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BOUNDS ON THE TRAINED VECTOR QUANTIZER DISTORTION MEASURED USING TRAINING DATA

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Abstract

Quantization can effectively reduce the huge amount of data with possibly small error (called quantization error). In designing a quantizer using a portion of the data as a training data, the training algorithm tries to find a codebook that minimizes the quantization error measured in the training data. It is known that, under several conditions, the minimized quantization error approaches the optimal error for the underlying distribution of the training data as the training data size increases. In this report, an upper bound for the minimized quantization error from the training data is derived as a function of the ratio of the training data size to the codebook size. This bound enables us to observe the convergence behavior of the trained quantizers as the training data size increases.

1. Introduction

Let F be a distribution function, k be a fixed integer, and $||\cdot||$ denote the L_2 norm on \mathbb{R}^k , k-dimensional Euclidean space. The optimal quantizer design problem for F is to minimize the quantization error defined by

$$D(C,F) := \int \min_{\mathbf{y} \in C} ||\mathbf{x} - \mathbf{y}||^2 dF(\mathbf{x})$$

over all possible choices of the set C in \mathcal{C}_n , where \mathcal{C}_n is the class of sets that contains n (or fewer) points in \mathbb{R}^k [1],[15].

Suppose that $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is a sequence of independent, identically distributed random variables taking values in \mathbb{R}^k with distribution F. Let ω denote a sample point in the underlying sample space R. For a set. $C \ (\in \mathcal{C}_n)$ and $\mathbf{X}_1^{\omega}, \ldots, \mathbf{X}_m^{\omega}$, we define the empirical error as

$$D(C, F_m^{\omega}) := \int \min_{\mathbf{y} \in C} |\mathbf{x} - \mathbf{y}||^2 dF_m^{\omega}(\mathbf{x}),$$

where F_m^{ω} is the empirical distribution function constructed by placing mass m^{-1} at each of the m points $\mathbf{X}_1^{\omega}, \ldots, \mathbf{X}_m^{\omega}$ [8]. In order to minimize D(C, F) with an unknown F, an inductive method that minimizes $D(C, F_m^{\omega})$ is usually considered based on the *empirical risk minimization* (ERM) principle [16]. Note that there exists a convergent subsequence of \mathbf{C}_m such that $(D(C_m, F))$ and $(D(C_m, F_m^{\omega}))$ converge to $\inf_{C \in \mathcal{C}_n} D(C, F)$ for almost every ω , where C_m satisfies $D(C_m, F_m^{\omega}) = \inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$ [1],[15]. If $\beta = 1$ (i.e., the set size n is equal to m), then for all sample points $\omega \in \mathbb{R}$, $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega}) = D(C^{\omega}, F_m^{\omega}) = 0$, since we can choose the set as $C^{\omega} = \{\mathbf{X}_1^{\omega}, \ldots, \mathbf{X}_n^{\omega}\}$ for each F_m^{ω} . In the special case when n = 1 and $m \geq 1$, we obtain the well known relation $E\{\inf_{C \in \mathcal{C}_n} D(C, F_m)\} = (m-1)/m \cdot \operatorname{Var}(\mathbf{X}_1)$, which implies that $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$ is a biased estimate of the variance.

In this report, upper bounds for the expectation of the empirically optimal error $E\{\inf_{C \in \mathcal{C}_n} D(C, F,)\}$ are introduced to observe the performance of C, with respect to m. Through the upper bounds, it will be shown that $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$ is a biased estimate of $\inf_{C \in \mathcal{C}_n} D(C, F)$. In evaluating the designed quantizer this fact often misleads the quantizer designers, since $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$ is usually less than $\inf_{C \in \mathcal{C}_n} D(C, F)$. A compensation method for this biased error based on the bounds will also be suggested.

2. Bound for the Empirically Optimal Error

In this chapter, upper bounds for $E\{\inf_{C \in C_n} D(C, F_m)\}$ are derived as a function of any n points $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and the corresponding partition $\{S_i\}_{i=1}^n$ in \mathbb{R}^k . The corresponding point of the region S_i is \mathbf{y}_i , and the partition size is n. Note that the partition is a finite, disjoint class $\{S_i\}_{i=1}^n$ whose union is \mathbb{R}^k (or includes the support of the density function of F). The partition that minimizes the error $\sum_{i=1}^n \int_{S_i} ||\mathbf{x} - \mathbf{y}_i||^2 dF(\mathbf{x})$ is called the Voronoi partition for $\mathbf{y}_1, \ldots, \mathbf{y}_n$. If we consider a vector \mathbf{W}_i^{ω} that is defined as

$$\mathbf{W}_{i}^{\omega} := \begin{cases} (00\cdots0)^{T}, & \text{if } m_{i}^{\omega} = 0\\ \frac{1}{m_{i}^{\omega}} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \mathbf{X}_{\ell}^{\omega}, & \text{otherwise,} \end{cases}$$
(2.1)

where $m_i^{\omega} := \sum_{\ell=1}^m I_{S_i}(\mathbf{X}_{\ell}^{\omega})$, for each S_i , then it is clear that

$$\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega}) \le \sum_{i=1}^n \int_{S_i} \|\mathbf{x} - \mathbf{W}_i^{\omega}\|^2 dF_m^{\omega}(\mathbf{x})$$
(2.2)

Let P_i denote the probability that \mathbf{X}_1 belongs to S_i , i.e., $P_i := P\{\mathbf{X}_1 \in S_i\} = \int_{S_i} dF$. Let an index set be $\mathcal{I} = \{i : P_i \neq 0, i = 1, ..., n\}$. Then (2.2) can be rewritten as

$$\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega}) \le \frac{1}{m} \sum_{i \in \mathcal{I}} \sum_{\ell=1}^m I_{S_i}(\mathbf{X}_{\ell}^{\omega}) ||\mathbf{X}_{\ell}^{\omega} - \mathbf{W}_i^{\omega}||^2$$
(2.3)

Proposition 1 Suppose that $E\{||\mathbf{X}_1||^2\} < \infty$. Then for any finite points $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and the corresponding partition $\{S_i\}$ in \mathbb{R}^k ,

$$E\{\inf_{C\in\mathcal{C}_n} D(C, F_m)\} \le \sum_{i\in\mathcal{I}} \int_{S_i} ||\mathbf{x} - \mathbf{y}_i||^2 dF(\mathbf{x}) \left[1 - \frac{1 - (1 - P_i)^m}{mP_i}\right]$$

Proof of Proposition 1: Taking expectations in (2.3) we have

$$E\{\inf_{C\in\mathcal{C}_n} D(C, F_m)\} \le \frac{1}{m} \int \sum_{\ell\in\mathcal{I}} \sum_{\ell=1}^m I_{S_\ell}(\mathbf{x}_\ell) ||\mathbf{x}_\ell - \mathbf{w}_i||^2 dF^m,$$
(2.4)

where dF^m denotes $dF(\mathbf{x}_1) \dots dF(\mathbf{x}_m)$, \int denotes a km-fold integral, and \mathbf{w}_i is the function of \mathbf{x}_ℓ 's as given in (2.1).

