Identification and Quantification of Nonlinear Behavior in a Disbonded Aluminum Honeycomb Panel using Single Degree-of-Freedom Models

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For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Douglas E. Adams
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Head of the Graduate Program Date
IDENTIFICATION AND QUANTIFICATION OF NONLINEAR BEHAVIOR IN
A DISBONDED ALUMINUM HONEYCOMB PANEL USING
SINGLE DEGREE-OF-FREEDOM MODELS

A Dissertation
Submitted to the Faculty
of
Purdue University
by
Eric R. Dittman

In Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

December 2013
Purdue University
West Lafayette, Indiana
This is dedicated to my wife who encouraged me to pursue another degree all while raising four kids, taking care of our home and earning a degree for herself. You are my inspiration.

The views expressed in this article are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government
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Lastly, I would like to thank my family for their love and support by allowing me to stay at work way too long and still loving me when I was home.
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ABSTRACT


There is not a complete understanding of how damage mechanisms in composite materials react to nondestructive testing inputs. A deeper understanding of composite damage mechanisms and their responses to external vibratory excitation is sought using nonlinear modeling of damping and stiffness characteristics related to the damage. Different types of damping and stiffness single degree-of-freedom models are presented with analysis of their behavior at specific harmonics of the primary resonance of the system. After an understanding of the approximate behaviors of the models is obtained, experimental tests on a damaged specimen are conducted. An aluminum honeycomb sandwich panel is damaged by applying a heat source to the top face sheet. The expansion of the heated area on the face sheet creates a disbond of the face sheet and the honeycomb core. This damaged panel is tested by exciting the damaged area with a shaker at known frequencies and amplitudes. The resulting responses of the panel are measured and compared with the nonlinear predictive models through a direct comparison of response amplitudes to the analytical solutions and by examination of restoring force curves. The disbonded aluminum honeycomb sandwich panel exhibited behavior similar to a pure quadratic stiffness as well as a smaller influence from a cubic stiffness. The quadratic stiffness is the result of the facesheet experiencing two distinct stiffness regimes, the first as the facesheet moves away from the core, and second as the facesheet presses into the core. The smaller cubic nonlinearity is thought to come from the additional stiffness imparted into the single degree-of-freedom system by the epoxy fillets that hold the facesheet to the
core. As the face sheet vibrates, the epoxy fillets contribute a small additional stiff-
ening as the displacement grows larger. It is also confirmed that the displacement of
the damaged area is able to be modeled using single degree-of-freedom models. This
enabled the use of single degree-of-freedom equations of motion, which simplified
the nonlinear analysis. Two further observations are made with regard to potential
damage detection applications. First, that lower frequency excitation of panels may
be able to excite the damaged areas more easily for nonlinear measurements than
higher frequency excitation. This is supported by the nonlinear analysis showing
that additional response peaks are more readily obtained through superharmonic ex-
citation than subharmonic excitation. Second, that smaller damage sizes have higher
quadratic and cubic stiffness coefficients, which produced relatively larger responses
at the primary resonance when excited at the superharmonic frequencies than the
larger damage sizes. These relatively larger responses may be a key in decreasing the
size of damage detectable using vibratory excitation.
1. INTRODUCTION

Composite materials are being used more often in many engineering applications. The high strength-to-weight ratio of these materials makes them ideal for most aerospace applications, and their use is also increasing in large-scale automotive manufacturing. The cost of manufacturing complex composite shapes is dropping to a point where they are within reach of most products. Yet, with all these advances, there is not a complete understanding of how damage mechanisms in composite materials respond to nondestructive measurements. While damage in composites is more frequently viewed as a nonlinear behavior in an otherwise healthy structure, and some proposed vibratory methods for locating damage use the nonlinear behavior of the damage area for identification [1–4], the physical meaning behind the nonlinearity is not completely understood. This work studies how a disbond in a composite honeycomb panel reacts to vibratory excitation, attempts to model the displacement of the damage as a single degree-of-freedom system, and how the measured nonlinearities are related to the physical behavior of the damage. It does so by first understanding how different solutions to nonlinear single degree-of-freedom models react to single frequency excitation. After testing three disbonded panels, the force restoration curves are examined for evidence of different types of nonlinearities. With the understanding gained from the force restoration curves, the response amplitudes are examined and compared to the solutions of the nonlinear SDOF solutions, and coefficients for the equation of motion are found. Finally, with the coefficients known and an equation of motion determined, the physical behavior of the damage is examined to understand the sources of the nonlinearities.
1.1 Background

In a healthy structural material, the forced response behavior of the material under structural loads is often assumed to be linear for small deformations. The linear nature of this forced response allows engineers to conduct sensitivity studies to gain an understanding of the material design performance under different tradeoffs using computer simulations. Recently, localized damage in structural materials has been described using nonlinear models. Many researchers have investigated how damage can be modeled as a nonlinearity and how the identification of nonlinearities in the material can assist in locating damage.

Vibration-based structural health monitoring techniques have been examined in depth. A summary of previous work was written by Doebling et al. [5]. In this work, the authors compiled many of the papers written on the subject to include stiffness changes and models that update the structural parameters for the identification of damaged areas.

Stiffness change in damaged materials has been the focus of many researchers. Chu and Shen [6] proposed using frequencies lower than the natural frequencies in a bilinear stiffness system to identify damage. The bilinear stiffness was modeled using two springs of different stiffnesses and then as two square-waves. The authors developed closed-form analytical solutions for forces at frequencies lower than the natural frequencies of the system. They were able to identify which harmonics of the forcing frequency would appear in the response of a cracked cantilevered beam. The analytical results were verified by numerical simulation, and they reported that further verification was to be done experimentally.

Ruotolo et al. [7] again focused on the nonlinear stiffness of a cracked cantilevered beam. The authors used a numerical method to simulate a bilinear stiffness in the beam. By focusing on how the crack opened and closed, the authors were able to improve on the simulation of a cracked beam and the results compared favorably to experimental results.
A bilinear analysis was also conducted by Tsyfansky and Berenevich [8] using a single-frequency excitation force on a damaged aircraft wing to examine the steady-state harmonic responses. Using a bilinear crack model, they were able to show that a greater sensitivity to the crack presence occurred at frequencies below the calculated natural frequency of the wing. This sensitivity was used to then locate the damage site in an aircraft wing. The authors were able to show that the use of this nonlinear behavior allowed for a ten-fold increase in damage sensitivity over previous linear procedures.

Andreas et al. [9] measured the nonlinear dynamics of a cracked cantilever beam under harmonic excitation. Using a bilinear frequency model, the primary resonance of the cracked beam was computed and confirmed. A sinusoidal force was applied to the tip of the beam with a varying frequency. The frequency was varied from 0.1 to 1 as a ratio of the primary resonance. Harmonic responses were seen at the primary resonance when the excitation frequency was near 1/4, 1/3, 1/2 and at 1. Other smaller harmonics were reported, but were not further investigated.

While the previous group of researchers examined stiffness changes in damage detection, another group examined how damping can be used for detection of damage. For example, Modena et al. [10] showed that changes in damping could be used to localize damage in structural materials. Through testing of an undamaged, and then damaged, concrete block, they were able to show that the natural frequency of the block would change very little for a large amount of damage. This finding implied that the stiffness of the material changed only marginally with the damage. However, they were able to show a 50 percent change in damping for the same level of damage. After further analysis, the authors were able to devise a modal friction damping method to identify the location of damage based on damping changes. Challenges in applying this method arose from the difficulty in obtaining the free mono-frequency vibrations of higher-frequency modes due to high modal densities.

Curadelli et al. [11] continued studying damping as a way of locating damage. The authors used a nonlinear damping and stiffness model to identify changes be-
tween an undamaged and damaged sample. The nonlinear model was a second-order system with damping and stiffness coefficients that were dependent on velocity and displacement, respectively. The data was then processed using a wavelet transform to decompose the multi-degree-of-freedom system into many single degree-of-freedom systems and to examine the changes in damping at a local level.

To better understand how damage in composite structural materials affects forced vibration response data, Los Alamos National Laboratory wrote a report in 2007. Farrar et al. [12] discuss in depth different methods for damage detection. Since damage can be modeled as a local nonlinearity, the authors study how a system behaves with a nonlinearity present. Different methods for detecting nonlinearity in structural systems were summarized, as well as the general uses and limitations of these methods for detecting nonlinear behavior. One of the arguments for the need to understand nonlinear behavior over a wide variety of materials and conditions was that nonlinear behavior does not generalize. Therefore, analysis techniques that may apply to one system may not apply for another. Also, it was noted that mathematical descriptions for nonlinear damped behavior due to damage were lacking.

A source of possible nonlinear damping is the viscoelastic nature of various epoxy adhesives used in the construction of composite structures. Gandhi and Chopra [13] discussed various models that can be used to simulate the complex behavior of the viscoelastic material. By combining a softening spring with a chain of springs of different rates and a dashpot style damper, the researchers were able to recreate the viscoelastic behavior of the elastomeric damper. The model was validated by comparing the numerical results against test values, which showed a good match over many different frequencies.

Nonlinear system identification presents a new area of research. The work on identifying and classifying the type of nonlinearities present in a system is vast. Kerschen et al. [14] summarized many of the techniques. The authors broke down the task of identifying nonlinearities into multiple methods: time-domain methods, frequency-
domain methods, modal methods, time-frequency methods, black-box modeling, and structural model updating.

One of the time-domain methods is the restoring force method. In a series of papers, Masri et al. [15–18] outlined a method to identify the types of nonlinearities encountered in a given system. They proposed using different types of restoring force curves to examine the nonlinearities. By using either random or swept-sine signals, any arbitrary nonlinearity was identified using the algorithm. By keeping the number of expansion terms small, the complexity of the calculations was kept to a reasonable level while still yielding excellent results. This method was later extended to multi-degree-of-freedom systems by estimating the pertinent mode shapes of the system. The multi-degree-of-freedom system was simulated using both deterministic and random excitation, producing excellent results. A final set of papers outlines an algorithm for applying the method on structural systems with dynamic loads. The algorithm was validated by applying the method to a calibration problem with realistic nonlinear structural characteristics. The algorithm produced a reduced-order nonlinear mathematical model that was able to reproduce the responses seen in the simulation data. Many other methods use the time domain to identify the nonlinearities, such as the NARMAX method [19; 20], the FORCEVIB method [21] and others [22–24].

While these techniques identified nonlinear behavior by examining behavior in the time domain, other researchers have worked to identify nonlinear behavior through methods in the frequency domain. Billings and Jones [25] developed a method to generate a General Frequency Response Function (GFRF). The GFRF is a recursive method for creating higher-order frequency response functions for nonlinear SDOF systems, much of it based on Volterra series expansions [26]. While the frequency response function contains only a single variable, frequency, the GFRF requires multiple frequencies to estimate the response of the nonlinear system. The number of frequencies needed is equal to the order of the nonlinearity. This method is able to show how the nonlinearities create response peaks at frequencies that are multiples
of the primary resonance of the system. This method was not able to identify the nonlinearities based on the frequency responses, but created the frequency responses based on the assumed nonlinearities. Adams and Allemang [27] use a feedback system to identify nonlinear parameters in the frequency domain. A separation between the linear and nonlinear portions of a multidegree-of-freedom system was created through a closed loop feedback system to include the nonlinear portion of the frequency response. This feedback loop is then included into the open linear system. Another method for parameter identification in the frequency domain was created by Adams [28]. By using an autoregressive model with exogenous inputs (ARX) in the frequency domain, the researcher was able to match parameters over a frequency range rather than to a single frequency. By matching over a frequency range, the model is able to be used as a non-parametric solution when linear models are used and a parametric model when nonlinear models are used. Another frequency domain method uses the dynamic stiffness equation of motion. Lee and Shin [29] based their work on a beam, and were able to identify multiple damage locations through numerical simulation and then experimental work.

A good review of modal methods for nonlinear system identification is given by Fan and Qiao [30]. The authors go through many of the methods that use modal parameters in nonlinear system identification and compare the different advantages and drawbacks to each approach. A few studies showed both the promise and difficulty in this method. Kosmatka and Ricles [31] used experimentally measured modes and frequencies to determine the changes in mass or stiffness values. A baseline was used from an analytical model to further enhance the accuracy, but reasonable results are obtained without it. Diaz Valdes and Soutis [32] used piezoceramic patches in the composite beams to excite the structure. From the changes in the modal frequencies that were measured from high frequency excitation sweeps, the researchers were able to identify delamination damage. Farrar et al. [33] measured the change in modal frequencies of the I-40 bridge over the Rio Grande. The authors were able to test a simulated fatigue crack in the bridge girders by cutting a crack in the girder web.
The cut reduced the overall bridge cross-section stiffness by 21 percent. The reduced stiffness was not found to reduce the measured modal frequencies significantly, making it impossible to detect damage using frequency shifts.

There have been previous attempts to identify nonlinear behavior in panels. Many have examined the response of the material due to high-frequency excitation. Ihn and Chang [34] used a pitch-catch method with piezoelectric actuators and sensors to locate damage in aircraft structures. Tsuda [35] used piezoelectric actuators and fiber Bragg gratings to monitor impact damage in carbon fiber reinforced composites. Giurgiutiu and Zagrai [36] used piezoelectric actuators and sensors on aging aircraft to locate multiple damage types using the electro-mechanical impedance technique. Mian et al. [37] used ultrasonic excitation to create friction in a fatigue crack. Infrared cameras then measured and located the increase in temperature at the crack as a means of damage detection. Work by Zumpano and Meo [38] detected damage in foam panels by tracking the harmonics and sidebands exhibited when testing at two frequencies, a low and high frequency. While the study presented in this dissertation will not focus on the effects of sidebands, it will be noted that sidebands will be observed in an unexpected nonlinear behavior. Brush [39] measured nonlinearities in sandwich composite panels. By measuring the restoring force of a fiberglass sandwich panel and approximating the panel by a single degree-of-freedom system, nonlinear stiffness and damping values were found by means of comparing restoring force curves. Difficulties arose due to the stiffness of the composite panels, most noticeably when the panels were constructed of composite materials that were reinforced with carbon fibers. The high level of structural stiffness required large amounts of excitation, which appeared to overwhelm the test set-up and shaker system. Resonances of the shaker-stinger set-up appeared in the data.

Based on this prior literature, and in order to fill the gap in understanding of damage behavior in sandwich composites under single frequency harmonic excitation, this research is taking the approach laid out in this section. Disbond damage in an aluminum honeycomb panel will be examined by exciting the damaged section with
harmonic forcing while measuring the acceleration responses of the facesheet. By examining the displacement responses of the facesheet and comparing these results to the analytical responses of nonlinear systems, it is expected that an understanding of how damage mechanisms influence the behavior of the facesheet under harmonic excitation will be identified.

1.2 Modeling

A disbonded facesheet in an aluminum honeycomb panel will have different vibratory tendencies than an undamaged panel. If the disbond is excited with enough force, the facesheet will experience two distinct vibratory regimes; when the facesheet is in contact with the honeycomb core and when it is not in contact with the honeycomb core. These two different response areas create a discontinuity in the vibratory response of the facesheet, as shown in Figure 1.1. The discontinuity between the two regimes can be approximated using a quadratic nonlinear stiffness term. Also, it is expected that this quadratic behavior, if observed, would be greater as the disbond area increases. As the disbond area increases, the area of contact between the two regimes will also increase. This increase in the contact area should lead to a strengthening of the observed quadratic stiffness behavior as as the damage size grows.
Another area in the disbond damage mechanism for potential nonlinear behavior is at the edges of the disbond. When the facesheet disbands from the honeycomb core, the tiny epoxy fillets that bind the two together are exposed to greater strains than when the panel was whole. A simplified example of the epoxy fillet that binds the facesheet to the honeycomb core is shown in Figure 1.2. As the disbonded facesheet vibrates up and down, the epoxy will see increased strains as the facesheet moves both away from and into the core. Also, the facesheet itself will experience deformation at the edge of the disbond as the facesheet vibrates. These two behaviors would be symmetrical in nature, acting in a similar fashion as the facesheet vibrates away from and into the honeycomb core. This symmetrical nature would need to be modeled as a cubic or a rectified quadratic nonlinearity. A rectified quadratic system is one where the quadratic term maintains the positive or negative magnitude of the displacement or velocity. It is expressed mathematically as $|u|u$ or $|\dot{u}|\ddot{u}$. The use of these rectified nonlinear quadratic and cubic terms is consistent with the results seen by Andreaus et al. in their work on beams.

In order to simplify the analysis of the nonlinear systems, the equations of motion will be approximated using only single degree-of-freedom models. A single degree-of-freedom model can be used in measuring the displacement of the center of the disbonded area if the rest of the system is secured properly and forcing of the disbonded area is constrained to the out of plane motion only. While these constraints may simplify the analysis, they do create limitations. A disbond in a honeycomb...
panel is not a single degree-of-freedom system. By attempting to model it as one limits the parameters that are needed to more fully describe the behavior of the system. It is expected that the single degree-of-freedom modeling will do an acceptable job of describing the behavior of the disbond under harmonic excitation, but it cannot fully account for all nonlinear behavior in the full disbond system. Understanding these limitations, nonlinear analyses of six different nonlinear single degree-of-freedom equations of motion will be accomplished. From the understanding of the analytical solutions to the nonlinear equations of motion, differences in the response amplitudes and phases to the forcing amplitudes will allow for discrimination of the types of nonlinearities that are seen in the disbonded panels.

1.3 Testing

Once the nonlinear responses to harmonic excitation are established, experimental data from damaged panels are needed. A proposed testing sequence is shown in Figure 1.3. The sequence starts by identifying the primary resonance through modal testing. With the primary resonance known, the shaker testing can begin at the three important frequencies, the primary resonance, one-half of the primary resonance, and one-third of the primary resonance. The test data can then be compared to the analytical results to identify the nonlinear behavior. It is important to ensure that the test instrumentation does not introduce any nonlinearities into the measurement system including the specimen and instrumentation. If this does occur, the nonlinearities due to the instrumentation must be fully accounted for when interpreting the resulting forced vibration response data. Therefore, it is important to also put an undamaged sample through many of the same tests to ensure that the nonlinear response characteristics seen in the damaged sample are from the damage and not the testing apparatus.
Figure 1.3. Proposed testing and analysis flowchart.
1.4 Data Analysis

The test data will then be analyzed using two methods. Restoring force curves will show how the forcing function and the displacement of the facesheet correspond to each other. Amplitudes and phases between the force and response can be seen and insight into how the system is responding can be gained. Finally, the responses to the excitation will be compared to the trends highlighted in the nonlinear analysis. Further distinctions will be made as to what types on nonlinearities are within the system by how the responses match the analytical results. This will also clarify the equation of motion for the system.

1.5 Coefficient Fitting and Results Confirmation

Once the equation of motion for the system is confirmed, the data will be fit to the coefficients by using a least-squares method. The magnitudes of the coefficients will be examined to find trends between the damage sizes. Finally, the displacement data will be replicated using the coefficients with the analytical solutions, and by using a numerical solver to determine the displacements. Comparisons of the different results will be examined for discrepancies. Once this is complete, the physical reasons will be sought to explain the nonlinear behavior of the damage.

1.6 Limitations of the Research

As previously noted, there are limitations of the approaches used in this research effort. While a single degree-of-freedom model may not capture all of the different behaviors seen in the damaged panels, this research is a first attempt to explore how the disbond damage reacts to harmonic excitation and how that relates to the physical behavior of the damage. Further efforts to incorporate modal parameters or to explore other damage mechanisms can expand on this effort, but in and of itself, this is an initial attempt to do so.
2. NONLINEAR EQUATIONS OF MOTION AND THEIR RESPONSE TO FORCED EXCITATION

As explained in the previous chapter, six different nonlinear models are initially chosen to be analyzed: cubic stiffness and damping, rectified quadratic stiffness and damping, and pure quadratic stiffness and damping models. There are many similarities in the setup of the different models and in the methods employed to analyze the models, but the results of the analyses provide qualitatively and quantitatively different results. The analyses are grouped into two main sections: analysis of stiffness nonlinearities and analysis of damping nonlinearities. It is through the differences in the results of the different analyses that the proposed identification of the nonlinear behavior in the damaged panels is accomplished. While unknown at this point, Figure 2.33 at the end of this section, uses the differences found in the analyses to create a decision flowchart for identifying the nonlinearities in the damaged panel. With this in mind, the nonlinear analysis is presented. All figures of responses in this chapter are created using arbitrary values for the coefficients of the equation of motion. The values were manipulated to highlight important features in the results.

2.1 Nonlinear Stiffness Models

Much work has been done in the literature on analyzing the oscillatory response of single degree-of-freedom systems using a cubic stiffness model, which is expressed using the Duffing equation. In this work, the Duffing equation is written as follows:

\[ \ddot{u} + 2\mu \dot{u} + \kappa u^3 + \omega_0^2 u = F \cos(\Omega t). \]  

(2.1)

The \( \ddot{u} \) term is an acceleration term. It does not have a mass coefficient because the mass term has been divided into the other terms in the equation of motion. The
next term is a viscous damping term, with $\mu$ being the coefficient of damping. The next term is the nonlinear term in this equation of motion. $\kappa$ is the cubic stiffness coefficient. The next term is the linear spring stiffness term, with $\omega_0$ representing the primary resonance of the system. $\omega_0$ can be rewritten as $\sqrt{\frac{k}{m}}$, which shows the spring stiffness constant $k$. The last term in the equation of motion is the forcing function, whose coefficient $F$ is the force normalized by the mass. A similar model is employed for two types of quadratic stiffness models:

\begin{align*}
\ddot{u} + 2\mu\dot{u} + \kappa|u|u + \omega_0^2 u &= F \cos(\Omega t) \\
\ddot{u} + 2\mu\dot{u} + \kappa u^2 + \omega_0^2 u &= F \cos(\Omega t). \quad (2.2) \\
\end{align*}

A review of previous work on the Duffing equation and similar analyses for quadratic stiffness models are provided here.

### 2.1.1 The Duffing Equation

Nayfeh documents a Method of Multiple Time Steps analysis of the Duffing equation in his books “Nonlinear Oscillations” [40] and “Problems in Perturbation” [41]. This section reviews his work as a refresher and as a means of comparing to the results of the analyses in this research. The analysis of the Duffing equation is grouped into two categories: response of the system when excited near primary resonance and response of the system when excited near harmonics of the primary resonance.

**Duffing Equation at Primary Resonance**

A harmonic excitation force at the same frequency as the primary resonance of a system produces large displacements in the system for very little force amplitude. With this characteristic in mind, the Duffing equation was analyzed at the primary resonance using the same order of $\varepsilon$ for the linear damping term, the nonlinear stiffness
term, and the excitation force. This ordering of the terms results in a modified Duffing equation of the following form:

\[ \ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon \kappa u^3 + \omega_0^2 u = \varepsilon F \cos(\Omega t) \]  

(2.4)

The Method of Multiple Time Scales introduces \( T_n \) as the \( n \)th time scale. Starting with \( T_0 = t \), the other times scales are orders of \( \varepsilon \) so that \( T_1 = \varepsilon t \), etc. It is assumed that the response of \( u \) could be rewritten as \( u = u_0 + \varepsilon u_1 \) and that the differentials could be rewritten as operators of the form \( \frac{d}{dt} = D_0 + \varepsilon D_1 \), where \( D_n \) is the derivative of the expression with respect to \( T_n \). By keeping only order \( \varepsilon^0 \) and \( \varepsilon^1 \) terms, Equation (2.4) is rewritten as:

\[ \varepsilon^0: D_0^2 u_0 + \omega_0^2 u_0 = 0 \]  

(2.5)

\[ \varepsilon^1: D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - 2\mu D_0 u_0 - \kappa u_0^3 + F \cos(\Omega t). \]  

(2.6)

Solving Equation (2.5) for \( u_0 \) yields:

\[ u_0 = A(T_1)e^{i\omega_0 T_0} + \bar{A}(T_1)e^{-i\omega_0 T_0} \]  

(2.7)

where \( \bar{A} \) is the complex conjugate of \( A \).

A detuning parameter, \( \sigma \), is introduced here. \( \sigma \), with a value in Hz, allows the analysis to understand how the response amplitude and phasing change as the forcing frequency changes in the neighborhood of a known value. Upon substitution of \( u_0 \) and the detuning parameter \( \Omega = \omega_0 + \varepsilon \sigma \) into Equation (2.6), the following expression is obtained:

\[ D_0^2 u_1 + \omega_0^2 u_1 = (-2\omega_0 A' - 2\mu \omega_0 A - 3\kappa A^2 \bar{A} + \frac{1}{2} F e^{i\sigma T_1} e^{i\omega_0 T_0} \]

\[ -\kappa A^3 e^{3i\omega_0 T_0} + c.c \]  

(2.8)
where $c.c.$ stands for complex conjugates. In order to solve for $A$, secular terms in (2.8) are set to zero. By letting $A = \frac{1}{2}ae^{i\beta}$, the following equation is obtained:

$$-\omega_0 a'e^{i\beta} + a\beta'\omega_0 e^{i\beta} - \mu\omega_0 ae^{i\beta} - \frac{3}{8}\kappa a^3 e^{i\beta} + \frac{1}{2}Fe^{i\sigma T_1} = 0 \quad (2.9)$$

where $a = a(T_1)$ and $\beta = \beta(T_1)$. The separation of this expression into real and imaginary parts and conversion of trigonometric functions yields the following coupled equations:

$$a' = -\mu a + \frac{F}{2\omega_0} \sin(\sigma T_1 - \beta) \quad (2.10)$$

$$a\beta' = \frac{3\kappa}{8\omega_0} a^3 - \frac{F}{2\omega_0} \cos(\sigma T_1 - \beta). \quad (2.11)$$

By letting $\gamma = \sigma T_1 - \beta$, the equations are written with the detuning parameter, $\sigma$, explicitly as follows:

$$a' = -\mu a + \frac{F}{2\omega_0} \sin(\gamma) \quad (2.12)$$

$$a\gamma' = \sigma a - \frac{3\kappa}{8\omega_0} a^3 + \frac{F}{2\omega_0} \cos(\gamma). \quad (2.13)$$

These equations are examined in the steady state to determine the response to the excitation force. Instead of solving for $a$ in terms of $\sigma$, the equation is solved for $\sigma$ in terms of $a$, the response amplitude:

$$\sigma = \frac{3\kappa}{8\omega_0} a^2 \pm \frac{1}{a} \sqrt{\frac{F^2}{4\omega_0^2} - \mu^2 a^2}. \quad (2.14)$$

This response with respect to the detuning parameter $\sigma$ is shown in Figures 2.1 and 2.2.

The response exhibits a backbone curve, which indicates that the maximum amplitude of the response moves away from the primary resonance with increasing force.
Figure 2.1. Amplitude of the response of a cubic stiffness EOM excited at the primary resonance vs. detuning parameter. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1$.

Figure 2.2. Amplitude of the response of a cubic stiffness EOM excited at the primary resonance vs. detuning parameter. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 10000$. 
amplitude and value of the cubic nonlinearity. This equation identifies important parameters within the response $a$. The maximum value of $a$ is identified as:

$$a_{\text{peak}} = \frac{F}{2\omega_0 \mu}.$$  \hfill (2.15)

This value is not dependent on the nonlinear coefficient $\kappa$ but rather is dependent on the excitation force $F$, the linear damping coefficient $\mu$, and the primary resonance $\omega_0$. The peak of $a$ occurs at:

$$\sigma_{\text{peak}} = \frac{3\kappa F^2}{32\omega_0^3 \mu^2}.$$  \hfill (2.16)

As $\kappa$ grows larger, the maximum response peak moves further away from the primary resonance. The same is true for increasing levels of forcing, although the same increase in forcing would generate a larger shift when compared to $\kappa$. It is important to note that not all parts of the backbone curve seen in Figure 2.2 are stable solutions. As the frequency increases, or the response moves from the left to the right in the figure, the response will increase up to the peak. Once the frequency increases beyond the peak frequency, the response will drop down to solution on the lower leg of the curve. When decreasing the frequency, or moving right to left on the curve, the response will stay on the lower solution until it reaches the point with no solution on the lower leg, when it will jump to the upper curve. Therefore, the stable solution is different if the frequency is increasing or decreasing.

The phasing component of the response, $\gamma$, is also isolated with respect to $\sigma$. Once again, it is more straightforward to solve for $\sigma$ as a function of $\gamma$.

$$\sigma = \frac{3\kappa F^2}{32\omega_0^3 \mu^2} \sin^2 \gamma - \mu \frac{\cos \gamma}{\sin \gamma}.$$  \hfill (2.17)

This response is seen in Figures 2.3 and 2.4. As seen in Figure 2.4, the phase at $\sigma_{\text{peak}}$ is always $-\frac{\pi}{2}$, but $\sigma_{\text{peak}}$ shifts with the value of $\kappa$. This shift creates a jump condition in the phase. This jump follows the same principles as outlined with the stable solutions seen in the response amplitude.
Figure 2.3. Phase of the response of a cubic stiffness EOM excited at the primary resonance vs. detuning parameter. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1$.

Figure 2.4. Phase of the response of a cubic stiffness EOM excited at the primary resonance vs. detuning parameter. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 10000$. 
For the current study, it is important to understand how the responses change as the forcing amplitude changes. As seen in Figure 2.5, the amplitude of the response of the system when excited at $\sigma_{\text{peak}}$ is a linear function of the force. The amplitude doubles when the force magnitude doubles. This result could be useful for damage detection; however, the ability to consistently excite the system at $\sigma_{\text{peak}}$, which is a function of the forcing value, leads to practical implementation issues. Examination of the change in response amplitude with a change in the force while maintaining the same excitation frequency will aid in identifying possible nonlinearities during a test. The relationship between the response amplitude $a$ and the force amplitude $F$ at $\sigma = 0$ is defined by the following equation:

$$F^2 = \frac{9}{16} \kappa^2 a^6 + 4\omega_0^2 \mu^2 a^2.$$  

(2.18)

This expression produces a relationship between $a$ and $F$ that is not proportional to $F$, and depends heavily on the values of the other coefficients. It will produce a response
that is smaller than the one at the peak detuning value, but an exact relationship is unknown without having a knowledge on the values of the other coefficients in the equation of motion.

Returning to Equation (2.8), after eliminating secular terms, the equation becomes:

\[ D_0^2 u_1 + \omega_0^2 u_1 = -\kappa A^3 e^{3\omega_0 T_0} + c.c. \]  

(2.19)

Solving for \( u_1 \) produces:

\[ u_1 = \frac{\kappa A^3}{8\omega_0^2} e^{3\omega_0 T_0} + c.c. \]  

(2.20)

Combining (2.7) and (2.20) yields the following first order solution:

\[ u = a \cos(\Omega t - \gamma) + \varepsilon \frac{\kappa A^3}{32\omega_0^2} \cos(3\Omega t - 3\gamma). \]  

(2.21)

This full first order solution shows a response at the excitation frequency and at three times the excitation frequency. The response at 3\( \Omega \) increases in a cubic manner when compared to the increase in response at \( \Omega \).

From this analysis, possible keys in identifying a cubic stiffness nonlinearity when the system is excited at the primary resonance are the backbone curve that is exhibited at the primary resonance and the response at three times the primary resonance. The backbone curve could be identified by exciting the system at a high forcing amplitude and then tracking the response amplitude as the forcing frequency is changed. The backbone curve will cause a sharp drop, or rise, as the frequency is changed.

**Duffing Equation at Harmonic Resonances**

In order to examine the Duffing equation near harmonic resonances, only the linear damping and nonlinear stiffness terms are grouped together as order \( \varepsilon \) terms, which yields:

\[ \ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon \kappa u^3 + \omega_0^2 u = F \cos(\Omega t). \]  

(2.22)
Once again, by separating the terms by their order of \( \varepsilon \), the following equations are generated:

\[
\varepsilon^0 : D_0^2 u_0 + \omega_0^2 u_0 = F \cos(\Omega t) \tag{2.23}
\]

\[
\varepsilon^1 : D_0^2 u_1 + \omega_0^2 u_1 = -2D_0D_1 u_0 - 2\mu D_0 u_0 - \kappa u_0^3. \tag{2.24}
\]

Solving Equation (2.23) for \( u_0 \) yields:

\[
u_0 = Ae^{i\omega_0 T_0} + \Lambda e^{i\Omega T_0} + \text{c.c.} \tag{2.25}\]

where \( \Lambda = \frac{F}{2(\omega_0^2 - \Omega^2)} \), \( u_0 \) is inserted into the \( \varepsilon^1 \) equation which simplifies to:

\[
D_0^2 u_1 + \omega_0^2 u_1 = (-2\omega_0 A' - 2\mu \omega_0 A - 3\kappa A^2 \bar{A} - 6\kappa A^2 \Lambda - 3\kappa \Lambda^3) e^{i\omega_0 T_0} + (-2\mu \Omega \Lambda - 6\kappa AA \Lambda - 3\kappa \Lambda^3) e^{i\Omega T_0} - \kappa \Lambda^3 e^{3i\Omega T_0} - 3\kappa A^2 L e^{i(\Omega - 2\omega_0) T_0} - \kappa A^3 e^{3i\omega_0 T_0} - 3\kappa A^2 \Lambda e^{i(\Omega + 2\omega_0) T_0} - 3\kappa AA^2 e^{i(2\Omega + \omega_0) T_0} - 3\kappa A^2 \bar{A} e^{i(2\Omega - \omega_0) T_0} + c.c. \tag{2.26}\]

To solve for \( A \), secular terms need to be eliminated; however, Equation (2.26) indicates that secular terms would arise when \( \Omega \approx 3\omega_0 \) and \( 3\Omega \approx \omega_0 \). Each of these situations is examined individually.

**Duffing Equation with Superharmonic Excitation**

When \( 3\Omega = \omega_0 + \varepsilon \sigma \) the secular terms are:

\[-2\omega_0 A' - 2\mu \omega_0 A - 3\kappa A^2 \bar{A} - 6\kappa A^2 \Lambda - \kappa A^3 e^{i\sigma T_1} = 0. \tag{2.27}\]
Figure 2.6. Amplitude of the response of the cubic stiffness EOM at the primary resonance when excited at 1/3 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 10$, $\Omega = \omega_0/3$, and $\kappa = 10000$.

Simplifying, separating the real and imaginary components, and making the equation autonomous produces the following two equations:

\begin{align*}
\dot{a} &= -\mu a - \frac{\kappa \Lambda^3}{\omega_0} \sin(\gamma) \quad (2.28) \\
\dot{a} \gamma &= \sigma a - \frac{3 \kappa}{8 \omega_0} a^3 - \frac{3 \kappa}{\omega_0} a \Lambda^2 - \frac{\kappa \Lambda^3}{\omega_0} \cos(\gamma). \quad (2.29)
\end{align*}

Solving the equations for $\sigma$ in terms of $a$ in the steady state produces:

\begin{equation}
\sigma = \frac{3 \kappa \Lambda^2}{\omega_0} + \frac{3 \kappa a^2}{8 \omega_0} \pm \sqrt{\frac{\kappa^2 \Lambda^6}{a^2 \omega_0^2} - \mu^2}. \quad (2.30)
\end{equation}

As with the excitation force at the primary resonance, the excitation force at one third the primary resonance leads to a backbone characteristic. As the force and/or the nonlinearity increases, the peak response occurs at an excitation frequency above
or below the superharmonic frequency. The backbone feature is not as noticeable as
the result seen previously, as shown in Figure 2.6. The backbone curve appears more
as a shift in the peak response amplitude. The solution for $a_{\text{peak}}$ reveals the following:

$$a_{\text{peak}} = \frac{\kappa \Lambda^3}{\omega_0 \mu}.$$  \hfill (2.31)

Unlike when exciting at the primary resonance frequency, when the cubic stiffness
model is excited at the superharmonic frequency, the amplitude of the response is
directly related to the value of the nonlinear stiffness coefficient. As the value of the
nonlinear stiffness increases, the response at $\omega_0$ increases. Substituting $a_{\text{peak}}$ back
into the equation yields:

$$\sigma_{\text{peak}} = \frac{3 \kappa \Lambda^2}{\omega_0} + \frac{3 \kappa^3 \Lambda^6}{8 \omega_0^3 \mu^2}.$$  \hfill (2.32)

The equation of $\sigma$ as a function of $\gamma$ also simplifies to:

$$\sigma = \frac{3 \kappa^3 \Lambda^6}{8 \omega_0^3 \mu^2} \sin^2 \gamma + \frac{3 \kappa \Lambda^2}{\omega_0} - \mu \frac{\cos \gamma}{\sin \gamma}.$$  \hfill (2.33)

This result is seen in Figure 2.7. The phase at $\sigma_{\text{peak}}$ is $-\frac{\pi}{2}$ with the phase curve
shifting to the right with increasing force amplitude and level of nonlinearity.

