-z+1}\right\}\ldots\end{array}\right]d\xi_1.
\end{aligned}
\tag{36}
$$

Equation (35) represents the conditional leftover in period $t+L+i$, when, among first $m+1$ cycles, z of them have demand that are greater than or equal to $S-M$, given that the total demand of the very last shorter cycle is less than $S-M$.

The expected backorders in period $t+L+i$ equal the cumulative sales from period t to period $t+L+i$ minus the part of sales that are satisfied from the physical inventory available (up to S), namely,

$$
\begin{aligned}
BO_{t+L+i} &= (m+1)\left[\int_0^{S-M}\xi f_T(\xi)d\xi+(S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
&+\left[\int_0^{S-M}\xi f_{n+i+1-T}(\xi)d\xi+(S-M)\int_{S-M}^{\infty} f_{n+i+1-T}(\xi)d\xi\right]-(S-HC_{t+L+i}) \quad \text{if } T-n \le i \le T-1,
\end{aligned}
$$

and, $BO(T,L,M,S)=\sum_{i=T-n}^{T-1} BO_{t+L+i}$.

If we let $y=S-M$, then

$$
\begin{aligned}
TC(T,L,y,S) &= [c-l-hm(T-n)+bn(m+1)]\left[\int_0^y \xi f_T(\xi)d\xi+y\int_y^{\infty} f_T(\xi)d\xi\right] \\
&+S[h(T-n)-bn]+(h+b)\sum_{i=T-n}^{T-1} HC_{t+L+i} \\
&-h\sum_{w=n+1}^{T}\left[\int_0^y \xi f_w(\xi)d\xi+y\int_y^{\infty} f_w(\xi)d\xi\right]+b\sum_{w=1}^{n}\left[\int_0^y \xi f_w(\xi)d\xi+y\int_y^{\infty} f_w(\xi)d\xi\right].
\end{aligned}
$$

To solve for the optimal S^* and y^* (or M^*), we should takes the first partial derivative of TC over S and y. For this purpose, we need to calculate the partial derivatives of HC first. For $T-n \le i \le T-1$,

$$\begin{cases} \dfrac{\partial HC_{t+L+i}}{\partial S} = P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}, \\ \dfrac{\partial HC_{t+L+i}}{\partial y} = -\bar{F}_{n+i+1-T}(y)P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\} \\ \qquad -(m+1)\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}. \end{cases}$$

Since $P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}$ is monotonically increasing in S for fixed y, so is $\dfrac{\partial HC_{t+L+i}}{\partial S}$; and both $-\bar{F}_{n+i+1-T}(y)P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\}$ and $-(m+1)\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}$ are monotonically increasing in y for fixed S, so is $\dfrac{\partial HC_{t+L+i}}{\partial y}$. Now,

$$\begin{aligned} \frac{\partial TC(T,L,y,S)}{\partial S} &= [h(T-n)-bn] + (h+b)\sum_{i=T-n}^{T-1}\frac{\partial HC_{t+L+i}}{\partial S} \\ &= [h(T-n)-bn] \\ &\quad +(h+b)\sum_{i=T-n}^{T-n-1}P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}. \end{aligned}$$

Hence, $\dfrac{\partial TC(T,L,y,S)}{\partial S}$ is monotonically increasing in S for any $y < S$, so that $\dfrac{\partial^2 TC(T,L,y,S)}{\partial S^2} \ge 0$, that is, $TC(T,L,y,S)$ is convex in S for fixed y. Moreover, if $h(T-n) > bn$, then $\dfrac{\partial TC(T,L,y,S)}{\partial S} > 0$, which implies that there exists no optimal solution in Case I, and we only need to consider Case II.

$$\begin{aligned} \frac{\partial TC(T,L,y,S)}{\partial y} &= [c-l-hm(T-n)+bn(m+1)]\bar{F}_T(y) - h\sum_{w=n+1}^{T}\bar{F}_w(y) + b\sum_{w=1}^{n}\bar{F}_w(y) \\ &\quad +(h+b)\sum_{i=T-n}^{T-1}\begin{Bmatrix} -\bar{F}_{n+i+1-T}(y)P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\} \\ -(m+1)\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\} \end{Bmatrix}. \end{aligned}$$

Similar to the analysis for the case of $L \le T$, if $l-c \ge (m+2)bn - h(m+1)(T-n)$, then $\dfrac{\partial TC(T,L,y,S)}{\partial y} \le 0$, it is better-off to increase y for given S, then $y \le \dfrac{S}{m+1}$ no longer holds, and we

must have $y > \dfrac{S}{m+1}$ in an optimal solution, still we only have to consider Case II.