In order to simplify the expansion, let $|\mathcal{I}| = n$. The proof in the case of $|\mathcal{I}| < n$ is also similar (details appear in [13]). Let B, $\subset \mathbb{R}^{km}$ be the m-fold Cartesian product of S_i 's, i.e., B, $= S_{i_{\nu,1}} \times \cdots \times S_{i_{\nu,m}}$, where $\nu = (i_{\nu,1}-1) + (i_{\nu,2}-1)n + \ldots + (i_{\nu,m}-1)n^{m-1}$ and $i_{\nu,j} \in \{1, 2, \ldots, n\}$. Then the right hand side of (2.4) can be rewritten as

$$\frac{1}{m}\sum_{\nu=0}^{n^{m}-1}\int_{B_{\nu}}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{x}_{\ell})\|\mathbf{x}_{\ell}-\mathbf{w}_{i}\|^{2}dF^{m}=:\frac{1}{m}\sum_{\nu=0}^{n^{m}-1}\sum_{i=1}^{n}\phi_{i,\nu},$$

where $\phi_{i,\nu} := \sum_{\ell=1}^{m} \int_{B_{\nu}} I_{S_i}(\mathbf{x}_{\ell}) \|\mathbf{x}_{\ell} - \mathbf{w}_i\|^2 dF^m$. If $m_{i,\nu}$ (:= the number of *i* in $\{i, j\}_{j=1}^{m}$ j is not zero, then the $m_{i,\nu}$ summands in $\phi_{i,\nu}$ remain non-zero, since $I_{S_i}(\mathbf{x}_{\ell})$ becomes *l* if $\mathbf{x}_{\ell} \in S_i$. By rearranging the parameters \mathbf{x}_{ℓ} , $\phi_{i,\nu}$ can be expanded as

$$\phi_{i,\nu} = \begin{cases} \prod_{\substack{j=1\\j\neq i}\\j\neq i}^{n} P_j^{m_{j,\nu}} \int_{S_i^{m_{i,\nu}}} \sum_{\ell=1}^{m_{i,\nu}} \|\mathbf{x}_\ell - \mathbf{w}_i\|^2 dF(\mathbf{x}_1) & dF(\mathbf{x}_{m_{i,\nu}}), \quad \text{if } m_{i,\nu} \neq 0\\ 0, & \text{otherwise.} \end{cases}$$

$$(2.5)$$

Here, $\mathbf{w}_i = m_{i,\nu}^{-1} \sum_{\ell=1}^{m_{i,\nu}} \mathbf{x}_\ell$ and $S_i^{m_{i,\nu}} := \times_{j=1}^{m_{i,\nu}} S_i$.

In the case when $mi_{,,} \neq 0$, we add

$$\prod_{\substack{j=1\\j\neq i}}^{n} P_{j}^{m_{j,\nu}} \int_{S_{i}^{m_{i,\nu}}} \sum_{\ell=1}^{m_{i,\nu}} \|\mathbf{w}_{i} - \mathbf{y}_{i}\|^{2} dF(\mathbf{x}_{1}) \cdots dF(\mathbf{x}_{m_{i,\nu}})$$
(2.6)

to both tie sides of (2.5), which results in

$$\phi_{i,\nu} + \prod_{\substack{j=1\\j\neq i}}^{n} P_{j}^{m_{j,\nu}} \int_{S_{i}^{m_{i,\nu}}} \sum_{\ell=1}^{m_{i,\nu}} ||\mathbf{w}_{i} - \mathbf{y}_{i}||^{2} dF(\mathbf{x}_{1}) \cdots dF(\mathbf{x}_{m_{i,\nu}})$$

$$= \prod_{j=1}^{n} P_{j}^{m_{j,\nu}} \cdot \frac{m_{i,\nu}}{P_{i}} \cdot \int_{S_{i}} ||\mathbf{x} - \mathbf{y}_{i}||^{2} dF(\mathbf{x}), \qquad (2.7)$$

since $\sum_{\ell=1}^{m_{i,\nu}} (\|\mathbf{x}_{\ell} - \mathbf{w}_{i}\|^{2} + \|\mathbf{w}_{i} - \mathbf{y}_{i}\|^{2}) = \sum_{\ell=1}^{m_{i,\nu}} \|\mathbf{x}_{\ell} - \mathbf{y}_{i}\|^{2}$

Moreover, (2.6) can be expanded as

$$\prod_{\substack{j=1\\j\neq i}}^{n} P_{j}^{m_{j,\nu}} \cdot \frac{1}{m_{i,\nu}} \cdot \int_{S_{i}^{m_{i,\nu}}} \left\| \sum_{\ell=1}^{m_{i,\nu}} (\mathbf{x}_{\ell} - \mathbf{y}_{i}) \right\|^{2} dF(\mathbf{x}_{1}) \cdots dF(\mathbf{x}_{m_{i,\nu}}) \\
\geq \prod_{j=1}^{n} P_{j}^{m_{j,\nu}} \cdot \frac{1}{P_{i}} \int_{S_{i}} \|\mathbf{x} - \mathbf{y}_{i}\|^{2} dF(\mathbf{x}),$$
(2.8)

where equality holds when $\mathbf{y}_i = \int_{S_i} \mathbf{x} dF(\mathbf{x}) / P_i$. From (2.7) and (2.8), we have

$$\phi_{i,\nu} \leq \prod_{j=1}^{n} P_{j}^{m_{j,\nu}} \cdot \frac{1}{P_{i}} (m_{i,\nu} - 1) \int_{S_{i}} ||\mathbf{x} - \mathbf{y}_{i}||^{2} dF(\mathbf{x}).$$

Note that since there are $m!/\prod_{j=1}^{n} m_{j,\nu}$ different ν 's in $\{0, \ldots, n^m - 1\}$ that yield the same values of $m_{1,\nu}, \ldots, m_{n,\nu}$, it follows that the joint distribution of the random variables m_1, \ldots, m_n $(m_i^{\omega} := \sum_{\ell=1}^{m} I_{S_1}(\mathbf{X}_{\ell}^{\omega}))$ is the multinomial distribution. Hence, $\sum_{\nu} \prod_j P_j^{m_{j,\nu}} m_{i,\nu} = mP_i$ and $\sum_{\nu} \prod_j P_j^{m_{j,\nu}} \gamma_{i,\nu} = 1 - (1 - P_i)^m$, where $\gamma_{i,\nu} := 1$ if $m_{i,\nu} \neq 0$, and 0 otherwise. Thus,

$$\sum_{\nu=0}^{n^m-1} \phi_{i,\nu} \le \left[m - \frac{1 - (1 - P_i)^m}{P_i} \right] \int_{S_i} ||\mathbf{x}_{\ell} - \mathbf{y}_i||^2 dF(\mathbf{x}).$$
(2.9)

The proposition follows from (2.4) and (2.9).

The next proposition will introduce a simple bound on $E\{\inf_{C \in C_n} D(C, F_m)\}$ as a function of m.