An examination of the change in response amplitude compared to changes in the
forcing function indicates a different response than when exciting at the primary
resonance. The amplitude is proportional to the forcing value cubed at the peak
detune parameter, as shown in Figure 2.8. Yet again, tracking the amplitude of the
response when the detuning parameter, $\sigma$, is set to zero is constrained by the values
of the other coefficients in the equation of motion. The response is determined by the
following equation:

$$9 \kappa^2 a^6 + 144 \kappa^2 \Lambda^2 a^4 + (64 \omega_0^2 \mu^2 + 576 \kappa^2 \Lambda^4) a^2 = 64 \kappa^2 \Lambda^6.$$  \hfill (2.34)

The amplitude of this response increases at a rate smaller than the cubic relationship
that was determined at the peak detuning value.
Figure 2.7. Phase ($\gamma$) of the response at the primary resonance when excited at 1/3 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 10$, $\Omega = \omega_0/3$, and $\kappa = 10000$.

Figure 2.8. Amplitude of the response at the primary resonance vs. forcing amplitude when excited at the peak detune value.
Returning to Equation (2.8) with the secular terms eliminated, the equation is:

\[
D_0^2 u_1 + \omega_0^2 u_1 = (-2\mu \Omega \Lambda - 6\kappa A \Lambda - 3\kappa \Lambda^2) e^{i \Omega T_0} \\
-\kappa A^3 e^{3\omega_0 T_0} - 3\kappa A^2 \Lambda e^{i(\Omega + 2\omega_0)T_0} - 3\kappa A^2 \Lambda e^{i(\Omega - 2\omega_0)T_0} \\
-3\kappa A \Lambda^2 e^{i(2\Omega + \omega_0)T_0} - 3\kappa A \Lambda^2 e^{i(2\Omega - \omega_0)T_0}.
\] (2.35)

Solving for \( u_1 \) results in:

\[
u_1 = \frac{\kappa A^3}{8\omega_0^2} e^{3\omega_0 T_0} + \frac{3\kappa A \Lambda^2}{4\Omega(\Omega + \omega_0)} e^{i(2\Omega + \omega_0)T_0} + \frac{6\kappa A \Lambda A}{\Omega^2 - \omega_0^2} e^{i \Omega T_0} \\
+ \frac{3\kappa A \Lambda A}{4\Omega(\Omega - \omega_0)} e^{i(2\Omega - \omega_0)T_0} + \frac{3\kappa \Lambda^3}{\Omega^2 - \omega_0^2} e^{i \Omega T_0} \\
+ \frac{3\kappa \Lambda^2 A}{4\Omega^2 - 4\Omega \omega_0 + 3\omega_0^2} e^{i(\Omega - 2\omega_0)T_0} + \frac{3\kappa \Lambda^2 A}{\Omega^2 + 4\Omega \omega_0 + 3\omega_0^2} e^{i(\Omega + 2\omega_0)T_0} + c.c. (2.36)
\]

Once again combining the two solutions yields the full first order solution:

\[
u = 2\Lambda \cos(\Omega t) + \alpha \cos(3\Omega t - \gamma) + \varepsilon \left( \frac{\kappa A^3}{32\omega_0^2} \cos(9\Omega t - 3\gamma) \right.
\]

\[+ \frac{3\kappa \Lambda^2}{4\Omega(\Omega + \omega_0)} \cos(5\Omega t - \gamma) + \frac{3\kappa^2 \Lambda}{\Omega^2 - \omega_0^2} \cos(\Omega t) + \frac{3\kappa \Lambda^2}{4\Omega(\Omega - \omega_0)} \cos(\Omega t + \gamma) \\
+ \frac{6\kappa \Lambda^3}{\Omega^2 - \omega_0^2} \cos(\Omega t) - \frac{4\mu \Lambda \Omega}{\Omega^2 - \omega_0^2} \sin(\Omega t) + \frac{3\kappa^2 \Lambda}{2(\Omega - 3\omega_0)(\Omega - \omega_0)} \cos(5\Omega t + 2\gamma) \\
+ \frac{3\kappa^2 \Lambda}{2(\Omega + 3\omega_0)(\Omega + \omega_0)} \cos(7\Omega t - 2\gamma)) \] (2.37)

From the total solution, the key points to use in the differentiation of the possible nonlinear models are once again the backbone curve that occurs at the primary resonance when the forcing frequency is one-third the primary resonance, and a quickly growing response at three times the primary resonance.
Duffing Equation with Subharmonic Excitation

Returning to Equation (2.26), the secular terms when $\Omega = 3\omega_0 + \varepsilon\sigma$ are set to zero.

\[-2k_0A' - 2\mu k_0 A - 3k A^2 \bar{A} - 6k A \Lambda^2 - 3k \Lambda A^2 e^{i\sigma T_i} = 0. \tag{2.38}\]

Simplifying, separating real and imaginary parts, and letting $\gamma = \sigma T_i - 2\beta$ yields:

\[a' = -\mu a - \frac{3k\Lambda}{4\omega_0}a^2 \sin \gamma \tag{2.39}\]
\[a\gamma' = \sigma a - \frac{9k\Lambda^2}{\omega_0}a - \frac{9k}{8\omega_0}a^3 - \frac{9k\Lambda}{4\omega_0}a^2 \cos \gamma. \tag{2.40}\]

Examination of the equations indicates that at steady state, the response $a$ can be zero. Special conditions are required for a non-zero response to occur. These conditions are:

\[\frac{\sigma}{\mu} - \left(\frac{\sigma^2}{\mu^2} - 63\right)^{1/2} \leq \frac{63k\Lambda^2}{4\omega_0\mu} \leq \frac{\sigma}{\mu} + \left(\frac{\sigma^2}{\mu^2} - 63\right)^{1/2} \tag{2.41}\]

with $\kappa > 0$. When $\kappa$ is larger, a larger value of $\sigma$ is needed to create a subharmonic response. The same is true as $F$ is increased. The difficulty of creating initial conditions that lead to this nontrivial solution lead to the zero solution for $a$ being the most likely scenario, and therefore does not create a response that can help identify the nonlinearity.

2.1.2 Quadratic Stiffness Models

Pure Quadratic Stiffness at Primary Resonance

As was done in the previous model excited at the primary resonance, the ordering of terms leads to the following equation:

\[\ddot{u} + 2\varepsilon\mu \dot{u} + \varepsilon k u^2 + \omega_0^2 u = \varepsilon F \cos(\Omega t). \tag{2.42}\]
The same procedure is followed as previously outlined with the Method of Multiple Scales, yielding the same result for \( u_0 \). After substitution of \( u_0 \) back into the approximate solution, the following equation is obtained:

\[
D_0^2 u_1 + \omega_0^2 u_1 = (-2i\omega_0 A' - 2i\mu \omega_0 A)e^{i\omega_0 T_0} - \kappa A^2 e^{2i\omega_0 T_0} - 2\kappa A \bar{A} + \frac{1}{2} F e^{i\Omega T_0} + c.c. \quad (2.43)
\]

When \( \Omega = \omega_0 + \varepsilon \sigma \) and the secular terms are set to zero, the following equation is obtained:

\[
-2i\omega_0 A' - 2i\mu \omega_0 A + \frac{1}{2} F e^{i\sigma T_1} = 0. \quad (2.44)
\]

Letting \( A = \frac{1}{2} ae^{i\beta} \) and \( \gamma = \sigma T_1 - \beta \), and grouping real and imaginary parts, the following two equations are obtained:

\[
a' = -\mu a + \frac{F}{2\omega_0} \sin \gamma \quad (2.45)
\]

\[
a'\gamma = \sigma a + \frac{F}{2\omega_0} \cos \gamma. \quad (2.46)
\]

Solving for \( a \) in terms of \( \sigma \) results in:

\[
a = \sqrt{\frac{F^2}{4\omega_0^2(\mu^2 + \sigma^2)}}. \quad (2.47)
\]

This response does not have the backbone that is seen in the cubic stiffness response, as seen in Figure 2.9. The phase of this response is plotted in Figure 2.10. The change in amplitude with respect to a change in force is plotted in Figure 2.11. The amplitude of the response at \( \omega_0 \) is proportional to the force level.

Returning to Equation (2.43) with the secular terms removed, the equation is solved for \( u_1 \):

\[
u_1 = \frac{\kappa A^2}{3\omega_0^2} e^{2i\omega_0 T_0} - \frac{2\kappa A \bar{A}}{\omega_0^2} + c.c. \quad (2.48)
\]
Figure 2.9. Amplitude of the response of the pure quadratic stiffness EOM at the primary resonance when excited at primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1$.

Figure 2.10. Phase of the response at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1$. 
Figure 2.11. Amplitude of the response at the primary resonance vs. forcing amplitude when excited at primary resonance.
Combining the $u_0$ and $u_1$ solutions yields the complete first order solution:

$$u = a \cos (\Omega t - \gamma) + \varepsilon \frac{\kappa a^2}{6\omega_0^2} \cos (2\Omega - 2\gamma) - \varepsilon \frac{\kappa a^2}{2\omega_0^2}.$$  \hfill (2.49)

The full first-order solution shows responses at the excitation frequency, twice the excitation frequency and a constant displacement response at zero frequency. The response at the excitation frequency increases linearly with the increase in the forcing amplitude. Both the response at twice the excitation frequency and the constant displacement increase with the square of the forcing amplitude increase, as seen in Figure 2.12.

The key differences highlighted in the pure quadratic stiffness equation excited at the primary resonance include a proportional increase in the response amplitude with an increase in the forcing amplitude, no backbone curve at the primary resonance, and a response at twice the primary resonance that increases at a quadratic rate with
an increase in the forcing amplitude. These are different than the aforementioned cubic stiffness.

**Pure Quadratic Stiffness at Harmonic Excitation**

The method here follows the same steps as in the above analysis for the slightly different equation of motion containing a quadratic stiffness:

\[
\ddot{u} + 2\varepsilon\mu\dot{u} + \varepsilon\kappa u^2 + \omega_0^2 u = F\cos(\Omega t),
\]

(2.50)

The difference is in the order of \(\varepsilon\) on the forcing term. As before, after separating and solving for \(u_0\), the same result is obtained as seen in Equation (2.25). Inserting this result into the \(\varepsilon^1\) equation yields:

\[
D_0^2u_1 + \omega_0^2u_1 = (-2\kappa\omega_0A' - 2\mu\omega_0A)e^{i\omega_0T_0} - 2\mu\Omega\Lambda e^{i\Omega T_0}
- \kappa A^2e^{2\omega_0T_0} - \kappa\Lambda^2e^{2\Omega T_0} - 2\kappa\Lambda\bar{A}
- 2\kappa\Lambda^2 - 2\kappa\Lambda\Lambda e^{i(\Omega + \omega_0)T_0} - 2\kappa\Lambda\Lambda e^{i(\Omega - \omega_0)T_0} + c.c.
\]

(2.51)

The equation shows that different secular terms arise for different conditions, most notably when \(2\Omega = \omega_0 + \varepsilon\sigma\) and \(\Omega = 2\omega_0 + \varepsilon\sigma\).

**Pure Quadratic Stiffness at 1/2 Primary Resonance**

When \(2\Omega = \omega_0 + \varepsilon\sigma\), the secular terms, after separating real and imaginary parts, indicate that:

\[
a' = -\mu a - \frac{\kappa\Lambda^2}{\omega_0}\sin\gamma
\]

(2.52)

\[
a'\gamma' = \sigma a - \frac{\kappa\Lambda^2}{\omega_0}\cos\gamma.
\]

(2.53)
This finding results in a peak without a backbone similar to the peak seen at primary resonance where:

$$a = \sqrt{\frac{\kappa^2 \Lambda^2}{\omega_0^2 (\mu^2 + \sigma^2)}}$$

which is shown in Figure 2.13. The change in phase with the detuning parameter is shown in Figure 2.14. As shown in Figure 2.15, the response amplitude at the primary resonance changes with the square of the forcing amplitude. With the secular terms set to zero, the $\varepsilon^1$ equation is solved for $u_1$:

$$u_1 = \frac{2\mu \Lambda \Omega}{\Omega^2 - \omega_0^2} e^{i\Omega T_0} + \frac{\kappa A^2}{3\omega_0^2} e^{2\omega_0 T_0} + \frac{2\kappa A}{\Omega (\Omega - 2\omega_0)} e^{i(\Omega - \omega_0) T_0}$$

$$+ \frac{2\kappa A}{\Omega (\Omega + 2\omega_0)} e^{i(\Omega + \omega_0) T_0} - \frac{2\kappa A A}{\omega_0} - \frac{2\kappa A^2}{\omega_0} + c.c. \quad (2.55)$$
Figure 2.14. Phase of the response at the primary resonance, excited at 1/2 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1$.

Figure 2.15. Amplitude of the response at the primary resonance vs. forcing amplitude when excited at 1/2 the primary resonance.
Figure 2.16. Amplitude of the response at twice the primary resonance vs. forcing amplitude when excited at 1/2 the primary resonance.

Once more, combining the $u_0$ and $u_1$ solutions yields the complete first order solution:

$$u = 2\Lambda \cos(\Omega t) + a \cos(2\Omega t - \gamma) + \varepsilon \left[ -\frac{4\mu \Lambda \Omega}{\Omega^2 - \omega_0^2} \sin(\Omega t) ight.$$

$$+ \frac{\kappa a^2}{6\omega_0^2} \cos(4\Omega t - 2\gamma) + \frac{4\kappa \Lambda}{\Omega(\Omega - 2\omega_0)} \cos((\Omega - \omega_0)t)$$

$$+ \frac{4\kappa \Lambda}{\Omega(\Omega + 2\omega_0)} \cos((\Omega + \omega_0)t) - \frac{\kappa a^2}{2\omega_0} - \frac{2\kappa \Lambda^2}{\omega_0} \Big].$$

(2.56)

With the full first order solution, another data point to examine is the amplitude at twice the primary resonance. The change in amplitude with respect to force amplitude is graphed in Figure 2.16. The growth in the response amplitude at harmonics of the primary resonance when excited at half the primary resonance is a quartic function of the change in force amplitude.

The important differentiators in this section are the relation of the response amplitude at the primary resonance with respect to the forcing amplitude, the lack of a backbone curve, and another response at twice the primary resonance. Another
difference between this nonlinearity and the cubic stiffness is the frequency of forcing. The forcing frequency is one-half the primary resonance, while the cubic stiffness has a superharmonic frequency of one-third the primary resonance.

**Pure Quadratic Stiffness at Twice Primary Resonance**

Analyzing the equations at twice the primary resonance leads to a zero response solution or a solution where \( \sigma = \frac{2\mu A^2}{\omega_0^2} \sin \gamma \cos \gamma \). This second solution requires forcing far away from twice the primary resonance, which leads to the zero response solution being the solution when excited at twice the primary resonance.

### 2.1.3 Rectified Quadratic Stiffness at Primary Resonance

A similar method is used to solve the Duffing equation as for the Quadratic Stiffness Model, but the Method of Multiple Time Scales is not used in this case. The magnitude of \( u \) in Equation (2.2) is more easily analyzed by using the Method of Averages. Two different variations of Equation (2.2) are needed, the first version for when the system is forced near the primary resonance and the second version for when the system is forced near the harmonics of that frequency.

**Rectified Quadratic Stiffness Model at Primary Resonance**

As with the Duffing Equation, orders of \( \varepsilon \) are inserted into the equation for the linear damping, nonlinear stiffness and the excitation force terms, yielding the following equation:

\[
\ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon \kappa |u|u + \omega_0^2 u = \varepsilon F \cos(\Omega t).
\]  

(2.57)

A solution to the differential equation is assumed:

\[
u = a(t) \cos[\omega_0 t + \beta(t)]
\]  

(2.58)
where \( a \) and \( \beta \) are functions of time. It is also assumed that since \( a \) and \( \beta \) vary slowly over time,

\[
\dot{u} = -\omega_0 a(t) \sin[\omega_0 t + \beta(t)]. \tag{2.59}
\]

Equation (2.58) was differentiated fully, yielding:

\[
\dot{u} = \dot{a} \cos(\omega_0 t + \beta) - a(\omega_0 + \dot{\beta}) \sin(\omega_0 t + \beta). \tag{2.60}
\]

Equating Equations (2.59) and (2.60) yielded the constraint equation for the system:

\[
\dot{a} \cos(\omega_0 t + \beta) - a \dot{\beta} \sin(\omega_0 t + \beta) = 0. \tag{2.61}
\]

The equation of motion is rewritten including Equations (2.58), (2.59) and:

\[
\ddot{u} = -\omega_0 \dot{a} \sin(\omega_0 t + \beta) - \omega_0 a(\omega_0 + \dot{\beta}) \cos(\omega_0 t + \beta). \tag{2.62}
\]

Also, \( \phi = \omega_0 t + \beta \) is inserted, yielding:

\[
-\omega_0 \dot{a} \sin \phi - \omega_0 a(\omega_0 + \dot{\beta}) \cos \phi - 2\varepsilon \mu \omega_0 a \sin \phi \\
+ \varepsilon \kappa a \cos \phi \sin \phi \sin \phi + \omega_0^2 a \cos \phi = \varepsilon F \cos \phi. \tag{2.63}
\]

Using the constraint Equation (2.61), \( \dot{a} \) and \( a \dot{\beta} \) are found:

\[
\dot{a} = -2\varepsilon \mu a \sin^2 \phi + \varepsilon \frac{\kappa}{\omega_0} a \cos \phi \sin \phi \sin \phi a \cos \phi |a \cos \phi| - \varepsilon \frac{F}{\omega_0} \cos(\Omega t) \sin \phi \tag{2.64}
\]

\[
a \dot{\beta} = -2\varepsilon \mu a \sin \phi \cos \phi + \varepsilon \frac{\kappa}{\omega_0} a \cos^2 \phi |a \cos \phi| - \varepsilon \frac{F}{\omega_0} \cos(\Omega t) \cos \phi. \tag{2.65}
\]

These are the un-averaged equations for the response amplitude and phase.
The final step is to average these equations over a period. Before this can be done, \( \Omega \) must be defined in terms of \( \phi \). Since the system is excited at the primary resonance, \( \Omega = \omega_0 + \varepsilon \sigma \). This condition is used to rewrite the equations in the form:

\[
\dot{a} = -2\varepsilon \mu a \sin^2 \phi + \varepsilon \frac{K}{\omega_0} \cos \phi \sin \phi |a\cos \phi|
\]

\[
-\varepsilon \frac{F}{\omega_0} \cos(\phi - \beta + \sigma T_1) \sin \phi
\]

\[
a \dot{\beta} = -2\varepsilon \mu a \sin \phi \cos \phi + \varepsilon \frac{K}{\omega_0} \cos^2 \phi |a\cos \phi|
\]

\[
-\varepsilon \frac{F}{\omega_0} \cos(\phi - \beta + \sigma T_1) \cos \phi
\]

where \( T_1 = \varepsilon t \). The equations are averaged by \( \phi \) over a period of \( 2\pi \), resulting in the following equations:

\[
\dot{a} = -\varepsilon \mu a - \varepsilon \frac{F}{2\omega_0} \sin(\beta - \sigma T_1)
\]

\[
a \dot{\beta} = \frac{4}{3\pi} \varepsilon \frac{\kappa}{\omega_0} a |a| - \varepsilon \frac{F}{2\omega_0} \cos(\beta - \sigma T_1).
\]

As before with the cubic stiffness equations, \( \gamma \) is set to \( \beta - \sigma T_1 \), resulting in:

\[
\dot{a} = -\varepsilon \mu a - \varepsilon \frac{F}{2\omega_0} \sin \gamma
\]

\[
a \dot{\gamma} = -\sigma a + \frac{4}{3\pi} \varepsilon \frac{\kappa}{\omega_0} a |a| - \varepsilon \frac{F}{2\omega_0} \cos \gamma.
\]

The equations are examined at a steady state condition by setting the derivatives equal to zero. The solution for \( \sigma \) in terms of \( a \) is more easily obtained than the inverse relationship:

\[
\sigma = \frac{4\kappa}{3\pi \omega_0} |a| \pm \frac{1}{a} \sqrt{F^2 - \frac{\mu^2 a^2}{4\omega_0^2}}.
\]

This relationship produces a backbone structure as demonstrated previously, and is seen in Figure 2.17. The term under the radical sign contains the key to finding the maximum value for \( a \).

\[
a_{\text{peak}} = \frac{F}{2\omega_0 \mu}.
\]
Figure 2.17. Amplitude of the response of the rectified quadratic stiffness EOM at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1000$. 
As with the cubic stiffness model, the peak value of $a$ does not depend on the value of the nonlinear stiffness term, but solely on the forcing term, the linear damping term, and the primary resonance.

After recognizing that all of these terms in $a_{\text{peak}}$ are positive, $a_{\text{peak}}$ is reinserted into Equation (2.72) to yield:

$$
\sigma_{\text{peak}} = \frac{2\kappa F}{3\pi \mu \omega_0^2}.
$$

As the nonlinear stiffness coefficient increases, the frequency of excitation that is required to reach the peak amplitude shifts above the primary resonance. $\gamma$ also changes with $\sigma$ as governed by the equation:

$$
\sigma = \mu \frac{\cos \gamma}{\sin \gamma} + \frac{4\kappa}{3\pi \omega_0^2} \frac{F}{2\mu \omega_0} |\sin \gamma|
$$

and seen in Figure 2.18. As in the previous example and here seen in Figure 2.19, the change in amplitude at the peak detune value is proportional to the force level. The
Figure 2.19. Amplitude of the response at the peak detune vs. forcing amplitude when excited at the primary resonance.
change in response at the primary resonance is more complicated, once again being
ddictated by the values of the coefficients of the EOM. The equation that dictates the
response at the primary resonance is:

\[ 16\kappa^2 a^4 + 9\pi^2 \omega_0^2 \mu^2 a^2 = \frac{9}{4} \pi^2 F^2. \] (2.76)

The change in response will not be proportional to the change in forcing amplitude,
and it will be less than the change seen at the peak detuning value.

The key feature of this nonlinearity is the backbone curve, similar to the one
seen in the cubic stiffness. A difference between the cubic stiffness response and
the rectified quadratic stiffness is the secondary response. The cubic stiffness non-
linearity produces a secondary response at three times the primary resonance, while
the rectified quadratic stiffness produces a secondary response at twice the primary
resonance.

**Rectified Quadratic Stiffness Model with Harmonic Excitation**

The same approach for harmonic excitation is used in this case when the frequency
of excitation is near the primary resonance. The order of the forcing function is not
changed to order epsilon since the force level needs to be greater to excite harmonic
responses. The equation of motion is:

\[ \ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon \kappa |u|u + \omega_0^2 u = F \cos(\Omega t). \] (2.77)

The assumed solution for \( u \) is of the same form used in the cubic stiffness problem:

\[ u = a(t) \cos[\omega_0 t + \beta(t)] + 2\Lambda \cos(\Omega t) \] (2.78)

where \( \Lambda = \frac{F}{2(\omega_0^2 - \Omega^2)} \). Following the same assumptions as previously used:

\[ \dot{u} = -\omega_0 a \sin(\omega_0 t + \beta) - \Omega \Lambda \sin(\Omega t). \] (2.79)
The full differential of (2.78) is:

\[ \dot{u} = \dot{a} \cos(\omega_0 t + \beta) - a(\omega_0 + \dot{\beta}) \sin(\omega_0 t + \beta) - \Omega \Lambda \sin(\Omega t). \]  

(2.80)

Equating (2.78) and (2.80) result in the constraint equation:

\[ \dot{a} \cos(\omega_0 t + \beta) - a \dot{\beta} \sin(\omega_0 t + \beta) = 0 \]  

(2.81)

which is the same as before.

Substituting all of the equations into the equation of motion including \( \phi = \omega_0 t + \beta \) and using the constraint equation generates two un-averaged equations describing the amplitude and phase:

\[ \dot{a} = -\frac{\Omega^2}{\omega_0} \Lambda \cos(\Omega t) \sin \phi - 2 \varepsilon \mu a \sin^2 \phi - 2 \varepsilon \mu \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \sin \phi \\
+ \omega_0 \Lambda \cos(\Omega t) \sin \phi + \varepsilon \frac{K}{\omega_0} a \cos \phi \sin \phi |a \cos \phi + \Lambda \cos(\Omega t)| \\
+ \varepsilon \frac{K}{\omega_0} \Lambda \cos(\Omega t) \sin \phi |a \cos \phi + \Lambda \cos(\Omega t)| - \frac{F}{\omega_0} \cos(\Omega t) \sin \phi \]  

(2.82)

\[ a \dot{\beta} = -\frac{\Omega^2}{\omega_0} \Lambda \cos(\Omega t) \cos \phi - 2 \varepsilon \mu a \sin \phi \cos \phi - 2 \varepsilon \mu \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \cos \phi \\
+ \omega_0 \Lambda \cos(\Omega t) \cos \phi + \varepsilon \frac{K}{\omega_0} a \cos^2 \phi |a \cos \phi + \Lambda \cos(\Omega t)| \\
+ \varepsilon \frac{K}{\omega_0} \Lambda \cos(\Omega t) \cos \phi |a \cos \phi + \Lambda \cos(\Omega t)| - \frac{F}{\omega_0} \cos(\Omega t) \cos \phi. \]  

(2.83)

Before the equations can be averaged, it is necessary to determine the frequency of the harmonic excitation. From previous work done by Neyfah (1979), it is known that quadratic non-linear models exhibit harmonic resonance at one half and twice the primary resonance. Using this knowledge, two frequencies of excitation are chosen for the analysis performed here.
Excitation at One Half the Primary Resonance

In order to excite the system at one half the primary resonance, $\Omega$ is set equal to $\frac{1}{2} \omega_0 + \varepsilon \sigma$. Since the excitation frequency has a period that is twice that of the primary resonance, Equations (2.82) and (2.83) are averaged over $4\pi$. The resulting averaging yields the following equations:

\[
\dot{a} = -\varepsilon \mu a + \varepsilon \frac{\kappa \Lambda^2}{4\omega_0} \sin(\beta - \sigma T_1) \frac{|a + \Lambda \cos(\frac{1}{2}(\beta - \sigma T_1))|}{a + \Lambda \cos(\frac{1}{2}(\beta - \sigma T_1))} \tag{2.84}
\]

\[
\dot{a} = \varepsilon \frac{\kappa \Lambda^2}{4\omega_0} \cos(\beta - \sigma T_1) \frac{|a + \Lambda \cos(\frac{1}{2}(\beta - \sigma T_1))|}{a + \Lambda \cos(\frac{1}{2}(\beta - \sigma T_1))} \tag{2.85}
\]

By letting $\gamma = \beta - \sigma T_1$, it is possible to solve for $\sigma$ in terms of $a$ at steady state.

\[
\sigma = \pm \sqrt{\frac{\kappa^2 \Lambda^4}{16 \omega_0^2 a^2} - \mu^2}. \tag{2.86}
\]

These conditions lead to:

\[
a_{\text{peak}} = \frac{\kappa \Lambda^2}{4\omega_0 \mu}. \tag{2.87}
\]

$a_{\text{peak}}$ occurs at $\sigma = 0$. The plots of the amplitude and phase change with respect to the detuning parameter $\sigma$ are seen in Figures 2.20 and 2.21. $a_{\text{peak}}$ is dependent on the nonlinear coefficient, unlike the value of $a$ when excited at the primary resonance. Also of interest is the relationship of the amplitude to the force value, which is plotted in Figure 2.22. The amplitude is proportional to the force value squared.

This analysis is not much different than the results from the pure quadratic stiffness. Therefore, it is important to use the differences seen when exciting the system at the primary resonance to help distinguish the pure quadratic stiffness and the rectified quadratic stiffness.
Figure 2.20. Amplitude of the response of the rectified quadratic stiffness EOM at the primary resonance when excited at 1/2 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1000$.

Figure 2.21. Phase of the response at the primary resonance when excited at 1/2 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $\kappa = 1000$. 
Figure 2.22. Amplitude of the response at the primary resonance vs. forcing amplitude when excited at 1/2 the primary resonance.
Excitation at Twice the Primary Resonance

In this case, \( \Omega \) was set equal to \( 2\omega_0 + \varepsilon \sigma \). Equations (2.82) and (2.83) were averaged over \( 2\pi \) to produce:

\[
\dot{a} = -\varepsilon \mu a + \varepsilon \frac{\kappa}{2\omega_0} a \Lambda \sin[2(\beta - \sigma T_1)] \frac{|a + \Lambda \cos(2(\beta - \sigma T_1))|}{a + \Lambda \cos(2(\beta - \sigma T_1))} \quad (2.88)
\]

\[
a \dot{\beta} = \varepsilon \frac{\kappa}{2\omega_0} a \Lambda \cos[2(\beta - \sigma T_1)] \frac{|a + \Lambda \cos(2(\beta - \sigma T_1))|}{a + \Lambda \cos(2(\beta - \sigma T_1))}. \quad (2.89)
\]

Yet again, \( \gamma = \beta - \sigma T_1 \) was introduced, generating:

\[
\dot{a} = -\varepsilon \mu a + \varepsilon \frac{\kappa}{2\omega_0} a \Lambda \sin(2\gamma) \frac{|a + \Lambda \cos(2\gamma)|}{a + \Lambda \cos(2\gamma)} \quad (2.90)
\]

\[
a \dot{\gamma} = -\sigma a + \varepsilon \frac{\kappa}{2\omega_0} a \Lambda \cos(2\gamma) \frac{|a + \Lambda \cos(2\gamma)|}{a + \Lambda \cos(2\gamma)}. \quad (2.91)
\]

An examination of the equations indicates that, at steady state, the value of \( a \) goes to zero unless \( \sigma = \pm \sqrt{\frac{\kappa^2 \Lambda^2}{4\omega_0^2} - \mu^2} \). Since achieving this condition is far from excitation at twice the primary resonance, the zero response condition is left as the only response.

2.2 Nonlinear Damping Models

Nonlinear damping models are also studied here because damage in composite materials may also introduce local nonlinear damping characteristics. Three different models of nonlinear damping are studied: a cubic damping model and two quadratic damping models.

\[
\ddot{u} + 2\mu \dot{u} + N\dot{u}^3 + \omega_0^2 u = F \cos(\Omega t) \quad (2.92)
\]

\[
\ddot{u} + 2\mu \dot{u} + N|\dot{u}| \dot{u} + \omega_0^2 u = F \cos(\Omega t) \quad (2.93)
\]

\[
\ddot{u} + 2\mu \dot{u} + Nu^2 + \omega_0^2 u = F \cos(\Omega t). \quad (2.94)
\]

To determine the possible responses to nonlinear damping in materials that are damaged in different ways, all models are analyzed here.
2.2.1 Cubic Nonlinear Damping at Primary Resonance

The same method used for the cubic stiffness model is used here. Equation (2.93) is rewritten as:

\[ \ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon N \dot{u}^3 + \omega_0^2 u = \varepsilon F \cos(\Omega t). \]  

(2.95)

Following the same steps as before yields a solution for \( u_0 \) that is the same as before:

\[ u_0 = A e^{i \omega_0 T_0} + \bar{A} e^{-i \omega_0 T_0}. \]  

(2.96)

Since the system is excited at the primary resonance, a detuning parameter is introduced in the equations:

\[ \Omega = \omega_0 + \varepsilon \sigma. \]  

(2.97)

Inserting this and \( u_0 \) into the \( \varepsilon \) equation, grouping the real and imaginary parts, and then converting to an autonomous equation produces the following equations:

\[ a' = -\mu a - \frac{3}{8} N \omega_0^2 a^3 + \frac{F}{2 \omega_0} \sin \gamma \]  

(2.98)

\[ a \gamma' = \sigma a + \frac{F}{2 \omega_0} \cos \gamma. \]  

(2.99)

Solving the equation for \( \sigma \) in terms of \( a \) at steady state leads to:

\[ \sigma = \pm \sqrt{\frac{F^2}{4 \omega_0^2 a^2} - \mu - \frac{3 N \omega_0^2 a^2}{8}}. \]  

(2.100)

This solution does not produce a backbone characteristic like the one exhibited in the Duffing equation analysis, as shown in Figure 2.23. The maximum response occurs at a detuning value of zero. Rewriting the solution for \( a \) with \( \sigma \) set to zero yields:

\[ a_{\text{peak}} = \sqrt{-4\mu + \sqrt{6NF^2 + 16\mu^2}}. \]  

(2.101)

\( \gamma \) at the peak response was equal to \(-\frac{\pi}{2}\) which is shown in phase plot in Figure 2.24.
Figure 2.23. Amplitude of the response of the cubic damping EOM at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$.

Figure 2.24. Phase of the response at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$. 
Upon analyzing the change in response amplitude with a change in force amplitude, a simple relationship between these two amplitudes is not obtained. Tracking the change in response amplitude with respect to a change in force amplitude does not simplify into a simple solution. The relationship is found to be:

\[ F^2 = 4\mu\omega_0^2a^2 + \frac{3}{2}N\omega_0^4a^4. \]  

(2.102)

This relationship dictates that the response is not proportional to the force change, and the response change will be smaller than the change in force.

The full first order solution is:

\[ u = a\cos(\Omega t - \gamma) - \varepsilon \frac{a^3\omega_0}{32}\sin(3\Omega t - 3\gamma). \]  

(2.103)

In determining if the potential cubic nonlinearity is stiffness or damping, the backbone curve, or lack thereof, is a key. Both cubic systems have responses at three times the primary resonance.

2.2.2 Forcing Frequency Away from Primary Resonance

When forcing the system at a frequency away from the primary resonance, the order of the forcing function must again be larger. Therefore, the model is rewritten to include the larger forcing term:

\[ \ddot{u} + 2\varepsilon\mu\dot{u} + \varepsilon N\dot{u}^3 + \omega_0^2u = F\cos(\Omega t). \]  

(2.104)

As before, the equation is split by orders of \( \varepsilon \) and solved in steps. The solution for \( u_0 \) is the same as before:

\[ u_0 = A e^{i\omega_0T_0} + \Lambda e^{i\Omega T_0} + c.c \]  

(2.105)
where $A$ is the response of the system at the primary resonance and $\Lambda = \frac{F}{2(\omega_0^2 - \Omega^2)}$

Substituting this solution into the $\varepsilon^1$ equation produces the following equation:

$$D_0^2 u_1 + \omega_0^2 u_1 = (-2i\omega_0 A' - 2\mu \omega_0 A - 6N \Omega^2 \omega_0 \Lambda^2 A - 3N \omega_0^3 A_2 \bar{A}) e^{i\omega_0 T_0} +$$

$$+ (-2\mu \Omega A - 3N \Omega^2 \Lambda^3 - 6N \Omega \omega_0^2 \Lambda \bar{A} \bar{A}) e^{i\Omega T_0} +$$

$$+ N \Omega^3 \Lambda^3 e^{3i\Omega T_0} + 3N \Omega \omega_0^2 \Lambda \bar{A}^2 e^{i(\Omega - 2\omega_0) T_0} +$$

$$+ N \omega_0^3 A^3 e^{3\omega_0 T_0} + 3N \Omega^2 \omega_0 \Lambda^2 A e^{i(2\Omega + \omega_0)} +$$

$$- 3N \Omega^2 \omega_0 \Lambda^2 \bar{A} e^{i(2\Omega - \omega_0) T_0} + 3N \Omega \omega_0^2 \Lambda A^2 e^{i(\Omega + 2\omega_0) T_0}. \quad (2.106)$$

Secular terms arose when $3\Omega = \omega_0 + \varepsilon \sigma$ and $\Omega = 3\omega_0 + \varepsilon \sigma$.