If $l - c < (m+2)bn - h(m+1)(T-n)$, we can rewrite above equation as

$$
\begin{aligned}
&\frac{\partial TC(T,L,y,S)}{\partial y} \\
&= [c - l - hm(T-n) + bn(m+1)] + [l - c + hm(T-n) - bn(m+1)]F_T(y) \\
&\quad -h(T-n) + h\sum_{w=n+1}^{T} F_w(y) + bn - b\sum_{w=1}^{n} F_w(y) \\
&\quad +(h+b)\sum_{i=T-n}^{T-1} \left\{ \begin{array}{l} -\bar{F}_{n+i+1-T}(y) P\left\{\sum_{q=1}^{m+1} \mathbb{S}_T^q \le S - y; 0 \le \mathbb{S}_T^q \le y\right\} \\ -(m+1)\cdot \bar{F}_T(y) P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S - y - \xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\} \end{array} \right\} \\
&= -(l-c) - h(m+1)(T-n) + (m+2)bn \\
&\quad +F_n(y)\left[\begin{array}{l} [l - c + hm(T-n) - bn(m+1)]\dfrac{F_T(y)}{F_n(y)} + h\sum_{w=n+1}^{T} \dfrac{F_w(y)}{F_n(y)} - b\sum_{w=1}^{n} \dfrac{F_w(y)}{F_n(y)} \\ \dfrac{(h+b)}{F_n(y)} \sum_{w=1}^{n} \left\{ \begin{array}{l} -\bar{F}_w(y) P\left\{\sum_{q=1}^{m+1} \mathbb{S}_T^q \le S - y; 0 \le \mathbb{S}_T^q \le y\right\} \\ -(m+1)\cdot \bar{F}_T(y) P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_w^{m+2} \le S - y - \xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_w^{m+2} \le y\right\} \end{array} \right\} \end{array} \right].
\end{aligned}
$$

Similar to the analysis for the case of $L \le T$, if $l - c \ge (m+1)bn - hm(T-n)$, then the last term in the bracket of the above equation is non-decreasing in y for a given S, and in this case,

$$
\begin{aligned}
\frac{\partial TC}{\partial y}\Big|_{y=0} &= -(l-c) - h(m+1)(T-n) + (m+2)bn - (h+b)n(m+2) \\
&= -(l-c) - h[(n+1)T + n] \\
&< 0,
\end{aligned}
$$

that is, if $(m+1)bn - hm(T-n) \le l - c < (m+2)bn - h(m+1)(T-n)$, there is a unique y^* such that $\dfrac{\partial TC}{\partial y}\Big|_{y=y^*} = 0$. As shown above, if $l - c \ge (m+2)(bn+\pi) - h(m+1)(T-n)$ or $h(T-n) > bn$, we only need to analyze Case II.

Case II)

In this case, since $j(S-M) \le S < (j+1)(S-M)$ (or $\dfrac{(j-1)S}{j} \le M < \dfrac{jS}{j+1}$), $0 \le j \le m$, the retailer could incur backorder or leftovers in each period $t + L + i$, $0 \le i \le T-1$, given that $(m+1)(S-M) > S$.

Consequently,

$$
\begin{cases}
HC_{t+L+i} = \sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \mathbb{S}_{n+i+1}^{m+1}\right]^+ \\
\qquad = \bar{F}_{n+i+1}(S-M)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - (S-M)\right]^+ \\
\qquad + \int_0^{S-j(S-M)} f_{n+i+1}(\xi)\sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \xi\right]^+ d\xi \\
\qquad + \int_{S-j(S-M)}^{S-M} f_{n+i+1}(\xi)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \xi\right]^+ d\xi \quad \text{if} \quad 0 \le i \le T-n, \\
HC_{t+L+i} = \sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \mathbb{S}_{n+i+1}^{m+2}\right]^+ \\
\qquad = \bar{F}_{n+i+1-T}(S-M)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - (S-M)\right]^+ \\
\qquad + \int_0^{S-j(S-M)} f_{n+i+1-T}(\xi)\sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \xi\right]^+ d\xi \\
\qquad + \int_{S-j(S-M)}^{S-M} f_{n+i+1-T}(\xi)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \xi\right]^+ d\xi \quad \text{if} \quad T-n \le i \le T-1,
\end{cases}
\tag{37}
$$

where $E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - (S-M)\right]^+$ (or $E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - (S-M)\right]^+$) and $E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \xi\right]^+$ (or $E_z\left[S - \sum_{q=1}^{m+1} \mathbb{S}_T^q - \xi\right]^+$) have similar expressions as those of (34) and (35). Again,

$$
\begin{cases}
BO_{t+L+i} = m\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
\qquad + \left[\int_0^{S-M} \xi f_{n+i+1}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1}(\xi)d\xi\right] - (S - HC_{t+L+i}) \quad \text{if } 0 \le i < T-1, \\
BO_{t+L+i} = (m+1)\left[\int_0^{S-M} \xi f_T(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_T(\xi)d\xi\right] \\
\qquad + \left[\int_0^{S-M} \xi f_{n+i+1-T}(\xi)d\xi + (S-M)\int_{S-M}^{\infty} f_{n+i+1-T}(\xi)d\xi\right] - (S - HC_{t+L+i}) \quad \text{if } T-n \le i < T-1,
\end{cases}
$$

and, $BO(T,L,M,S) = \sum_{i=0}^{T-1} BO_{t+L+i}$.