Proposition 2 Suppose that $E\{||\mathbf{X}_1||^2\} < \infty$. Then for any set $B \in \mathcal{C}_n$,

$$E\{\inf_{C \in \mathcal{C}_n} D(C, F_m)\} \le D(B, F) \left(1 - \frac{1}{m}\right).$$
(2.10)

Proof of Proposition 2: Consider a codebook $B = \{y_1, \dots, y_n\} \in C_n$ and the corresponding Voronoi partition $\{S_i\}_{i=1}^n$. Then D(B, F), the quantizer distortion for B, can be rewritten as

$$D(B,F) = \sum_{i=1}^{n} \int_{S_{i}} ||\mathbf{x} - \mathbf{y}_{i}||^{2} dF(\mathbf{x})$$
(2.11)

for a distribution function F. In a similar manner in (2.1), if we consider a vector \mathbf{W}_{i}^{ω} that is defined as

$$\mathbf{W}_{i}^{\omega} := \begin{cases} (00\cdots0)^{T}, & \text{if } m_{i}^{\omega} = 0\\ \frac{1}{m_{i}^{\omega}} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \mathbf{X}_{\ell}^{\omega}, & \text{otherwise,} \end{cases}$$
(2.12)

where $m_i^{\omega} := \sum_{\ell=1}^m I_{S_i}(\mathbf{X}_{\ell}^{\omega})$, for each S_i , then it is clear that

$$\inf_{C \in \mathcal{C}_{n}} D(C, F_{m}^{\omega}) = \inf_{C \in \mathcal{C}_{n}} \frac{1}{m} \sum_{\ell=1}^{m} \min_{\mathbf{y} \in C} \|\mathbf{X}_{\ell}^{\omega} - \mathbf{y}\|^{2} \\
\leq \frac{1}{m} \sum_{i=1}^{n} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \|\mathbf{X}_{\ell}^{\omega} - \mathbf{W}_{i}^{\omega}\|^{2},$$
(2.13)

where $F_m^{\boldsymbol{u}}(\mathbf{x}) := m^{-1} \sum_{\ell=1}^m I_{(-\infty,\mathbf{x}]}(\mathbf{X}_\ell^{\boldsymbol{\omega}})$ and $-\infty := (-\infty, \cdots, -\infty)^T$. The last term in (2.13) can be expanded as follows.

$$\frac{1}{m}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})||\mathbf{X}_{\ell}^{\omega}-\mathbf{W}_{i}^{\omega}||^{2}$$

$$= \frac{1}{m}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})||(\mathbf{X}_{\ell}^{\omega}-\mathbf{y}_{i})+(\mathbf{y}_{i}-\mathbf{W}_{i}^{\omega})||^{2}$$

$$= \frac{1}{m}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})||\mathbf{X}_{\ell}^{\omega}-\mathbf{y}_{i}||^{2}$$

$$+\frac{1}{m}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})\left[2(\mathbf{X}_{\ell}^{\omega})^{T}\mathbf{y}_{i}-2(\mathbf{X}_{\ell}^{\omega})^{T}\mathbf{W}_{i}^{\omega}-\mathbf{y}_{i}^{T}\mathbf{y}_{i}+\mathbf{W}_{i}^{T}\mathbf{W}_{i}\right]$$

$$= \frac{1}{m}\sum_{i=1}^{n}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})||\mathbf{X}_{\ell}^{\omega}-\mathbf{y}_{i}||^{2}-\frac{1}{m}\sum_{i=1}^{n}m_{i}^{\omega}||\mathbf{W}_{i}^{\omega}-\mathbf{y}_{i}||^{2}.$$
(2.14)

In (2.14), if $m_i^{\omega} \neq 0$, then

$$m_{i}^{\omega} \|\mathbf{W}_{i}^{\omega} - \mathbf{y}_{i}\|^{2} = \frac{1}{m_{i}^{\omega}} \left\| \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})(\mathbf{X}_{\ell}^{\omega} - \mathbf{y}_{i}) \right\|^{2}$$
$$= \frac{1}{m_{i}^{\omega}} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \|\mathbf{X}_{\ell}^{\omega} - \mathbf{y}_{i}\|^{2} + \frac{1}{m_{i}^{\omega}} \alpha_{i}^{\omega}, \qquad (2.15)$$

where

$$\alpha_i^* := 2I_{S_i}(\mathbf{X}_1^{\omega})I_{S_i}(\mathbf{X}_2^{\omega})(\mathbf{X}_1^{\omega} - \mathbf{y}_i)(\mathbf{X}_2^{\omega} - \mathbf{y}_i) + \cdots + 2I_{S_i}(\mathbf{X}_{m-1}^{\omega})I_{S_i}(\mathbf{X}_m^{\omega})(\mathbf{X}_{m-1}^{\omega} - \mathbf{y}_i)(\mathbf{X}_m^{\omega} - \mathbf{y}_i),$$

and $m_i^{\omega} ||\mathbf{W}_i^{\omega} - \mathbf{y}_i||^2 = 0$ otherwise. Hence, by changing m_i^{ω} to *m* in the last term of (2.15) and from (2.14), we obtain a relation

$$\frac{1}{m} \sum_{i=1}^{n} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \|\mathbf{X}_{\ell}^{\omega} - \mathbf{y}_{i}\|^{2} - \frac{1}{m} \sum_{i=1}^{n} m_{i}^{\omega} \|\mathbf{W}_{i}^{\omega} - \mathbf{y}_{i}\|^{2} \\
\leq \frac{1}{m} \sum_{i=1}^{n} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \|\mathbf{X}_{\ell}^{\omega} - \mathbf{y}_{i}\|^{2} \\
- \left[\frac{1}{m^{2}} \sum_{i=1}^{n} \sum_{\ell=1}^{m} I_{S_{i}}(\mathbf{X}_{\ell}^{\omega}) \|\mathbf{X}_{\ell}^{\omega} - \mathbf{y}_{i}\|^{2} + \frac{1}{m^{2}} \sum_{i=1}^{n} \alpha_{i}^{\omega}\right].$$
(2.16)

Note that $E\{\sum_{i=1}^{n} \alpha_i\} \ge 0$, where the equality holds when $\mathbf{y}_i = \int_{S_i} \mathbf{x} dF(\mathbf{x}) / \int_{S_i} dF$, and

$$E\left\{\frac{1}{m}\sum_{\ell=1}^{m}I_{S_{i}}(\mathbf{X}_{\ell}^{\omega})\|\mathbf{X}_{\ell}^{\omega}-\mathbf{y}_{i}\|^{2}\right\}=D(B,F).$$
(2.17)

Therefore, from (2.13), (2.14), and (2.16),

$$E\left\{\inf_{C\in\mathcal{C}_n} D(C, F_m)\right\} \le D(B, F)\left(1 - \frac{1}{m}\right).$$
(2.18)

From Proposition 2, $E\{\inf_{C \in \mathcal{C}_n} D(C, F_m)\} \leq \inf_{C \in \mathcal{C}_n} D(C, F)(1-1/m)$. Therefore, for a finite *m*, we obtain

$$E\{\inf_{C\in\mathcal{C}_n} D(C, F_m)\} < \inf_{C\in\mathcal{C}_n} D(C, F),$$
(2.19)

which implies that $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$ is a biased estimate of $\inf_{C \in \mathcal{C}_n} D(C, F)$.

3. Asymptotic Bound based on Root Lattices

In this chapter, asymptotic bounds for $E\{\inf_{C \in C_n} D(C, F_m)\}$ are suggested for an absolutely continuous F. In order to derive an asymptotic bound, we consider root lattices [11]. Let the points of a R-dimensional lattice \mathcal{L}_k ($\subset \mathbb{R}^k$) be denoted by $\mathbf{y}_j, j \in \mathbb{Z}$. The closure of the ith Voronoi region of the lattice \mathcal{L}_k is the convex polytope H_i defined as

$$\mathrm{Hi} := \{ \mathbf{x} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{y}_i\|^2 \le \|\mathbf{x} - \mathbf{y}_j\|^2, \text{ for all } \mathbf{y}_j \in \mathcal{L}_k \}, \text{ for } i \in \mathbb{Z}.$$