**Forcing Frequency at One Third Primary Resonance**

When $3\Omega = \omega_0 + \varepsilon \sigma$, the secular terms in Equation (2.106) reduce to the following equations after separating real and imaginary parts:

$$a' = (-\mu - 3N \Omega^2 \Lambda^2) a - \frac{3}{8} N \omega_0^2 a^3 + \frac{N \Omega^3 \Lambda^3}{\omega_0} \cos \gamma \quad (2.107)$$

$$a\gamma' = \sigma a - \frac{N \Omega^3 \Lambda^3}{\omega_0} \sin \gamma. \quad (2.108)$$

Solving the equations for $\sigma$ in terms of $a$ at steady state leads to the following:

$$\sigma = \pm \sqrt{\frac{N^2 \Omega^6 \Lambda^6}{\omega_0^2 a^2} - \mu - 3N \Omega^2 \Lambda^2 - \frac{3}{8} N \omega_0^2 a^2}. \quad (2.109)$$

Once again there is no backbone characteristic to the response, as seen in Figure 2.25, indicating that the maximum response will be found at $\omega_0$. The maximum response is found to be:

$$a_{\text{peak}} = \sqrt{-\frac{4}{3\omega_0^2} \left( \frac{\mu}{N} + 3\Omega^2 \Lambda^2 \right) + \frac{16}{9\omega_0^4} \left( \frac{\mu}{N} + 3\Omega^2 \Lambda^2 \right)^2 + \frac{8N \Omega^6 \Lambda^6}{3\omega_0^3}}. \quad (2.110)$$
Figure 2.25. Amplitude of the response of the cubic damping EOM at the primary resonance when excited at 1/3 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$.

Figure 2.26. Phase of the response at the primary resonance when excited at 1/3 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$. 
The phase of the response at $\omega_0$ is $\gamma = 0$, with the remainder of the curve seen in Figure 2.26. The response will increase at a near cubic rate compared to the increase in force, but the relationship cannot be determined exactly due to the dependence on the unknown coefficient values.

Returning to Equation (2.106) with the secular terms eliminated, the equation is solved for $u_1$:

$$u_1 = -\frac{iA^3N\omega_0}{8}e^{i\omega_0T_0} + \frac{i\Lambda\Omega}{\Omega^2 - \omega_0^2}e^{i\Omega T_0}(2\mu + 3\Lambda^2\Omega^2 + 6A\Lambda N\omega_0^2) - \frac{3A^2\Lambda^2N\omega_0^2}{(\Omega + 3\omega_0)(\Omega + \omega_0)}e^{i(\Omega + 2\omega_0)T_0} - \frac{3A\Lambda^2N\omega_0^2}{4(\Omega + \omega_0)}e^{i(2\Omega + \omega_0)T_0} + \frac{3A\Lambda^2N\omega_0^2}{4(\Omega - \omega_0)}e^{i(2\Omega - \omega_0)T_0} - \frac{3A^2\Lambda^2N\omega_0^2}{(\Omega - 3\omega_0)(\Omega - \omega_0)}e^{i(\Omega - 2\omega_0)T_0} + c.c. \quad (2.111)$$

Combining this solution with $u_0$ produces the complete first order solution of:

$$u = 2\Lambda \cos(\Omega t) + a \cos(3\Omega t - \gamma) + \varepsilon \left( \frac{Na^3\omega_0}{32} \sin(9\Omega t - 3\gamma) - \frac{\Lambda\Omega}{\Omega^2 - \omega_0^2}(2\mu + 3\Lambda^2\Omega^2 + \frac{3}{2}a^2N\omega_0^2)\sin(\Omega t) + \frac{3a\Lambda^2N\omega_0^2}{4(\Omega + \omega_0)}\sin(5\Omega t - \gamma) + \frac{3a\Lambda^2N\omega_0^2}{4(\Omega - \omega_0)}\sin(\Omega t + \gamma) + \frac{3a^2\Lambda N\omega_0^2}{2(\Omega + 3\omega_0)(\Omega + \omega_0)}\sin(7\Omega t - 2\gamma) - \frac{3a^2\Lambda N\omega_0^2}{2(\Omega - 3\omega_0)(\Omega - \omega_0)}\sin(5\Omega t + 2\gamma) \right). \quad (2.112)$$

Once again, the important keys for identifying a cubic damping nonlinearity are the lack of a backbone curve in the response at the primary resonance when the system is excited at one-third the primary resonance, and a response at three times the primary resonance.
Forcing Frequency at Three Times Primary Resonance

When \( \Omega = 3\omega_0 + \varepsilon \sigma \), the secular terms change. When worked as previously described, the terms yield:

\[
a' = -(\mu + 3N\Omega^2\Lambda^2)a - \frac{3}{8}N\omega_0^2a^3 + \frac{3}{4}N\Omega\omega_0\Lambda a^2 \cos \gamma \tag{2.113}
\]
\[
a\gamma' = \sigma a - \frac{9}{4}N\Omega\omega_0\Lambda a^2 \sin \gamma. \tag{2.114}
\]

At steady state, it can be seen that \( a = 0 \). Further exploration of other solutions for \( a \) at steady state yield only situations that are not under consideration for the current system.

2.2.3 Quadratic Nonlinear Damping

With two different models for quadratic damping, two different analysis methods are employed. When analyzing the rectified damping model, the Method of Averaging is used. The Method of Multiple Time Scales is used to examine the pure quadratic damping model. An analysis of the pure quadratic model is given first.

Pure Quadratic Non-linear Damping at Primary Resonance

Similar to other analyses in this work, the order of the forcing function is set to \( \varepsilon \) along with the damping terms. The following model was obtained using this approach:

\[
\ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon Nu^2 + \omega_0^2 u = \varepsilon F \cos(\Omega t). \tag{2.115}
\]
After splitting the equation into terms of \( \varepsilon \) and solving for \( u_0 \), which is the same as previous solutions at the primary resonance, the secular terms in the \( \varepsilon^1 \) equation are eliminated. This results in the following solutions:

\[
a' = -\mu a - \frac{F}{2\omega_0} \sin(\sigma T_1 - \beta)
\]

(2.116)

\[
a \beta' = -\frac{F}{2\omega_0} \cos(\sigma T_1 - \beta).
\]

(2.117)

Setting \( \gamma = \sigma T_1 - \beta \) and then solving for \( a \) in terms of \( \sigma \) produces:

\[
a = \sqrt{\frac{F^2}{4\omega_0^2(\mu^2 + \sigma^2)}}.
\]

(2.118)

This is the same response as exhibited by the Quadratic Stiffness model that is plotted in Figures 2.9 and 2.10. The change in response for a change in force level is also the same, as plotted in Figure 2.11.

Continuing the analysis, \( u_1 \) is found after the secular terms have been set to zero. The solution for \( u_1 \) was:

\[
u_1 = -\frac{1}{3} N A^2 e^{2\omega_0 T_0} - 2 N A \tilde{A} + c.c.
\]

(2.119)

Combining the \( u_0 \) and \( u_1 \) solution yields the complete first order solution:

\[
u = a \cos(\Omega t - \gamma) + \varepsilon \left[ -\frac{N a^2}{6} \cos(2\Omega t - 2\gamma) - \frac{N a^2}{2} \right].
\]

(2.120)

These responses change in the same manner as the response changes seen in the pure quadratic stiffness model.

**Pure Quadratic Damping away from Primary Resonance**

To examine the response of the system away from excitation at the primary resonance, the model is modified slightly. The force level needed to be larger, so it is no
longer modeled as an order \( \varepsilon \) term. After splitting the equation based on orders of \( \varepsilon \) and solving for \( u_0 \), which is the same as in other analyses away from the primary resonance, the \( \varepsilon^1 \) equation becomes:

\[
D_0^2 u_1 + \omega_0^2 u_1 = (-2i\omega_0 A' - 2\mu \omega_0 A)e^{i\omega_0 T_0} - 2i\mu \Omega \Lambda e^{i\Omega T_0} + N\omega_0^2 A' e^{2i\omega_0 T_0} + 2N\omega_0 \Omega \Lambda A e^{i(\Omega + \omega_0) T_0} - 2N\omega_0 \Omega \Lambda \bar{A} e^{i(\Omega - \omega_0) T_0} + N\Omega^2 \Lambda^2 e^{2i\Omega T_0} - 2N\omega_0^2 A \bar{A} - 2N\Omega^2 \Lambda^2 + c.c. \tag{2.121}
\]

When \( 2\Omega = \omega_0 + \varepsilon\sigma \) and \( \Omega = 2\omega_0 + \varepsilon\sigma \), it can be seen that additional secular terms are created. Each of these cases is examined separately.

**Excitation at One-Half Primary Resonance**

When \( 2\Omega = \omega_0 + \varepsilon\sigma \), the secular terms are set to zero:

\[
-2i\omega_0 A' - 2\mu \omega_0 A + N\Omega^2 \Lambda^2 e^{i\sigma T_1} = 0. \tag{2.122}
\]

Following the same steps as previously outlined, the amplitude of the response is found to be:

\[
a = \sqrt{\frac{N^2 \Omega^4 \Lambda^4}{\omega_0^2 (\mu^2 + \sigma^2)}}. \tag{2.123}
\]

This response exhibits no backbone characteristic, with a maximum response at \( \sigma = 0 \). The change in response with changes in force is shown in Figure 2.27. The response amplitude is proportional to the square of the force level.

Returning back to the \( \varepsilon^1 \) equation, it is solved for \( u_1 \) with the secular terms set to zero, yielding:

\[
u_1 = -\frac{1}{3} N A^2 e^{2i\omega_0 T_0} + \frac{2i\mu \Omega \Lambda}{\Omega^2 - \omega_0^2} e^{i\Omega T_0} - \frac{2N\omega_0 \Lambda A}{\Omega + 2\omega_0} e^{i(\Omega + \omega_0) T_0} + \frac{2N\omega_0 \Lambda \bar{A}}{\Omega - 2\omega_0} e^{i(\Omega - \omega_0) T_0} - 2NA \bar{A} - 2N\Omega^2 \Lambda^2 \frac{1}{\omega_0^2} + c.c. \tag{2.124}
\]
Figure 2.27. Amplitude of response vs. forcing input at primary resonance, excited at 1/2 primary resonance.
Combining the $u_0$ and $u_1$ solutions together creates the complete first order solution:

$$u = 2\Lambda \cos(\Omega t) + a \cos(2\Omega t - \gamma) + \varepsilon\left[-\frac{Na^2}{6} \cos(4\Omega t - 2\gamma) - \frac{4\mu\Lambda}{\Omega^2 - \omega_0^2} \sin(\Omega t) - \frac{2N\omega_0\Lambda a}{\Omega + 2\omega_0} \cos(3\Omega t - \gamma) + \frac{2N\omega_0\Lambda a}{\Omega - 2\omega_0} \cos(\Omega t + \gamma) - \frac{Na^2}{2} - \frac{2N\Lambda^2\Omega^2}{\omega_0^2}\right].$$

(2.125)

While the responses from the pure quadratic damping model are slightly different than the pure quadratic stiffness model, the trends in the response amplitudes due to a change in forcing amplitude are the same. It appears that another tool is necessary to help distinguish the pure quadratic stiffness from the pure quadratic damping.

### 2.2.4 Rectified Quadratic Damping at Primary Resonance

Once again, this analysis follows the analysis of the Rectified Quadratic Stiffness. Therefore, many steps will be skipped in order to reach the solution faster.

To start, a solution to the differential equation is assumed,

$$u = a(t) \cos[\omega_0 t + \beta(t)].$$

(2.126)

Following the previous steps, a constraint equation is found:

$$\dot{a} \cos(\omega_0 t + \beta) - a\dot{\beta} \sin(\omega_0 t + \beta) = 0.$$  

(2.127)

Substituting derivatives of $u$ into the equation of motion, and then using that result and Equation (2.127), $\dot{a}$ and $a\dot{\beta}$ are found:

$$\dot{a} = -2\varepsilon\mu a \sin^2 \phi - \varepsilon Na \sin^2 \phi |\omega_0 a \sin \phi|$$

$$-\varepsilon \frac{F}{\omega_0} \cos(\Omega t) \sin \phi$$

(2.128)

$$a\dot{\beta} = -2\varepsilon\mu a \sin \phi \cos \phi - \varepsilon Na \sin \phi \cos \phi |\omega_0 a \sin \phi|$$

$$-\varepsilon \frac{F}{\omega_0} \cos(\Omega t) \cos \phi$$

(2.129)
where $\phi = \omega_0 t + \beta$.

This result is then averaged over $2\pi$, yielding:

$$\dot{a} = -\varepsilon \mu a - \varepsilon \frac{4}{3\pi} Na|a\omega_0| - \varepsilon \frac{F}{2\omega_0} \sin(\beta - \sigma T_1) \tag{2.130}$$

$$a\dot{\beta} = -\varepsilon \frac{F}{2\omega_0} \cos(\beta - \sigma T_1). \tag{2.131}$$

Letting $\gamma = \beta - \sigma T_1$ generates:

$$\dot{a} = -\varepsilon \mu a - \varepsilon \frac{4}{3\pi} Na|a\omega_0| - \varepsilon \frac{F}{2\omega_0} \sin \gamma \tag{2.132}$$

$$a\dot{\gamma} = -\varepsilon \sigma a - \varepsilon \frac{F}{2\omega_0} \cos \gamma. \tag{2.133}$$

Solving for $\sigma$ in terms of $a$ at steady state leads to:

$$\sigma = \pm \sqrt{\frac{F^2}{4\omega_0^2 a^2} - (\mu + \frac{4}{3\pi} Na|a\omega_0|)^2}. \tag{2.134}$$

This response does not exhibit the backbone curve (refer to Figure 2.28) that is evident in the responses for the stiffness nonlinearities. The phase of the response also does not experience a jump, as seen in Figure 2.29. Solving for the maximum response yields:

$$a_{peak} = -3\pi \mu \omega_0 + \sqrt{24 FN\pi \omega_0 + 9\pi^2 \mu^2 \omega_0^2} \over 8N\omega_0^2. \tag{2.135}$$

The change in the response amplitude is smaller than the change in the force level.

Without a backbone curve, the rectified quadratic damping is different from the other quadratic models in the change in response amplitude compared to the change in forcing amplitude. The response amplitude at the primary resonance of the rectified quadratic damping model is not proportional to the forcing amplitude, unlike the other quadratic models.
Figure 2.28. Amplitude of the response of the rectified quadratic damping EOM at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$.

Figure 2.29. Phase of the response at the primary resonance when excited at the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$. 
Forcing Frequency Away from Primary Resonance

For this case, a solution to the differential equation is assumed,

\[ u = a(t) \cos[\omega_0 t + \beta(t)] + 2\Lambda \cos(\Omega t). \]  
(2.136)

Following the same steps as before results in the following equations:

\[
\dot{a} = -2\frac{\Omega^2}{\omega_0}\Lambda \cos(\Omega t) \sin \phi - 2\varepsilon \mu a \sin^2 \phi - 4\varepsilon \mu \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \sin \phi \\
-\varepsilon N a \sin^2 \phi |\omega_0 a \sin \phi + 2\Omega \Lambda \sin(\Omega t)| \\
-2\varepsilon N \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \sin \phi |\omega_0 a \sin \phi + 2\Omega \Lambda \sin(\Omega t)| \\
+2\omega_0 \Lambda \cos(\Omega t) \cos \phi - F \frac{\omega_0}{\omega_0} \cos(\Omega t) \sin \phi 
\]  
(2.137)

\[
a\dot{\beta} = -2\frac{\Omega^2}{\omega_0} \Lambda \cos(\Omega t) \cos \phi - 2\varepsilon \mu a \sin \phi \cos \phi - 4\varepsilon \mu \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \cos \phi \\
-\varepsilon N a \sin \phi \cos \phi |\omega_0 a \sin \phi + 2\Omega \Lambda \sin(\Omega t)| \\
-2\varepsilon N \frac{\Omega}{\omega_0} \Lambda \sin(\Omega t) \cos \phi |\omega_0 a \sin \phi + 2\Omega \Lambda \sin(\Omega t)| \\
+2\omega_0 \Lambda \cos(\Omega t) \cos \phi - F \frac{\omega_0}{\omega_0} \cos(\Omega t) \cos \phi 
\]  
(2.138)

where \( \phi = \omega_0 t + \beta \). As seen in the other quadratic damping model, unique behavior occurs when the system is excited at one half the primary resonance and at twice the primary resonance.

Forcing Frequency at One Half Primary Resonance

When \( 2\Omega = \omega_0 + \varepsilon \sigma \), Equations (2.137) and (2.138) are averaged over \( 4\pi \):

\[
\dot{a} = -\mu a - \frac{N\Omega^2\Lambda^2}{\omega_0} \sin(\beta - 2\sigma T_1) \left| \frac{\sin(\frac{1}{2}(\beta - 2\sigma T_1))}{\sin(\frac{1}{2}(\beta - 2\sigma T_1))} \right| 
\]  
(2.139)

\[
a\dot{\beta} = -\frac{N\Omega^2\Lambda^2}{\omega_0} \cos(\beta - 2\sigma T_1) \left| \frac{\sin(\frac{1}{2}(\beta - 2\sigma T_1))}{\sin(\frac{1}{2}(\beta - 2\sigma T_1))} \right| 
\]  
(2.140)
By letting $\gamma = \beta - 2\sigma T_1$, the equations are rewritten in autonomous form:

$$\dot{a} = -\mu a - \frac{N\Omega^2\Lambda^2}{\omega_0} \sin \gamma \left| \sin \left( \frac{1}{2} \gamma \right) \right| \sin \left( \frac{1}{2} \gamma \right)$$

$$a\dot{\gamma} = -2\sigma a - \frac{N\Omega^2\Lambda^2}{\omega_0} \cos \gamma \left| \sin \left( \frac{1}{2} \gamma \right) \right| \sin \left( \frac{1}{2} \gamma \right).$$

Solving for $a$ in terms of $\sigma$ leads to:

$$a = \pm \sqrt{\frac{N^2\Lambda^4\Omega^4}{\omega_0^2 (\mu^2 + 4\sigma^2)}}.$$ 

This response also does not have a backbone characteristic as seen in Figure 2.30. The phase of the response also does not experience a jump, as seen in Figure 2.31. The change in response amplitude versus the change in forcing amplitude is graphed in Figure 2.32. The change in response for a change in forcing level exhibits a quadratic relationship.
Figure 2.31. Phase of the response at the primary resonance when excited at 1/2 the primary resonance. Coefficient values are $\omega_0 = 12$, $\mu = 1$, $F = 5$, and $N = 1$.

Figure 2.32. Amplitude of the response at the primary resonance vs. forcing amplitude when excited at 1/2 the primary resonance.
These responses are very similar to the other quadratic models. Once again, it is necessary to use information from the excitation at the primary resonance to help distinguish the models.

**Forcing Frequency at Twice the Primary Resonance**

When $\Omega = 2\omega_0 + \varepsilon\sigma$, Equations (2.137) and (2.138) are averaged over $2\pi$ to yield:

\[
\dot{a} = -\varepsilon\mu a + \varepsilon Na\Lambda \Omega \sin(2\beta - \sigma T) \frac{|\sin(2\beta - \sigma T)|}{\sin(2\beta - \sigma T)} \quad (2.144)
\]
\[
a\dot{\beta} = \varepsilon Na\Lambda \Omega \cos(2\beta - \sigma T) \frac{|\sin(2\beta - \sigma T)|}{\sin(2\beta - \sigma T)} \quad (2.145)
\]

At steady state, $a$ goes to zero, unless $\sigma = -2\mu \cot \gamma$. Once again, since the second solution requires forcing away from twice the primary resonance, the zero response solution is the only one considered for this research.

**2.3 Combined Solutions**

As will be shown in a following section, test data shows a strong indication of a pure quadratic stiffness. There is also an indication of a cubic nonlinearity, but it did not match any of the cubic models that have been presented. From this, it is thought that the cubic nonlinearity is of a different order in $\varepsilon$ from the quadratic stiffness. This section will analyze two equations of motion with a different order on the cubic term.

**2.3.1 Quadratic Stiffness and Cubic Stiffness**

The equation of motion for this section includes both a quadratic stiffness term and a cubic stiffness term, with the cubic term having a higher order of $\varepsilon$:

\[
\ddot{u} + 2\varepsilon \mu \dot{u} + \varepsilon \kappa u^2 + \varepsilon^2 N u^3 + \omega_0^2 u = F \cos(\Omega t).
\quad (2.146)
\]
With the cubic term being a second-order term, it does not change the first-order solutions found in the pure quadratic stiffness section. The cubic stiffness term would appear in a second-order approximation, but the first-order approximation fits very well when excited at the primary resonance and at one half the primary resonance, as will be shown later. With this understanding, the solution when $3\Omega = \omega_0$ is focused on. Recalling Equation (2.51), the secular terms are set to zero:

$$-2i\omega_0 A' - 2i\omega_0 A = 0. \quad (2.147)$$

Solving for $A$ yields:

$$A = A(T_2)e^{-\mu T_1} + B(T_2). \quad (2.148)$$

As $T_1$ goes to $\infty$, the first term goes to zero, but the second term could be stable at steady-state conditions.

With the secular terms eliminated, the equation becomes:

$$D_0^2 u_1 + \omega_0^2 u_1 = -2\mu \Omega \Lambda e^{i \Omega T_0} - \kappa A^2 e^{2i\omega_0 T_0}$$

$$-2\kappa A\Lambda e^{i(\omega_0 + \Omega)T_0} - 2\kappa A\Lambda e^{i(\omega_0 - \Omega)T_0}$$

$$-\kappa A^2 e^{2\Omega T_0} - 2\kappa A\bar{A} - 2\kappa \Lambda^2 + c.c. \quad (2.149)$$

Solving Equation (2.149) for $u_1$ generates:

$$u_1 = \frac{8i\mu \Lambda \Omega}{(4\Omega^2 - \omega_0^2)(\Omega^2 - \omega_0^2)} e^{i \Omega T_0} + \frac{2\kappa A^2}{3\omega_0^2(4\Omega^2 - \omega_0^2)} e^{2i\omega_0 T_0}$$

$$+ \frac{\kappa \Lambda^2}{4\Omega^2 - \omega_0^2} e^{2i \Omega T_0} + \frac{2\kappa A \Lambda}{\Omega(4\Omega^2 - \omega_0^2)(\Omega + 2\omega_0)} e^{i(\omega_0 + \Omega)T_0}$$

$$+ \frac{2\kappa A \Lambda}{\Omega(4\Omega^2 - \omega_0^2)(\Omega - 2\omega_0)} e^{i(\omega_0 - \Omega)T_0} - \frac{2\kappa A\bar{A}}{\omega_0^2} - \frac{2\kappa \Lambda^2}{\omega_0^2}. \quad (2.150)$$

Taking the solutions in Equations (2.150) and (2.25) and substituting them into the $\varepsilon^2$ equation produces an equation with many terms. Since this is a second-order
approximation, only the secular terms are sought. Setting the secular terms to equal zero yields:

\[
0 = -2i\omega_0 \dot{A} + \mu^2 A + \frac{4\kappa^2 A^2 \bar{A}}{\omega_0^2} + \frac{4\kappa^2 A \Lambda^2}{\omega_0^2} - \frac{2\kappa^2 \Lambda^3}{4\Omega^2 - \omega_0^2} - \frac{4\kappa^2 A \bar{A}}{\Omega(4\Omega^2 - \omega_0^2)(\Omega - 2\omega_0)} - \frac{4\kappa^2 A^2 \bar{A}}{3\omega_0^2(4\Omega^2 - \omega_0^2)} - \frac{4\kappa^2 A^2 \bar{A}}{\Omega(4\Omega^2 - \omega_0^2)(\Omega + 2\omega_0)} - 3NA^2 \bar{A} - N\Lambda^3 - 6N\Lambda A^2
\]

(2.151)

where \( \dot{()} \) indicates a derivative in the \( T_2 \) time scale. After manipulation of the previous equation to separate the real and imaginary parts and letting \( B = \frac{1}{2}be^{i\beta} \), remembering that \( A = A(T_2)e^{-\mu T_1} + B(T_2) \), the steady state condition is examined. Since \( A(T_2)e^{-\mu T_1} \) goes to zero at steady state, \( \beta \) is found to equal 0, \( \pi \), ... at steady state unless \( N = -\frac{2\kappa^2}{4\Omega^2 - \omega_0^2} \). Provided \( N \) does not meet the aforementioned equality, then \( b \) can be solved for.

Solving for \( b \) is difficult due to the order of \( b \) in the equation. However, \( N \) can be identified as the slope of a line:

\[
\frac{\mu^2 b}{2} + \frac{\kappa^2 b^3}{2\omega_0^2} + \frac{2\kappa^2 \Lambda^2 b}{\omega_0^2} - \frac{4\kappa^2 \Lambda^2 b}{(4\Omega^2 - \omega_0^2)(\Omega^2 - 4\omega_0^2)} - \frac{\kappa^2 b^3}{6\omega_0^2(4\Omega^2 - \omega_0^2)} - \frac{2\kappa^2 \Lambda^3}{4\Omega^2 - \omega_0^2} = N\left(\frac{3b^3}{8} + 3\Lambda^2 b + \Lambda^3\right)
\]

(2.152)

where \( \beta = 0 \). If \( \beta = \pi \) than the equation simplifies to:

\[
\frac{\mu^2 b}{2} + \frac{\kappa^2 b^3}{2\omega_0^2} + \frac{2\kappa^2 \Lambda^2 b}{\omega_0^2} - \frac{4\kappa^2 \Lambda^2 b}{(4\Omega^2 - \omega_0^2)(\Omega^2 - 4\omega_0^2)} - \frac{2\kappa^2 \Lambda^3}{6\omega_0^2(4\Omega^2 - \omega_0^2)} + \frac{2\kappa^2 \Lambda^3}{4\Omega^2 - \omega_0^2} = N\left(\frac{3b^3}{8} + 3\Lambda^2 b - \Lambda^3\right).
\]

(2.153)
The correct phase will be addressed in a later section. By organizing the previous equation as the slope of a line, coefficient fitting will be easier. Least-squares fitting is efficient when the data is presented in a linear fashion.

### 2.3.2 Quadratic Stiffness and Cubic Damping

The equation of motion for this study is as follows:

\[
\dddot{u} + 2\varepsilon\mu\ddot{u} + \varepsilon\kappa u^2 + \varepsilon^2 N\dot{u}^3 + \omega_0^2 u = F\cos(\Omega t). \tag{2.154}
\]

The subsequent analysis follows the same steps as in the previous section up to the \(\varepsilon^2\) equation. Substituting Equations (2.150) and (2.25) into the new \(\varepsilon^2\) equation and setting the secular terms to zero yields the following equation:

\[
0 = -2\kappa\omega_0\dot{A} + \mu^2 A + \frac{4\kappa^2 A^2\dot{A}}{\omega_0^2} + \frac{4\kappa^2 AA^2}{\omega_0^2} - \frac{2\kappa^2 A^3}{4\Omega^2 - \omega_0^2} - \frac{4\kappa^2 AA^2}{\Omega(4\Omega^2 - \omega_0^2)(\Omega - 2\omega_0)} - \frac{4\kappa^2 A^2\dot{A}}{3\omega_0^2(4\Omega^2 - \omega_0^2)} - \frac{4\kappa^2 AA^2}{\Omega(4\Omega^2 - \omega_0^2)(\Omega + 2\omega_0)} - 3N\omega_0^3 A^2\dot{A} - Nt\Omega^3 A^3 - 6Nt\Omega^2\omega_0 AA^2. \tag{2.155}
\]

Notice that the difference between the previous secular terms and the current secular terms exists in the last three terms. Once again, manipulating the equation in the same manner as before generates a new equation with \(N^2\) as the slope of a line, as seen:

\[
-\frac{4\kappa^4 A^6}{\omega_0^2} + \left(\frac{\mu^2 b}{2\omega_0} + \frac{\kappa^2 b^3}{2\omega_0^3}\right) + \frac{2\kappa^2 A^2b}{\omega_0^2} - \frac{4\kappa^2 A^2b}{\omega_0^2(4\Omega^2 - \omega_0^2)(\Omega^2 - 4\omega_0^2)} - \frac{\kappa^2 b^3}{6\omega_0^3(4\Omega^2 - \omega_0^2)^2} = N^2\left(\frac{\Omega^6 A^6}{\omega_0^2} - 9\Omega^4 A^4b^2 - \frac{9}{4}\Omega^2\Lambda^2\omega_0^2b^4 - \frac{9}{64}\omega_0^4b^6\right). \tag{2.156}
\]

Although this slope is \(N^2\), the same method for data fitting applies here as did for the cubic stiffness term.
While neither of these two cubic and quadratic models were solved for the amplitude of the response at the primary resonance frequency, the cubic coefficient was isolated as the slope of a line. The points of each line are composed of the amplitude of the response at the primary resonance, the forcing amplitude, and other combinations of the various coefficients of the equation of motion.

2.4 Summary

From the nonlinear analysis that is performed in this section, patterns in the responses of the systems to sinusoidal excitation have emerged. When excited at the primary resonance, all nonlinear models demonstrate a response at the primary frequency and a few models were able to track the response at twice the primary frequency, while others exhibited a secondary response at three times the primary resonance. A summary of the changes in the response amplitude for a change in force level when excited at the primary resonance is listed in Table 2.1. It is also seen that when excited at one half or one third the primary resonance, the response amplitude at the excitation frequency exhibits a linear relationship with respect to force level. All models also demonstrate a response that occurs at the primary resonance. The amplitude of the response at the primary resonance differs between the models, especially when compared to a changing force level. A summary of the changes in the response amplitude that are observed versus the change in force level when the systems are excited at the superharmonic frequency is given in Table 2.2. Tables 2.3 and 2.4 show the phase of the responses when excited at the primary resonance and the superharmonic frequencies, respectively.

With the exception of the cubic models, all of the responses at the primary resonance, when excited at 1/2 the primary resonance, exhibit a quadratic relationship with the forcing amplitude. The differences in response characteristics and frequencies of excitation are sufficient enough to create a decision flowchart to identify the nonlinear characteristics in the systems, as seen in Figure 2.33.
Table 2.1. Summary of nonlinear responses due to excitation at the primary resonance.

<table>
<thead>
<tr>
<th>Nonlinear Type</th>
<th>Change in the response amplitude at the primary resonance compared to the change in the forcing amplitude</th>
<th>Change in the response amplitude at twice or three times the primary resonance compared to the change in the forcing amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic Stiffness</td>
<td>Linear 1 to 1 at $\sigma_{\text{peak}}$ Unidentified at primary resonance, but less than at $\sigma_{\text{peak}}$</td>
<td>Cubic increase on change seen at the primary resonance</td>
</tr>
<tr>
<td>Quadratic Stiffness</td>
<td>Linear 1 to 1</td>
<td>Quadratic</td>
</tr>
<tr>
<td>Rectified Stiffness</td>
<td>Linear 1 to 1 at $\sigma_{\text{peak}}$ Unidentified at primary resonance, but less than at $\sigma_{\text{peak}}$</td>
<td>Unidentified</td>
</tr>
<tr>
<td>Cubic Damping</td>
<td>Unidentified, but less than a linear 1 to 1 ratio</td>
<td>Cubic increase on change seen at the primary resonance</td>
</tr>
<tr>
<td>Quadratic Damping</td>
<td>Linear 1 to 1</td>
<td>Quadratic</td>
</tr>
<tr>
<td>Rectified Damping</td>
<td>Unidentified, but less than a linear 1 to 1 ratio</td>
<td>Unidentified</td>
</tr>
</tbody>
</table>
Table 2.2. Summary of nonlinear responses due to excitation at the superharmonic frequency.

<table>
<thead>
<tr>
<th>Nonlinear Effect</th>
<th>Change in the response amplitude at the primary resonance compared to the change in the forcing amplitude</th>
<th>Change in the response amplitude at twice or three times the primary resonance compared to the change in the forcing amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic Stiffness</td>
<td>Cubic at $\sigma_{\text{peak}}$ Unidentified at primary resonance, but less than a cubic response</td>
<td>Nine times increase at $\sigma_{\text{peak}}$, or a cubic increase on the increase at the primary resonance</td>
</tr>
<tr>
<td>Quadratic Stiffness</td>
<td>Quadratic</td>
<td>Quartic</td>
</tr>
<tr>
<td>Rectified Stiffness</td>
<td>Quadratic</td>
<td>Unidentified</td>
</tr>
<tr>
<td>Cubic Damping</td>
<td>Unidentified, but less than cubic</td>
<td>A Cubic increase on the increase at the primary resonance</td>
</tr>
<tr>
<td>Quadratic Damping</td>
<td>Quadratic</td>
<td>Quartic</td>
</tr>
<tr>
<td>Rectified Damping</td>
<td>Quadratic</td>
<td>Unidentified</td>
</tr>
</tbody>
</table>

Table 2.3. Summary of nonlinear response phasing due to excitation at the primary resonance.

<table>
<thead>
<tr>
<th>Nonlinear Effect</th>
<th>Phase of the Response at the Primary Resonance</th>
<th>Phase of the Response at 2 or 3 Times the Primary Resonance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic Stiffness</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Quadratic Stiffness</td>
<td>$\pi/2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Rectified Stiffness</td>
<td>$\pi/2$</td>
<td>Unidentified</td>
</tr>
<tr>
<td>Cubic Damping</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Quadratic Damping</td>
<td>$\pi/2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Rectified Damping</td>
<td>$\pi/2$</td>
<td>Unidentified</td>
</tr>
</tbody>
</table>
Table 2.4. Summary of nonlinear response phasing due to excitation at the superharmonic frequencies.

<table>
<thead>
<tr>
<th></th>
<th>Phase of the Response at the Primary Resonance</th>
<th>Phase of the Response at 2 or 3 Times the Primary Resonance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic Stiffness</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Quadratic Stiffness</td>
<td>$\pi/2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Rectified Stiffness</td>
<td>$\pi/2$</td>
<td>Unidentified</td>
</tr>
<tr>
<td>Cubic Damping</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Quadratic Damping</td>
<td>$\pi/2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Rectified Damping</td>
<td>$\pi/2$</td>
<td>Unidentified</td>
</tr>
</tbody>
</table>
Figure 2.33. Testing and decision flowchart for determining nonlinearities in the damaged panel.
3. EXPERIMENTAL METHODOLOGY

3.1 Aluminum Honeycomb Sandwich Material

When selecting a material and damage type to use in characterizing nonlinear damage mechanisms, a balance between different desired outcomes of the testing was sought. If one were to go back and examine the analysis of the nonlinear responses with respect to harmonic excitation, it was observed that a lower primary resonance leads to larger response amplitudes. Also, it should be possible to damage the material that is selected in a repeatable manner. Ideally, the damage mechanism that is selected will also be limited to a single, definable mechanism so that any interactions between different types of damages are eliminated. An aluminum honeycomb sandwich material was chosen, as shown in Figure 3.1, because it fulfilled a number of these desired outcomes. Aluminum honeycomb sandwiches are currently in use in advanced aerospace applications. The lower stiffness of the aluminum when compared to a carbon fiber panel led to a lower primary resonance, which was desirable for the reason mentioned above. The sandwich construction of the material is composite in nature, leading to many of the same difficulties in detecting and assessing damage mechanisms as seen in carbon fiber materials. The bonding between multiple layers gives the material high internal damping, which is a characteristic common to many different types of composite materials. The combination of high stiffness and low weight, with relatively lower cost when compared to a carbon fiber reinforced panel, made it an ideal material to choose for conducting these tests.