In this case, similar to (37),

$$
\begin{aligned}
HC_{t+L-1} &= \sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \mathbb{S}_n^{m+1}\right]^+ \\
&= \bar{F}_n(S-M)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - (S-M)\right]^+ + \int_0^{S-j(S-M)} f_n(\xi)\sum_{z=0}^{j} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \xi\right]^+ d\xi \\
&\quad + \int_{S-j(S-M)}^{S-M} f_n(\xi)\sum_{z=0}^{j-1} E_z\left[S - \sum_{q=1}^{m} \mathbb{S}_T^q - \xi\right]^+ d\xi.
\end{aligned}
$$

Finally, letting $y = S - M$, we get

$$TC(T,L,y,S)=[c-l+Lb]\left[\int_0^y \xi f_T(\xi)d\xi + y\int_y^\infty f_T(\xi)d\xi\right]+(lTu-bTS)$$
$$+(h+b)\sum_{i=0}^{T-1} HC_{t+L+i} + b\sum_{w=1}^{T}\left[\int_0^y \xi f_w(\xi)d\xi + y\int_y^\infty f_w(\xi)d\xi\right].$$

To solve for the optimal S^* and y^* (or M^*), we take the partial derivative of *TC* over *S* and *y*. For this purpose, we need to calculate the partial derivatives of *HC* first. For $0 \le i < T-n$,

$$\begin{cases} \dfrac{\partial HC_{t+L+i}}{\partial S} = P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\}, \\ \dfrac{\partial HC_{t+L+i}}{\partial y} = -\bar{F}_{n+i+1}(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\} \\ \qquad -m\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m-1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\}. \end{cases}$$

Since $P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\}$ is monotonically increasing in *S* for fixed *y*, so is $\dfrac{\partial HC_{t+L+i}}{\partial S}$; and both $-\bar{F}_{n+i+1}(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\}$ and $-m\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m-1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\}$ are monotonically increasing in *y* for fixed *S*, so is $\dfrac{\partial HC_{t+L+i}}{\partial y}$.

For $T-n \le i \le T-1$,

$$\begin{cases} \dfrac{\partial HC_{t+L+i}}{\partial S} = P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}, \\ \dfrac{\partial HC_{t+L+i}}{\partial y} = -\bar{F}_{n+i+1-T}(y)P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-y; 0 \le \mathbb{S}_T^q \le y\right\} \\ \qquad -(m+1)\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S-y-\xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\}. \end{cases}$$

Also, in this case, $\dfrac{\partial HC_{t+L+i}}{\partial S}$ is monotonically increasing in *S* for fixed *y*, and $\dfrac{\partial HC_{t+L+i}}{\partial y}$ is monotonically increasing in *y* for fixed *S*. Finally,

$$\begin{cases} \dfrac{\partial HC_{t+L-1}}{\partial S} = P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_n^{m+1} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_n^{m+1} \le y\right\}, \\ \dfrac{\partial HC_{t+L-1}}{\partial y} = -\bar{F}_n(y) P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q \le S - y; 0 \le \mathbb{S}_T^q \le y\right\} \\ \qquad -m \cdot \bar{F}_T(y) P\left\{\sum_{q=1}^{m-1} \mathbb{S}_T^q + \mathbb{S}_n^{m+1} \le S - y - \xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_n^{m+1} \le y\right\}. \end{cases}$$