Here we lat $\mathbf{y}_1 = (0 \cdots 0)^T$, thus H_1 includes the origin \mathbf{y}_1 . Then $G(\mathcal{L}_k)$, the normalized second moment of H_i is defined as

$$G(\mathcal{L}_k) := \frac{1}{k} \frac{\int_{H_i} \|\mathbf{x} - \mathbf{y}_i\|^2 d\mathbf{x}}{V(H_i)^{1/\rho}},$$

where p ::= k/(k+2) and $V(H_i) := \int_{H_1} dx$ is the volume of H_i [9]. Note that all H_i , $i \in \mathbb{Z}$, have the same shape. Thus, the normalized second moments and the volumes of H_i are all the same. Conway and Sloane have calculated the second moments of various lattices that yield values close to $\inf_{\mathcal{L}_k} G(\mathcal{L}_k)$ for various dimensions, where the infimum is taken over all R-dimensional lattices [5],[6, Table I]. For example, the hexagonal lattice, which is equivalent to the lattice A_2 , is the optimal lattice in 2-dimensions. In the 3-dimensional case, the D_3^{\perp} lattice (or equivalently the lattice A_3^{\perp}) is a body-centerel4 cubic lattice and optimal in 3-dimensions [3]. Furthermore, Conway and Sloane proposed a lower bound for $\inf_{\mathcal{L}_k} G(\mathcal{L}_k)$ [7]. To summarize, it is known that $\inf_{\mathcal{L}_2} G(\mathcal{L}_2) = G(A_2) = 0.08018 \cdots$ and $\inf_{\mathcal{L}_3} G(\mathcal{L}_3) = G(D_3^{\perp}) =$ 0.07854... For the definitions of the lattices see [5].

Let d be the diameter of the convex polytope H_1 defined as $d := \sup\{||\mathbf{x} - \mathbf{y}|| : \mathbf{x}, \mathbf{y} \in H_1\}$. Note that $d < \infty$. Consider a sequence of cubes

$$U_t := \times_{i=1}^k \left[-\frac{1}{2} - dt, \frac{1}{2} + dt \right], \text{ for } t = 0, 1, 2, \dots$$

and the Lebesque measure μ . $\mu(H_1) =: u$, where u is a non-zero constant and $\mu(U_t) = (1+2dt)^k$. Suppose that the lattice \mathcal{L}_k satisfies $H_1 \supset U_0$.

Now we consider the number of H_i 's that have non-zero measure intersections with the cube U_k . Let $N_t :=$ number of H_i , $i \in \mathbb{Z}$, such that $\mu(H_i \cap U_t) \neq 0$ and $N_t^I :=$ number of H_i , $i \in \mathbb{Z}$, such that $\mu(H_i \cap U_t) = u$. Note that $N_t \geq N_t^I$. Then it is clear that, for $t \in \mathbb{N}$,

$$\mu(U_{t-1}) \le N_t^I u \le \mu(U_t),$$

$$\mu(U_t) \le N_t u \le \mu(U_{t+1}).$$

Let t_n be the largest integer such that $n - N_{t_n} \ge 0$, then $N_{t_n} \le n < N_{t_n+1}$ and $t_n \to \infty$ as $n \to \infty$. Thus, we have a relation

$$\frac{N_{t_n}^I}{n} > \frac{N_{t_n}^I}{N_{t_n+1}} \ge \frac{\mu(U_{t_n-1})}{\mu(U_{t_n+2})} \\ = \left[\frac{1+2d(t_n-1)}{1+2d(t_n+2)}\right]^k \to 1, \text{ as } n \to \infty.$$

Since $N_{t_n}^I/n \le N_{t_n}/n \le I$, for $n \in \mathbb{N}$, we obtain

$$\lim_{n \to \infty} \frac{N_t}{n} = \lim_{n \to \infty} \frac{N_t^I}{n} = I.$$
(3.1)

The following lemma shows that the quantization error has an asymptotic upper bound that is a function of $G(\mathcal{L}_k)$ based on (3.1).

Lemma 1 Suppose that \mathbf{X}_1 is uniformly distributed over the cube $U := \times_{i=1}^{k} [-1/2, 1/2]$. Then there exists a sequence of sets C_n such that

$$\limsup_{n \to a} n^{2/k} D(C_n, F) \le k G(\mathcal{L}_k)$$

Proof of Lemma 1: Consider a sequence of equivalent lattices $\mathcal{L}_{k,t}$, t = 0, 1, 2, ..., where the k generator vectors are obtained by multiplying $(1 + 2dt)^{-1}$ to the generator vectors of \mathcal{L}_k . Note that $G(\mathcal{L}_k) = G(\mathcal{L}_{k,t})$, for $t \in \mathbb{N}$. Let $H_{i,t}$ denote the convex polytope for $\mathbf{y}_{i,t} \in \mathcal{L}_{k,t}$ in the same manner as H_i is a polytope for $\mathbf{y}_i \in \mathcal{L}_k$. Then, following the earlier methodology we can derive the result (3.1) in the same way. Note that, since

$$\mu(U) \le n\mu(H_{1,t}) = \frac{n}{N_t^I} \left[N_t^I \mu(H_{1,t}) \right] \le \frac{n}{N_t^I} \mu(U), \text{ for } t = 2, 3, \dots$$

and from (3.1),

$$\lim_{n \to \infty} n\mu(H_{1,t_n}) = \mu(U). \tag{3.2}$$

There is a sequence of C_n such that

$$n^{2/k}D(C_n, F) \leq n^{2/k} \sum_{i=1}^n \int_{H_{i,t_n} \cap U} \frac{\|\mathbf{x} - \mathbf{y}_i\|^2}{\mu(U)} d\mathbf{x}$$

$$\leq kG(\mathcal{L}_k) n^{2/k} N_{t_n} \left[\mu(H_{1,t_n}) \right]^{1/\rho} / \mu(U)$$

$$\leq kG(\mathcal{L}_k) \left[n\mu(H_{1,t_n}) \right]^{1/\rho} / \mu(U),$$

from $N_{t_n} \leq n$. Thus from (3.2),

$$\limsup_{n \to \infty} n^{2/k} D(C_n, F) \le k G(\mathcal{L}_k) \cdot \left[\mu(U) \right]^{2/k}$$

holds. This completes the proof.

In a s milar manner as in Lemma 1, we can construct a sequence of C_n satisfying

$$\limsup_{n \to ca} n^{2/k} D(C_n, F) \le k G(\mathcal{L}_k) ||f||_{\rho},$$
(3.3)

under certain conditions on f, the density function of X_1 , where

$$||f||_
ho := \left[\int f^
ho({f x}) d{f x}
ight]^{1/
ho}$$

The conditions on f will be introduced in the following theorems and corollary. It is clear that $\limsup_{n\to\infty} n^{2/k} \inf_{C\in\mathcal{C}_n} D(C, F) \leq J_k ||f||_{\rho}$ from (3.3), where $J_k := \inf_{\mathcal{L}_k} kG(\mathcal{L}_k)$. However, if Gersho's conjecture [9] is correct, then

$$\lim_{n \to \infty} n^{2/k} \inf_{C \in \mathcal{C}_n} D(C, \mathbf{F}) = J_k ||\mathbf{f}||_{\rho}$$

which tells us the asyinptotically optimal quantizer performance [4],[5].

We now express the upper bound in Proposition 1 as a function of $J_k ||f||_{\rho}$ asymptotically in n.

Theorem 1 Suppose that \mathbf{X}_1 has a density function f with compact support and f is bounded on \mathbb{R}^k . Then

$$\limsup_{n \to \infty} n^{2/k} E\{\inf_{C \in \mathcal{C}_n} D(C, F_{m_n})\} \le J_k ||f||_{\rho} \left(1 - \frac{1 - e^{-\beta}}{\beta}\right), \tag{3.4}$$

where β is a constant $(1 \leq \beta < \infty)$ and m_n is a sequence of n such that $m_n/n - \beta$ is $n - \infty$.