3.2 Damage Modes

The aluminum honeycomb sandwich material is comprised of two sheets of aluminum bonded to the opposite sides of an aluminum honeycomb core. The bonding
Figure 3.1. Aluminum honeycomb sandwich panel comprised of a facesheet, honeycomb core and backsheet, bonded together with a toughened epoxy adhesive.
material can be a variety of different adhesives or epoxies. Unlike a fiber-matrix composite sandwich material, the aluminum sheets are not susceptible to delamination within the sheet material; however, the aluminum sandwich material can exhibit disbands between the sheet and honeycomb, as well as core crushing. Core crushing damage can be inflicted with an impact tower; however, while this type of damage may appear invisible in a fiber-matrix composite sandwich, the impact can cause plastic deformation of the core and face sheet in an aluminum sandwich material. The disbond damage is the best choice for testing a damage mechanism in the aluminum sandwich material due to the repeatability and isolation of the damage mechanism.

3.3 Damage Creation

In order to create a disbond between the facesheet and honeycomb core, the difference in thermal properties of the aluminum and the epoxy adhesive is exploited. The sandwich panel is placed in a freezer for a minimum of 24 hours. The panel is then removed and a heat source, in the form of a torch, is applied to the center of the panel. Thermal expansion of the facesheet in a small area creates a large enough thermal stress to fracture the frozen, and now brittle, bonding material. The diameter of the disbond is controlled by placing a ring of ice at the edge of the desired disbond area. The ice acts as a heat sink, thereby limiting the area of thermal expansion to the desired area of the disbond.

In order to determine the true size of the disbond, a dial indicator is mounted to a flat steel table. The panels are run under the dial indicator until a small change in displacement is measured, as shown in Figure 3.2. This small displacement is less than 30 thousandths of an inch at maximum displacement. The disbond is marked as the displacement changes from the undamaged part. The change in surface height of the damaged section can be attributed to small plastic deformations of the face sheet when the thermal stresses break the bond and the heated aluminum is allowed to deform, or from internal stresses created during the initial construction of the
Figure 3.2. Dial indicator set-up for use in damage size measurement; the damaged area is determined by measuring a change in honeycomb panel thickness.
sandwich material. An informal tap test is conducted to map out the damaged area and confirm the results of the dial gage measurements. The tap test relies on changes in the audible response of a material to locate damage, and is considered the current standard in damage detection [42]. The change in stiffness, added vibratory interactions, and other acoustic differences indicate potential damage. The tap test confirms the dial gage measurements on damage size. Thermographic testing is also performed on the panels. Thermography is not able to provide any further insight on the size of the disbonds due to the high level of thermal conductivity of the aluminum material. The ice ring has proven successful in limiting the diameter of the disbond and in maintaining a circular shape of the disbond. Three panels are damaged with different sized disbonds: 101, 63.5 and 25.0 millimeter disbonds. The different sizes of disbonds allows for the tracking of the nonlinearities as the damage increases in size.

3.4 Fixture

Due to the nature of the single degree-of-freedom modeling that was conducted in the previous chapter, proper fixturing of the panels is important. Brush in his Master’s Thesis [39] attempts to use a clamping system in order to restrain a composite honeycomb panel and simulate a single degree-of-freedom model. The author admitted that, at times, this set-up was not effective over the range of testing parameters. In order to remove some of the variability in a clamping fixture, a new method for securing a honeycomb panel is used in this research. A Pierson Workholding SmartVac II vacuum chuck, which is seen in Figure 3.3, created a secure way in which to mount a honeycomb panel. By using compressed shop air, the vacuum chuck uses a venturi effect to create a distributed vacuum boundary condition beneath the part. Sizing of the vacuum area is controlled through the use of gaskets in the chuck. The evenly distributed suction force on the back face of the composite honeycomb panel allows the top sheet to be modeled as a plate (face sheet) bonded to an elastic foundation
Figure 3.3. Pierson Workholding SmartVac II vacuum chuck.
3.5 Excitation of the Composite Panel

Having secured the panel with the fixture to facilitate the use of a single degree-of-freedom model, the excitation approach is now described in order to complete the experimental set-up. A K2007E01 Smart Shaker from The Modal Shop is used to excite the damage location of the honeycomb panel. This shaker is able to produce up to 10 lbf of force by receiving an external signal of up to 1 Volt in a frequency range from DC to 9,000 Hz. This shaker system is oriented in a vertical manner above the panel, as seen in Figure 3.4. This orientation of the shaker allows the vacuum
chuck to be secured to a large steel tabletop, thereby minimizing any outside noise sources.

3.6 Testing Setup

A PCB T288D01 impedance head is used to gather acceleration and force measurements. The impedance head has a sensitivity of 100.5 mV/g for the accelerometer and 99.49 mV/lbf for force. The impedance head is attached securely to the center of both the damaged and undamaged panels with cyanoacrylate adhesive, as seen in Figure 3.5. The impedance head is attached to the shaker unit by way of a nylon stinger. The force part of the impedance head measures the actual force as applied by the shaker and the accelerometer side of the impedance head measures the acceleration of the panel face.

Since the testing occurs at the primary resonance of the damaged area and at various superharmonics of this frequency, modal impact testing is conducted to determine the primary resonance. The modal impact tests are conducted with all components
included in the test set-up using a PCB 086D80 mini-impact hammer. The PCB 086D80 mini-impact hammer is able to apply an impact up to 50 lbf with a sensitivity of 100 mV/lbf. This approach is used to identify the primary resonance of the real system rather than trying to tune the real system to an ideal system. The added mass of the impedance head and shaker system has a tendency to decrease the primary resonance of the real system, while any stiffness provided by the shaker unit serves to increase the measured natural frequency. All panels are impacted, while secured by the vacuum chuck, in 34 locations and the acceleration responses are measured by the impedance head. The vacuum chuck is also impacted in twelve locations with the undamaged panel attached to determine if the vacuum chuck contributes to the response of the shaker system. The acceleration data are then processed to identify the primary resonances of all panels and the vacuum chuck.

The undamaged panel exhibits a natural frequency of 1161.2 Hz, which is the first bending mode, as seen in Figure 3.6. The 101 millimeter damaged panel shows a primary resonance of 299 Hz, which is a mode in which only the damaged area is deforming, as seen in Figure 3.7. The other two damaged panels show the same mode shape as the 101 millimeter damaged panel, at 360 Hz and 468 Hz for the 63.5 mm and 25.0 mm damaged panels, respectively. The vacuum table does not demonstrate any appreciable response at the frequencies of interest for studying the panel response.
Figure 3.7. Primary mode shape of the damaged honeycomb panel.
To further refine the value of the primary resonance of the damaged panels, a frequency sweep is performed on each damaged panel. Identifying the frequency where the panel exhibits the largest response better locates the primary resonance, as well as identifying any potential nonlinear backbone features as discussed in analysis of the different nonlinear models. The frequency sweeps show that more accurate values for the primary resonances are 296.0 Hz, 359.5 Hz, and 467.5 Hz for the different panels. None of the frequency sweeps showed indications of nonlinear backbone curves.

### 3.7 Shaker Data Analysis

Piezoelectric accelerometers can be affected by noise at low frequencies. This noise can then be compounded if the acceleration data are to be integrated to velocity or displacement data. A two-prong approach for dealing with the low-frequency noise is used in this study. The first approach is to use a high-pass filter system to remove the low-frequency noise. The second method is to only integrate the values of the acceleration data at the frequencies of interest.

The first method is employed with the goal of numerically integrating the acceleration data for the purpose of creating restoring force curves. These restoring force curves plot the displacement versus the force needed to create said displacement. Doing so allows for quick visual checks of phasing, hardening or softening stiffness, and other information on damping.

A Butterworth filter is created in Matlab using the built-in “f-design” filter creation tool. The acceleration signal is passed through the filter twice, once in the forward time direction and once in the backward time direction, with the intent to remove any phasing changes caused by the filter. This is done using the “filtfilt” tool in Matlab. After the acceleration data is filtered, a cumulative numerical integrator called “cumtrapz” is used to estimate the velocity data. Another round of filtering is conducted on the velocity data, and then another numerical integration is performed to estimate the displacement data. A final filtering of the displacement data is done to
complete the process. This approach does allow direct integration of the acceleration data, but the filtering may remove important constant value information.

The second approach takes advantage of the single-frequency excitation of the system and the response of the system at frequencies identified by the nonlinear analysis. The raw acceleration data are transformed into the frequency domain by way of a Discrete Fourier Transform (DFT). Plotting the frequency domain response shows the frequencies of the system response as well as the strength of the response. By selecting the peak values of the frequency domain response, transforming these values back to acceleration values, and then dividing twice by the frequency at which these values occurred, the acceleration data is transformed to displacement data. The displacement data no longer contains the phase information of the response, but this is recognized as 180 degrees out of phase with the acceleration data.

While this method addresses the issues with low frequency noise by only integrating at the frequencies of interest, it introduces other potential issues. By selecting only the peak values from the frequency domain response, some of the system response at frequencies near the frequency of interest is lost. However, since the analysis of the nonlinear models identifies the different nonlinearities by tracking the changes in the panel response at discrete frequencies as the forcing amplitude is changed, the losses should not affect the response trends since the losses should be similar across the DFTs. This method was also applied to the force data without integrating. All DFTs are performed by using the “fft” program in MatLab.

3.8 Epoxy Testing

The manufacture of the aluminum honeycomb panels uses an epoxy adhesive to secure the two sheets to the honeycomb core. If the epoxy plays a role in the vibratory response of the damaged areas, an understanding of the material properties of the epoxy is important. To accomplish this objective, the dynamic moduli of the epoxy are found by testing the epoxy by way of a Dynamic Mechanical Analysis (DMA).
The dynamic moduli form a set of moduli that inform on the storage and loss behavior of visco-elastic materials. The storage and loss moduli are related by the following equation:

\[ G = G' + iG'' \]  \hspace{1cm} (3.1)

where \( G' \) is the storage modulus and \( G'' \) is the loss modulus. In a purely elastic material, the loss modulus would be zero and the strain would be in phase with the stress, where in a pure viscous material the storage modulus would be zero and the strain would be 90 degrees out of phase with the stress. The DMA tester conducts a series of dynamic tests across a range of frequencies at a constant temperature and stress amplitude, or at a constant frequency across a range of temperatures, that determines the dynamic moduli.

A TA Instruments AR-G2 Rheometer is used to find the dynamic moduli of the panel epoxy. Testing is conducted at room temperature with the frequency range of 10 to 100 Hz using a constant stress amplitude of 1.00 MPa. Results of the shear storage moduli and the shear loss moduli are seen in Figures 3.8 and 3.9. The storage and loss moduli grow as the frequency is increased, but the loss modulus shows a greater percentage growth than the storage modulus. This is more clearly seen in Figure 3.10. The loss modulus is growing at a faster rate than the storage modulus, but is still approximately an order of magnitude smaller than the storage modulus at 100 Hz.

These results are typical for an epoxy with a glass transition temperature of 180 Celsius. Of note for the purposes of this study is the increasing loss modulus as the frequency increases. This behavior may play a role in the nonlinear response of the disbonded panel.

3.9 Testing Regime

The harmonic testing of the panels is conducted with the same arrangement as the modal impact testing. The shaker is driven by an Agilent 33250A 80 MHz Func-
Figure 3.8. Shear storage modulus across the tested frequency range.

Figure 3.9. Shear loss modulus across the tested frequency range.
Figure 3.10. Shear storage and loss moduli across the tested frequency range.
Table 3.1. Testing parameters of the damaged panels.

<table>
<thead>
<tr>
<th>Damage Size in mm</th>
<th>Testing Frequency in Hz</th>
<th>Forcing Amplitude in N</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>467.5</td>
<td>9.0 - 28</td>
</tr>
<tr>
<td></td>
<td>233.8</td>
<td>7.7 - 45</td>
</tr>
<tr>
<td></td>
<td>155.8</td>
<td>9.1 - 40</td>
</tr>
<tr>
<td>63.5</td>
<td>359.5</td>
<td>6.9 - 27</td>
</tr>
<tr>
<td></td>
<td>179.8</td>
<td>8.4 - 47</td>
</tr>
<tr>
<td></td>
<td>119.8</td>
<td>7.7 - 41</td>
</tr>
<tr>
<td>101</td>
<td>296.0</td>
<td>5.4 - 31</td>
</tr>
<tr>
<td></td>
<td>148.0</td>
<td>1.5 - 18</td>
</tr>
<tr>
<td></td>
<td>98.67</td>
<td>1.5 - 18</td>
</tr>
</tbody>
</table>

A National Instruments NI 9234 4-Channel analog input module was used to collect the data. The NI 9234 has a built in anti-aliasing filter that automatically adjusts for the chosen sampling frequency. The three frequencies chosen are the primary resonance, one half the primary resonance, and one third the primary resonance. Each of these differed for each damage size, from 467.5 Hz as a high to 98.67 Hz on the low end.

3.10 Undamaged Panel Testing

An undamaged panel is tested at the same frequencies as the damaged panels. This is done to see if the testing system adds any frequency content and/or nonlinearities.
to the testing responses. The undamaged panel is tested at an input amplitude that is approximately the median excitation amplitude used during testing at that frequency. Table 3.2 shows the input amplitudes used at the frequency of testing. The forcing amplitude is reported here in mVs and not Newtons due to an observed phenomenon that makes reporting Newtons less accurate in these tests. The phenomenon will be explained later in the section. The panel response data is transformed to the frequency domain and then is examined.

The response data show that in addition to a response at the excitation frequency, there are responses at higher frequencies. The responses appear to be side bands of the forcing frequency off of an unknown higher frequency. This additional frequency content is not seen in the testing of the damaged panels at magnitudes that approach the magnitudes of the responses at the excitation frequency. Examples of the additional frequency content are seen in Figures 3.11, 3.12 and 3.13. A plot of the frequency response of the 101 mm damaged panel excited at 98.67 Hz is shown in Figure 3.14.

It is hypothesized that the higher frequency content comes from the nylon stinger that is attached to the shaker. In order to test this hypothesis, a stiffer steel stinger

Table 3.2. Input amplitudes for each testing frequency of the undamaged panel.

<table>
<thead>
<tr>
<th>Frequency of Test (Hz)</th>
<th>Input Forcing Amplitude (mV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.67</td>
<td>300</td>
</tr>
<tr>
<td>119.8</td>
<td>300</td>
</tr>
<tr>
<td>148.0</td>
<td>300</td>
</tr>
<tr>
<td>155.8</td>
<td>300</td>
</tr>
<tr>
<td>179.8</td>
<td>300</td>
</tr>
<tr>
<td>233.8</td>
<td>300</td>
</tr>
<tr>
<td>296.0</td>
<td>30</td>
</tr>
<tr>
<td>359.5</td>
<td>30</td>
</tr>
<tr>
<td>467.5</td>
<td>30</td>
</tr>
</tbody>
</table>
Figure 3.11. Additional frequency content seen on the undamaged panel with a nylon stinger excited at 98.67 Hz; note the sidebands off the second high peak around 800 Hz.

Figure 3.12. Additional frequency content seen on the undamaged panel with a nylon stinger excited at 179.8 Hz; note the sidebands off the second high peak around 750 Hz.
Figure 3.13. Additional frequency content seen on the undamaged panel with a nylon stinger excited at 233.8 Hz; note the sidebands off the second high peak around 950 Hz.

Figure 3.14. Frequency response of 101 mm damaged panel excited at 98.67 Hz; note the relatively larger magnitudes of the responses at the excitation frequency and three times the primary resonance.
Figure 3.15. Additional frequency content seen on the undamaged panel with a steel stinger excited at 98.67 Hz; note that the second high peak is around 1200 Hz.

is attached to the shaker and the undamaged panel, and a test is run at a frequency of 98.67 Hz using the same input amplitude as the previous test at that frequency. The frequency content of the stiffer stinger test is seen in Figure 3.15. The higher frequency peak has moved higher in frequency with the stiffer steel stinger. This implies that the higher frequency content does come from the stinger.

The reason why the stinger generates this additional frequency content on the undamaged panel and not on the damaged panel is seen in examining the force data from these tests. The force data for the two undamaged panel tests are seen in Figure 3.16. The force levels measured by the impedance head are 7.1 Newtons for the nylon stinger and 6.7 Newtons for the steel stinger. The force data for the test of the 101 mm damaged panel at the same input amplitude is seen in Figure 3.17. The force level for the damaged panel is 8.1 Newtons. Even though all three tests have the input, the test on the damaged panel measures a higher force level than either of the two tests on the undamaged panel. The lower force measurements on the undamaged panels
Figure 3.16. Excitation force as measured on undamaged panels using (top) nylon and (bottom) steel stingers.

Figure 3.17. Excitation force as measured on 101 mm damaged panel using nylon stinger.
can be attributed to a buckling of the stinger during the testing on the undamaged panel. The buckling of the stinger decreases the force available to excite the panel. The buckling also adds a higher frequency input to the system, which is seen in the frequency content of the undamaged panel tests. This loss of applied force due to buckling is the reason that mV was used to organize the setup of the tests, since the force level from the same input varied between the damaged and undamaged panels.
4. DATA ANALYSIS

The testing that was conducted on the damaged panels is analyzed with two different methods. The first method is an examination of the data as constructed in restoring force curves. Restoring force curves relate the displacement and velocity data of the test with the force required to generate the displacements and velocities. Examining these relationships can reveal insight into the storage and loss behavior of the system. The second method is to examine the trends and changes in the response amplitudes as the forcing amplitude changes. These trends are then compared to the trends in the analyses from Chapter 2.

4.1 Restoring Force Curves

Before any integration of the acceleration data, the acceleration and force data is examined in the time domain. Figure 4.1 shows an example of the raw acceleration and force data. The acceleration and force curves appear to be in phase with each other. This is the same for all sets of acceleration and forcing data at all amplitudes, frequencies and damage sizes. When the acceleration response of a SDOF system is in phase with the forcing function, the system is dominated by the mass of the system and not the damping or stiffness. If the system was dominated by the stiffness, the acceleration would be out of phase with the forcing, which in turn places the displacement in phase with the forcing. With the acceleration of the facesheet being in phase with the forcing, the mass of the impedance head is dominating the response behavior of the system. The mass of the impedance head and stinger total 28.0 grams, which is magnitudes greater than the estimated mass of the facesheet.

In order to continue with the restoring force analysis, the effect of the impedance head mass needs to be removed. To do so, the mass of the impedance head is multi-
Figure 4.1. Raw Acceleration and raw force data from testing at 179.8 Hz on a 63.5 mm damaged honeycomb panel plotted over time. This figure depicts the acceleration data and the force data as being in phase with each other.
plied by the measured acceleration, creating an impedance head force. The measured force is then subtracted from the impedance head force, generating a new force without the influence of the impedance head mass. The resultant force is then divided by the mass of the impedance head to return the data back to an acceleration. This is seen in the following equation:

\[
a_{\text{corrected}} = \frac{a_{\text{original}} \cdot m_{\text{impedancehead}} - F_{\text{measured}}}{m_{\text{impedancehead}}}. \tag{4.1}
\]

The resultant data is once again plotted against the force data, as seen in Figure 4.2. The acceleration is now 180 degrees out of phase with the forcing data.

The shape of the resultant acceleration data is different than the raw acceleration data. A comparison of the frequency content of the two signals is seen in Figure 4.3. While the magnitude of the peak at the forcing frequency (148.0 Hz) has changed, the peaks at the other frequencies have not changed in magnitude appreciably. This
Figure 4.3. Comparison of the frequency content of the raw acceleration signal (top) and the modified signal (bottom).
Figure 4.4. Restoring force curve at 467.5 Hz on 25.0 mm damaged panel, low forcing amplitude.

shows that the nonlinear responses, or the responses at frequencies other than the forcing frequency, have not been eliminated or altered with this method.

Restoring force curves are created for all data by passing the acceleration data through a high-pass filter and integrating the signal twice to obtain displacement data. When the panels were excited at the primary resonance, the displacement versus force curve is very linear at low forcing levels, around 8 Newtons, and becomes slightly less linear at higher forcing levels, around 20 Newtons, as additional frequency content alters the figure. Figures 4.4, 4.5 and 4.6 show the curves at low forcing levels. Figures 4.7, 4.8 and 4.9 show the curves at high forcing levels. The curves exhibit a small phase difference between the displacement and the forcing function, which is seen as the ellipse in the curve. The linear nature of the curves at their respective primary resonances suggest that there is a quadratic stiffness or damping nonlinearity, as identified in Table 2.1.

Examination of the restoring force curves when the panels are excited at one-half the primary resonance reveals different behavior. Figures 4.10, 4.11 and 4.12 show
Figure 4.5. Restoring force curve at 359.5 Hz on 63.5 mm damaged panel, low forcing amplitude.

Figure 4.6. Restoring force curve at 296.0 Hz on 101 mm damaged panel, low forcing amplitude.
Figure 4.7. Restoring force curve at 467.5 Hz on 25.0 mm damaged panel, high forcing amplitude.

Figure 4.8. Restoring force curve at 359.5 Hz on 63.5 mm damaged panel, high forcing amplitude.
Figure 4.9. Restoring force curve at 296.0 Hz on 101 mm damaged panel, high forcing amplitude.
the curves at low forcing levels. The curves at low forcing exhibit a figure eight type loop. Figures 4.13, 4.14 and 4.15 show the curves at high forcing levels. As the forcing amplitudes increase, the loops tend to shift to a position away from the origin. The curves, both with low and high forcing, generated from excitation at 148.0 Hz do not show the same strong loop characteristic that the other panels show. Close examination of the 148.0 Hz curve at high forcing does show a small figure eight pattern, but it is not very pronounced. The figure eight patterns in these curves show two distinct stiffness regions, with one stiffness being greater than the other. The figure eight shape also indicates a response at twice the forcing frequency that is 90 degrees out of phase with the forcing function. This behavior indicates a quadratic nonlinear stiffness.

Examination of the restoring force curves when the panels are excited at one-third the primary resonance reveals different behavior. Figures 4.16, 4.17 and 4.18 show the curves at low forcing levels. The restoring force curves show a small hardening stiffness at low forcing levels. The curve at 155.8 Hz shows much less of this trend
Figure 4.11. Restoring force curve at 179.8 Hz on 63.5 mm damaged panel, low forcing amplitude.

Figure 4.12. Restoring force curve at 148.0 Hz on 101 mm damaged panel, low forcing amplitude.
Figure 4.13. Restoring force curve at 233.8 Hz on 25.0 mm damaged panel, high forcing amplitude.

Figure 4.14. Restoring force curve at 179.8 Hz on 63.5 mm damaged panel, high forcing amplitude.
Figure 4.15. Restoring force curve at 148.0 Hz on 101 mm damaged panel, high forcing amplitude.

Figure 4.16. Restoring force curve at 155.8 Hz on 25.0 mm damaged panel, low forcing amplitude.
Figure 4.17. Restoring force curve at 119.8 Hz on 63.5 mm damaged panel, low forcing amplitude.

Figure 4.18. Restoring force curve at 98.67 Hz on 101 mm damaged panel, low forcing amplitude.
at low forcing if it is seen at all. Figures 4.19, 4.20 and 4.21 show the curves at high forcing levels. All three curve have similarities in appearance. Small figure eight type loops are visible, but there are two crossover points rather than one seen in the previous grouping of restoring force curves.

In summary, the restoring force curves illuminate some possible behaviors in the panels. When excited at the primary resonance, the panels demonstrate a linear type behavior with a small phase difference with the forcing function. When excited at one-half the primary resonance, the panels show figure eight style loops. These loops exhibit two distinct stiffnesses, one as the force grows more positive and another as it grows more negative. When excited at one-third the primary resonance, the panels show faint signs of a hardening stiffness. Not all panels showed this trend, as the 25.0 mm damaged panel did not demonstrate this feature.
Figure 4.20. Restoring force curve at 119.8 Hz on 63.5 mm damaged panel, high forcing amplitude.

Figure 4.21. Restoring force curve at 98.67 Hz on 101 mm damaged panel, high forcing amplitude.
4.2 Nonlinear Response Trends

The second method of analysis is used to examine the changes in the response amplitudes due to the changes in forcing amplitude. As was explained in Chapter 3, this is done by isolating the responses in the frequency domain and tracking the changes in the peaks. The changes in the response peaks are examined across the three panels when excited at the primary resonance and the superharmonic frequencies.

4.2.1 Trends when Excited at the Primary Resonance

As the panels are excited at the primary resonance, the amplitudes of the responses were tracked at three frequencies: the forcing frequency, twice the forcing frequency, and three times the forcing frequency. These frequencies were chosen based on the analyses from Chapter 2 that highlighted the responses at the primary resonance, and the possibility of responses at twice or three times the primary resonance, based on the order of the nonlinearity. Figure 4.22 shows the amplitude of the response at the primary resonance for the three panels as the forcing amplitude is increased. The three panels demonstrate a different stiffness between them. The 25.0 mm damaged panel has the highest stiffness while the 101 mm damaged panel has the lowest stiffness. This is expected since as the damage size increases, the primary resonance decreases. This also makes sense physically since the larger disbonds have less stiffening support from the honeycomb core.

When the data are normalized by the value of the initial force amplitude and response amplitude, it is possible to track the change in response amplitude as a multiple of the change in forcing amplitude. Figure 4.23 shows the normalized curves. All three response curves lie upon one another and increase in a near one-to-one rate with the increasing force amplitudes. Looking back to Table 2.1, it is seen that only the pure quadratic stiffness and pure quadratic damping produce this type of increase in the response at the primary resonance without changing the forcing frequency.
Figure 4.22. Response amplitude at the primary resonance when excited at the primary resonance vs. forcing amplitude.

Figure 4.23. Normalized response amplitude at the primary resonance when excited at the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal one-to-one proportionality.
Examination of the responses at twice the primary resonance illuminates the nature of the nonlinear behavior of the system. Figure 4.24 shows the normalized responses of the response peaks at twice the primary resonance. The normalized curves closely follow the quadratic shape indicative of the quadratic stiffness or quadratic damping analyses. The larger damage sizes more closely follow the trend line when compared to the 25.0 mm damage size.

The changes in the response at three times the forcing frequency are also charted, as seen in Figure 4.25. No discernible trends are seen in the data at three times the primary resonance. The displacement values are very small, on the order of tens of nanometers, especially when compared to the amplitude at twice the primary resonance, which is on the order of tens of micrometers. From this data, it is concluded that there is not a strong response peak at frequencies three times the primary resonance. Figures 4.26 and 4.27 show the response spectrum of the 63.5 mm panel excited at the primary resonance for low and high forcing. Response peaks can be
Figure 4.25. Response amplitude at three times the primary resonance when excited at the primary resonance vs. forcing amplitude.

Figure 4.26. Frequency response on the 63.5 mm honeycomb panel excited at the primary resonance, at low forcing.
Figure 4.27. Frequency response on the 63.5 mm honeycomb panel excited at the primary resonance, at high forcing.
seen at the primary resonance (359.5 Hz), twice the primary resonance (719.0) and three times the primary resonance (1079 Hz) in both the low and high forcing examples. However, the response peaks at the primary resonance and twice the primary resonance show considerable growth, around a magnitude, between the two plots, the peak at three times the primary resonance does not appear to change.

The trends in the responses when excited at the primary resonance indicate the presence of a quadratic nonlinearity and no indication of a cubic nonlinearity. The changes in the response peaks do not distinguish if it is a quadratic stiffness or a quadratic damping nonlinearity.

4.2.2 Trends when Excited at One-Half the Primary Resonance

The responses at three frequencies are again tracked as the forcing changes: the forcing frequency, or one-half the primary resonance, the primary resonance, and twice the primary resonance. The trend of the response at the forcing frequency is a near one-to-one increase with the increase in forcing, which is expected based on the prior analysis. This is seen in Figure 4.28. The analytical analyses for all quadratic nonlinearities indicate that the response at the primary resonance should increase in a quadratic nature compared to the increase in the forcing amplitude. Figure 4.29 shows the normalized changes in the response amplitudes. The response curves appear to follow the quadratic curve, with the larger damage curves more closely following the trend when compared to the 25.0 mm damage.

The response at twice the primary resonance is charted in Figure 4.30. The trend is difficult to see. The response amplitudes do not change much over the initial force changes, but then increase rapidly after that. The quartic curve, from the analytical solution, does not fall over much of the data, however, the rapid increase in response amplitude similar to the quartic curve can be seen once the forcing amplitude is sufficiently large. This implies that the response at twice the primary resonance when excited at one-half the primary resonance may follow a quartic relationship.
Figure 4.28. Normalized response amplitude at one-half the primary resonance when excited at one-half the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal one-to-one proportionality.
Figure 4.29. Normalized response amplitude at the primary resonance when excited at one-half the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal quadratic proportionality.
Figure 4.30. Normalized response amplitude at twice the primary resonance when excited at one-half the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal quartic proportionality.
with the forcing amplitude, but sufficient forcing is needed for this response peak to become large enough to distinguish it from the noise floor. Also, the smaller damage sizes show a stronger relationship to the possible quartic increase than the larger damage size.

The changes in response amplitudes combined with the restoring force curves indicate that the damaged panels exhibit behavior that is consistent with pure quadratic, or bilinear, stiffness nonlinearities. This conclusion is made after comparing the trends in the response data with the analytical analysis following the trends as indicated from the nonlinear analysis for quadratic stiffness and damping behavior. The two distinct stiffness regions seen in the restoring force curves indicate that the quadratic nonlinearity is a stiffness nonlinearity. There has not been any strong indicators of a cubic nonlinearity up to this point.

### 4.2.3 Trends when Excited at One-Third the Primary Resonance

The responses at three frequencies are again tracked as the forcing changes: forcing frequency, primary resonance, and three times the primary resonance. The trend of the response at the forcing frequency is a near one-to-one increase with the increase in forcing, which is expected based on the analytical analysis. This is seen in Figure 4.31. The response at the primary resonance is seen in Figure 4.32. The red line shows the expected cubic relationship that would be the maximum response as indicated by the nonlinear analysis. None of the panels show a trend that increases as quickly as the cubic curve. In a change from the results seen at one-half the primary resonance, the larger diameter damaged panel shows less similarity to the approximate nonlinear solution of a cubic nonlinearity than the smaller damaged panels.

Figure 4.33 shows the response changes at three times the primary resonance. Once again, the smaller damaged panels show greater changes in response amplitude than the larger panels. They do not come close to the nine times increase that the
Figure 4.31. Normalized response amplitude at one-third the primary resonance when excited at one-third the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal one-to-one proportionality.

Figure 4.32. Normalized response amplitude at the primary resonance when excited at one-third the primary resonance vs. normalized forcing amplitude. The red line denotes the ideal cubic proportionality.
Figure 4.33. Normalized response amplitude at three times the primary resonance when excited at one-third the primary resonance vs. normalized forcing amplitude.
analytical nonlinear analysis would indicate, but the increase is very large for the 25.0 mm panel.

The response curves at the primary resonance do not follow the trends from Table 2.2. As was seen in the restoring force analysis, there is reason to expect a cubic nonlinearity, however, the responses generated from excitation at one-third the primary resonance do not follow the trends established by the single nonlinearity models from Chapter 2. The lack of a direct match on the cubic nonlinearity combined with the strong matches with a quadratic stiffness nonlinearity leads to reconsider the $\varepsilon$ weighting on the equations of motion. The stronger matching of the quadratic stiffness leads to the following formulations on the equation of motion for the system:

\[
\ddot{u} + 2\varepsilon\mu \dot{u} + \varepsilon ku^2 + \varepsilon^2 Nu^3 + \omega_0^2 u = F \cos(\Omega t) \quad (4.2) \\
\ddot{u} + 2\varepsilon\mu \dot{u} + \varepsilon ku^2 + \varepsilon^2 N\dot{u}^3 + \omega_0^2 u = F \cos(\Omega t) \quad (4.3)
\]

where the order of $\varepsilon$ stays the same for the linear damping and quadratic stiffness, but is increased on the cubic stiffness and damping terms. As was shown in Chapter 2, a second order solution has been solved for and organized into a form that is easy to use a least-squares method to find a value of the coefficient of $N$. The results of the coefficient fitting will be seen in Chapter 5.

In summary, the restoring force curves, in conjunction with the tracking of the changes in the response amplitude with respect to the changes in the forcing amplitude, have illuminated the different nonlinear behaviors in the damaged panels. The restoring force curves have highlighted the strong indications of a quadratic stiffness in the panels. The figure eight loop, together with the response trends at the primary resonance, follow the results of the analytical analysis from Chapter 2. The restoring force curves also showed signs of a cubic nonlinearity, however neither the restoring force curves nor the amplitude trends could positively identify it as a stiffness or damping nonlinearity.
5. COEFFICIENT FITTING AND DATA REPLICATION

With the identification of a pure quadratic stiffness and a smaller cubic nonlinearity in the equation of motion, fitting the coefficients of the equations of motion to the data sets is the next step. A least-squares method is used to fit the data to the equations of motion. Once the equation of motion coefficients are identified through the least-squares fitting, replication of the test data using both the nonlinear solutions and numerical ordinary differential equation (ODE) solvers is next. The replicated results are compared to the actual test data and reasons for any mismatches are identified.

5.1 Coefficient Fitting

As stated previously, the coefficient fitting is done using a least-squares method. All information on the least-squares method presented here comes from Rao and Toutenburg’s book, “Linear Models: Least Squares and Alternatives” [43]. The least-squares method is a method that is easily applied to an overdetermined problem, such as data fitting where there are more data points to fit than coefficients to solve for. The least-squares method works to minimize the total squared error between the model and the data:

\[ S = \sum_{i=1}^{n} r_i^2 \]  

(5.1)

where \( S \) is the objective to minimize, \( r_i \) is the residual between the model point and the data point and \( n \) is the number of data points. The residual can be defined as:

\[ r_i = y_i - f(x_i, \beta) \]  

(5.2)
where $\beta$ is the coefficient or coefficients of interest. Since the objective is to minimize $S$, this is accomplished by taking a partial derivative of $S$ with respect to the coefficient of interest $\beta$ and then setting this equal to zero and solving for $\beta$:

$$\frac{\partial S}{\partial \beta} = 2 \sum_{i=1}^{n} r_i \frac{\partial r_i}{\partial \beta} = 0.$$  \hfill (5.3)

With the basics of the least-squares method presented, it is important to know how well this method has encapsulated the data. For this purpose, $R^2$ is used as a measure of how closely the coefficient found using the least-squares method matches the data. $R^2$ is a way of understanding how much of the variance in the data can be ascribed to the model. Because of this a perfect fit would have an $R^2$ value of 1, since 100% of the variance in the data can be attributed to the model and an $R^2$ value of 0 would mean that none of the variance in the data can be attributed to the model. In this work, the definition of $R^2$ is:

$$R^2 \equiv 1 - \frac{SS_{res}}{SS_{tot}}$$  \hfill (5.4)

where:

$$SS_{res} = \sum_{i=1}^{n} (y_i - f_i)^2$$  \hfill (5.5)

$$SS_{tot} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$  \hfill (5.6)

and with $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$. This relationship measures the ratio between the difference in the modeled data and the actual data, and the variance in the data. If the difference between the model and the actual data is the same as the variance, then the model only fits the data as well as a simple mean. Another way to understand $R^2$ is with the variance of the data. As the variance of the data gets smaller, or as the data approaches a simple mean value, the fit of the model needs to improve to achieve the same $R^2$ value.
Table 5.1. Mass, linear stiffness and $R^2$ values by damage size for the corrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Mass (gm)</th>
<th>Stiffness (N/m)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>4.5</td>
<td>38,500</td>
<td>0.9999</td>
</tr>
<tr>
<td>63.5</td>
<td>4.9</td>
<td>24,900</td>
<td>0.9992</td>
</tr>
<tr>
<td>101</td>
<td>5.3</td>
<td>18,300</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

Coefficient fitting was done on all data sets in two ways. The first fitting was done on the data after it was corrected for the mass of the impedance head. Since the restoring force curves were established with this data, the coefficient fitting was first performed on this data. The second fitting was performed on the uncorrected data. This was done to see what effects, if any, appeared in the coefficient values.

The coefficient fitting was done sequentially on the values. This may not be the best method for fitting the parameters of the equations of motion to the experimental data. Simultaneous fitting of the coefficients is another way to approach the problem, with merits that are different than the sequential fitting done here. However, due to the isolated nature of the responses at different frequencies when excited at the primary resonance and superharmonics, sequential fitting was identified as the approach for this research.