Therefore,

$$\begin{aligned} \frac{\partial TC(T,L,y,S)}{\partial S} &= (h+b)\sum_{i=0}^{T-1} \frac{\partial HC_{t+L+i}}{\partial S} - bT \\ &= (h+b)\sum_{i=0}^{T-n-1} P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\} \\ &\quad +(h+b)\sum_{i=T-n}^{T-1} P\left\{\sum_{q=1}^{m+1} \mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\} - bT, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial TC(T,L,y,S)}{\partial y} &= [c - l + bL]\bar{F}_T(y) + b\sum_{w=1}^{T} \bar{F}_w(y) + (h+b)\sum_{i=0}^{T-1} \frac{\partial HC_{t+L+i}}{\partial y} \\ &= [c - l + b(L+1) + \pi]\bar{F}_T(y) + b\sum_{w=1}^{T-1} \bar{F}_w(y) \\ &\quad +(h+b)\sum_{i=0}^{T-n-1} \begin{Bmatrix} -\bar{F}_{n+i+1}(y) P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q \le S - y; 0 \le \mathbb{S}_T^q \le y\right\} \\ -m \cdot \bar{F}_T(y) P\left\{\sum_{q=1}^{m-1} \mathbb{S}_T^q + \mathbb{S}_{n+i+1}^{m+1} \le S - y - \xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1}^{m+1} \le y\right\} \end{Bmatrix} \\ &\quad +(h+b)\sum_{i=T-n}^{T-1} \begin{Bmatrix} -\bar{F}_{n+i+1-T}(y) P\left\{\sum_{q=1}^{m+1} \mathbb{S}_T^q \le S - y; 0 \le \mathbb{S}_T^q \le y\right\} \\ -(m+1) \cdot \bar{F}_T(y) P\left\{\sum_{q=1}^{m} \mathbb{S}_T^q + \mathbb{S}_{n+i+1-T}^{m+2} \le S - y - \xi; 0 \le \mathbb{S}_T^q \le y, 0 \le \mathbb{S}_{n+i+1-T}^{m+2} \le y\right\} \end{Bmatrix}, \end{aligned}$$

Hence, $\dfrac{\partial TC(T,L,y,S)}{\partial S}$ is monotonically increasing in S for any fixed $y < S$, so that $\dfrac{\partial^2 TC(T,L,y,S)}{\partial S^2} \ge 0$,

that is, $TC(T,L,y,S)$ is convex in S for fixed y. Moreover, $\dfrac{\partial TC(T,L,y,S)}{\partial S}\Big|_{S=0} = -bT < 0$, and

$\dfrac{\partial TC(T,L,y,S)}{\partial S}\Big|_{S=+\infty} = hT > 0$. Hence, for both case I and case II, there is a unique S^* that is optimal

for any fixed y. This proves Theorem 8.1. Also,

$$
\begin{aligned}
\frac{\partial TC(T,L,y,S)}{\partial y} &= [c-l+bL]\bar{F}_T(y) + b\sum_{w=1}^{T}\bar{F}_w(y) + (h+b)\sum_{i=0}^{T-1}\frac{\partial HC_{t+L+i}}{\partial y} \\
&= [c-l+bL]\bar{F}_T(y) + b\sum_{w=1}^{T}\bar{F}_w(y) \\
&+(h+b)\sum_{i=0}^{T-n-1}\left\{\begin{array}{l} -\bar{F}_{n+i+1}(y)\left[\begin{array}{l} P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q \le S-y; 0\le \mathbb{S}_T^q \le y\right\} \\ +m\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m-1}\mathbb{S}_T^q \le S-2y; 0\le \mathbb{S}_T^q \le y\right\}\end{array}\right] \\ -\int_0^y f_{n+i+1}(\xi)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q \le S-\xi; 0\le \mathbb{S}_T^q \le y\right\}d\xi \end{array}\right\} \\
&+(h+b)\sum_{i=T-n}^{T-1}\left\{\begin{array}{l} -\bar{F}_{n+i+1-T}(y)\left[\begin{array}{l} P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-y; 0\le \mathbb{S}_T^q \le y\right\} \\ +(m+1)\cdot\bar{F}_T(y)P\left\{\sum_{q=1}^{m}\mathbb{S}_T^q \le S-2y; 0\le \mathbb{S}_T^q \le y\right\}\end{array}\right] \\ -\int_0^y f_{n+i+1-T}(\xi)P\left\{\sum_{q=1}^{m+1}\mathbb{S}_T^q \le S-\xi; 0\le \mathbb{S}_T^q \le y\right\}d\xi \end{array}\right\}.
\end{aligned}
$$

If $l-c \ge (L+T)b$, then $\frac{\partial TC(T,L,y,S)}{\partial y} \le 0$, so that optimal $M^* \le 0$, that is, $M^* = -\infty$. Now, similar to Case I we can show that for $Lb \le l-c \le (L+T)b$, there is a unique y^* such that $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$. Notice that in Case I, for $(m+1)bn - hm(T-n) \le l-c < (m+2)bn - h(m+1)(T-n)$, we have shown that there is a unique y^* such that $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$, since $T > n$, $Lb = (mT+n) > (m+1)nb - hm(T-n)$;

We can conclude that for $Lb \le l-c \le (L+T)b$, $Lb \le l-c \le (L+T)b$, there is a unique y^* such that $\frac{\partial TC}{\partial y}|_{y=y^*} = 0$. This completes the proof of Theorem 8. Q.E.D.