Proof of Theorem 1: Let B be a cube that contains the support off and is defined by $B := \times_{i=1}^{k} [a, b]$ (C \mathbb{R}^{k}), where a and b are finite. Consider a partition of B into 2^{kq} cubes B_{ℓ} such that $\mu(B_{\ell}) = [(b - a)/2^{q}]^{k} =: v, \ell = 1, \ldots, 2^{kq}$. Define a simple function f_{q} as

$$f_q(\mathbf{x}) := \sum_{\ell=1}^{2^{kq}} p_\ell I_{B_\ell}(\mathbf{x}),$$

where $p_{\ell} := \sup_{\mathbf{x} \in B_{\ell}} f(\mathbf{x})$. Then since the sequence $(f_q(\mathbf{x}))_q$ is monotonic and $\lim_{q \to \infty} f_q(\mathbf{x}) = f(\mathbf{x})$ a.e., it follows that $\int f_q(\mathbf{x}) d\mathbf{x} \to 1$ [10, p.112]. From [10, p.96], the expectation of the empirically optimal error $E\{\inf_{C \in C_n} D(C, F_m)\}$ satisfies the relation:

$$E\{\inf_{C\in\mathcal{C}_{n}}D(C,F_{m})\}$$

$$=\frac{1}{m}\int\cdots\int\inf_{C\in\mathcal{C}_{n}}\sum_{i=1}^{m}\min_{\mathbf{y}\in C}\|\mathbf{x}_{i}-\mathbf{y}\|^{2}f(\mathbf{x}_{1})\cdots f(\mathbf{x}_{m})d\mathbf{x}_{1}\cdots d\mathbf{x}_{m}$$

$$\leq \frac{1}{m}\int\cdots\int\operatorname{Ged}_{n}\sum_{i=1}^{m}\min_{\mathbf{y}\in C}\|\mathbf{x}_{i}-\mathbf{y}\|^{2}f_{q}(\mathbf{x}_{1})\cdots f_{q}(\mathbf{x}_{m})d\mathbf{x}_{1}\cdots d\mathbf{x}_{m}.$$
(3.5)

Now consider partitions for each cube Be. Assign

$$n_{\ell} := \left\lfloor n \cdot \frac{(p_{\ell})^{\rho}}{\sum_{j=1}^{2^{k_q}} (p_j)^{\rho}} \right\rfloor$$
(3.6)

points to each *Be*. Suppose that $n_{\ell} \ge 1$ if $p_{\ell} \ne 0$ for $n \ge no$, where n_0 is a positive integer. In (3.6), $\lfloor c \rfloor$, $e \in \mathbb{R}$ is the largest integer less than or equal to e and n_{ℓ}/n $(p_{\ell})^{\rho}/\sum_{j=1}^{2^{kq}} (p_j)^{\rho}$, as $n \to \infty$. For each cube *Be*, make polytopes $H_{i,\ell}$ and the corresponding points $\mathbf{y}_{i,\ell}$ in the same way as in Lemma 1 for the $p_{\ell} \ne 0$ cases. Note that $H_{i,\ell}$ and $\mathbf{y}_{i,\ell}$ are functions of n_{ℓ} (or $t_{n_{\ell}}$). However, for simplicity we omit n_{ℓ} in this notation. The lattice for B_{ℓ} is a coset of the scaled lattice, which is equivalent to the lattice \mathcal{L}_k . Let $N_{\ell} :=$ number of $H_{i,\ell}$, $i \in \mathbb{Z}$, such that $\mu(H_{i,\ell} \cap B_{\ell}) \ne 0$ and suppose that $\mu(H_{i,\ell} \cap B_{\ell}) \ne 0$, for $i = 1, 2, \ldots, N_{\ell}$.

Using $H_{i,\ell}$, $i \in \mathbb{Z}$ and $\theta = 1, \ldots, 2^{kq}$, make a partition $\{S_{i,\ell}\}$ of B, where $S_{i,\ell} \subset (H_{i,\ell} \cap B_\ell)$ and $\mu(S_{i,\ell}) = \mu(H_{i,\ell} \cap B_\ell)$, for $i = 1, 2, \ldots, N_\ell$, and $S_{i,\ell} = \emptyset$ for $i \in \{1, 2, \ldots, n_\ell\} - \{1, 2, \ldots, N_\ell\}$. Then, from Proposition 1 and (3.5), we obtain a relation

$$E\{\inf_{C\in\mathcal{C}_n} D(C, F_{m_n})\} \leq \sum_{\ell=1}^{2^{kq}} \sum_{i\in\mathcal{I}_\ell} \alpha_{i,\ell} \int_{S_{i,\ell}} \|\mathbf{x} - \mathbf{y}_{i,\ell}\|^2 f_q(\mathbf{x}) d\mathbf{x}$$
(3.7)

where $\mathcal{I}_{\ell} := \{\mathbf{I} : P_{j,\ell} \neq 0, j = 1, \dots, n_{\ell}\}, P_{i,\ell} := \int_{S_{i,\ell}} f_q(\mathbf{x}) d\mathbf{x} / \int f_q(\mathbf{x}) d\mathbf{x}$, for all θ , and

$$\alpha_{i,\ell} := \left[1 - \frac{1 - (1 - P_{i,\ell})^{m_n}}{m_n P_{i,\ell}} \right], \text{ for } i \in \mathcal{I}_{\ell}.$$

Note that $\alpha_{i,\ell} < l$ and $\mathcal{I}_{\ell} = \emptyset$ if $p_{\ell} = 0$. Since $n_{\ell}\mu(H_{i,\ell}) \to v$ as $n_{\ell} \to \infty$, for $i = 1, ..., n_{\ell}$, from (3.2), we have

$$\begin{split} \limsup_{n \to \infty} n^{1/\rho} \int_{S_{i,\ell}} \|\mathbf{x} - \mathbf{y}_{i,\ell}\|^2 f_q(\mathbf{x}) d\mathbf{x} \\ &\leq \lim_{n \to \infty} n^{1/\rho} p_\ell \int_{H_{i,\ell}} \|\mathbf{x} - \mathbf{y}_{i,\ell}\|^2 d\mathbf{x} \\ &= n \lim_{n \to \infty} \left(\frac{n}{n_\ell}\right)^{1/\rho} \left[n_\ell \mu(H_{i,\ell})\right]^{1/\rho} p_\ell \left[\frac{\int_{H_{i,\ell}} \|\mathbf{x} - \mathbf{y}_{i,\ell}\|^2 d\mathbf{x}}{\left[\mu(H_{i,\ell})\right]^{1/\rho}}\right] \\ &= k G(\mathcal{L}_k) \left[\sum_{j=1}^{2^{kq}} (p_j)^\rho v\right]^{1/\rho} = k G(\mathcal{L}_k) \|f_q\|_{\rho}, \text{ for all } i \text{ and } 0. \end{split}$$

Thus, multiplying the right hand side of (3.7)by $n^{2/k}$ and taking $n \to \infty$ yields

$$\limsup_{n \to \infty} n^{2/k} \sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \alpha_{i,\ell} \int_{S_{i,\ell}} ||\mathbf{x} - \mathbf{y}_{i,\ell}||^2 f_q(\mathbf{x}) d\mathbf{x}$$

$$\leq k G(\mathcal{L}_k) ||f_q||_{\rho} \limsup_{n \to \infty} \frac{1}{n} \left(\sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \alpha_{i,\ell} \right).$$
(3.8)