The equations of motion discussed at the end of Chapter 4, Equations 4.2 and 4.3, use coefficients that are mass normalized. In the present work, the mass of the system is unknown. In order to calculate more accurate coefficients, a mass needs to be determined for each panel. The primary resonance of each of the damaged panels has been determined empirically, which in a SDOF system model is the square root of the stiffness divided by the mass. As shown previously in Figure 4.22, the stiffness of the panels decreases as the damage area increases. By taking the slope of the line and the primary resonance, a working mass, or approximate mass of the facesheet as incorporated into the SDOF model, can be obtained. Tables 5.1 and 5.2 list the stiffnesses, masses and the $R^2$ values of the stiffnesses.
Table 5.2. Mass, linear stiffness and $R^2$ values by damage size for the uncorrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Mass (gm)</th>
<th>Stiffness (N/m)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>7.9</td>
<td>68,100</td>
<td>0.9996</td>
</tr>
<tr>
<td>63.5</td>
<td>6.9</td>
<td>35,000</td>
<td>0.9985</td>
</tr>
<tr>
<td>101</td>
<td>6.1</td>
<td>21,300</td>
<td>0.9999</td>
</tr>
</tbody>
</table>
The first difference of note is the differences in the working mass of the system. When using the corrected data, the mass increases slightly as the damage area increases. This seems logical since as the damage area increases, more of the facesheet is available to move in the system, thereby increasing the mass that the system must move. With the uncorrected data, the mass decreases as the damage area increases, which is counter to intuition.

The next big difference is the stiffnesses of the panels. The uncorrected data demonstrates a much higher stiffness in the panels. The stiffness does decrease as the damage area increases in both corrected and uncorrected data. However, the jump in stiffness between the two sets of data decreases as the damage increases. The ratios between the corrected data masses and the uncorrected data masses is replicated in the corrected versus uncorrected stiffnesses. Both sets of data display high $R^2$ values.

With the working mass of each panel and its associated stiffness identified, the next coefficient to be determined is the linear damping coefficient $\mu$. This is isolated by analyzing the data from the panels excited at their primary resonances. The response amplitude of the damage is $\frac{F}{2\omega_0\mu}$, as defined in Equation 2.47 for excitation of a quadratic stiffness or damping nonlinearity at the primary resonance. Since $F$ can be determined by dividing the forcing amplitude by the working mass, and the primary resonance is known, $\mu$ can be determined by using the least-squares method. Tables 5.3 and 5.4 list the values for $\mu$ and their corresponding $R^2$ values. The values of the linear damping coefficients for both data sets are the same. The $R^2$ values for the linear damping coefficient $\mu$ are the same as the $R^2$ values for the linear stiffness

Table 5.3. Linear damping and $R^2$ values by damage size for the corrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Damping (N.s/m)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>1470</td>
<td>0.9999</td>
</tr>
<tr>
<td>63.5</td>
<td>1130</td>
<td>0.9992</td>
</tr>
<tr>
<td>101</td>
<td>930.0</td>
<td>0.9998</td>
</tr>
</tbody>
</table>
Table 5.4. Linear damping and $R^2$ values by damage size for the uncorrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Damping (N.s/m)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>1470</td>
<td>0.9996</td>
</tr>
<tr>
<td>63.5</td>
<td>1130</td>
<td>0.9986</td>
</tr>
<tr>
<td>101</td>
<td>930.0</td>
<td>0.9999</td>
</tr>
</tbody>
</table>
Table 5.5. Quadratic stiffness and $R^2$ values by damage size for the corrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Quadratic Stiffness ($N/m^2$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>3.500*10^6</td>
<td>0.9895</td>
</tr>
<tr>
<td>63.5</td>
<td>937.0*10^6</td>
<td>0.9611</td>
</tr>
<tr>
<td>101</td>
<td>140.0*10^6</td>
<td>0.9022</td>
</tr>
</tbody>
</table>

Table 5.6. Quadratic stiffness and $R^2$ values by damage size for the uncorrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Quadratic Stiffness ($N/m^2$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>6,420*10^6</td>
<td>0.9875</td>
</tr>
<tr>
<td>63.5</td>
<td>1,300*10^6</td>
<td>0.9560</td>
</tr>
<tr>
<td>101</td>
<td>186.0*10^6</td>
<td>0.9903</td>
</tr>
</tbody>
</table>

The coefficient $k$. Since both $\mu$ and $\kappa$ were obtained using the same linear relationship between the amplitude data and the forcing data, it is expected that the $R^2$ values would be the same while the coefficient values would be different.

Moving to the data acquired while exciting the panels at one half the primary resonance, the quadratic stiffness coefficient is estimated next. The amplitude of the system response at the primary resonance while exciting at one half the primary resonance is $\kappa\Lambda^2/\mu_0\omega$. Fitting this model solution to the data provides high $R^2$ values, seen in Tables 5.5 and 5.6.

The $R^2$ values for $\kappa$ show that approximately 90% or greater of variability in the data can be attributed to the value of $\kappa$. It is interesting to note that as the damage size increased, the confidence in the value of $\kappa$ decreased on the corrected data. Looking back to Figures 4.10-4.15, the restoring force curves for excitation at one-half the primary resonance showed a similar trend with the distinctive figure eight loop being much more pronounced on the smaller damage sizes and less so on the 101 mm damaged panel. This is different from what was seen in Figure 4.29,
Table 5.7. Cubic stiffness and $R^2$ values by damage size for the corrected data with a cubic phase of 0.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Stiffness ($N/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>$3,560 \times 10^9$</td>
<td>0.9828</td>
</tr>
<tr>
<td>63.5</td>
<td>$589.0 \times 10^9$</td>
<td>0.9975</td>
</tr>
<tr>
<td>101</td>
<td>$93.00 \times 10^9$</td>
<td>0.8672</td>
</tr>
</tbody>
</table>

Table 5.8. Cubic damping and $R^2$ values by damage size for the corrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Damping ($N \cdot s^3/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>$7,420 \times 10^6$</td>
<td>0.8645</td>
</tr>
<tr>
<td>63.5</td>
<td>$5,140 \times 10^6$</td>
<td>0.9884</td>
</tr>
<tr>
<td>101</td>
<td>$187.0 \times 10^6$</td>
<td>0.9930</td>
</tr>
</tbody>
</table>

where the larger damage panels matched the quadratic curve better than the smaller damage. Also of note is the decreasing value of $\kappa$ as the damage size increases. While this behavior is expected in the linear stiffness, it is not anticipated in the quadratic stiffness.

The last coefficient to fit is the cubic nonlinearity. The restoring force curves at one-third the primary resonance, see Figures 4.16-4.21, indicate that the cubic nonlinearity is a stiffness nonlinearity. Both the cubic stiffness solution as well as the cubic damping solution are fit to the data to compare against one another. Once again using a least-squares method, the models were fit to the data, with corrected data seen in Tables 5.7 and 5.8, and uncorrected data seen in Tables 5.9 and 5.10.

The cubic stiffness model fits the data much better than the cubic damping model. In two of the three damping cases in both data sets, the coefficients are imaginary, which does not fit the real valued SDOF model. The stiffness coefficients are all real and positive, which matches with the curves seen in Figures 4.16-4.21. The $R^2$ values are similar across the stiffness and damping models, but the real values of the stiffness
Table 5.9. Cubic stiffness and $R^2$ values by damage size for the uncorrected data with a cubic phase of 0.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Stiffness ($N/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>12,400*10^9</td>
<td>0.9852</td>
</tr>
<tr>
<td>63.5</td>
<td>1,140*10^9</td>
<td>0.9974</td>
</tr>
<tr>
<td>101</td>
<td>129.0*10^9</td>
<td>0.9079</td>
</tr>
</tbody>
</table>

Table 5.10. Cubic damping and $R^2$ values by damage size for the uncorrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Damping ($N \cdot s^3/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>30,900*10^6</td>
<td>0.8634</td>
</tr>
<tr>
<td>63.5</td>
<td>10,000*10^6i</td>
<td>0.9969</td>
</tr>
<tr>
<td>101</td>
<td>324.0*10^6i</td>
<td>0.9979</td>
</tr>
</tbody>
</table>
Table 5.11. Cubic stiffness and $R^2$ values by damage size for the corrected data with a cubic phase of $\pi$.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Stiffness ($N/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>$5,080*10^9$</td>
<td>-0.6539</td>
</tr>
<tr>
<td>63.5</td>
<td>$663.0*10^9$</td>
<td>0.9956</td>
</tr>
<tr>
<td>101</td>
<td>$-77.30*10^9$</td>
<td>0.6070</td>
</tr>
</tbody>
</table>

Table 5.12. Cubic stiffness and $R^2$ values by damage size for the uncorrected data with a cubic phase of $\pi$.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Cubic Stiffness ($N/m^3$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>$28,700*10^9$</td>
<td>0.1468</td>
</tr>
<tr>
<td>63.5</td>
<td>$1,280*10^9$</td>
<td>0.9957</td>
</tr>
<tr>
<td>101</td>
<td>$-88.00*10^9$</td>
<td>0.6026</td>
</tr>
</tbody>
</table>

model leads to the stiffness being the root of the cubic nonlinearity. Once again, the models fit the data better for the smaller damage sizes. The uncorrected data has $R^2$ values that are very similar to the corrected data, with only the 101 mm damage showing a change, with the uncorrected value fitting slightly better.

This cubic stiffness model was solved with the phase of the response at the primary resonance equal to 0. Fitting the data to the cubic stiffness model with a phase of $\pi$ generates the results seen in Tables 5.11 and 5.12. The $R^2$ values and the coefficient values do not provide as strong a fit when the phase is $\pi$ as when the phase is 0. The 63.5 mm damaged panel is unique in that the response data fits the cubic damping and both cubic stiffness models with high $R^2$ values.

A final fitting is done on the data sets with the intention of estimating a value for $\varepsilon$. This is done by examining the response amplitudes at twice the primary resonance
Table 5.13. Values of $\varepsilon$ by damage size for the corrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Value of $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>0.4224</td>
</tr>
<tr>
<td>63.5</td>
<td>0.4906</td>
</tr>
<tr>
<td>101</td>
<td>0.6410</td>
</tr>
</tbody>
</table>

Table 5.14. Values of $\varepsilon$ by damage size for the uncorrected data.

<table>
<thead>
<tr>
<th>Damage (mm)</th>
<th>Value of $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0</td>
<td>0.0310</td>
</tr>
<tr>
<td>63.5</td>
<td>0.1993</td>
</tr>
<tr>
<td>101</td>
<td>0.1735</td>
</tr>
</tbody>
</table>

when excited at the primary resonance. Combining Equations 2.47 and 2.49 yields the following equations for the response amplitude at twice the primary resonance:

$$\text{Response amplitude} = \varepsilon \frac{\kappa F^2}{8\mu^2\omega_0^4}.$$  \hspace{1cm} (5.7)

Using the least-squares method for estimating values for $\varepsilon$ generates the values in Tables 5.13 and 5.14.

The size of $\varepsilon$ is much smaller in the uncorrected data than in the corrected data. Results presented later in this chapter will show that 0.03 is a good estimate for the size of $\varepsilon$.

5.2 Data Replication

Now that the coefficients of Equation 4.2 have been estimated, the coefficients are used to try to recreate the restoring force curves from Chapter 4. The recreated force restoration curves are done for both the corrected and uncorrected data sets. This is done to confirm the results of the analysis and coefficient fitting.
5.2.1 Replication at Primary Resonance

To create a restoring force curve from the analytical solution at primary resonance, the solution of the pure quadratic stiffness at the primary resonance is recalled:

\[ u = a \cos(\Omega t + \beta) \]  \hspace{1cm} (5.8)
\[ a = \frac{F}{2\omega_0 \mu} \]  \hspace{1cm} (5.9)
\[ \beta = \frac{\pi}{2} \]  \hspace{1cm} (5.10)

By inserting the coefficients as determined by the previous fits, the displacement is generated with respect to time. Plotting this versus the forcing, which is \( F \cos(\Omega t) \), yields the restoring force curves seen in Figures 5.1 and 5.2. Since the response curve is 90 degree out of phase with the forcing function, the restoring force curve is an ellipse, seen in both plots. This does not match the restoring force curves at the primary resonance seen in Figure 4.5.
Figure 5.2. Restoring force curve based on analytical analysis with uncorrected data at 359.5 Hz with forcing amplitude of 8.5 N.
The equation of motion, Equation 4.2, is entered into a numerical ODE solver in Matlab to generate a restoring force curve as another source for checking the analysis. The numerical ODE solver used is the “ode45” solver, which uses a Runge-Kutta Method to solve the displacement of the system. The solver is given a zero displacement, zero velocity initial condition to start from. The resultant restoring force curve is seen in Figures 5.3 and 5.4.

The numerical solutions match very well with the analytical restoring force curves in shape, which means they do not match the experimental restoring force curves. This is explained by the mass of the impedance head. Since the mass of the impedance head is many times larger than the mass of the facesheet in the damage area, the mass of the impedance head drives the phase of the response. The mass of the impedance head overwhelms the natural response phasing, which at the primary resonance would be normally driven by the damping.

Another possible explanation for the phase difference is that the forcing function was not actually forcing at the primary resonance. Recalling Figure 2.10, if the
Figure 5.4. Restoring force curve based on ODE solver with uncorrected data at 359.5 Hz with forcing amplitude of 8.5 N.
forcing frequency is below the primary resonance, the phase of the response would go to $\pi$, or in other words, it would match the uncorrected displacement phasing. If that were the case, then the amplitude of the response would also need to change to accommodate the move off frequency. Recalling Figure 2.9, the amplitude would be a small fraction of what is expected. The amplitude of the experimental restoring force curves is much smaller than the analytical or numerical restoring force curves, which creates support for the forcing being off the primary resonance. However, the restoring force curves for data taken at one-half and one-third the primary resonance were also 180 degrees out of phase with the forcing before correcting the data. The analytical results do not generate a condition in which the displacement at the forcing frequency at these superharmonic resonances would be out of phase with the forcing. This gives more weight to the hypothesis that the mass of the impedance head drives the phasing and not the possibility of forcing the system off the primary resonance.

5.2.2 Replication at One-Half the Primary Resonance

Repeating the same process outlined in the previous section and recalling:

$$u = 2\Lambda \cos(\Omega t) + a \cos(2\Omega t + \beta)$$  
$$a = \frac{\kappa \Lambda^2}{\omega_0 \mu}$$  
$$\beta = \frac{\pi}{2}$$

generates Figures 5.5 and 5.6.

These curves have the same figure eight shapes as Figures 4.10 - 4.15 seen in Chapter 4 for data taken at one-half the primary resonance. Here is the evidence that supports the idea that the impedance head mass is the main cause of the phase of the response at the forcing frequency. The experimental restoring force curves were corrected for the impedance head mass, which then allowed them to match the analytical restoring force curves. Without the mass correction, the experimental
Figure 5.5. Restoring force curve based on analytical analysis with corrected data at 179.8 Hz with forcing amplitude of 8.5 N.

Figure 5.6. Restoring force curve based on analytical analysis with uncorrected data at 179.8 Hz with forcing amplitude of 8.5 N.
restoring force curves would be 180 degrees out of phase with the forcing, which is not a possibility in the analytical solution for the response at the forcing frequency. Only the secondary response at the primary resonance will shift phase based on whether the forcing frequency is at or slightly away from the superharmonic frequency.

The numerical solution restoring force curves are seen in Figures 5.7 and 5.8. The numerical solutions both show the noticeable figure eight pattern as seen in the experimental restoring force curves and the analytical solutions. The uncorrected data curves do a better job matching the amplitude of the experimental restoring force curves than the corrected data, but misses with the cross-over point on the figure eight. In creating the numerical solutions, the order of $\varepsilon$ had to be adjusted for the solver to produce the curves seen in the figures. The corrected data needed $\varepsilon^5$ on the damping and quadratic stiffness for the figure eight to match the experimental restoring force curves, while the uncorrected data needed only $\varepsilon^2$ for the curves at 179.8 Hz. The numerical curve at 233.8 Hz for the uncorrected data did not need the order of $\varepsilon$ to be changed, as seen in Figure 5.9. The value of $\varepsilon$ in the 25.0 mm panel
Figure 5.8. Restoring force curve based on ODE solver with uncorrected data at 179.8 Hz with forcing amplitude of 8.5 N.

Figure 5.9. Restoring force curve based on ODE solver with uncorrected data at 233.8 Hz with forcing amplitude of 8.5 N.
with uncorrected data is 0.0310, which is very close to the value obtained from the higher orders of $\varepsilon$ needed to produce better fitting curves.

### 5.2.3 Replication at One-Third the Primary Resonance

For this section, the numerical solutions will be presented first. Following the steps in the other sections for numerical solutions, the solutions at one-third the primary resonance are seen in Figures 5.10 and 5.11. The first thing of note is the amplitude of the numerical solutions is less than the experimental restoring force curves. These numerical curves have about half the amplitude of the experimental curves. Also of note is the lack of a distinct cubic stiffening shape to either numerical solution. While the corrected data curve may show hints of this, it is more likely an illusion due to the minor phase lag seen in the curve. This brings into doubt if this stiffening effect was seen in the experimental restoring force curves or not. Recalling Figures 4.16
Figure 5.11. Restoring force curve based on ODE solver with uncorrected data at 119.8 Hz with forcing amplitude of 8.5 N.
through 4.18, the shape of the numerical solutions does indeed appear more like the experimental restoring force curves than initially thought.

The analytical solution was more problematic. As was previously mentioned, solving the equation for the response at the primary resonance when excited at one-third the primary resonance does not yield an easy to use solution. Also, due to the order of the $b$, it yields three potential solutions for the amplitude of the response. After examination of the three solutions, two are found to generate complex conjugate solutions for the response amplitude. These solutions are dismissed. By solving it numerically with the known coefficients and a force of 8.5 Newtons, the response amplitude is of the magnitude of $10^{-3}$. This is larger than the magnitude of the response at the forcing frequency. The only way to achieve a response magnitude comparable to that seen during testing is to multiply the value of $b$ by $\varepsilon^2$, or around 0.0009. Doing so generates a restoring force curve seen in Figure 5.12. Without the $\varepsilon^2$ factor, the restoring force curve exhibits a large “S” shape seen in Figure 5.13.
Figure 5.13. Restoring force curve based on analytical analysis with uncorrected data at 179.8 Hz with forcing amplitude of 8.5 N.
It is possible that, with the solution for $b$ being a second-order solution, the need to change its magnitude by a factor of $\varepsilon^2$ is necessary. It is also possible that the solution for the cubic nonlinearity is not complete. The conclusion that the disbonded panel has a cubic stiffness nonlinearity was based on the excellent fitting of the cubic stiffness coefficient to the experimental data and the close resemblance of the numerical solution to the experimental restoring force curves using the said coefficient. However, the analytical solution does not match the experimental or numerical solutions even though the analytical restoring force curves are based on the fitting with a high $R^2$ value. A possible explanation for the large difference could be in the error of computer numbers is a complex equation. The analytical solution for the response amplitude of the cubic stiffness has many terms with powers up to 12 in addition to a large divisor term and cubic roots. It may be possible that computer rounding errors in all the terms have led to a large overall error on the magnitude of the response amplitude.

5.3 Physical Meaning

With the consistent data supporting the existence of the quadratic stiffness nonlinearity, the physical behavior behind the nonlinearity is sought. A quadratic stiffness is one where the system reacts to one stiffness value over a certain period of the cycle, and another different stiffness over another part of the cycle. As it was discussed in the Introduction, this behavior can be seen when the facesheet vibrates away from and into the honeycomb core, as seen in Figure 5.14. As the facesheet presses into the core, the motion of the facesheet is resisted by not only the inherent stiffness of the facesheet metal, but also the added stiffness of the core. When the face sheet separates from the core, its motion is only resisted by the stiffness in the facesheet material. This simple model explains the physical behavior behind the quadratic stiffness. Recalling Table 5.6 and Figures 4.24 and 4.29, the experimental data fit the analytical solution better as the damage size increases. This connects well with the
Figure 5.14. Example of motion in a disbonded panel that leads to quadratic stiffness.
understanding of the contact stiffness proposed in the Introduction. As the disbond area increases, there is a greater area for the facesheet to experience the discontinuous stiffness, which also increases the nonlinear responses seen in the experimental data. While it was thought that the increase in disbond size would also create an increase in the nonlinear coefficient, this is not true. The quadratic stiffness coefficient decreases in value as the damage increases, which is what also occurs with the linear stiffness. Even though the nonlinear stiffness coefficient decreases in size as the damage increases, the nonlinear responses fit better as the damage increases.

A physical behavior behind the cubic stiffness is not as easily explained. The cubic stiffness is a symmetric stiffness, meaning that the stiffness is increasing with displacement both into and away from the honeycomb core. It must also be remembered that this stiffness is not able to exhibit its nature when the panel is intact. From these ideas, it is thought that the most likely source for this cubic behavior must lie in the epoxy bonding material and the manner in which it secures the facesheet to the core and the small facesheet deformation experienced at the edge of the disbond. Figure 5.15 shows a simplified model of the epoxy fillet that holds the facesheet to the core. Since the cubic stiffness is small compared to the quadratic stiffness and much smaller than the linear stiffness, the additional stiffness added by the epoxy fillet and small facesheet deformations would fit that scale. This is also supported by the better fit of the cubic stiffness to the smaller damaged panels. With the exposed epoxy fillets in the smaller damage sizes reacting to similar displacements as the displacements

Figure 5.15. Simplified cross-section of a honeycomb panel showing how the epoxy fillet binds the facesheet to the core.
Figure 5.16. Demonstration of the epoxy fillet and facesheet deforming and adding stiffness as the displacement of the facesheet increases.

in larger damage sizes, the fillets would be able to contribute more to the added stiffness. Also, the facesheet deformation at the edges of the disbonds would also be of a greater degree than on the larger disbonds areas. Figure 5.16 demonstrates how the edge effects can add additional stiffness as the displacement increases. For very small displacements of the facesheet, the epoxy fillet and facesheet do not deform
much. The curve of the fillet allows for small displacements without much change in the physical shape. As the facesheet displaces more, the curve of the fillet straightens out and stiffens as the facesheet creates larger local deformations. This is where the additional stiffness seen in the cubic stiffness could come from.

5.4 Recap

The experimental data was fit to the analytical solutions with excellent results. High $R^2$ values were achieved for all the coefficients of the equation of motion. From the data fitting of the cubic nonlinearity, the cubic stiffness once again appeared the most likely solution for the cubic nonlinearity. Replication of the restoring force curve further reinforced the identification of the coefficients. The only area in which the recreated restoring force curves did not confirm the coefficient fitting was with the analytical restoring force curves for excitation at one-third the primary resonance. It is thought that the mismatch occurs from errors in the computation of large and small values.
6. SUMMARY

The harmonic response characteristics of a disbonded aluminum honeycomb panel to single-frequency harmonic excitation have been measured and identified. The disbonded panels exhibit responses at the primary resonance when excited at one-half and one-third the primary resonance. These responses are related to a quadratic stiffness and a cubic stiffness, respectively. The quadratic stiffness was identified based on the changes in the panel response when excited at the primary resonance and at one-half the primary resonance. The response peaks at the primary resonance and at twice the primary resonance exhibited the types of increase in response seen in the analytical solutions of a quadratic stiffness equation of motion. This match, as well as the evidence seen from the restoring force curves that showed two stiffness regions, created a strong indication of a quadratic stiffness in the disbonded panel. While there were indications of a cubic nonlinearity, the experimental response peaks did not follow the trends predicted by the analytical techniques. Due to the strong match of the quadratic stiffness responses, the order of the cubic nonlinearity was changed in the nonlinear equations of motion. New analytical solutions were found for the now smaller contribution of the cubic nonlinearity, either a cubic stiffness or cubic damping. The new analytical solutions were not compared directly to the response peaks, but rather they were used in a least-squares coefficient fitting. The cubic stiffness solution had higher $R^2$ values for the fitting than the cubic damping. The cubic damping solution generated imaginary valued coefficients for the equation of motion. Since the equation of motion is a real valued equation, it was determined that cubic damping was not likely present.

After the terms in the equation of motion were identified, a least-squares method was applied to all coefficients to better understand the contributions of each term, and for validation of the tests. The $R^2$ values of all coefficients were very high,
from 0.87 to 0.99, giving a high degree of confidence in the fitting. Validation of the coefficients was done using both an analytical and numerical approach, which provided an interesting comparison. The analytical solution and the numerical ODE solver gave very similar restoring force curves at the primary resonance, where the response amplitude was 90 degrees out of phase with the forcing amplitude. The restoring force curve at the primary resonance was initially 180 degrees out of phase with the forcing function. This was determined to be due to the large mass of the impedance head. The mass of the impedance head was so much larger than the mass of the facesheet that the phase of the response was dominated by the mass, or acceleration term. Even after correcting for the mass, the phase of the response was still determined by the impedance head. The restoring force curves from the analytical solution at one-half the primary resonance matched the experimental restoring force curves very well. Both showed a strong response at the forcing frequency and a response at twice the forcing frequency that was 90 degrees out of phase with the forcing frequency. The numerical solution also showed a good correlation with the experimental restoring force curves, giving further support to the determination of a quadratic stiffness in the disbonded panels. The analytical and numerical results at one-third the primary resonance did not generate as strong of an agreement with the experimental restoring force curves. While the numerical results exhibited many of the characteristics of the experimental restoring force curves, they did not match as well as the quadratic curves did. Also, the analytical results were only able to match in amplitude after manipulating the magnitude of the response amplitude at the primary resonance. This does not generate the same level of confidence in the cubic stiffness as in the quadratic stiffness, even with a coefficient fitting that yielded excellent results. However, the closer match between the numerical solution and the experimental restoring force curves does support the cubic stiffness claim.

The quadratic stiffness was caused by the two different stiffnesses encountered by the disbonded facesheet as it vibrated. One stiffness was experienced by the facesheet as it moved away from the honeycomb core, and the second greater stiffness occurred
as the facesheet was pressed into the core. This dual stiffness nature can be modeled as a quadratic stiffness in conjunction with the linear stiffness.

The cubic stiffness is different. The data showed that the cubic stiffness was not as strong as the quadratic stiffness, so the physical reasoning behind the cubic stiffness also needed to be less significant. Also, the cubic stiffness has the symmetric nature of a cubic term. These considerations led to thinking that the epoxy that holds the facesheet to the core and small deformations of the facesheet near the edges of the disbonds as the sources of the nonlinearity. Further work could be conducted to see how well this hypothesis fits the data.

An important observation for the detection and analysis of disbonds in the aluminum honeycomb panel is the relative sizes of the nonlinearities as compared to the size of the damage. The smaller damage sizes exhibited higher values for the quadratic and cubic stiffnesses when compared to the panels with larger damage sizes. This was not expected, as it was thought that the larger the damage, the greater the nonlinearity. Also, the magnitudes of the response peaks at the primary resonance when excited at the superharmonic frequencies are very close in size across the damage sizes. Since the smaller damage panels have smaller response peaks at the forcing frequency, the response peaks at the primary resonance are therefore larger in comparison. This was seen in the restoring force curves at one-half the primary resonance in the size of the loops in the figure eights. This could imply that using the superharmonic frequencies to look for damage may be more effective for smaller damage than larger. Further investigation in this is warranted.

6.1 Contributions

This work has demonstrated that, in controlled experimental conditions, a disbonds damage mechanism in an aluminum honeycomb panel can be modeled using a nonlinear single degree-of-freedom representation. The disbonds damage creates a quadratic stiffness nonlinearity due to the two different stiffness regimes of the facesheet press-
ing into and away from the honeycomb core. There is a cubic stiffness nonlinearity in the disbond damage, but it is small compared to the quadratic stiffness nonlinearity.

6.2 Future Work

Future work could include using this research method with other damage types to examine how different damage mechanisms can be measured using single degree-of-freedom models. This would further expand the ability to use superharmonic excitation to classify the damage mechanism in real world practice. Also, examination of the same disbond damage could be done without the impedance head as the measurement device. The mass of the impedance head dominated some of the response phasing. By exciting the panel away from the damage and using a small accelerometer, or a laser vibrometer, it could be possible to generate a more detailed understanding of the response phase, as well as add clarity to the cubic nonlinearity. Lastly, an important area for future work would be finite element modeling of the disbonded panel to examine the behavior primarily at the edges of the disbond. Since it is thought that the epoxy fillets and facesheet deformation are the sources of the cubic stiffness, finite element modeling could be used to explore these concepts.
LIST OF REFERENCES
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APPENDICES
Appendix A: Coefficient Fitting and Restoring Force Recreation with Corrected and Uncorrected Data MATLAB code

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This code gathers the shaker test data, corrects it for the phasing of the acceleration signal, then uses a least-squares method to fit the data to the equation of motion. Once the coefficients of the equation of motion are determined, the code uses the analytical models and a numerical solver to recreate restoring force curves to confirm the identification of the proper equation of motion.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;
cle;

%% Loading the data from the 101mm damaged panel at the primary resonance
for ii = 1:12
    cnt=num2str(ii);
datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Oct_2_4inch\296_amp\data' cnt '.mat'];
    scan_data1(ii)=load(datafile);
    force=scan_data1(ii).data(:,2)*4.44822162;
    massImp=0.028;
    newData=(scan_data1(ii).data(:,1)*9.81*massImp-force)/massImp;
    temp = abs(fft(newData,640000));
    Dft1(:,ii)=temp;
    Force_Dft1(:,ii)=abs(fft(scan_data1(ii).data(:,2),640000));
    [C,I]= max(Dft1(:,1));
    omega1=1/640000*6400;
    peak_294(ii)=max(Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*pi)^2;
    peak_590(ii)=max(Dft1((2*omega1-10)/6400*640000:(2*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*2*pi)^2;
    force_peak1(ii)=max(Force_Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*4.44822162;
end

%% Loading the data from the 101mm damaged panel at one-third the primary resonance
for ii = 1:19
    cnt=num2str(ii);

datafile = ['C:\Users\editman\Documents\MATLAB\Data_Aq\Oct_2_4inch\99_amp\data\'\n    cnt '.mat'];

scan_data2(ii)=load(datafile);
force=scan_data2(ii).data(:,2)*4.44822162;

massImp=0.028;
newData=(scan_data2(ii).data(:,1)*9.81*massImp-force)/massImp;

temp=abs(fft(newData,640000));
Dft2(:,ii)=temp;
Force_Dft2(:,ii)=abs(fft(scan_data2(1,ii).data(:,2),640000));

[C,I]=max(Dft2(:,1));
omega2=I/640000*6400;

peak_98(ii)=max(Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii))\n    /3200*9.81/(2*pi*omega2)^2;
peak_198(ii)=max(Dft2((2*omega2-10)/6400*640000:(2*omega2+10)/6400*640000,ii))\n    /3200*9.81/(2*pi*(omega2+10)^2);
peak_295(ii)=max(Dft2((3*omega2-10)/6400*640000:(3*omega2+10)/6400*640000,ii))\n    /3200*9.81/(2*pi*(3*omega2)^2);
force_peak2(ii)=max(Force_Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii))\n    /3200*4.44822162);
end

%% Loading the data from the 101mm damaged panel at one-half the primary resonance
for ii = 1:19
    cnt=num2str(ii);
    datafile = ['C:\Users\editman\Documents\MATLAB\Data_Aq\Oct_2_4inch\148_amp\data\'\n       cnt '.mat'];

    scan_data3(ii)=load(datafile);
    force=scan_data3(ii).data(:,2)*4.44822162;

    massImp=0.028;
    newData=(scan_data3(ii).data(:,1)*9.81*massImp-force)/massImp;

    temp=abs(fft(newData,640000));
    Dft3(:,ii)=temp;
    Force_Dft3(:,ii)=abs(fft(scan_data3(1,ii).data(:,2),640000));

    [C,I]=max(Dft3(:,1));
    omega3=I/640000*6400;

    peak_147(ii)=max(Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii))\n        /3200*9.81/(2*pi*omega3)^2);
peak_2952(ii)=max(Dft3((2*omega3-10)/6400*640000:(2*omega3+10)/6400*640000,ii) /3200*9.81/(2*pi*2*omega3)^2);
peak_590(ii)=max(Dft2(588/6400*65536:592/6400*65536,ii)/3200*9.81/(2*pi*592)^2);
force_peak3(ii)=max(Force_Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii) /3200*4.44822162);

%% Coefficient Fitting based on the analytical results
clc;

w0_4inch=mean([omega1 3*omega2 2*omega3])*2*pi;
omega_H4inch=w0_4inch/2;
omega_T4inch=w0_4inch/3;
num=length(peak_294);
for ii =1:num
    top_k_4inch(ii)=peak_294(ii)*force_peak1(ii);
    denom_k_4inch(ii)=peak_294(ii)^2;
end
% This is the fitting for the stiffness of the damaged area
k_4inch=sum(top_k_4inch)/sum(denom_k_4inch);
ybar_k_4inch=mean(force_peak1);
SStot_k_4inch=sum((force_peak1-ybar_k_4inch).^2);
SSres_k_4inch=sum((force_peak1-k_4inch*peak_294).^2);
Rsquared_k_4inch=1-SSres_k_4inch/SStot_k_4inch;

mass_4inch=k_4inch/(w0_4inch^2);
F_4inch_PR=force_peak1/mass_4inch;
num=length(peak_294);
for ii =1:num
    top_mu_4inch(ii)=peak_294(ii)*F_4inch_PR(ii);
    denom_mu_4inch(ii)=F_4inch_PR(ii)^2;
end
% This is the fitting of the linear damping term
beta=sum(top_mu_4inch)/sum(denom_mu_4inch);
mu_4inch=1/(2*w0_4inch*beta);
ybar_mu_4inch=mean(peak_294);
SStot_mu_4inch=sum((peak_294-ybar_mu_4inch).^2);
SSres_mu_4inch=sum((peak_294-beta*F_4inch_PR).^2);
Rsquared_mu_4inch=1-SSres_mu_4inch/SStot_mu_4inch;

F_4inch_HPR=force_peak3/mass_4inch;
Lam_HPR=F_4inch_HPR/(2*(w0_4inch^2-omega_H4inch^2));
num = length(peak_2952);
for ii=1:num
    top_kappa_4inch(ii) = peak_2952(ii)*Lam_HPR(ii)^2;
    denom_kappa_4inch(ii) = Lam_HPR(ii)^4;
end
% This is the fitting of the quadratic stiffness term at one-half the primary resonance
beta = sum(top_kappa_4inch)/sum(denom_kappa_4inch);

kappa_4inch = beta*w0_4inch*mu_4inch;
ybar_kappa_4inch = mean(peak_2952);
SStot_kappa_4inch = sum((peak_2952-ybar_kappa_4inch).^2);
SSres_kappa_4inch = sum((peak_2952-beta*Lam_HPR.^2).^2);

Rsquared_kappa_4inch = 1- SSres_kappa_4inch / SStot_kappa_4inch;
num = length(peak_590);
for ii=1:num
    top_kappa_4inch2(ii) = peak_590(ii)*F_4inch_PR(ii)^2/(24*w0_4inch^4*mu_4inch^2);
    denom_kappa_4inch2(ii) = (F_4inch_PR(ii)^2/(24*w0_4inch^4*mu_4inch^2))^2;
end
% This is the fitting of the quadratic stiffness term at the primary resonance. This is done to solve for an estimate of epsilon
kappa_4inch2 = sum(top_kappa_4inch2)/sum(denom_kappa_4inch2);
ybar_kappa_4inch2 = mean(peak_590);
SStot_kappa_4inch2 = sum((peak_590-ybar_kappa_4inch2).^2);
SSres_kappa_4inch2 = sum((peak_590-kappa_4inch2*F_4inch_PR.^2/(24*w0_4inch^4*mu_4inch^2)).^2);

Rsquared_kappa_4inch2 = 1- SSres_kappa_4inch2 / SStot_kappa_4inch2;

%% This section fits the cubic stiffness and cubic damping analytical solutions to the data
F_4inch_TPR = force_peak2/mass_4inch;
Lam_TPR = F_4inch_TPR/((2*(w0_4inch^2-omega_T4inch^2))/

a = peak_2952;
y = mu_4inch^2*a/2+kappa_4inch^2*a.*3/((2*w0_4inch^2)+2*kappa_4inch^2*a.*Lam_TPR.^2/(w0_4inch^2))
    -2*kappa_4inch^2*a.*Lam_TPR.^2/(omega_T4inch*(4*omega_T4inch^2-w0_4inch^2))*(omega_T4inch^2-w0_4inch))
    -2*kappa_4inch^2*a.*Lam_TPR.^2/(omega_T4inch*(4*omega_T4inch^2-w0_4inch^2))*(omega_T4inch^2-w0_4inch))
    -kappa_4inch^2*a.*3/(6*w0_4inch^2*(4*omega_T4inch^2-w0_4inch^2))...
    -2*kappa_4inch^2*Lam_TPR.^3/(4*omega_T4inch^2-w0_4inch^2);
x = 3/6*a.*3^3*a.*Lam_TPR.^2+Lam_TPR.^3;
% this stiffness value is based on the phase of the cubic response being equal to 0
\[ N_{\text{stiff \_4}} = \frac{\sum (x \cdot y)}{\sum (x^2)}; \]

\[ y_{\text{bar \_Nstiff \_4inch}} = \text{mean}(y); \]
\[ S_{\text{Stot \_Nstiff \_4inch}} = \sum ((y - y_{\text{bar \_Nstiff \_4inch}})^2); \]
\[ S_{\text{Res \_Nstiff \_4inch}} = \sum ((y - N_{\text{stiff \_4}} \cdot x)^2); \]
\[ R_{\text{squared \_N\_4inch}} = 1 - \frac{S_{\text{Res \_Nstiff \_4inch}}}{S_{\text{Stot \_Nstiff \_4inch}}}; \]

\[ y_{\text{bad}} = \mu_{\text{4inch}}^2 \cdot a / (2 \cdot w_{\text{0\_4inch}}^2) + \kappa_{\text{4inch}}^2 \cdot a^3 / (2 \cdot w_{\text{0\_4inch}}^3) + 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}}^2) - 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}} \cdot \omega_{\text{T4inch}} \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2) \cdot (\omega_{\text{T4inch}} - 2 \cdot w_{\text{0\_4inch}}) - 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}} \cdot \omega_{\text{T4inch}} \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2) \cdot (\omega_{\text{T4inch}} + 2 \cdot w_{\text{0\_4inch}}) - \kappa_{\text{4inch}}^2 \cdot a^3 / (6 \cdot w_{\text{0\_4inch}}^3 \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2) + 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^3 \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2)); \]

\[ x_{\text{bad}} = 3/8 \cdot a^3 + 3 \cdot a \cdot \Lambda_{\text{TPR}}^2 - \Lambda_{\text{TPR}}^3; \]

% this stiffness value is based on the phase of the cubic response being
% equal to pi
\[ N_{\text{stiff \_4bad}} = \frac{\sum (x_{\text{bad}} \cdot y_{\text{bad}})}{\sum (x_{\text{bad}}^2)}; \]

\[ y_{\text{bar \_Nstiff \_4inchbad}} = \text{mean}(y_{\text{bad}}); \]
\[ S_{\text{Stot \_Nstiff \_4inchbad}} = \sum ((y_{\text{bad}} - y_{\text{bar \_Nstiff \_4inchbad}})^2); \]
\[ S_{\text{Res \_Nstiff \_4inchbad}} = \sum ((y_{\text{bad}} - N_{\text{stiff \_4bad}} \cdot x_{\text{bad}})^2); \]
\[ R_{\text{squared \_N\_4inchbad}} = 1 - \frac{S_{\text{Res \_Nstiff \_4inchbad}}}{S_{\text{Stot \_Nstiff \_4inchbad}}}; \]

\[ y_{\text{d}} = (\mu_{\text{4inch}}^2 \cdot a / (2 \cdot w_{\text{0\_4inch}}^2) + \kappa_{\text{4inch}}^2 \cdot a^3 / (2 \cdot w_{\text{0\_4inch}}^3) + 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}}^2) - 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}} \cdot \omega_{\text{T4inch}} \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2) \cdot (\omega_{\text{T4inch}} - 2 \cdot w_{\text{0\_4inch}}) - 2 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a / (w_{\text{0\_4inch}} \cdot \omega_{\text{T4inch}} \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2) \cdot (\omega_{\text{T4inch}} + 2 \cdot w_{\text{0\_4inch}}) - \kappa_{\text{4inch}}^2 \cdot a^3 / (6 \cdot w_{\text{0\_4inch}}^3 \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2)) + 4 \cdot \kappa_{\text{4inch}}^2 \cdot \Lambda_{\text{TPR}}^3 \cdot (4 \cdot \omega_{\text{T4inch}}^2 - w_{\text{0\_4inch}}^2))^2 - 9/4 \cdot \omega_{\text{T4inch}}^4 \cdot \Lambda_{\text{TPR}}^4 \cdot a^2 - 9/64 \cdot w_{\text{0\_4inch}}^4 \cdot \Lambda_{\text{TPR}}^4 \cdot a^6; \]

\[ x_{\text{d}} = \omega_{\text{T4inch}}^6 \cdot \Lambda_{\text{TPR}}^6 / (w_{\text{0\_4inch}}^2) - 9 \cdot \omega_{\text{T4inch}}^4 \cdot \Lambda_{\text{TPR}}^4 \cdot a^2 - 9/4 \cdot \omega_{\text{T4inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a^4 \cdot w_{\text{0\_4inch}}^2 - 9/64 \cdot w_{\text{0\_4inch}}^4 \cdot a^6; \]

% this is the fit for a cubic damping term
\[ N_{\text{damp \_4}} = \frac{\sum (x_{\text{d}} \cdot y_{\text{d}})}{\sum (x_{\text{d}}^2)}; \]
\[ \text{realND4} = \sqrt{N_{\text{damp \_4}}}; \]

\[ y_{\text{bar \_Ndamp \_4inch}} = \text{mean}(y_{\text{d}}); \]
\[ S_{\text{Stot \_Ndamp \_4inch}} = \sum ((y_{\text{d}} - y_{\text{bar \_Ndamp \_4inch}})^2); \]
\[ S_{\text{Res \_Ndamp \_4inch}} = \sum ((y_{\text{d}} - N_{\text{damp \_4}} \cdot x_{\text{d}})^2); \]
\[ R_{\text{squared \_Ndamp \_4inch}} = 1 - \frac{S_{\text{Res \_Ndamp \_4inch}}}{S_{\text{Stot \_Ndamp \_4inch}}}; \]

%% This section repeats the previous work but on data from the 63mm damaged panel for ii = 1:31
cnt = num2str(ii);
datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_25_25inch_360_amp\data' cnt '.mat'];

scan_data4(ii)=load(datafile);
force=scan_data4(ii).data(:,2)*4.44822162;

massImp=0.028;

newData=(scan_data4(ii).data(:,1)*9.81*massImp-force)/massImp;

temp=abs(fft(newData,640000));
Dft4(:,ii)=temp;
Force_Dft4(:,ii)=abs(fft(scan_data4(1,ii).data(:,2),640000));
[C,I]=max(Dft4(:,1));
omega1=I/640000*6400;

peak_360(ii)=max(Dft4((omega1-10)/6400:omega1+10)/640000,ii))/3200*9.81/(omega1*pi)^2;
peak_720(ii)=max(Dft4((2*omega1-10)/6400:2*omega1+10)/640000,ii))/3200*9.81/(omega1*pi)^2;
force_peak4(ii)=max(Force_Dft4((omega1-10)/6400:omega1+10)/640000,ii))/3200*4.44822162;

end
for ii = 1:26
cnt = num2str(ii);
datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_25_25inch_180_amp\data' cnt '.mat'];

scan_data5(ii)=load(datafile);
force=scan_data5(ii).data(:,2)*4.44822162;

massImp=0.028;

newData=(scan_data5(ii).data(:,1)*9.81*massImp-force)/massImp;

temp=abs(fft(newData,640000));
Dft5(:,ii)=temp;
Force_Dft5(:,ii)=abs(fft(scan_data5(1,ii).data(:,2),640000));
[C,I]=max(Dft5(:,1));
omega1=I/640000*6400;

peak_180(ii)=max(Dft5((omega2-10)/6400:omega2+10)/640000,ii))/3200*9.81/(2*pi*omega2)^2;
peak_3602(ii)=\text{max}(\text{Dft5}((2\cdot\omega_2 - 10)/6400\cdot640000:(2\cdot\omega_2 + 10)/6400\cdot640000,\ ii)/3200\cdot9.81/(2\cdot\pi\cdot2\cdot\omega_2)^2); \\

force_peak5(ii)=\text{max}(\text{Force_Dft5}((\omega_2 - 10)/6400\cdot640000:(\omega_2 + 10)/6400\cdot640000,\ ii)/3200\cdot4.44822162); \\

end

for ii = 1:26
cnt=num2str(ii);
datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_25_25inch_120_amp\ data'\ cnt '.mat'];
scan_data6(ii)=load(datafile);
force=scan_data6(ii).data(:,2)\cdot4.44822162;
massImp=0.028;
newData=(scan_data6(ii).data(:,1)\cdot9.81\cdot\text{massImp} - force)/\text{massImp};

temp=abs(\text{fft}(\text{newData},640000));
Dft6(:,ii)=temp;
Force_Dft6(:,ii)=abs(\text{fft}(\text{scan_data6}(1,\ ii).data(:,2),640000));
[C,I]=\text{max}(\text{Dft6}(:,1));
\omega_3=I/640000\cdot6400;
peak_120(ii)=\text{max}(\text{Dft6}((\omega_3 - 10)/6400\cdot640000:(\omega_3 + 10)/6400\cdot640000,\ ii)/3200\cdot9.81/(2\cdot\pi\cdot\omega_3)^2); \\
peak_240(ii)=\text{max}(\text{Dft6}((2\cdot\omega_3 - 10)/6400\cdot640000:(2\cdot\omega_3 + 10)/6400\cdot640000,\ ii)/3200\cdot9.81/(2\cdot\pi\cdot2\cdot\omega_3)^2); \\
peak_3603(ii)=\text{max}(\text{Dft6}((3\cdot\omega_3 - 10)/6400\cdot640000:(3\cdot\omega_3 + 10)/6400\cdot640000,\ ii)/3200\cdot9.81/(2\cdot\pi\cdot3\cdot\omega_3)^2); \\

force_peak6(ii)=\text{max}(\text{Force_Dft6}((\omega_3 - 10)/6400\cdot640000:(\omega_3 + 10)/6400\cdot640000,\ ii)/3200\cdot4.44822162); \\
end

clc;

\omega_{0.25\text{inch}}=\text{mean}([\omega_\text{a1} \cdot2\cdot\omega_2 \cdot3\cdot\omega_3])\cdot2\cdot\pi; \\
\omega_{\text{H25inch}}=\omega_{0.25\text{inch}}/2; \\
\omega_{\text{T25inch}}=\omega_{0.25\text{inch}}/3; \\
num=\text{length}(\text{peak}_360);

for ii=1:num

top_k_{25\text{inch}}(ii)=\text{peak}_360(ii)\cdot\text{force_peak4}(ii); \\
denom_k_{25\text{inch}}(ii)=\text{peak}_360(ii)^2;
end

\[ k_{25\text{inch}} = \frac{\text{sum}(\text{top}_k_{25\text{inch}})}{\text{sum}(\text{denom}_k_{25\text{inch}})}; \]

\[ y_{\text{bar}}_{k_{25\text{inch}}}=\text{mean}(\text{force}_\text{peak}4); \]
\[ S_{\text{Stot}}_{k_{25\text{inch}}}=\text{sum}((\text{force}_\text{peak}4 - y_{\text{bar}}_{k_{25\text{inch}}})^2); \]
\[ S_{\text{Sres}}_{k_{25\text{inch}}}=\text{sum}(((\text{force}_\text{peak}4 - k_{25\text{inch}} \times \text{peak}_360)^2); \]

\[ R_{\text{Squared}}_{k_{25\text{inch}}}=1-\frac{S_{\text{Sres}}_{k_{25\text{inch}}}}{S_{\text{Stot}}_{k_{25\text{inch}}}}; \]

\[ \text{mass}_{25\text{inch}}=\frac{k_{25\text{inch}}}{(w_{0_{25\text{inch}}})^2}; \]

\[ F_{25\text{inch}_{\text{PR}}} = \frac{\text{force}_\text{peak}4}{\text{mass}_{25\text{inch}}}; \]

\[ \text{num}=\text{length}(\text{peak}_360); \]

\[ \text{for } \text{ii}=1:\text{num} \]
\[ \text{top}_{\mu_{25\text{inch}}}(\text{ii})=\text{peak}_360(\text{ii}) \times F_{25\text{inch}_{\text{PR}}}(\text{ii}); \]
\[ \text{denom}_{\mu_{25\text{inch}}}(\text{ii})=F_{25\text{inch}_{\text{PR}}}(\text{ii})^2; \]
\[ \beta=\frac{\text{sum}(\text{top}_{\mu_{25\text{inch}}})}{\text{sum}(\text{denom}_{\mu_{25\text{inch}}})}; \]
\[ \mu_{25\text{inch}}=\frac{1}{2 \times w_{0_{25\text{inch}}} \times \beta}; \]
\[ y_{\text{bar}}=\text{mean}(\text{peak}_360); \]
\[ S_{\text{Stot}}=\text{sum}((\text{peak}_360 - y_{\text{bar}})^2); \]
\[ S_{\text{Sres}}=\text{sum}(((\text{peak}_360 - \beta \times F_{25\text{inch}_{\text{PR}}})^2); \]
\[ R_{\text{Squared}}_{\mu_{25\text{inch}}}=1-\frac{S_{\text{Sres}}}{S_{\text{Stot}}}; \]

\[ F_{25\text{inch}_{\text{HPR}}}=\frac{\text{force}_\text{peak}5}{\text{mass}_{25\text{inch}}}; \]
\[ \text{Lam}_{H25}=\frac{F_{25\text{inch}_{\text{HPR}}}}{(2 \times (w_{0_{25\text{inch}}})^2 - \omega_{H25\text{inch}}^2)); \]
\[ \text{num}=\text{length}(\text{peak}_360); \]

\[ \text{for } \text{ii}=1:\text{num} \]
\[ \text{top}_{\kappa_{25\text{inch}}}(\text{ii})=\text{peak}_360(\text{ii}) \times \text{Lam}_{H25}(\text{ii})^2; \]
\[ \text{denom}_{\kappa_{25\text{inch}}}(\text{ii})=\text{Lam}_{H25}(\text{ii})^4; \]
\[ \beta=\frac{\text{sum}(\text{top}_{\kappa_{25\text{inch}}})}{\text{sum}(\text{denom}_{\kappa_{25\text{inch}}})}; \]
\[ \kappa_{25\text{inch}}=\beta \times w_{0_{25\text{inch}}} \times \mu_{25\text{inch}}; \]
\[ y_{\text{bar}}=\text{mean}(\text{peak}_360); \]
\[ S_{\text{Stot}}=\text{sum}((\text{peak}_360 - y_{\text{bar}})^2); \]
\[ S_{\text{Sres}}=\text{sum}(((\text{peak}_360 - \beta \times \text{Lam}_{H25})^2); \]
\[ R_{\text{Squared}}_{\kappa_{25\text{inch}}}=1-\frac{S_{\text{Sres}}}{S_{\text{Stot}}}; \]

\[ \text{num}=\text{length}(\text{peak}_720); \]

\[ \text{for } \text{ii}=1:\text{num} \]
\[ \text{top}_{\kappa_{25\text{inch}}2}(\text{ii})=\text{peak}_720(\text{ii}) \times F_{25\text{inch}_{\text{PR}}}(\text{ii})^2/(24 \times w_{0_{25\text{inch}}}^4 \times \mu_{25\text{inch}}^4); \]
\[ \text{denom}_{\kappa_{25\text{inch}}2}(\text{ii})=(F_{25\text{inch}_{\text{PR}}}(\text{ii})^2/(24 \times w_{0_{25\text{inch}}}^4 \times \mu_{25\text{inch}}^4))^2; \]
\[ \text{end} \]
kappa_25inch2 = sum(top_kappa_25inch2) / sum(denom_kappa_25inch2);
ybar_kappa_25inch2 = mean(peak_720);
SStot_kappa_25inch2 = sum((peak_720 - ybar_kappa_25inch2).^2);
SSres_kappa_25inch2 = sum((peak_720 - kappa_25inch2 * F_25inch_PR ./ (24 * w0_25inch.^4 * mu_25inch.^2)).^2);
Rsquared_kappa_25inch2 = 1 - SSres_kappa_25inch2 / SStot_kappa_25inch2;

%%
F_25inch_TPR = force_peak6 / mass_25inch;
Lam_TPR = F_25inch_TPR / (2 * (w0_25inch.^2 - omega_T25inch.^2));
a2 = peak_3603;
y25 = mu_25inch.^2 * a2 ./ 2 + 2 * kappa_25inch.^2 * a2 ./ 2 * Lam_TPR ./ (w0_25inch.^2) * (omega_T25inch - 2 * w0_25inch) *
    -2 * kappa_25inch.^2 * a2 .* Lam_TPR ./ (omega_T25inch * (4 * omega_T25inch.^2 - w0_25inch.^2) * (omega_T25inch + 2 * w0_25inch)) *
    -2 * kappa_25inch.^2 * a2 .* Lam_TPR ./ (omega_T25inch * (4 * omega_T25inch.^2 - w0_25inch.^2) * (omega_T25inch - 2 * w0_25inch)) *
    -kappa_25inch.^2 * a2 .* Lam_TPR ./ (6 * w0_25inch.^2 * (4 * omega_T25inch.^2 - w0_25inch.^2)) *
    -2 * kappa_25inch.^2 * a2 .* Lam_TPR ./ (4 * omega_T25inch.^2 - w0_25inch.^2) .* x25 bad = 3/8 * a2.^3 + 3 * a2 .* Lam_TPR .* 2 * Lam_TPR .* 3;
N_stiff_25bad = sum(x25bad .* y25bad) / sum(x25bad.^2);

ybar_N_25inch = mean(y25);
SStot_N_25inch = sum((y25 - ybar_N_25inch).^2); SSres_N_25inch = sum((y25 - N_stiff_25 * x25).^2);
Rsquared_N_25inch = 1 - SSres_N_25inch / SStot_N_25inch;

yd25 = (mu_25inch.^2 * a2 ./ (2 * w0_25inch) + kappa_25inch.^2 * a2 .* Lam_TPR ./ (w0_25inch.^2) * (omega_T25inch - 2 * w0_25inch) *
    -2 * kappa_25inch.^2 * a2 .* Lam_TPR ./ (omega_T25inch * (4 * omega_T25inch.^2 - w0_25inch.^2) * (omega_T25inch + 2 * w0_25inch)) *
    -2 * kappa_25inch.^2 * a2 .* Lam_TPR ./ (omega_T25inch * (4 * omega_T25inch.^2 - w0_25inch.^2) * (omega_T25inch - 2 * w0_25inch)) *
    -kappa_25inch.^2 * a2 .* Lam_TPR ./ (6 * w0_25inch.^2 * (4 * omega_T25inch.^2 - w0_25inch.^2)) .* 2 * x25 bad = 3/8 * a2.^3 + 3 * a2 .* Lam_TPR .* 2 * Lam_TPR .* 3;
N_stiff_25bad = sum(x25bad .* y25bad) / sum(x25bad.^2);

ybar_N_25inchbad = mean(y25bad);
SStot_N_25inchbad = sum((y25bad - ybar_N_25inchbad).^2); SSres_N_25inchbad = sum((y25bad - N_stiff_25bad * x25bad).^2);
Rsquared_N_25inchbad = 1 - SSres_N_25inchbad / SStot_N_25inchbad;

xd25 = omega_T25inch.^6 * Lam_TPR * 6 ./ (w0_25inch.^2) - 9 .* omega_T25inch * 4 .* Lam_TPR * 4 .* a2 .* 2 - ...
9/4.*\omega_{T25\text{inch}}^2 \cdot \text{Lam}\_\text{TPR} \cdot 2 \cdot a_2 \cdot 4 \cdot w_0\_25\text{inch}^2 - 9/64 \cdot w_0\_25\text{inch}^4 \cdot a_2 \cdot 6;

N\_\text{damp}\_25 = \text{sum}(xd25 \cdot yd25)/\text{sum}(xd25^2);

\text{realND25} = \sqrt{N\_\text{damp}\_25};

ybar\_\text{Ndamp}\_25\text{inch} = \text{mean}(yd25);

SStot\_\text{Ndamp}\_25\text{inch} = \text{sum}((yd25 - ybar\_\text{Ndamp}\_25\text{inch})^2);

SSres\_\text{Ndamp}\_25\text{inch} = \text{sum}((yd25 - N\_\text{damp}\_25 \cdot xd25)^2);

Rsquared\_\text{Ndamp}\_25\text{inch} = 1 - SSres\_\text{Ndamp}\_25\text{inch}/SStot\_\text{Ndamp}\_25\text{inch};

\%\% Repeating once more for the 25mm damaged panel

for ii = 1:7
    cnt = num2str(ii);
    datafile = ['C:\Users\editman\Documents\MATLAB\Data Aq\Apr_24_1 inch_476_amp\data' cnt '.mat'];
    scan_data7(ii) = load(datafile);
    force = scan_data7(ii).data(:,2)*4.44822162;
    massImp = 0.028;
    newData = (scan_data7(ii).data(:,1)*9.81*massImp-force)/massImp;
    temp = abs(fft(newData,64000));
    Dft7(:,ii) = temp;
    Force_Dft7(:,ii) = abs(fft(scan_data7(1,ii).data(:,2),64000));
    [C,I] = max(Dft7(:,1));
    omega1 = I/640000*6400;
    peak_476(ii) = max(Dft7(((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*9.81/(omega1^2*pi));
    peak_952(ii) = max(Dft7((2*omega1-10)/6400*640000:(2*omega1+10)/6400*640000,ii))/3200*9.81/(omega1^4*pi));
    force_peak7(ii) = max(Force_Dft7(((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*4.44822162);
end

for ii = 1:6
    cnt = num2str(ii);
    datafile = ['C:\Users\editman\Documents\MATLAB\Data Aq\Apr_24_1 inch_234_amp\data' cnt '.mat'];
    scan_data8(ii) = load(datafile);
    force = scan_data8(ii).data(:,2)*4.44822162;
massImp = 0.028;

newData = (scan_data8(ii).data(:,1) * 9.81 * massImp - force) / massImp;

temp = abs(fft(newData,640000));
Dft8(:,ii) = temp;
Force_Dft8(:,ii) = abs(fft(scan_data8(1,ii).data(:,2),640000));

[C, I] = max(Dft8(:,1));
omega2 = 1 / 640000 * 6400;

peak_234(ii) = max(Dft8((omega2 - 10) / 6400 * 640000:(omega2 + 10) / 6400 * 640000, ii) / 3200 * 9.81 / (2 * pi * omega2) ^ 2);
peak_4762(ii) = max(Dft8((2 * omega2 - 10) / 6400 * 640000:(2 * omega2 + 10) / 6400 * 640000, ii) / 3200 * 9.81 / (2 * pi * 2 * omega2) ^ 2);

force_peak8(ii) = max(Force_Dft8((omega2 - 10) / 6400 * 640000:(omega2 + 10) / 6400 * 640000, ii) / 3200 * 4.44822162);
end

for ii = 1:6
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_24_1 inch_156 amp\data' ' ' cnt ' .mat '];
    scan_data9(ii) = load(datafile);
    force = scan_data9(ii).data(:,2) * 4.44822162;

    massImp = 0.028;
    newData = (scan_data9(ii).data(:,1) * 9.81 * massImp - force) / massImp;

    temp = abs(fft(newData,640000));
    Dft9(:,ii) = temp;
    Force_Dft9(:,ii) = abs(fft(scan_data9(1,ii).data(:,2),640000));

    [C, I] = max(Dft9(:,1));
    omega3 = 1 / 640000 * 6400;

    peak_156(ii) = max(Dft9((omega3 - 10) / 6400 * 640000:(omega3 + 10) / 6400 * 640000, ii) / 3200 * 9.81 / (2 * pi * omega3) ^ 2);
peak_312(ii) = max(Dft9((2 * omega3 - 10) / 6400 * 640000:(2 * omega3 + 10) / 6400 * 640000, ii) / 3200 * 9.81 / (2 * pi * 2 * omega3) ^ 2);
peak_4763(ii) = max(Dft9((3 * omega3 - 10) / 6400 * 640000:(3 * omega3 + 10) / 6400 * 640000, ii) / 3200 * 9.81 / (2 * pi * 3 * omega3) ^ 2);

    force_peak9(ii) = max(Force_Dft9((omega3 - 10) / 6400 * 640000:(omega3 + 10) / 6400 * 640000, ii) / 3200 * 4.44822162);
end
clc;

w0_1inch=mean([omega1 2*omega2 3*omega3])*2*pi;
omega_H1inch=w0_1inch/2;
omega_T1inch=w0_1inch/3;

num=length(peak_476);

for ii=1:num
    top_k_1inch(ii)=peak_476(ii)*force_peak7(ii);
    denom_k_1inch(ii)=peak_476(ii)^2;
end

k_1inch=sum(top_k_1inch)/sum(denom_k_1inch);
ybar_k_1inch=mean(force_peak7);
SStot_k_1inch=sum((force_peak7-ybar_k_1inch).^2);
SSres_k_1inch=sum((force_peak7-k_1inch*peak_476).^2);
Rsquared_k_1inch=1-SSres_k_1inch/SStot_k_1inch;
mass_1inch=k_1inch/(w0_1inch^2);
F_1inch_PR=force_peak7/mass_1inch;

num=length(peak_476);

for ii=1:num
    top_mu_1inch(ii)=peak_476(ii)*F_1inch_PR(ii);
    denom_mu_1inch(ii)=F_1inch_PR(ii)^2;
end

beta=sum(top_mu_1inch)/sum(denom_mu_1inch);
mu_1inch=1/(2*w0_1inch*beta);
ybar=mean(peak_476);
SStot=sum((peak_476-ybar).^2);
SSres=sum((peak_476-beta*F_1inch_PR).^2);
Rsquared_mu_1inch=1-SSres/SStot;
F_1inch_HPR=force_peak8/mass_1inch;
Lam_H1=F_1inch_HPR/(2*(w0_1inch^2-omega_H1inch^2));

num=length(peak_4762);

for ii=1:num
    top_kappa_1inch(ii)=peak_4762(ii)*Lam_H1(ii)^2;
    denom_kappa_1inch(ii)=Lam_H1(ii)^4;
end

beta=sum(top_kappa_1inch)/sum(denom_kappa_1inch);
kappa_1inch=beta*w0_1inch*mu_1inch;
ybar=mean(peak_4762);
\[
\text{SStot} = \text{sum}((\text{peak}_{4762} - \text{ybar})^2);
\]
\[
\text{SSres} = \text{sum}((\text{peak}_{4762} - \beta \cdot \text{Lam}_H1)^2);
\]
\[
\text{Rsquared}_\text{kappa}_{1\text{inch}} = 1 - \text{SSres} / \text{SStot}.
\]
\[
\text{real_damping} = [\mu_{\text{linch}} \ \mu_{25\text{inch}} \ \mu_{4\text{inch}}] \cdot [\text{mass}_{\text{linch}} \ \text{mass}_{25\text{inch}} \ \text{mass}_{4\text{inch}}];
\]
\[
\text{real_stiff} = [w_{0\text{inch}}^2 \ w_{0\text{inch}}^2 \ w_{0\text{inch}}^2] \cdot [\text{mass}_{\text{linch}} \ \text{mass}_{25\text{inch}} \ \text{mass}_{4\text{inch}}];
\]
\[
\text{real_kappa} = [\kappa_{\text{linch}} \ \kappa_{25\text{inch}} \ \kappa_{4\text{inch}}] \cdot [\text{mass}_{\text{linch}} \ \text{mass}_{25\text{inch}} \ \text{mass}_{4\text{inch}}];
\]
\[
\text{num} = \text{length} (\text{peak}_{952});
\]
\[
\text{for } ii = 1: \text{num}
\quad \text{top} \_\text{kappa}_{1\text{inch}} (ii) = \text{peak}_{952} (ii) \cdot F_{1\text{inch} \cdot \text{PR}} (ii)^2 / (24 \cdot w_{0\text{inch}}^4 \cdot \mu_{1\text{inch}}^2);
\quad \text{denom} \_\text{kappa}_{1\text{inch}} (ii) = (F_{1\text{inch} \cdot \text{PR}} (ii)^2 / (24 \cdot w_{0\text{inch}}^4 \cdot \mu_{1\text{inch}}^2))^2;
\end{aligned}
\]
\[
\kappa_{1\text{inch}} = \text{sum} (\text{top} \_\text{kappa}_{1\text{inch}}) / \text{sum} (\text{denom} \_\text{kappa}_{1\text{inch}});
\]
\[
\text{ybar} \_\text{kappa}_{1\text{inch}} = \text{mean} (\text{peak}_{952});
\]
\[
\text{SStot}_{\text{kappa}_{1\text{inch}}} = \text{sum} ((\text{peak}_{952} - \text{ybar} \_\text{kappa}_{1\text{inch}})^2);
\]\n\[
\text{SSres}_{\text{kappa}_{1\text{inch}}} = \text{sum} ((\text{peak}_{952} - \kappa_{1\text{inch}} \cdot F_{1\text{inch} \cdot \text{PR}}^2 / (24 \cdot w_{0\text{inch}}^4 \cdot \mu_{1\text{inch}}^2))^2);
\]
\[
\text{Rsquared}_{\text{kappa}_{1\text{inch}}} = 1 - \text{SSres}_{\text{kappa}_{1\text{inch}}} / \text{SStot}_{\text{kappa}_{1\text{inch}}}.
\]
\[
F_{1\text{inch} \cdot \text{TPR}} = \text{force}_{\text{peak}9} / \text{mass}_{\text{linch}};
\]
\[
\text{Lam}_{\text{TPR}} = F_{1\text{inch} \cdot \text{TPR}} / (2 \cdot (w_{0\text{inch}}^2 - \omega_{T1\text{inch}}^2));
\]
\[
a_3 = \text{peak}_{4763};
\]
\[
\omega_{\text{TPR}} = \omega_{T1\text{inch}};
\]
\[
\begin{aligned}
y_1 &= \mu_{\text{linch}}^2 \cdot a_3 / 2 + \kappa_{\text{linch}}^2 \cdot a_3^3 / 2 \cdot w_{0\text{inch}}^2 + 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (w_{0\text{inch}}^2) \\
&\quad - 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (\omega_{T1\text{inch}} (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2) - 2) \\
&\quad - 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (\omega_{T1\text{inch}} (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2) + 2) \\
&\quad - \kappa_{\text{linch}}^2 \cdot a_3^3 / 2 \cdot w_{0\text{inch}}^2 \\
&\quad + 2 \cdot \kappa_{\text{linch}}^2 \cdot \text{Lam}_{\text{TPR}}^3 / (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2);
\end{aligned}
\]
\[
x_1 = 3 / 8 \cdot a_3^3 + 3 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 + \text{Lam}_{\text{TPR}}^3;
\]
\[
\text{N}_{\text{stiff}1} = \text{sum} (x_1 \cdot y_1) / \text{sum} (x_1^2);
\]
\[
\text{ybar} \_\text{N}_{\text{linch}} = \text{mean} (y_1);
\]
\[
\text{SStot}_{\text{N}_{\text{linch}}} = \text{sum} ((y_1 - \text{ybar} \_\text{N}_{\text{linch}})^2);
\]
\[
\text{SSres}_{\text{N}_{\text{linch}}} = \text{sum} ((y_1 - \text{N}_{\text{stiff}1} \cdot x_1)^2);
\]
\[
\text{Rsquared}_{\text{N}_{\text{linch}}} = 1 - \text{SSres}_{\text{N}_{\text{linch}}} / \text{SStot}_{\text{N}_{\text{linch}}}.
\]
\[
\begin{aligned}
y_{1\text{bad}} &= \mu_{\text{linch}}^2 \cdot a_3 / 2 + \kappa_{\text{linch}}^2 \cdot a_3^3 / 2 \cdot w_{0\text{inch}}^2 + 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (w_{0\text{inch}}^2) \\
&\quad - 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (\omega_{T1\text{inch}} (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2) - 2) \\
&\quad - 2 \cdot \kappa_{\text{linch}}^2 \cdot a_3 \cdot \text{Lam}_{\text{TPR}}^2 / (\omega_{T1\text{inch}} (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2) + 2) \\
&\quad - \kappa_{\text{linch}}^2 \cdot a_3^3 / 2 \cdot w_{0\text{inch}}^2 \\
&\quad + 2 \cdot \kappa_{\text{linch}}^2 \cdot \text{Lam}_{\text{TPR}}^3 / (4 \cdot \omega_{T1\text{inch}}^2 - w_{0\text{inch}}^2)
\end{aligned}
x1bad = 3/8 * a3 .* Lam_TPR .* 2 - Lam_TPR .* 3;

N_stiff_1bad = sum(x1bad .* y1bad) / sum(x1bad .* y1bad);

ybar_N_1inchbad = mean(y1bad);
SStot_N_1inchbad = sum((y1bad - ybar_N_1inchbad).^2);
SSres_N_1inchbad = sum((y1bad - N_stiff_1bad * x1bad).^2);
Rsquared_N_1inchbad = 1 - SSres_N_1inchbad / SStot_N_1inchbad;

yd1 = (mu_1inch^2 * a3 ./ (2 * w0_1inch) + kappa_1inch^2 * a3 .* Lam_TPR .* (4 * omega_T1inch^2 - w0_1inch^2) .* (omega_T1inch + w0_1inch)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 ./ (w0_1inch .* omega_T1inch .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* Lam_TPR .* a3 .* (4 * omega_T1inch^2 - w0_1inch^2)) ...
    - 2 * kappa_1inch^2 * a3 .* (6 * w0_1inch^2)/(4 * omega_T1inch^2 - w0_1inch^2));

N_damp_1 = sum(xd1 .* yd1) / sum(xd1 .* yd1);
realND1 = sqrt(N_damp_1);
ybar_Ndamp_1inch = mean(yd1);
SStot_Ndamp_1inch = sum((yd1 - ybar_Ndamp_1inch).^2);
SSres_Ndamp_1inch = sum((yd1 - N_damp_1 * xd1).^2);
Rsquared_Ndamp_1inch = 1 - SSres_Ndamp_1inch / SStot_Ndamp_1inch;

%% This section recreates the restoring force curves using the analytical solutions for the response due to excitation at the primary resonance
eps1inch = kappa_1inch^2 / kappa_1inch;
eps25inch = kappa_25inch^2 / kappa_25inch;
eps4inch = kappa_4inch^2 / kappa_4inch;
eps = mean([eps1inch eps25inch eps4inch]);
F = 8.5;
t = 0:0.00001:2;
a_1inch_PR = (F / mass_1inch) ./ (2 * w0_1inch * mu_1inch);
a_25inch_PR = (F / mass_25inch) ./ (2 * w0_25inch * mu_25inch);
a_4inch_PR = (F / mass_4inch) ./ (2 * w0_4inch * mu_4inch);
u_1inch_PR = a_1inch_PR .* cos(w0_1inch .* t + pi/2) + eps * kappa_1inch * a_1inch_PR^2 / (6 * w0_1inch^2) .* cos(2 * w0_1inch .* pi) * eps * kappa_1inch * a_1inch_PR^2 / (2 * w0_1inch^2);
u_25inch_PR = a_25inch_PR .* cos(w0_25inch .* t + pi/2) + eps * kappa_25inch * a_25inch_PR^2 / (6 * w0_25inch^2) .* cos(2 * w0_25inch .* pi) * eps * kappa_25inch * a_25inch_PR^2 / (2 * w0_25inch^2);
u_4inch_PR = a_4inch_PR .* cos(w0_4inch .* t + pi/2) + eps * kappa_4inch * a_4inch_PR^2 / (6 * w0_4inch^2) .* cos(2 * w0_4inch .* pi) * eps * kappa_4inch * a_4inch_PR^2 / (2 * w0_4inch^2);
F_1inch_PR = F .* cos(w0_1inch .* t);
F_25inch_PR = F .* cos(w0_25inch .* t);
F_4inch_PR = F .* cos(w0_4inch .* t);

figure;
plot(u_1inch_PR, F_1inch_PR);
xlabel('Displacement in m');
ylabel('Force in N');
%% This section recreates the restoring force curves using a numerical
% solver at the primary resonance