Now, the partial second derivatives $\partial^2 \left[\left[1 - (1 - x)^m \right] / mx \right] / ax^2 > 0$, for x > 0 and $m = 3, 4, \dots$ Hence, under the constraint $\sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} P_{i,\ell} = 1$, the term inside parenthesis in (3.81 is bounded as

$$\sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \alpha_{i,\ell} \le \sum_{\ell=1}^{2^{kq}} n_{\ell} - \frac{N^2}{m_n} \left[1 - \left(1 - \frac{1}{N} \right)^{m_n} \right], \tag{3.9}$$

for $n \ge n_0$ if $P_{i,\ell} = 1/N$, where $N := \sum_{\ell=1}^{2^{kq}} N_\ell$ $(N_\ell = |\mathcal{I}_\ell|)$ [12]. Note that $n^{-1} \sum_{\ell=1}^{2^{kq}} n_\ell \to 1$ and $N/n \to 1$, as $n \to \infty$, from (3.1). Thus, since $m_n/n \to \beta$ as n increases, dividing (3.9) by n and taking $n \to \infty$ yields

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \alpha_{i,\ell} \leq \operatorname{nlime} \frac{1}{n} \left[\sum_{\ell=1}^{2^{kq}} n_{\ell} - \frac{N^2}{m_n} \left[1 - \left(1 - \frac{1}{N} \right)^{m_n} \right] \right]$$
$$= 1 - \lim_{n \to \infty} \left(\frac{n}{m_n} \right) \left(\frac{N}{n} \right)^2 \left[1 - \left(1 - \frac{1}{N} \right)^{n(\overline{m}_n/n)} \right]$$
$$= 1 - \frac{1}{\beta} \left[1 - \lim_{r \to \infty} \left(1 - \frac{1}{r} \right)^{r \cdot \beta} \right]$$
$$= 1 - \frac{1}{\beta} (1 - e^{-\beta}).$$

Since $[f(\mathbf{x})]^{\rho}$ is also bounded, $[f_q(\mathbf{x})]^{\rho} \to [f(\mathbf{x})]^{\rho}$, a.e., it follows that $||f_q||_{\rho} \to ||f||_{\rho}$. This completes the proof.

Corollary 1 Suppose that $E\{||\mathbf{X}_1||^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$ and f is bounded on \mathbb{R}^k . Then the inequality (3.4) holds.

Proof of Corollary 1: Consider an increasing sequence of cubes $B^1 \subset B^2 \subset \cdots \subset B^s$. For a constant $0 < \eta < 1$, assign (1 - q)n points to the cube B^s and ηn points to \overline{B}^s , which is the complement of B^s . Then from [4, Theorem 2], there exists a sequence such that

$$\lim_{\delta \neq 0 \text{ of } n \to \infty} \lim_{n \to \infty} n^{2/k} \int_{\bar{B}^s} \min_{\mathbf{y} \in C_{\eta n}} \|\mathbf{x} - \mathbf{y}\|^2 dF(\mathbf{x}) = 0.$$

Hence, by letting $\eta \to 0$, we obtain the corollary. Note that from the assumption and Holder's inequality, $||f||_{\rho} < \infty$ for any $\epsilon > 0$ and $k \in \mathbb{N}$.

From Theorem 1 (or Corollary 1), the asymptotic bound has two parts; $J_k ||f||_{\rho}$ and $[1-(1-e^{-\beta})/\beta]$. The second part is concerned with the ratio β and, regardless off and the dimension k, β is an important factor in the convergence of $\inf_{C \in C_n} D(C, F_m^{\omega})$. The curve of $[1 - (1 - e^{-\beta})/\beta]$ with respect to β is depicted in Fig. 3.1. As we can see in this figure, as β decreases, the bias between $\inf_{C \in C_n} D(C, F_m^{\omega})$ and $\inf_{C \in C_n} D(C, F)$ increases monotonically. We can also see that $1/\beta \approx (1 - e^{-\beta})/\beta$ for $\beta = 4, 5, \ldots$.



Figure 3.1: The curve of $\left[1 - (1 - e^{-\beta})/\beta\right]$ with respect to the ratio $\beta = m/n$.

Note that the term $(1 - e^{-\beta})/\beta$ in Theorem 1 is independent of the distribution type and the vector dimension. In the next theorem, a better bound will he introduced, where the term is expressed as a function of f as well as the ratio β .

Theorem 2 Under the assumptions in Theorem 1,

$$\limsup_{n \to 03} n^{2/k} E\{\inf_{C \in \mathcal{C}_n} D(C, F_{m_n})\} \le J_k ||f||_{\rho} \left[1 - \frac{\zeta - \xi(\beta)}{\beta}\right],$$

where

$$\begin{aligned} \zeta &:= \| f \|_{2\rho-1}^{2\rho-1} / \| f \|_{\rho}^{2\rho}, \\ \xi(\beta) &:= \int f^{2\rho-1}(\mathbf{x}) e^{-f^{1-\rho}(\mathbf{x}) \| f \|_{\rho}^{\rho\beta}} d\mathbf{x} / \| f \|_{\rho}^{2\rho}, \end{aligned}$$
(3.10)

for $\left[\zeta - \xi(\beta) \right] / \beta < 1$.

Proof of Theorem 2: From (3.7) and (3.8), we have

$$\limsup_{n \to \infty} n^{2/k} E\{\inf_{C \in \mathcal{C}} D(C, F_{m_n})\} \le k G(\mathcal{L}_k) ||f||_{\rho} \left(\limsup_{\substack{9 = 03 \ n \to \infty}} \sup_{n \to \infty} -\sum_{\ell=1}^{2^{k_q}} \sum_{i \in \mathcal{I}_\ell} \alpha_{i,\ell}\right).$$
(3.11)

Let $N_{\ell}^{I} := \{j : H_{j,\ell} \subset B_{\ell}, j \in \mathbb{Z}\}$, then the term inside parenthesis in (3.11) can be expanded as

$$\limsup_{q \to 03} \limsup_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \alpha_{i,\ell}$$

=
$$\limsup_{q \to \infty} \limsup_{n \to \infty} \left[\frac{1}{n} \sum_{\ell=1}^{2^{kq}} N_{\ell} - \frac{1}{m_n n} \sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \frac{1}{P_{i,\ell}} + \frac{1}{m_n n} \sum_{\ell=1}^{2^{kq}} \sum_{i \in \mathcal{I}_{\ell}} \frac{(1 - P_{i,\ell})^{m_n}}{P_{i,\ell}} \right]$$

$$\leq 1 - \liminf_{q \to \infty} \liminf_{n \to \infty} \frac{n}{m_n} \sum_{\ell=1}^{2^{kq}} \left(\frac{N_\ell}{n}\right)^2 \frac{1}{p_\ell v} + \limsup_{q \to \infty} \limsup_{n \to \infty} \frac{n}{m_n} \sum_{\ell=1}^{2^{kq}} \left(\frac{N_\ell}{n}\right)^2 \frac{(1 - p_\ell v/N_\ell^I)^{m_n}}{p_\ell v}$$
(3.12)

since $n^{-1} \sum_{\ell=1}^{2^{kq}} N_\ell \leq 1$, $N_\ell^I \leq N_\ell$, and $p_\ell v/N_\ell \leq P_{i,\ell} \leq p_\ell v/N_\ell^I$. Since $N_\ell^I/n + (p_\ell)^{\rho} / \sum_{j=1}^{2^{kq}} (p_j)^{\rho}$ and $N_\ell/n \to (p_\ell)^{\rho} / \sum_{j=1}^{2^{kq}} (p_j)^{\rho}$, the second term of (3.12) becomes