F1test = F* \cos((w0_1inch)*t);
F25test = F* \cos((w0_25inch)*t);
F4test = F* \cos((w0_4inch)*t);

[T_1inch_PR, Y_1inch_PR] = ode45(@rigid_1inch_PR, t, [0 0]);
[T_25inch_PR, Y_25inch_PR] = ode45(@rigid_25inch_PR, t, [0 0]);
[T_4inch_PR, Y_4inch_PR] = ode45(@rigid_4inch_PR, t, [0 0]);

%% This section recreates the restoring force curves using the analytical
% solutions for the response due to excitation at one-half the primary resonance

omega_1inch_HPR = w0_1inch / 2;
omega_25inch_HPR = w0_25inch / 2;
omega_4inch_HPR = w0_4inch / 2;

Lam_1inch_HPR = F / mass_1inch / ((2*(w0_1inch)^2 - (omega_1inch_HPR)^2));
Lam_25inch_HPR = F / mass_25inch / ((2*(w0_25inch)^2 - (omega_25inch_HPR)^2));
Lam_4inch_HPR = F / mass_4inch / ((2*(w0_4inch)^2 - (omega_4inch_HPR)^2));

a_1inch_HPR = kappa_1inch * Lam_1inch_HPR^2 / (w0_1inch * mu_1inch);
a_25inch_HPR = kappa_25inch * Lam_25inch_HPR^2 / (w0_25inch * mu_25inch);
a_4inch_HPR = kappa_4inch * Lam_4inch_HPR^2 / (w0_4inch * mu_4inch);

u_1inch_HPR = 2 * Lam_1inch_HPR * \cos(omega_1inch_HPR*t) + a_1inch_HPR * \cos(w0_1inch*t + pi/2);
u_25inch_HPR = 2 * Lam_25inch_HPR * \cos(omega_25inch_HPR*t) + a_25inch_HPR * \cos(w0_25inch*t + pi/2);
u_4inch_HPR = 2 * Lam_4inch_HPR * \cos(omega_4inch_HPR*t) + a_4inch_HPR * \cos(w0_4inch*t + pi/2);

F_1inch_HPR = F* \cos(omega_1inch_HPR*t);
F_25inch_HPR = F* \cos(omega_25inch_HPR*t);
F_4inch_HPR = F* \cos(omega_4inch_HPR*t);

figure;
plot(u_1inch_HPR, F_1inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');
ylabel('Force in N');
figure;
plot(u_25inch_HPR,F_25inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(u_4inch_HPR,F_4inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');
freqt=0:0.005:100000.499;

% This section recreates the restoring force curves using the analytical
% solutions for the response due to excitation at one-third the primary resonance

omega_1inch_TPR = w0_1inch/3;
omega_25inch_TPR = w0_25inch/3;
omega_4inch_TPR = w0_4inch/3;

Lam_1inch_TPR = F/mass_1inch/(2*(w0_1inch^2 - (omega_1inch_TPR)^2));
Lam_25inch_TPR = F/mass_25inch/(2*(w0_25inch^2 - (omega_25inch_TPR)^2));
Lam_4inch_TPR = F/mass_4inch/(2*(w0_4inch^2 - (omega_4inch_TPR)^2));

F_1inch_TPR = F*cos(omega_1inch_TPR*t);
F_25inch_TPR = F*cos(omega_25inch_TPR*t);
F_4inch_TPR = F*cos(omega_4inch_TPR*t);
\[
\kappa_{1\text{inch}}^2 \cdot (4 - 48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 12 \cdot w_0_{1\text{inch}}^2) \cdot 2 + \sqrt{(
\omega_{1\text{inch}_\text{TPR}}^2 - \ldots
4 \cdot w_0_{1\text{inch}}^2 \cdot 3 \cdot (-9 \cdot N_{\text{stiff}_1} \cdot w_0_{1\text{inch}}^2 \cdot (-4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + w_0_{1\text{inch}}^2) + \ldots
kappa_{1\text{inch}}^2 \cdot (4 - 48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 12 \cdot w_0_{1\text{inch}}^2) \cdot 3 \cdot (-4 \cdot \mu_{1\text{inch}}^6 - 72 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^4 \cdot N_{\text{stiff}_1} \cdot \ldots
432 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^6 \cdot N_{\text{stiff}_1} \cdot \ldots
945 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^6 \cdot N_{\text{stiff}_1} \cdot \ldots
4 \cdot w_0_{1\text{inch}}^2 + \ldots
768 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + 17 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 2 \cdot w_0_{1\text{inch}}^2 + \ldots
19556 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (256 + 4349 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
30708 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (1024 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
12 \cdot (64 + 1052 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 5339 \cdot \omega_{1\text{inch}_\text{TPR}}^4) \cdot w_0_{1\text{inch}}^2 + \ldots
48 \cdot (20 + 443 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot w_0_{1\text{inch}}^2 + 2752 \cdot w_0_{1\text{inch}}^2 + \ldots
12 \cdot \kappa_{1\text{inch}}^2 \cdot \omega_{1\text{inch}_\text{TPR}}^4 \cdot w_0_{1\text{inch}}^2 \cdot (-16 \cdot \mu_{1\text{inch}}^2 \cdot (4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + w_0_{1\text{inch}}^2) + \ldots
17 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + \ldots
3 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (512 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (127 + 1100 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
12144 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 10677 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (w_0_{1\text{inch}}^2) \cdot (-234) + \ldots
8 \cdot (56 + 661 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot w_0_{1\text{inch}}^2 + 848 \cdot w_0_{1\text{inch}}^2 + \ldots
12 \cdot \kappa_{1\text{inch}}^2 \cdot \omega_{1\text{inch}_\text{TPR}}^4 \cdot w_0_{1\text{inch}}^2 \cdot (4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
17 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + \ldots
48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
17 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + \ldots
4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
kappa_{1\text{inch}}^2 \cdot (4 - 48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 12 \cdot w_0_{1\text{inch}}^2) \cdot 2 + \sqrt{(
\omega_{1\text{inch}_\text{TPR}}^2 - \ldots
4 \cdot w_0_{1\text{inch}}^2 \cdot 3 \cdot (2 \cdot \kappa_{1\text{inch}}^2 + 4 \cdot N_{\text{stiff}_1} \cdot \omega_{1\text{inch}_\text{TPR}}^2 - \ldots
N_{\text{stiff}_1} \cdot w_0_{1\text{inch}}^2) \cdot (-9 \cdot N_{\text{stiff}_1} \cdot w_0_{1\text{inch}}^2) \cdot (-4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + w_0_{1\text{inch}}^2) + \ldots
kappa_{1\text{inch}}^2 \cdot (4 - 48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 12 \cdot w_0_{1\text{inch}}^2) \cdot 2 + \sqrt{(
\omega_{1\text{inch}_\text{TPR}}^2 - \ldots
4 \cdot w_0_{1\text{inch}}^2 \cdot 3 \cdot (-9 \cdot N_{\text{stiff}_1} \cdot w_0_{1\text{inch}}^2) \cdot (-4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + w_0_{1\text{inch}}^2) + \ldots
kappa_{1\text{inch}}^2 \cdot (4 - 48 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 12 \cdot w_0_{1\text{inch}}^2) \cdot 3 \cdot (4 \cdot \mu_{1\text{inch}}^6 + 72 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^4 \cdot N_{\text{stiff}_1} + \ldots
432 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^6 \cdot N_{\text{stiff}_1} + \ldots
945 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot \mu_{1\text{inch}}^6 \cdot N_{\text{stiff}_1} + \ldots
4 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + w_0_{1\text{inch}}^2) \cdot 3 \cdot \ldots
16 \cdot \kappa_{1\text{inch}}^2 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (1024 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
768 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + 17 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
3 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (256 + 4349 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + \ldots
19556 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (2 + \ldots
30708 \cdot \omega_{1\text{inch}_\text{TPR}}^2 + 106043 \cdot \omega_{1\text{inch}_\text{TPR}}^2 \cdot (w_0_{1\text{inch}}^2) \cdot (-234) + \ldots
\ldots}
12* (64 + 1052* omega_1inch_TPR^2 + 5339* omega_1inch_TPR^4) * w0_1inch^8 
48* (20 + 443* omega_1inch_TPR^2) * w0_1inch^10 + 2752 * w0_1inch^12) + ...

12 * kappa_1inch^4 * Lam_1inch_TPR^4 * w0_1inch^2 * (4* omega_1inch_TPR^4 - 17 * omega_1inch_TPR^2 * w0_1inch^2 + 4* w0_1inch^4) 
+ 12* (64 + 1052* omega_1inch_TPR^2 + 5339* omega_1inch_TPR^4) * w0_1inch^8 
48* (20 + 443* omega_1inch_TPR^2) * w0_1inch^10 + 2752 * w0_1inch^12) + ...

214* omega_1inch_TPR^4 * (127 + 1100* omega_1inch_TPR^2) * w0_1inch^2 + (128 + ... 
8* (56 + 661* omega_1inch_TPR^2) * w0_1inch^6 + 848* w0_1inch^8)) - ...

b_25inch = (2^(1/3) * (omega_25inch_TPR - 2 * w0_25inch) * (omega_25inch_TPR + 2 * w0_25inch) * (omega_25inch_TPR^2 - 6 * w0_25inch^2) * (omega_25inch_TPR^2 - 4 * w0_25inch^2)) 
+ 2 * w0_25inch * (-9* N_stiff_25 * w0_25inch^2 + (-4 * omega_25inch_TPR^2 + w0_25inch^2) + ...

kappa_25inch^2 * (4 - 48* omega_25inch_TPR^2 + 12* w0_25inch^2)) * (mu_25inch^2 - ...

6 * Lam_25inch_TPR^2 * N_stiff_25 * w0_25inch^2 * (4* omega_25inch_TPR^4 + 17 * omega_25inch_TPR^2 - 4 * w0_25inch^4) * ...

4* kappa_25inch^2 * Lam_25inch_TPR^2 * (4* omega_25inch_TPR^4 + 2* omega_25inch_TPR^2 + w0_25inch^2) * ...

4* w0_25inch^4) + 2 * (1/3) * (-3 * Lam_25inch_TPR^3 * w0_25inch^2 * (-omega_25inch_TPR^2 - 4* w0_25inch^2)) * ...

4 * N_stiff_25 * omega_25inch_TPR^2 + N_stiff_25 * w0_25inch^2) * (-9* N_stiff_25 * w0_25inch^2) * ...

kappa_25inch^2 * 4 * (omega_25inch_TPR^2 + 2 * w0_25inch^2) + sqrt((...

4 * w0_25inch^2) * 3* (-9* N_stiff_25 * w0_25inch^2) * ...

kappa_25inch^2 * (4 - 48* omega_25inch_TPR^2 + 12* w0_25inch^2)) * 3 * (-4 * mu_25inch^2 - 72* Lam_25inch_TPR^2 + mu_25inch^4 * N_stiff_25 ...

432* Lam_25inch_TPR^4 * mu_25inch^2 * N_stiff_25^2 - ...

945 * Lam_25inch_TPR^6 * N_stiff_25^3 * w0_25inch^6) * (4* omega_25inch_TPR^2 - 17 * omega_25inch_TPR^2 + w0_25inch^2) + ...

4* w0_25inch^4) * 3 + kappa_25inch^2 * Lam_25inch_TPR^4 * (1024* omega_25inch_TPR^12 - ...

768 * omega_25inch_TPR^8 * (2 + 17 * omega_25inch_TPR^2) * w0_25inch^2 + ...

3 * omega_25inch_TPR^4 * (256 + 4349* omega_25inch_TPR^2 + ...

19556* omega_25inch_TPR^4) * w0_25inch^4 - (128 + 3264 + ...

omega_25inch_TPR^2 + ...

30708 * omega_25inch_TPR^4 + 106043 * omega_25inch_TPR^6) * w0_25inch^6 + ...

12 * (64 + 1052* omega_25inch_TPR^2 + 5339* omega_25inch_TPR^4) * w0_25inch^8 
48* (20 + 443* omega_25inch_TPR^2) * w0_25inch^10 + 2752 * w0_25inch^12) + ...

12* kappa_25inch^4 * Lam_25inch_TPR^4 * w0_25inch^2 * (4* omega_25inch_TPR^4 - 17 * omega_25inch_TPR^2 + w0_25inch^2) + ...

4* w0_25inch^4) * (-16* mu_25inch^2 * (4* omega_25inch_TPR^4 - (2 + 17* omega_25inch_TPR^2) * w0_25inch^2 + 4* w0_25inch^4)) + ...

3 * Lam_25inch_TPR^2 * N_stiff_25 * (512 * omega_25inch_TPR^8 - ...)
\[4 \cdot \omega_{25\text{inch\_TPR}}^4 \cdot (127 + 1100 \cdot \omega_{25\text{inch\_TPR}}^2 \cdot w_{0\_25\text{inch}}^2 + (128 + \ldots
\begin{align*}
&2144 \cdot \omega_{25\text{inch\_TPR}}^2 + 10677 \cdot \omega_{25\text{inch\_TPR}}^4) \cdot w_{0\_25\text{inch}}^4 - \ldots \\
&8 \cdot (56 + 661 \cdot \omega_{25\text{inch\_TPR}}^2 \cdot w_{0\_25\text{inch}}^6 + 848 \cdot w_{0\_25\text{inch}}^8) \\
&12 \cdot \kappa_{25\text{inch\_TPR}} \cdot \Lambda_{25\text{inch\_TPR}}^2 \cdot w_{0\_25\text{inch}}^4 \cdot (4 \cdot \omega_{25\text{inch\_TPR}}^4 \cdot w_{0\_25\text{inch}}^2 + \ldots \\
&4 \cdot \omega_{25\text{inch\_TPR}}^2 \cdot (4 \cdot \mu_{25\text{inch\_TPR}}^4 \cdot (4 \cdot \omega_{25\text{inch\_TPR}}^4 - (2 + 17 \cdot \omega_{25\text{inch\_TPR}}^2) \cdot w_{0\_25\text{inch}}^2 + \ldots \\
&48 \cdot \Lambda_{25\text{inch\_TPR}}^2 \cdot \mu_{25\text{inch\_TPR}}^2 \cdot N_{\text{stiff\_25}} \cdot (4 \cdot \omega_{25\text{inch\_TPR}}^4 - (2 + 17 \cdot \omega_{25\text{inch\_TPR}}^2) \cdot w_{0\_25\text{inch}}^2 + \ldots \\
&4 \cdot \omega_{25\text{inch\_TPR}}^4 \cdot N_{\text{stiff\_25}} \cdot (204 \cdot \omega_{25\text{inch\_TPR}}^4 - 92 \cdot w_{0\_25\text{inch}}^2 + \ldots \\
&\omega_{25\text{inch\_TPR}}^4 \cdot (1 + 876 \cdot w_{0\_25\text{inch}}^2) \cdot (1/3));
\]
\[ b_{4\text{inch}} = \left(2^{1/3} \cdot (2^{4\text{inch}_{\text{TPR}} - 2 \cdot w_{0\text{\text{-}4inch}}}) \cdot (\text{omega}_{4\text{\text{-}inch}_{\text{TPR}} + 4 \cdot \text{omega}_{4\\text{\text{-}inch}_{\text{TPR}}}'' + 2 w_{0\text{\text{-}4inch}}) \right) \cdot \left( -9 \cdot \text{N\_stiff}_{4} \cdot w_{0\text{\text{-}4inch}}' \cdot \left( -4 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + w_{0\text{\text{-}4inch}} ight)'' - 2 \cdot w_{0\text{-4inch}}' \right) + ... \\
\text{kappa}_{4\text{\text{-}inch}}' \cdot \left(4 - 48 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + 12 \cdot w_{0\text{-4inch}}'' \right) \cdot \left(\text{mu}_{4\text{\text{-}inch}}'', 4 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}' \cdot \left(2 \cdot \text{omega}_{4\\text{\text{-}inch}_{\text{TPR}}}' \cdot \left( -9 \cdot \text{N\_stiff}_{4} \cdot w_{0\text{\text{-}4inch}}'' \cdot \left( -4 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + w_{0\text{\text{-}4inch}} ight)'' +\right) \right) \cdot \left(4 - 48 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + 12 \cdot w_{0\text{-4inch}}'' \right) + ... \\
\text{sqrt}\left(\left(\text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' \cdot \text{omega}_{4\\text{\text{-}inch}_{\text{TPR}}}' \cdot \left( -9 \cdot \text{N\_stiff}_{4} \cdot w_{0\text{-4inch}}'' \cdot \left( -4 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + w_{0\text{-4inch}}'' \right) +\right) \right)\right) \cdot \left( \left(4 - 48 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + 12 \cdot w_{0\text{-4inch}}'' \right) + ... \\
\text{kappa}_{4\text{\text{-}inch}}' \cdot \left(4 - 48 \cdot \text{omega}_{4\text{\text{-}inch}_{\text{TPR}}}'' + 12 \cdot w_{0\text{-4inch}}'' \right) + ...
\[
4 \cdot w_{0\text{4inch}}^2 \cdot (-9 \cdot N_{\text{stiff\_4}} \cdot w_{0\text{4inch}}^2 \cdot (-4 \cdot \omega_{\text{4inch\_TPR}}^2 + w_{0\text{4inch}}^2) + \ldots
\]
\[
kappa_{\text{4inch}}^2 \cdot ((4 - 48 \cdot \omega_{\text{4inch\_TPR}}^2 + 12 \cdot w_{0\text{4inch}}^2) \cdot (-9 \cdot N_{\text{stiff\_4}} \cdot w_{0\text{4inch}}^2 \cdot (-4 \cdot \omega_{\text{4inch\_TPR}}^2 + w_{0\text{4inch}}^2) + \ldots
\]
\[
17 \cdot \omega_{\text{4inch\_TPR}}^2 \cdot w_{0\text{4inch}}^2 + 4 \cdot w_{0\text{4inch}}^2) \cdot 3 - \ldots
\]
\[
16 \cdot \kappa_{\text{4inch}}^2 \cdot \omega_{\text{4inch\_TPR}}^2 \cdot (1024 \cdot \omega_{\text{4inch\_TPR}}^4 - \ldots
\]
\[
768 \cdot \omega_{\text{4inch\_TPR}}^4 \cdot (2 + 17 \cdot \omega_{\text{4inch\_TPR}}^2) \cdot w_{0\text{4inch}}^2 + \ldots
\]
\[
3 \cdot \omega_{\text{4inch\_TPR}}^4 \cdot (256 + 4349 \cdot \omega_{\text{4inch\_TPR}}^2 + \ldots
\]
\[
19556 \cdot \omega_{\text{4inch\_TPR}}^6 \cdot w_{0\text{4inch}}^4 - (128 + 3264 \cdot \omega_{\text{4inch\_TPR}}^2 + \ldots
\]
\[
30708 \cdot \omega_{\text{4inch\_TPR}}^8 \cdot w_{0\text{4inch}}^6 + \ldots
\]
\[
12 \cdot (64 + 1052 \cdot \omega_{\text{4inch\_TPR}}^2 + 5339 \cdot \omega_{\text{4inch\_TPR}}^4 + 128 \cdot 3264 \cdot \omega_{\text{4inch\_TPR}}^6 + \ldots
\]
\[
12 \cdot \kappa_{\text{4inch}}^4 \cdot \omega_{\text{4inch\_TPR}}^4 \cdot w_{0\text{4inch}}^4 \cdot (4 \cdot \omega_{\text{4inch\_TPR}}^4 - 17 \cdot \omega_{\text{4inch\_TPR}}^2 \cdot w_{0\text{4inch}}^2 + \ldots
\]
\[
12 \cdot \kappa_{\text{4inch}}^2 \cdot \omega_{\text{4inch\_TPR}}^2 \cdot w_{0\text{4inch}}^2 \cdot (4 \cdot \omega_{\text{4inch\_TPR}}^4 - 17 \cdot \omega_{\text{4inch\_TPR}}^2 \cdot w_{0\text{4inch}}^2 + \ldots
\]
\[
8 \cdot \omega_{\text{4inch\_TPR}}^4 \cdot (56 + 661 \cdot \omega_{\text{4inch\_TPR}}^2 + 848 \cdot \omega_{\text{4inch\_TPR}}^4) - \ldots
\]
\[
u_{1\text{inch\_TPR}} = 2 \cdot \lambda_{1\text{inch\_TPR}} \cdot \cos(\omega_{1\text{inch\_TPR}} \cdot t) + b_{1\text{inch}} \cdot \cos(w_{0\text{1inch}} \cdot t + \pi);
\]
\[
u_{25\text{inch\_TPR}} = 2 \cdot \lambda_{25\text{inch\_TPR}} \cdot \cos(\omega_{25\text{inch\_TPR}} \cdot t) + b_{25\text{inch}} \cdot \cos(w_{0\text{25inch}} \cdot t + \pi);
\]
\[
u_{25\text{inch\_TPR\_temp}} = 2 \cdot \lambda_{25\text{inch\_TPR}} \cdot \cos(\omega_{25\text{inch\_TPR}} \cdot t) + 3.69512 \cdot 10^{-7} \cdot \cos(w_{0\text{25inch}} \cdot t + \pi);
\]
\[
u_{4\text{inch\_TPR}} = 2 \cdot \lambda_{4\text{inch\_TPR}} \cdot \cos(\omega_{4\text{inch\_TPR}} \cdot t) + b_{4\text{inch}} \cdot \cos(w_{0\text{4inch}} \cdot t + \pi);
\]
\[
\text{figure;}
\]
\[
\text{plot(u_{1\text{inch\_TPR}}, F_{1\text{inch\_TPR}});}
\]
\[
\text{xlabel('Displacement in m');}
\]
\[
\text{ylabel('Force in N');}
\]
\[
\text{figure;}
\]
\[
\text{plot(u_{25\text{inch\_TPR}}, F_{25\text{inch\_TPR}});}
\]
\[
\text{xlabel('Displacement in m');}
\]
\[
\text{ylabel('Force in N');}
\]
\[
\text{figure;}
\]
\[
\text{plot(u_{4\text{inch\_TPR}}, F_{4\text{inch\_TPR}});}
\]
\[
\text{xlabel('Displacement in m');}
\]
\[
\text{ylabel('Force in N');}
\]
\[
\text{figure;}
\]
\[
\text{plot(u_{25\text{inch\_TPR\_temp}}, F_{25\text{inch\_TPR}});}
\]
xlabel('Displacement in m');
ylabel('Force in N');

%%
[T_1inch_TPR,Y_1inch_TPR]=ode45(@rigid_1inch_TPR,t,[0 0]);
[T_25inch_TPR,Y_25inch_TPR]=ode45(@rigid_25inch_TPR,t,[0 0]);
[T_4inch_TPR,Y_4inch_TPR]=ode45(@rigid_4inch_TPR,t,[0 0]);

figure;
plot(Y_1inch_TPR(100001:200001,1),F_1inch_TPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_25inch_TPR(100001:200001,1),F_25inch_TPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_4inch_TPR(100001:200001,1),F_4inch_TPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This code is the same as the previous code, but does not correct the data
% for the phase as the previous one did.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;
clc;

%%
for ii = 1:12

cnt=num2str(ii);

datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Oct_2_4inch\294_amp\data' cnt '.mat'];

scan_data1(ii)=load(datafile);

force=scan_data1(ii).data(:,2)*4.44822162;

massImp=0.028;

newData1=scan_data1(ii).data(:,1)*9.81;

temp=abs(fft(newData1,640000));

Dft1(:,ii)=temp;

Force_Dft1(:,ii)=abs(fft(scan_data1(1,ii).data(:,2),640000));

[C,I]=max(Dft1(:,ii));

omega1=I/640000*6400;

peak_294(ii)=max(Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*sqrt(2));

peak_590(ii)=max(Dft1((2*omega1-10)/6400*640000:(2*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*2*sqrt(2));

force_peak1(ii)=max(Force_Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*4.44822162;
end

%%
for ii = 1:19
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data\Aq\Oct_2_4inch\99_\amp\data' cnt '.mat'];
    scan_data2(ii)=load(datafile);
    force=scan_data2(ii).data(:,2)*4.44822162;
    massImp=0.028;
    newData2=scan_data2(ii).data(:,1)*9.81;
    temp=abs(fft(newData2,640000));
    Dft2(:,ii)=temp;
    Force_Dft2(:,ii)=abs(fft(scan_data2(1,ii).data(:,2),640000));
    [C,I]=max(Dft2(:,ii));
    omega2=I/640000*6400;
    peak_98(ii)=max(Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii)
                 /3200*9.81/(2*pi*omega2)^2);
    peak_198(ii)=max(Dft2((2*omega2-10)/6400*640000:(2*omega2+10)/6400*640000,ii)
                 /3200*9.81/(2*pi*198)^2);
    peak_295(ii)=max(Dft2((3*omega2-10)/6400*640000:(3*omega2+10)/6400*640000,ii)
                 /3200*9.81/(2*pi*3*omega2)^2);
    peak_590(ii)=max(Dft2((588/6400*65536:592/6400*65536,ii)/3200*9.81/(2*pi*592)^2);
    force_peak2(ii)=max(Force_Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii)
                /3200*4.44822162);
end

%%
for ii = 1:19
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data\Aq\Oct_2_4inch\148_\amp\data' cnt '.mat'];
    scan_data3(ii)=load(datafile);
    force=scan_data3(ii).data(:,2)*4.44822162;
    massImp=0.028;
    newData3=scan_data3(ii).data(:,1)*9.81;
    temp=abs(fft(newData3,640000));
    Dft3(:,ii)=temp;
Force_Dft3(:,ii)=abs.fft(scan_data3(:,ii).data(:,2),640000));

[C,I]= max(Dft3(:,1));
omega3=1/(640000+6400;)

peak_147(ii)=max(Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii)/3200*9.81/(2*pi*omega3)~2);

peak_2952(ii)=max(Dft3((2*omega3-10)/6400*640000:(2*omega3+10)/6400*640000,ii)/3200*9.81/(2*pi*2*omega3)~2);

peak_5902(ii)=max(Dft2(588/6400*65536:592/6400*65536,ii)/3200*9.81/(2*pi*592)~2);

force_peak3(ii)=max(Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii)/3200*4.44822162);

end

%%
clc;

w0_4inch=mean([omega1 3*omega2 2*omega3])*2*pi;
omega_H4inch=w0_4inch/2;
omega_T4inch=w0_4inch/3;
num=length(peak_294);

for ii=1:num
    top_k_4inch(ii)=peak_294(ii)*force_peak1(ii);
    denom_k_4inch(ii)=peak_294(ii)^2;
end

k_4inch=sum(top_k_4inch)/sum(denom_k_4inch);
ybar_k_4inch=mean(force_peak1);
SStot_k_4inch=sum((force_peak1-ybar_k_4inch).^2);
SSres_k_4inch=sum((force_peak1-k_4inch*peak_294).^2);
Rsquared_k_4inch=1-SSres_k_4inch/SStot_k_4inch;
mass_4inch=k_4inch/(w0_4inch^2);
F_4inch_PR=force_peak1/mass_4inch;
num=length(peak_294);

for ii=1:num
    top_mu_4inch(ii)=peak_294(ii)*F_4inch_PR(ii);
    denom_mu_4inch(ii)=F_4inch_PR(ii)^2;
end

beta=sum(top_mu_4inch)/sum(denom_mu_4inch);
mu_4inch=1/(2*w0_4inch*beta);
ybar_mu_4inch=mean(peak_294);
SStot_mu_4inch=sum((peak_294-ybar_mu_4inch).^2);
SSres_mu_4inch=sum((peak_294-beta*F_4inch_PR).^2);
Rsquared_mu_4inch=1-SSres_mu_4inch/SStot_mu_4inch;
\[
F_{\text{4inch\_HPR}} = \text{force\_peak3}\bigg(/\text{mass\_4inch}\bigg) ; \\
\text{Lam\_HPR} = \frac{F_{\text{4inch\_HPR}}}{(2*(w_{\text{0\_4inch}}^2-\omega_{\text{T4inch}}^2))} ; \\
\text{num} = \text{length}(\text{peak\_2952}) ; \\
\text{for } ii=1:\text{num} \\
\quad \text{top\_kappa\_4inch}(ii) = \text{peak\_2952}(ii) * (\text{Lam\_HPR}(ii)^2) ; \\
\quad \text{denom\_kappa\_4inch}(ii) = (\text{Lam\_HPR}(ii)^4) ; \\
\text{end} \\
\beta = \frac{\text{sum}(\text{top\_kappa\_4inch})}{\text{sum}(\text{denom\_kappa\_4inch})} ; \\
\text{kappa\_4inch} = \frac{\beta * w_{\text{0\_4inch}} * \mu_{\text{4inch}}}{\text{ybar\_kappa\_4inch} = \text{mean}(\text{peak\_2952})} ; \\
\text{SStot\_kappa\_4inch} = \text{sum}((\text{peak\_2952} - \text{ybar\_kappa\_4inch})^2) ; \\
\text{SSres\_kappa\_4inch} = \text{sum}((\text{peak\_2952} - \beta * (\text{Lam\_HPR})^2)^2) ; \\
\text{Rsquared\_kappa\_4inch} = 1 - \frac{\text{SSres\_kappa\_4inch}}{\text{SStot\_kappa\_4inch}} ; \\
\text{num} = \text{length}(\text{peak\_590}) ; \\
\text{for } ii=1:\text{num} \\
\quad \text{top\_kappa\_4inch2}(ii) = \text{peak\_590}(ii) * (\frac{F_{\text{4inch\_PR}}}{(24*w_{\text{0\_4inch}}*4*\mu_{\text{4inch}}^2)})^2 ; \\
\quad \text{denom\_kappa\_4inch2}(ii) = \frac{(F_{\text{4inch\_PR}})^2}{(24*w_{\text{0\_4inch}}*4*\mu_{\text{4inch}}^2)^2} ; \\
\text{end} \\
\text{kappa\_4inch2} = \frac{\text{sum}(\text{top\_kappa\_4inch2})}{\text{sum}(\text{denom\_kappa\_4inch2})} ; \\
\text{ybar\_kappa\_4inch2} = \text{mean}(\text{peak\_590}) ; \\
\text{SStot\_kappa\_4inch2} = \text{sum}((\text{peak\_590} - \text{ybar\_kappa\_4inch2})^2) ; \\
\text{SSres\_kappa\_4inch2} = \text{sum}((\text{peak\_590} - \text{kappa\_4inch2} * (\frac{F_{\text{4inch\_PR}}}{(24*w_{\text{0\_4inch}}*4*\mu_{\text{4inch}}^2)})^2)^2) ; \\
\text{Rsquared\_kappa\_4inch2} = 1 - \frac{\text{SSres\_kappa\_4inch2}}{\text{SStot\_kappa\_4inch2}} ; \\
F_{\text{4inch\_TPR}} = \frac{\text{force\_peak2}}{\text{mass\_4inch}} ; \\
\text{Lam\_TPR} = \frac{F_{\text{4inch\_TPR}}}{(2*(w_{\text{0\_4inch}}^2-\omega_{\text{T4inch}}^2))} ; \\
a = \text{peak\_295} ; \\
y = \mu_{\text{4inch}}^2 \cdot \frac{a}{2} * (2*w_{\text{0\_4inch}}^2 + 2 * \text{kappa\_4inch}^2 + 2 * a \cdot \text{Lam\_TPR}^2) ; \\
\text{x} = 3/8 * \text{a} * \text{Lam\_TPR}^2 + 3 * \text{Lam\_TPR} + \text{Lam\_TPR}^3 ; \\
\text{N\_stiff\_4} = \frac{\text{sum}(\text{x} \cdot y)}{\text{sum}(\text{x} \cdot 2)} ; \\
\text{ybar\_Nstiff\_4inch} = \text{mean}(\text{y}) ; \\
\text{SStot\_Nstiff\_4inch} = \text{sum}((\text{y} - \text{ybar\_Nstiff\_4inch})^2) ; \\
\text{SSres\_Nstiff\_4inch} = \text{sum}((\text{y} - \text{N\_stiff\_4inch} \cdot \text{x})^2) ; \\
\text{Rsquared\_N\_4inch} = 1 - \frac{\text{SSres\_Nstiff\_4inch}}{\text{SStot\_Nstiff\_4inch}} ;
\[
y_{bad} = \mu_{4\text{inch}}^2 \cdot a \cdot \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{2 \cdot w_{0\text{inch}}^2} + \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{(\omega_{T4\text{inch}}^2 - 2w_{0\text{inch}}^2)} \cdot \frac{1}{(4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2)(\omega_{T4\text{inch}} + 2w_{0\text{inch}})} - \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{6 \cdot w_{0\text{inch}}^3(4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2)} + 2 \cdot \frac{\kappa_{4\text{inch}}^2 \cdot \func{Lam}_\text{TPR}^3}{4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2} ;
\]

\[
x_{bad} = 3/8 \cdot a^3 + 3 \cdot a \cdot \func{Lam}_\text{TPR}^2 - \func{Lam}_\text{TPR}^3 ;
\]

\[
N_{\text{stiff}_{4\text{bad}}} = \frac{\text{sum}(x_{bad} \cdot y_{bad})}{\text{sum}(x_{bad}^2)} ;
\]

\[
y_{bar_{N_{\text{stiff}_{4\text{bad}}}}} = \text{mean}(y_{bad}) ;
\]

\[
S_{\text{Stot}_{N_{\text{stiff}_{4\text{inch}}}}} = \text{sum}((y_{bad} - y_{bar_{N_{\text{stiff}_{4\text{inch}}}}})^2) ;
\]

\[
S_{\text{Res}_{N_{\text{stiff}_{4\text{inch}}}}} = \text{sum}((y_{bad} - N_{\text{stiff}_{4\text{bad}}} \cdot x_{bad})^2) ;
\]

\[
R^2_{N_{\text{stiff}_{4\text{inch}}}} = 1 - \frac{S_{\text{Res}_{N_{\text{stiff}_{4\text{inch}}}}}}{S_{\text{Stot}_{N_{\text{stiff}_{4\text{inch}}}}}} ;
\]

\[
y_{d} = \mu_{4\text{inch}}^2 \cdot a \cdot \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{2 \cdot w_{0\text{inch}}^2} + \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{(\omega_{T4\text{inch}}^2 - 2w_{0\text{inch}}^2)} \cdot \frac{1}{(4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2)(\omega_{T4\text{inch}} - 2w_{0\text{inch}})} - \frac{\kappa_{4\text{inch}}^2 \cdot a \cdot \func{Lam}_\text{TPR}^2}{6 \cdot w_{0\text{inch}}^3(4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2)} + 2 \cdot \frac{\kappa_{4\text{inch}}^2 \cdot \func{Lam}_\text{TPR}^3}{4\omega_{T4\text{inch}}^2 - w_{0\text{inch}}^2} ;
\]

\[
x_{d} = \omega_{T4\text{inch}}^6 \cdot \func{Lam}_\text{TPR}^6 / w_{0\text{inch}}^2 - 9 \cdot \omega_{T4\text{inch}}^4 \cdot \func{Lam}_\text{TPR}^4 \cdot a^2 - 9/4 \cdot \omega_{T4\text{inch}}^2 \cdot \func{Lam}_\text{TPR}^2 \cdot a^4 \cdot w_{0\text{inch}}^2 - 9/64 \cdot w_{0\text{inch}}^4 \cdot a^6 ;
\]

\[
N_{\text{damp}_{4}} = \frac{\text{sum}(x_{d} \cdot y_{d})}{\text{sum}(x_{d}^2)} ;
\]

\[
\text{real}_{4\text{damp}} = \sqrt{N_{\text{damp}_{4}}} ;
\]