$$\liminf_{q \to \infty} \left[\frac{1}{\beta} \sum_{\ell=1}^{2^{kq}} (p_\ell)^{2\rho-1} v \right] / \left[\sum_{j=1}^{2^{kq}} (p_j)^{\rho} v \right]^2 = \frac{\zeta}{\beta},$$

and the third term becomes

$$\begin{split} \lim_{n \to \infty} \sup_{q \to \infty} \left[\frac{1}{\beta} \sum_{\ell=1}^{2^{kq}} (p_{\ell})^{2\rho-1} v \right] \lim_{n \to \infty} \left(1 - \frac{p_{\ell} v}{N_{\ell}^{I}} \right)^{\beta n} / \left[\sum_{j=1}^{2^{kq}} (p_{j})^{\rho} v \right]^{2} \\ &= \limsup_{q \to \infty} \left[\frac{1}{\beta} \sum_{\ell=1}^{2^{kq}} (p_{\ell})^{2\rho-1} v \right] e^{-\left[p_{\ell}^{1-\rho} \sum_{j=1}^{2^{kq}} (p_{j})^{\rho} v \beta \right]} / \left[\sum_{j=1}^{2^{kq}} (p_{j})^{\rho} v \right]^{2} = \frac{\xi(\beta)}{\beta}. \end{split}$$

Note that we have a relation between Theorem 1 and Theorem 2 as follows.

$$\limsup_{n \to \infty} n^{2/k} E\{\inf_{C \in \mathcal{C}_n} D(C, F_{m_n})\} \leq J_k ||f||_{\rho} \left[1 - \frac{\zeta - \xi(\beta)}{\beta}\right] \leq J_k ||f||_{\rho} \left(1 - \frac{1 - e^{-\beta}}{\beta}\right).$$
(3.13)

Under the assumption that $||f||_{2\rho-1} < \infty$, Theorem 2 can also be extended to a more generally bounded f in the same manner as in Corollary 1. Furthermore, suppose that F is an arbitrary distribution function expressed as $F(\mathbf{x}) = \alpha F_{ac}(\mathbf{x}) + (1 - \alpha)F_{s}(\mathbf{x})$ for some $\alpha \in [0, 1]$, where F_{ac} is absolutely continuous and F_{s} is singular with respect to μ . Then from [4], the asymptotic upper bound is given by

$$\alpha J_k ||f||_{\rho} \left[1 - \frac{\zeta - \xi(\beta)}{\beta} \right]$$

Note that $\zeta \geq 1$ from the Schwarz inequality. Since $f^{2\rho-1}(\mathbf{x})e^{-f^{1-\rho}(\mathbf{x})||f||_{\rho}^{\rho}} \to 0$ as $\beta \to \infty$, $\lim_{\beta\to\infty} \xi(\beta) = 0$ from Lebesgue's dominated convergence theorem [10, p.110]. In other words, $\xi(\beta)/\beta$ converges to zero at a faster rate than β^{-1} , it follows that, for relatively large β , $[\zeta - \xi(\beta)]/\beta \approx \zeta/\beta$. (Note that a similar approximation for the Theorem 1 case is already illustrated in Fig. 3.1.) Hence, we can obtain an approximate upper bound as

$$\limsup_{n \to \infty} n^{2/k} E\{ \inf_{C \in \mathcal{C}_n} D(C, F_{m_n}) \} \le J_k ||f||_\rho \left(1 - \frac{\zeta}{\beta} \right).$$
(3.14)

for relatively large β , if follows that the constant ζ , in conjunction with β , is the dominant term of the bound.

Suppose that J is an $n \ge n$ matrix and is invertable. Consider a random vector $\mathbb{Z} = JX$. If X has a density function f_X then \mathbb{Z} has the density function as

$$f_{\mathbb{Z}}(\mathbf{z}) = f_{\mathbf{X}}(J^{-1}\mathbf{z})/|\det J|.$$
(3.15)

Hence, $\zeta_{\mathbb{Z}}$ for \mathbb{Z} satisfies the following relation

$$\begin{aligned} \zeta_{\mathbb{Z}} &= \left[\int f_{\mathbb{Z}}^{2\rho-1}(\mathbf{z}) d\mathbf{z} \right] / \left[\int f_{\mathbb{Z}}^{2\rho-1}(\mathbf{z}) d\mathbf{z} \right]^2 \\ &= \left[\int f_{\mathbf{X}}^{2\rho-1}(\mathbf{x}) |\det J|^{-2\rho+2} d\mathbf{x} \right] / \left[\int f_{\mathbf{X}}^{2\rho-1}(\mathbf{x}) |\det J|^{-\rho+1} d\mathbf{x} \right]^2 = \zeta_{\mathbf{X}} \end{aligned}$$
(3.16)

Therefore ζ is invariant under a linear transformation of \mathbf{X}_1 , e.g., independent. of the variance of the input distribution or the correlation inside the vector \mathbf{X}_1 .

4. Examples of ζ

We next introduce examples involving the constant ζ for several different distributions to observe the effect of j on the bound.

For a uniform density function, it is clear that $\zeta = 1$ and $\xi(\beta) = e^{-\beta}$. This result is identical with that of Theorem 1. From (3.13), the maximal bound of Theorem 2 is obtained when f is a uniform density function.

For the generalized exponential distribution, including the Laplace and normal distributions, ζ is derived. The generalized exponential density function with the parameters s and η (s, $\eta > 0$) is defined as

$$f_{s,\eta}(\mathbf{x}) := rac{\eta^{k/s}}{V_k \Gamma(1+k/s)} \ e^{-\eta \langle \mathbf{x}
angle^s},$$

where $\langle \cdot \rangle$ denotes an arbitrary norm on \mathbb{R}^k , and V_k is the volume of the k-dimensional unit sphere defined by $V_k := \int_{\langle \mathbf{x} \rangle \leq 1} d\mathbf{x}$. For example, if s = 1 and the norm is $|\cdot|$, then the distribution is the Laplace distribution; if s = 2 and the norm is $||\cdot||$, then the distribution is the normal distribution.

Suppose that f is the generalized exponential density function. Then from (3.10), we obtain

$$\begin{aligned} \zeta_{s,k} &:= \||f_{s,\eta}\|_{2\rho-1}^{2\rho-1}/\|f_{s,\eta}\|_{\rho}^{2\rho} \\ &= \frac{(\rho^2\eta)^{k/s}}{V_k\Gamma(1+k/s)} \int e^{-(2\rho-1)\eta(\mathbf{x})^s} d\mathbf{x}. \end{aligned}$$

If $2\rho - 1 > 0$, then

$$\int e^{-(2\rho-1)\eta \langle \mathbf{x} \rangle^s} d\mathbf{x} = \frac{V_k \Gamma(1+k/s)}{\left[\eta(2\rho-1) \right]^{k/s}}.$$

Therefore,

$$\zeta_{s,k} = \left(\frac{k^2}{k^2 - 4}\right)^{k/s}$$
, for $k = 3, 4, \dots$

The normal distribution (s = 2) has lower values of $\zeta_{s,k}$ than the Laplace distribution case (s = 1). Furthermore, for a given distribution, increasing vector dimension decreases $\zeta_{s,k}$.

5. Application: Training Vector Quantizer

Suppose that we have L points $\mathbf{x}_1, \ldots, \mathbf{x}_L$ as data. The data could be a part of the images or the the speech spectral vectors. Depending on the application, the data size can be arbitrary large. The optimal quantizer design problem for $\mathbf{x}_1, \ldots, \mathbf{x}_L$ consists of finding a set $C \in C_n$ that minimizes the time-average error given by

$$\frac{1}{L} \sum_{\ell=1}^{L} ||\mathbf{x}_{\ell} - Q_C(\mathbf{x}_{\ell})||^2.$$
(5.1)

In this error, the surjective map $Q_C: \mathbb{R}^k \ \ \ C$ is defined as

$$Q_C(\mathbf{x}) := \arg\min_{\mathbf{y}\in C} ||\mathbf{x} - \mathbf{y}||^2,$$

which is called the vector quantizer. The finite set C is called the *codebook*.

There exists at least one codebook that minimizes the time-average error (5.1). Finding such an optimal codebook is, however, difficult, especially for large L, since the search complexity increases dramatically as L gets large [2]. Note that, in order to search for an optimal codebook, we should compare the n^{L} distortions of (5.1) in the worst case, since the number of possible codebooks is n^{L} . In order to reduce the search complexity, first, we generally use a portion of the data, which is called training data, in other words, we search a codebook that is optimal for a finite and relatively small data. However, the search complexity can still be high. Thus, instead of a full search for the optimum, the *K*-means clustering algorithm and the Kohonen learning algorithm, which cluster the training data, have been proposed [2] to find a suboptimal codebook for the training data.

Now, suppose that the time-average error in (5.1) can be rewritten (modeled) as $D(C, F_L^{\omega'}), \omega' \in \Omega$. From the Strong Law of Large Numbers [10],

$$\lim_{L \to \infty} D(C, F_L^{\omega'}) = D(C, \mathbf{F}) \text{ for almost every } \omega',$$

for a given codebook C. Therefore, the optimal quantizer design problem can be regarded as finding a codebook C that minimizes D(C, F) for F if L is infinitely large. Furthermore, $\mathbf{X}_{1}^{\omega}, \ldots, \mathbf{X}_{m}^{\omega}$ are also regarded as a training data and m is the training data size. Then $\inf_{C \in \mathcal{C}_{n}} D(C, F_{m}^{\omega})$ is the trained error for the training data. Therefore, the upper bounds in Theorem 1 (or Theorem 2) can be used to describe the performance of the trained vector quantizer.

An example of the bound for a normal distribution is illustrated in Fig. 5.1 with a sequence of numerica results of $\inf_{C \in C_n} D(C, F_m^{\omega})$, for $\beta = 4 \sim 30$, where the K-means algorithm was employed for the numerical results. If \mathbf{X}_1 has the normal distribution, then the upper bound in (3.14) is given by

$$J_k \|f\|_{\rho} \left(1 - \frac{\zeta}{\beta}\right) = 2\pi J_k \rho^{-(k+2)/2} (\det \mathbf{S})^{1/k} \left(1 - \frac{\zeta_{2,k}}{\beta}\right), \tag{5.2}$$



Figure 5.1: An example of the bounds for the normal distribution case (Bound 1 is from Theorem 1, Bound 2 is from (5.2), $\{\det S\}^{1/k} = 1$, n = 512, and k = 3).



Figure 5.2: Compensated trained error curves using $(1 - \zeta_{2,3}/\beta)$ (n = 512).

where S is the auto-covariance matrix of \mathbf{X}_1 . This bound is denoted as Bound 2 in Fig. 5.1, where k = 3, n = 512 and $2\pi J_k \rho^{-(k+2)/2} (\det \mathbf{S})^{1/k} = 5.3092 \cdots$. The numerical results increases as β increases. At $\beta = 5120$, the numerical error becomes 5.184 and is approximately equal to the error for F. The bound in Theorem 1 is also illustrated in Fig. 5.1 (Bound 1). Note that the bound in (5.2), Bound 2, is better than Bound 1, for relatively large β .

As shown in Fig. 5.1, the ratio $\beta = m/n$ dominates the performance of the trained vector quantizer; the bias increases as β decreases. In other words, the trained error for a given set of training data is usually much less than $\inf_{C \in C_n} D(C, F)$, especially for small β . From this fact, we often overestimate the performance of the trained quantizer. Therefore, it is general procedure to check the performance of a trained quantizer for the validating data [14]. In order to alleviate the bias, from (3.14), we can consider an error

$$\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega}) / \left(1 - \frac{\zeta}{\beta}\right)$$
(5.3)

instead of $\inf_{C \in \mathcal{C}_n} D(C, F_m^{\omega})$, where the trained error is compensated for the bias. Note that

 $E\{\inf_{C \in \mathcal{C}_n} D(C, F_m)\}$ is a lower bound of $\inf_{C \in \mathcal{C}_n} D(C, F)$. However, the expectation of (5.3) is tighter than $E\{\inf_{C \in \mathcal{C}_n} D(C, F_m)\}$. This compensated error can prevent a wrong estimation due to the biased trained error. Note that, in (5.3), we can simply set $\zeta = 1$ for unknown input distributions. In Fig.5.2, the trained error curve (n = 512) of Fig.5.1 are compensated using the error in (5.3).

List of References

- [1] E. A. ABAYA AND G. L. WISE, Convergence of vector quantizers with applications to optimal quantization, SIAM J. Appl. Math., 44 (1984), pp. 183–189.
- [2] M. R. ANDERBERG, Clustering Analysis for Applications, Academic, New York 1973.
- [3] E. S. BARNES AND N. J. A. SLOANE, The optimal lattice quantizer in three dimensions, SIAM J. Alg. Disc. Meth., 4 (1983), pp. 30-41.
- [4] J. A. BUCKLEW AND G. L. WISE, Multidimensional asymptotic quantization theory with rth power distortion measures, IEEE Trans. Inf. Theory, 28 (1982), pp. 239-247.
- [5] J. H. CONWAY, AND N. J. A. SLOANE, Voronoi regions of lattices. second moments of polytopes, and quantization, IEEE Trans. Inf. Theory, 28 (1982), pp. 211-226.
- [6] ——On the voronoi regions of certain lattices, SIAM J. Alg. Disc. Meth., 5 (1984), pp. 294-305.
- [7] ——-A lower bound on the average error of vector quantzzers, IEEE Trans. Inf. Theory, 31 (1985), pp. 106-109.
- [8] P. GAENSSLER AND W. STUTE, (1979) Empirical processes: A survey of results for independent and identically distributed random variables, Ann. Prob., 7 (1979), pp. 193-243.
- [9] A. GERSHO, (1979) Asymptotically optimal block quantization, IEEE Trans. Inf. Theory, 25 (1979), pp. 373-380.
- [10] P. R. HALMOS, Measure Theory, Springer-Verlag, New York, 1974
- [11] J. E. HUMPHREYS, Introduction to Lie Algebras and Representation Theory, Springer-Verlag. New York. 1972.
- [12] D. G. LUENBERGER Linear and Nonlinear Programming, Addison-Wesley, New York, 1984, 2nd. ed.
- [13] D. S. KIM, Performance of vector quantzeers and zts relation to traznzng-set szze, Ph.D. dissertation, Dep. of Control and Instrumentation, Seoul National Univ. Seoul, Korea, 1994.
- [14] D. S. KIM, T. KIM, AND S. U. LEE, On testing trained vector quantizer codebooks, IEEE Trans. Image Processing, 6 (1997), pp. 398-406.
- [15] D. FOLLARD. Strong consistency of k-means clustering, Ann. Stat., 9 (1981), pp. 135-140.
- [16] V. VAPNIK, Estimation of Dependences Based on Empirical Data, Springer-Verlag, New York, 1982.