\[
y_{bar_{\text{ndamp}_{4}}} = \text{mean}(y_{d}) ;
\]

\[
S_{\text{Stot}_{\text{ndamp}_{4}}} = \text{sum}((y_{d} - y_{bar_{\text{ndamp}_{4}}})^2) ;
\]

\[
S_{\text{Res}_{\text{ndamp}_{4}}} = \text{sum}((y_{d} - N_{\text{damp}_{4}} \cdot x_{d})^2) ;
\]

\[
R^2_{\text{ndamp}_{4}} = 1 - \frac{S_{\text{Res}_{\text{ndamp}_{4}}}}{S_{\text{Stot}_{\text{ndamp}_{4}}}} ;
\]

```matlab
%%
for ii = 1:31
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data_Aq\Apr_25_25inch_360_amp\data' cnt '.mat '];
    scan_data4(ii)=load(datafile);
    force=scan_data4(ii).data(:,2)*4.44822162;
    massImp = 0.028;
    newData4=scan_data4(ii).data(:,1)*9.81;
    temp=abs(fft(newData4,640000));
    Dft4(:,ii)=temp;
    Force_Dft4(:,ii)=abs(fft(scan_data4(1,ii).data(:,2),640000));
    [C,I]=max(Force_Dft4(:,ii));
    omega1=I/640000+6400;
    peak_360(ii)=max(Force_Dft4((omega1-10)/640000:omega1+10)/640000,ii))/3200/9.81/(omega1*2*pi)^2;
end
```
peak_720(ii) = \max(Dft4((2*omega1 -10)/6400*640000:(2*omega1 +10)/6400*640000,ii)) /3200*9.81/(omega1*4*pi)^2; 

force_peak4(ii) = \max(Force_Dft4((omega1 -10)/6400*640000:(omega1 +10)/6400*640000,ii))/3200*4.44822162;
end

for ii = 1:26
    cnt = num2str(ii);
    datafile = ['C:\ Users\ editman\ Documents\ MATLAB\ Data Aq\ Apr_25_25inch_180_amp\ data' cnt '.mat'];
    scan_data5(ii) = load(datafile);
    force = scan_data5(ii).data(:,2) * 4.44822162;
    massImp = 0.028;
    newData5 = scan_data5(ii).data(:,1) * 9.81;
    temp = abs(fft(newData5,640000));
    Dft5(:,ii) = temp;
    Force_Dft5(:,ii) = abs(fft(scan_data5(1,ii).data(:,2),640000));
    [C,I] = max(Dft5(:,1));
    omega2 = I/640000*6400;
    peak_180(ii) = \max(Dft5((omega2 -10)/6400*640000:(omega2 +10)/6400*640000,ii)) /3200*9.81/(2*pi*omega2)^2;
    peak_3602(ii) = \max(Dft5((2*omega2 -10)/6400*640000:(2*omega2 +10)/6400*640000,ii)) /3200*9.81/(2*pi*2*omega2)^2;
    force_peak5(ii) = \max(Force_Dft5((omega2 -10)/6400*640000:(omega2 +10)/6400*640000,ii)) /3200*4.44822162;
end

for ii = 1:26
    cnt = num2str(ii);
    datafile = ['C:\ Users\ editman\ Documents\ MATLAB\ Data Aq\ Apr_25_25inch_180_amp\ data' cnt '.mat'];
    scan_data5(ii) = load(datafile);
    force = scan_data5(ii).data(:,2) * 4.44822162;
    massImp = 0.028;
    newData5 = scan_data5(ii).data(:,1) * 9.81;
    temp = abs(fft(newData5,640000));
    Dft5(:,ii) = temp;
    Force_Dft5(:,ii) = abs(fft(scan_data5(1,ii).data(:,2),640000));
    [C,I] = max(Dft5(:,1));
    omega2 = I/640000*6400;
    peak_180(ii) = \max(Dft5((omega2 -10)/6400*640000:(omega2 +10)/6400*640000,ii)) /3200*9.81/(2*pi*omega2)^2;
    peak_3602(ii) = \max(Dft5((2*omega2 -10)/6400*640000:(2*omega2 +10)/6400*640000,ii)) /3200*9.81/(2*pi*2*omega2)^2;
    force_peak5(ii) = \max(Force_Dft5((omega2 -10)/6400*640000:(omega2 +10)/6400*640000,ii)) /3200*4.44822162;
peak_240 (ii) = \max (Dft6((2*omega3 -10)\div 6400*640000:(2*omega3 +10)\div 6400*640000 , ii) /3200*9.81/(2*\pi*2*omega3)^2);

peak_3603 (ii) = \max (Dft6((3*omega3 -10)\div 6400*640000:(3*omega3 +10)\div 6400*640000, ii) /3200*9.81/(2*\pi*3*omega3)^2);

force_peak6 (ii) = \max (Force_Dft6((omega3 -10)\div 6400*640000:(omega3 +10)\div 6400*640000, ii) /3200*4.44822162);

end

clc;
w0_25inch = \text{mean}([omega1 2*omega2 3*omega3]) * 2*\pi;
omega_H25inch = w0_25inch /2;
omega_T25inch = w0_25inch /3;
num = \text{length}(peak_360);
for ii = 1: num
    top_k_25inch(ii) = peak_360(ii) * force_peak4(ii);
    denom_k_25inch(ii) = peak_360(ii)^2;
end

k_25inch = \text{sum}(top_k_25inch) / \text{sum}(denom_k_25inch);
ybar_k_25inch = \text{mean}(force_peak4);
SSres_k_25inch = \text{sum}((force_peak4 - ybar_k_25inch)^2); SSot_k_25inch = \text{sum}((force_peak4 - k_25inch * peak_360)^2);
Rsquared_k_25inch = 1 - SSres_k_25inch / SSot_k_25inch;
mass_25inch = k_25inch / (w0_25inch^2);
F_25inch_PR = force_peak4 / mass_25inch;
num = \text{length}(peak_360);
for ii = 1: num
    top_mu_25inch(ii) = peak_360(ii) * F_25inch_PR(ii);
    denom_mu_25inch(ii) = F_25inch_PR(ii)^2;
end

beta = \text{sum}(top_mu_25inch) / \text{sum}(denom_mu_25inch);
ybar = \text{mean}(peak_360);
SSot = \text{sum}((peak_360 - ybar)^2); SSres = \text{sum}((peak_360 - beta*F_25inch_PR)^2);
Rsquared_mu_25inch = 1 - SSres / SSot;
F_25inch_HPR = force_peak5 / mass_25inch;
Lam_H25 = F_25inch_HPR / (2*(w0_25inch^2 - omega_H25inch^2));
num = \text{length}(peak_360);
for ii = 1: num
\[
\text{top}_k\text{appa}_25\text{inch}(ii) = \text{peak}_{3602}(ii) \cdot \text{Lam}_{H25}(ii)^2;
\]
\[
\text{denom}_k\text{appa}_25\text{inch}(ii) = \text{Lam}_{H25}(ii)^4;
\]
\[
\text{end}
\]
\[
\text{beta} = \frac{\text{sum}(\text{top}_k\text{appa}_25\text{inch})}{\text{sum}(\text{denom}_k\text{appa}_25\text{inch})};
\]
\[
\text{kappa}_25\text{inch} = \beta \cdot \text{w0}_25\text{inch} \cdot \mu_{25\text{inch}};
\]
\[
\text{ybar} = \text{mean}(\text{peak}_{3602});
\]
\[
\text{SS}_{\text{tot}} = \text{sum}((\text{peak}_{3602} - \text{ybar})^2);
\]
\[
\text{SS}_{\text{res}} = \text{sum}((\text{peak}_{3602} - \beta \cdot \text{Lam}_{H25})^2);
\]
\[
\text{Rsquared}_{\text{kappa}_25\text{inch}} = 1 - \frac{\text{SS}_{\text{res}}}{\text{SS}_{\text{tot}}};
\]
\[
\text{num} = \text{length}(\text{peak}_{720});
\]
\[
\text{for } ii = 1: \text{num}
\]
\[
\text{top}_k\text{appa}_25\text{inch2}(ii) = \text{peak}_{720}(ii) \cdot \text{F}_{25\text{inch} \text{PR}}(ii)^2/(24 \cdot \text{w0}_{25\text{inch}}^4 \cdot \mu_{25\text{inch}}^2)
\]
\[
\text{denom}_k\text{appa}_25\text{inch2}(ii) = (\text{F}_{25\text{inch} \text{PR}}(ii)^2/(24 \cdot \text{w0}_{25\text{inch}}^4 \cdot \mu_{25\text{inch}}^2))^2;
\]
\[
\text{end}
\]
\[
\text{kappa}_25\text{inch2} = \frac{\text{sum}(\text{top}_k\text{appa}_25\text{inch2})}{\text{sum}(\text{denom}_k\text{appa}_25\text{inch2})};
\]
\[
\text{ybar}_k\text{appa}_25\text{inch2} = \text{mean}(\text{peak}_{720});
\]
\[
\text{SS}_{\text{tot}_k\text{appa}_25\text{inch2}} = \text{sum}((\text{peak}_{720} - \text{ybar}_k\text{appa}_25\text{inch2})^2);
\]
\[
\text{SS}_{\text{res}_k\text{appa}_25\text{inch2}} = \text{sum}((\text{peak}_{720} - \text{kappa}_25\text{inch2} \cdot \text{F}_{25\text{inch} \text{PR}}^2/(24 \cdot \text{w0}_{25\text{inch}}^4 \cdot \mu_{25\text{inch}}^2))^2);
\]
\[
\text{Rsquared}_{\text{kappa}_25\text{inch2}} = 1 - \frac{\text{SS}_{\text{res}_k\text{appa}_25\text{inch2}}}{\text{SS}_{\text{tot}_k\text{appa}_25\text{inch2}}};
\]
\[
\text{F}_{25\text{inch} \text{TPR}} = \frac{\text{force}_{\text{peak6}}}{\text{mass}_{25\text{inch}}};
\]
\[
\text{Lam}_{\text{TPR}} = \frac{\text{F}_{25\text{inch} \text{TPR}}}{2 \cdot (\text{w0}_{25\text{inch}} - \omega_{T25\text{inch}}^2)^2};
\]
\[
\text{a2} = \text{peak}_{3603};
\]
\[
\text{y25} = \mu_{25\text{inch}}^2 \cdot a2/2 + \text{kappa}_25\text{inch} \cdot a2 \cdot 3/(2 \cdot \text{w0}_{25\text{inch}}^2) + 2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(2 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 4 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 6 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 8 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 10 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
\text{x25} = 3/8 \cdot a2 \cdot 3 \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2 \cdot \text{Lam}_{\text{TPR}} \cdot 2
\]
\[
\text{N}_{\text{stiff}_25} = \frac{\text{sum}(\text{x25} \cdot \text{y25})}{\text{sum}(\text{x25}^2)};
\]
\[
\text{ybar}_{\text{N}_{\text{25\text{inch}}}} = \text{mean}(\text{y25});
\]
\[
\text{SS}_{\text{tot}_{\text{N}_{\text{25\text{inch}}}}} = \text{sum}((\text{y25} - \text{ybar}_{\text{N}_{\text{25\text{inch}}}})^2);
\]
\[
\text{SS}_{\text{res}_{\text{N}_{\text{25\text{inch}}}}} = \text{sum}((\text{y25} - \text{N}_{\text{stiff}_25} \cdot \text{x25})^2);
\]
\[
\text{Rsquared}_{\text{N}_{\text{25\text{inch}}}} = 1 - \frac{\text{SS}_{\text{res}_{\text{N}_{\text{25\text{inch}}}}}}{\text{SS}_{\text{tot}_{\text{N}_{\text{25\text{inch}}}}}};
\]
\[
\text{y25bad} = \mu_{25\text{inch}}^2 \cdot a2/2 + \text{kappa}_25\text{inch} \cdot a2 \cdot 3/(2 \cdot \text{w0}_{25\text{inch}}^2) + 2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(2 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 4 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 6 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 8 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 10 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 12 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 14 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 16 \cdot \text{w0}_{25\text{inch}}^2)
\]
\[
-2 \cdot \text{kappa}_25\text{inch} \cdot a2 \cdot \text{Lam}_{\text{TPR}} \cdot 2/(\omega_{T25\text{inch}}^2 \cdot 18 \cdot \text{w0}_{25\text{inch}}^2)
\]
\begin{align*}
+2\kappa_{25\text{inch}}^2\Lambda_{\text{TPR}}.3/(4\omega_{\text{T25inch}}^2w_{0,25\text{inch}}^2) ; \\
x_{25\text{bad}}=3/8a_2.3+a_2.\Lambda_{\text{TPR}}.2-Lam_{\text{TPR}}.3; \\
N_{\text{stiff,25bad}}=\sum(x_{25\text{bad}}.y_{25\text{bad}})/\sum(x_{25\text{bad}}.2); \\
y_{\text{bar,N,25inchbad}}=\text{mean}(y_{25\text{bad}}); \\
SStot_{\text{N,25inchbad}}=\sum((y_{25\text{bad}}-y_{\text{bar,N,25inchbad}}.2)); \\
SSres_{\text{N,25inchbad}}=\sum((y_{25\text{bad}}-N_{\text{stiff,25bad}}x_{25\text{bad}}).2); \\
Rsquared_{\text{N,25inchbad}}=1-SSres_{\text{N,25inchbad}}/SStot_{\text{N,25inchbad}}; \\
y_{25}=\mu_{25\text{inch}}^2.a_2.2*(w_{0,25\text{inch}}.2+a_2.3)/(2.3-w_{0,25\text{inch}}^2) + \\
2.kappa_{25\text{inch}}^2.Lam_{\text{TPR}}.2.a_2.2/w_{0,25\text{inch}}^2; \\
x_{25}=omega_{\text{T25inch}}^2.Lam_{\text{TPR}}.2.w_{0,25\text{inch}}^2; \\
N_{\text{damp,25}}=\sum(x_{25}.y_{25})/\sum(x_{25}.2); \\
realND25=sqrt(N_{\text{damp,25}}); \\
y_{\text{bar,Ndamp,25inch}}=\text{mean}(y_{25}); \\
SStot_{\text{Ndamp,25inch}}=\sum((y_{25}-y_{\text{bar,Ndamp,25inch}}.2)); \\
SSres_{\text{Ndamp,25inch}}=\sum((y_{25}-N_{\text{damp,25}}x_{25}).2); \\
Rsquared_{\text{Ndamp,25inch}}=1-SSres_{\text{Ndamp,25inch}}/SStot_{\text{Ndamp,25inch}}; \\
\end{align*}

\begin{verbatim}
for ii = 1:7
cnt=num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_24_1inch_476_amp\data'
              'cnt '.mat'];
    scan_data7(ii)=load(datafile);
    force=scan_data7(ii).data(:,2)*4.44822162;
    massImp=0.028;
    newData7=scan_data7(ii).data(:,1)*9.81;
    temp=abs(fftnewData7,64000);
    Dft7(:,ii)=temp;
    Force_Dft7(:,ii)=abs(fftnewData7(1,ii).data(:,2),64000));
    [C,I]= max(Dft7(:,1));
    omega1=1/640000*64000; 
    peak_476(ii)=max(Dft7((omega1-10)/640000*(omega1+10)/640000,ii))
                  /3200*9.81/(omega1*2*\pi)^2;
    peak_952(ii)=max(Dft7((2*omega1-10)/640000*(2*omega1+10)/640000,ii))
                  /3200*9.81/(omega1*4*\pi)^2;
\end{verbatim}
for ii = 1:6
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_24_1inch_234_amp\data', cnt '.mat'];
    scan_data8(ii) = load(datafile);
    force = scan_data8(ii).data(:,2) * 4.4882162;
    massImp = 0.028;
    newData8 = scan_data8(ii).data(:,1) * 9.81;
    temp = abs(fft(newData8, 640000));
    Dft8(:, ii) = temp;
    Force_Dft8(:, ii) = abs(fft(scan_data8(1, ii).data(:,2), 640000));
    [C, I] = max(Dft8(:, 1));
    omega2 = I / 640000 * 6400;
    peak_234(ii) = max(Dft8((omega2 - 10) / 6400:640000:omega2 + 10) / 640000, ii) / 3200 * 9.81 / (2*pi*omega2)^2;
    peak_4762(ii) = max(Dft8((2*omega2 - 10) / 6400:640000:2*omega2 + 10) / 640000, ii) / 3200 * 9.81 / (2*pi*2*omega2)^2;
    force_peak8(ii) = max(Force_Dft8((omega2 - 10) / 6400:640000:omega2 + 10) / 640000, ii) / 3200 * 4.4882162;
end

for ii = 1:6
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_24_1inch_156_amp\data', cnt '.mat'];
    scan_data9(ii) = load(datafile);
    force = scan_data9(ii).data(:,2) * 4.4882162;
    massImp = 0.028;
    newData9 = scan_data9(ii).data(:,1) * 9.81;
    temp = abs(fft(newData9, 640000));
    Dft9(:, ii) = temp;
    Force_Dft9(:, ii) = abs(fft(scan_data9(1, ii).data(:,2), 640000));
    [C, I] = max(Dft9(:, 1));
    omega3 = I / 640000 * 6400;
    peak_156(ii) = max(Dft9((omega3 - 10) / 6400:640000:omega3 + 10) / 640000, ii) / 3200 * 9.81 / (2*pi*omega3)^2;
    peak_312(ii) = max(Dft9((2*omega3 - 10) / 6400:640000:2*omega3 + 10) / 640000, ii) / 3200 * 9.81 / (2*pi*2*omega3)^2;
    peak_4763(ii) = max(Dft9((3*omega3 - 10) / 6400:640000:3*omega3 + 10) / 640000, ii) / 3200 * 9.81 / (2*pi*3*omega3)^2;
force_peak9(ii) = \text{max}(\text{Force}_\text{Dft9}((\omega_3 - 10)/6400*640000:(\omega_3 + 10)/6400*640000, ii) / \text{3200*4.44822162});
end

clc;

w0_1inch = \text{mean}([\omega_1 \times 2 \times \omega_2 \times 3 \times \omega_3]) + \times 2 \times \text{pi};
omega_{H1inch} = w0_{1inch}/2;
omega_{T1inch} = w0_{1inch}/3;
num = \text{length}(\text{peak}_476);
for ii = 1: num
  top_k_1inch(ii) = \text{peak}_476(ii) \times \text{force}_\text{peak7}(ii);
  denom_k_1inch(ii) = \text{peak}_476(ii)^2;
end
k_1inch = \text{sum}(\text{top}_k_{1inch})/\text{sum}(\text{denom}_k_{1inch});
ybar_k_1inch = \text{mean}(\text{force}_\text{peak7});
SS_{tot_k_1inch} = \text{sum}((\text{force}_\text{peak7} - ybar_k_{1inch})^2);
SS_{res_k_1inch} = \text{sum}((\text{force}_\text{peak7} - k_{1inch} \times \text{peak}_476)^2);
Rsquared_k_1inch = 1 - \text{SSres}_k_{1inch}/\text{SStot}_k_{1inch};
mass_1inch = k_{1inch}/(w0_{1inch}^2); F_{1inch\_PR} = \text{force}_\text{peak7}/\text{mass}_1inch;
num = \text{length}(\text{peak}_476);
for ii = 1: num
  top_mu_1inch(ii) = \text{peak}_476(ii) \times F_{1inch\_PR}(ii);
  denom_mu_1inch(ii) = F_{1inch\_PR}(ii)^2;
end
beta = \text{sum}(\text{top}_mu_{1inch})/\text{sum}(\text{denom}_mu_{1inch});
mu_1inch = 1/(2 \times w0_{1inch} \times \beta);
ybar = \text{mean}(\text{peak}_476);
SS_{tot} = \text{sum}((\text{peak}_476 - ybar)^2);
SS_{res} = \text{sum}((\text{peak}_476 - \beta \times F_{1inch\_PR})^2);
Rsquared_mu_1inch = 1 - \text{SSres}/\text{SStot};
F_{1inch\_HPR} = \text{force}_\text{peak8}/\text{mass}_1inch;
Lam_{H1} = F_{1inch\_HPR}/(2 \times (w0_{1inch}^2 - \omega_{H1inch}^2));
num = \text{length}(\text{peak}_476);
for ii = 1: num
  top_kappa_1inch(ii) = \text{peak}_4762(ii) \times Lam_{H1}(ii)^2;
  denom_kappa_1inch(ii) = Lam_{H1}(ii)^4;
end
\[
\begin{align*}
\beta &= \frac{\text{sum}(\text{top}_kappa_{1inch})}{\text{sum}(\text{denom}_kappa_{1inch})}; \\
\kappa_{1inch} &= \beta \cdot w_{0_{1inch}} \cdot \mu_{1inch}; \\
ybar &= \text{mean}(\text{peak}_{4762}); \\
\text{SSStot} &= \text{sum}((\text{peak}_{4762} - ybar)^2); \\
\text{SSres} &= \text{sum}((\text{peak}_{4762} - \beta \cdot \text{Lam}_H_{1inch}^2)^2); \\
\text{Rsquared}_{\kappa_{1inch}} &= 1 - \frac{\text{SSres}}{\text{SSStot}}; \\
\text{real}_\text{damping} &= [\mu_{1inch} \ \mu_{25inch} \ \mu_{4inch}] \cdot [\text{mass}_{1inch} \ \text{mass}_{25inch} \ \text{mass}_{4inch}]; \\
\text{real}_\text{stiff} &= [w_{0_{1inch}}^2 \ w_{0_{25inch}}^2 \ w_{0_{4inch}}^2] \cdot [\text{mass}_{1inch} \ \text{mass}_{25inch} \ \text{mass}_{4inch}]; \\
\text{real}_\text{kappa} &= [\kappa_{1inch} \ \kappa_{25inch} \ \kappa_{4inch}] \cdot [\text{mass}_{1inch} \ \text{mass}_{25inch} \ \text{mass}_{4inch}]; \\
\text{num} &= \text{length}(\text{peak}_{952}); \\
\text{for} \ ii=1:\text{num} \\
\text{top}_kappa_{1inch2}(ii) &= \text{peak}_{952}(ii) \cdot F_{1inch\_PR}(ii)^2/(24 \cdot w_{0_{1inch}}^4 \mu_{1inch}^2); \\
\text{denom}_kappa_{1inch2}(ii) &= (F_{1inch\_PR}(ii)^2/(24 \cdot w_{0_{1inch}}^4 \mu_{1inch}^2))^2; \\
\end{align*}
\]
\[-kappa_{\text{inch}}^2 \cdot a_{3,}^3 \cdot \left( \frac{3}{6 \cdot w_{0,\text{inch}}^2 \cdot (4 \cdot \omega_{\text{T,inch}}^2 - w_{0,\text{inch}}^2)} \right) \]
\[+ 2 \cdot kappa_{\text{inch}}^2 \cdot \frac{\Lambda_{\text{TPR}}}{4 \cdot \omega_{\text{T,inch}}^2 - w_{0,\text{inch}}^2} ;\]
\[x_{\text{1bad}} = \frac{3}{8} \cdot a_{3,}^3 + 3 \cdot a_{3,} \cdot \Lambda_{\text{TPR}}^2 - \Lambda_{\text{TPR}}^3;\]
\[N_{\text{stiff,1bad}} = \frac{\text{sum}(x_{\text{1bad}}, y_{\text{1bad}})}{\text{sum}(x_{\text{1bad}}^2)};\]
\[ybar_{\text{N,1inchbad}} = \text{mean}(y_{\text{1bad}});\]
\[SS_{\text{tot,N,1inchbad}} = \text{sum}((y_{\text{1bad}} - ybar_{\text{N,1inchbad}})^2);\]
\[SS_{\text{res,N,1inchbad}} = \text{sum}((y_{\text{1bad}} - N_{\text{stiff,1bad}} \cdot x_{\text{1bad}})^2);\]
\[R_{\text{sq,1bad}} = 1 - \frac{SS_{\text{res,N,1inchbad}}}{SS_{\text{tot,N,1inchbad}}};\]
\[yd_1 = \left( \mu_{\text{inch}}^2 \cdot a_{3,} + \frac{kappa_{\text{inch}}^2 \cdot a_{3,}^3}{6 \cdot w_{0,\text{inch}}^3 \cdot (4 \cdot \omega_{\text{T,inch}}^2 - w_{0,\text{inch}}^2)} \right)^2 - 4 \cdot \frac{kappa_{\text{inch}}^4 \cdot \Lambda_{\text{TPR}}^6}{w_{0,\text{inch}}^2};\]
\[xd_1 = \frac{\omega_{\text{T,inch}}^6 \cdot \Lambda_{\text{TPR}}^6}{w_{0,\text{inch}}^2} - 9 \cdot \omega_{\text{T,inch}}^4 \cdot \Lambda_{\text{TPR}}^4 \cdot a_{3,}^2 - \frac{9 \cdot \omega_{\text{T,inch}}^2 \cdot \Lambda_{\text{TPR}}^2 \cdot a_{3,}^4 \cdot w_{0,\text{inch}}^2}{4 \cdot \omega_{\text{T,inch}}^2 \cdot a_{3,} \cdot w_{0,\text{inch}}^2 - 9/64 \cdot w_{0,\text{inch}}^4 \cdot a_{3,}^6;\]
\[N_{\text{damp,1}} = \frac{\text{sum}(xd_1 \cdot yd_1)}{\text{sum}(xd_1^2)};\]
\[\text{realND1} = \sqrt{N_{\text{damp,1}}};\]
\[ybar_{\text{Ndamp,1inch}} = \text{mean}(yd_1);\]
\[SS_{\text{tot,Ndamp,1inch}} = \text{sum}((yd_1 - ybar_{\text{Ndamp,1inch}})^2);\]
\[SS_{\text{res,Ndamp,1inch}} = \text{sum}((yd_1 - N_{\text{damp,1}} \cdot xd_1)^2);\]
\[R_{\text{sq,Ndamp,1inch}} = 1 - \frac{SS_{\text{res,Ndamp,1inch}}}{SS_{\text{tot,Ndamp,1inch}}};\]

%% Panels at PR
\[\varepsilon_{\text{1inch}} = \frac{kappa_{\text{inch}}^2}{kappa_{\text{inch}}};\]
\[\varepsilon_{25inch} = \frac{kappa_{25inch}^2}{kappa_{25inch}};\]
\[\varepsilon_{4inch} = \frac{kappa_{4inch}^2}{kappa_{4inch}};\]
\[\varepsilon = \text{mean}([\varepsilon_{\text{1inch}}, \varepsilon_{25inch}, \varepsilon_{4inch}]);\]
\[F = 8.5;\]
\[t = 0:0.00001:2;\]
\[a_{\text{1inch,PR}} = \frac{F}{\text{mass}_{\text{1inch}}} / (2 \cdot w_{0,\text{inch}} \cdot \mu_{\text{inch}});\]
\[a_{25inch,PR} = \frac{F}{\text{mass}_{25inch}} / (2 \cdot w_{0,25inch} \cdot \mu_{25inch});\]
\[a_{4inch,PR} = \frac{F}{\text{mass}_{4inch}} / (2 \cdot w_{0,4inch} \cdot \mu_{4inch});\]
\[u_{\text{1inch,PR}} = a_{\text{1inch,PR}} \cdot \cos(w_{0,\text{inch}} \cdot t - \frac{\pi}{2});\]
\[u_{25inch,PR} = a_{25inch,PR} \cdot \cos(w_{0,25inch} \cdot t - \frac{\pi}{2});\]
\[u_{4inch,PR} = a_{4inch,PR} \cdot \cos(w_{0,4inch} \cdot t - \frac{\pi}{2});\]
\[F_{\text{1inch,PR}} = F \cdot \cos(w_{0,\text{inch}} \cdot t);\]
\[F_{25inch,PR} = F \cdot \cos(w_{0,25inch} \cdot t);\]
\[F_{4inch,PR} = F \cdot \cos(w_{0,4inch} \cdot t);\]
\[\text{figure};\]
\[\text{plot}(u_{\text{1inch,PR}}, F_{\text{1inch,PR}});\]
\[\text{xlabel('Displacement in m')};\]
\[\text{ylabel('Force in N')};\]
\[\text{figure};\]
\[\text{plot}(u_{25inch,PR}, F_{25inch,PR});\]
\[\text{xlabel('Displacement in m')};\]
ylabel('Force in N');

figure;
plot(u_4inch_PR,F_4inch_PR);
xlabel('Displacement in m');
ylabel('Force in N');

%%

[T_1inch_PR,Y_1inch_PR]=ode45(@rigid_1inch_PR2,t,[0 0]);
[T_25inch_PR,Y_25inch_PR]=ode45(@rigid_25inch_PR2,t,[0 0]);
[T_4inch_PR,Y_4inch_PR]=ode45(@rigid_4inch_PR2,t,[0 0]);

figure;
plot(Y_1inch_PR(100001:200001,1),F_1inch_PR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_25inch_PR(100001:200001,1),F_25inch_PR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_4inch_PR(100001:200001,1),F_4inch_PR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

%% Panels at 1/2 PR
omega_1inch_HPR = w0_1inch /2;
omega_25inch_HPR = w0_25inch /2;
omega_4inch_HPR = w0_4inch /2;

Lam_1inch_HPR = F/mass_1inch/(2*(w0_1inch^2-(omega_1inch_HPR)^2));
Lam_25inch_HPR = F/mass_25inch/(2*(w0_25inch^2-(omega_25inch_HPR)^2));
Lam_4inch_HPR = F/mass_4inch/(2*(w0_4inch^2-(omega_4inch_HPR)^2));

a_1inch_HPR = kappa_1inch*Lam_1inch_HPR^2/(w0_1inch*mu_1inch);
a_25inch_HPR = kappa_25inch*Lam_25inch_HPR^2/(w0_25inch*mu_25inch);
a_4inch_HPR = kappa_4inch*Lam_4inch_HPR^2/(w0_4inch*mu_4inch);

u_1inch_HPR = 2*Lam_1inch_HPR*cos(omega_1inch_HPR*t)+a_1inch_HPR*cos(w0_1inch*t+pi/2);
u_25inch_HPR = 2*Lam_25inch_HPR*cos(omega_25inch_HPR*t)+a_25inch_HPR*cos(w0_25inch*t+pi/2);
u_4inch_HPR = 2*Lam_4inch_HPR*cos(omega_4inch_HPR*t)+a_4inch_HPR*cos(w0_4inch*t+pi/2);

F_1inch_HPR = F*cos(omega_1inch_HPR*t);
F_25inch_HPR = F*cos(omega_25inch_HPR*t);
F_4inch_HPR = F*cos(omega_4inch_HPR*t);

figure;
plot(u_1inch_HPR,F_1inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(u_25inch_HPR,F_25inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(u_4inch_HPR,F_4inch_HPR);
xlabel('Displacement in m');
ylabel('Force in N');

[T_1inch_HPR , Y_1inch_HPR ]=ode45(@rigid_1inch_HPR2 ,t , [0 0]);
[T_25inch_HPR , Y_25inch_HPR ]=ode45(@rigid_25inch_HPR2 ,t , [0 0]);
[T_4inch_HPR , Y_4inch_HPR ]=ode45(@rigid_4inch_HPR2 ,t , [0 0]);

figure;
plot(Y_1inch_HPR(100001:200001,1) , F_1inch_HPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_25inch_HPR(100001:200001,1) , F_25inch_HPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

figure;
plot(Y_4inch_HPR(100001:200001,1) , F_4inch_HPR(100001:200001));
xlabel('Displacement in m');
ylabel('Force in N');

%% Panels at 1/3 PR

omega_1inch_TPR = w0_1inch /3;
omega_25inch_TPR = w0_25inch /3;
omega_4inch_TPR = w0_4inch /3;

Lam_1inch_TPR=F/mass_1inch/(2*(w0_1inch^2-(omega_1inch_TPR)^2));
Lam_25inch_TPR=F/mass_2inch/(2*(w0_25inch^2-(omega_25inch_TPR)^2));
Lam_4inch_TPR=F/mass_4inch/(2*(w0_4inch^2-(omega_4inch_TPR)^2));

F_1inch_TPR=F*cos(omega_1inch_TPR*t);
F_25inch_TPR=F*cos(omega_25inch_TPR*t);
F_4inch_TPR=F*cos(omega_4inch_TPR*t);

b_1inch=(2^(1/3)*(-2*(omega_1inch_TPR - 2*w0_1inch)*(omega_1inch_TPR +...2*w0_1inch)*(-9*N_stiff_1*w0_1inch^2*(-4*omega_1inch_TPR^2 + w0_1inch^2)^2) +...6*Lam_1inch_TPR^2*N_stiff_1*w0_1inch^2*(-4*omega_1inch_TPR^4 + 17*omega_1inch_TPR^2*w0_1inch^2)^2) +...4*kappa_1inch*w0_1inch^4) + 2^((1/3)*(-3*Lam_1inch_TPR^3*w0_1inch^2*(-4*Lam_1inch_TPR^2 - 4*w0_1inch^2)*(-4*w0_1inch^2)^2) +...4*N_stiff_1*omega_1inch_TPR^2 + N_stiff_1*w0_1inch^2)^2*(-9*N_stiff_1*w0_1inch^2)*...kappa_1inch*w0_1inch^2*(4 - 48*omega_1inch_TPR^2 + 12*w0_1inch^2)^2 +...sqrt((omega_1inch_TPR^2 -...4*w0_1inch^2)^3) + 9*N_stiff_1*w0_1inch^2*(-4*omega_1inch_TPR^2 + w0_1inch^2)^2) +...kappa_1inch*w0_1inch^2*(4 - 48*omega_1inch_TPR^2 + 12*w0_1inch^2)^3*(-4*mu_1inch^6 - 72*Lam_1inch_TPR^2*mu_1inch^4*N_stiff_1 +...432*Lam_1inch_TPR^4*mu_1inch^2*N_stiff_1^2 -...96*N_stiff_1*omega_1inch_TPR^6 + N_stiff_1^2*w0_1inch^6*(4*omega_1inch_TPR^4 - 17*omega_1inch_TPR^2*w0_1inch^2 +...4*w0_1inch^4)^3 - 16*kappa_1inch^6*Lam_1inch_TPR^6*(1024*omega_1inch_TPR^12 -...768*omega_1inch_TPR^8 + 2 + 17*omega_1inch_TPR^2*w0_1inch^2 +...3*omega_1inch_TPR^4*(256 + 4349*omega_1inch_TPR^2 +...1855*omega_1inch_TPR^4)*w0_1inch^4 - (128 + 3264*omega_1inch_TPR^2 +...30708*omega_1inch_TPR^4 + 106043*omega_1inch_TPR^6)*w0_1inch^6 +...
\[
12 \ast (64 + 1052 \ast \omega_{\text{inch TPR}}^2 + 5339 \ast \omega_{\text{inch TPR}}^4) \ast w_{\text{inch}}^8 - ... \\
48 \ast (20 + 443 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^{10} + 2752 \ast w_{\text{inch}}^{12} + ... \\
12 \ast \kappa_{\text{inch}}^4 \ast \lambda_{\text{inch TPR}}^4 \ast w_{\text{inch}}^{12} - 17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + ... \\
4 \ast w_{\text{inch}}^{-4} \ast (-16 \ast \mu_{\text{inch}}^2 \ast \omega_{\text{inch TPR}}^2 \ast \omega_{\text{inch TPR}}^4 - 2 + ... \\
17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + 4 \ast w_{\text{inch}}^{-4})^2 \ast w_{\text{inch}}^2 + ... \\
3 \ast \lambda_{\text{inch TPR}}^2 \ast N_{\text{stiff 1}} \ast (512 \ast \omega_{\text{inch TPR}}^8 - ... \\
4 \ast \omega_{\text{inch TPR}}^4 \ast (127 + 1100 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^2 + (128 + ... \\
2144 \ast \omega_{\text{inch TPR}}^2 + 10677 \ast \omega_{\text{inch TPR}}^4) \ast w_{\text{inch}}^4 - ... \\
8 \ast (56 + 661 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^6 + 848 \ast w_{\text{inch}}^8) - ... \\
12 \ast \kappa_{\text{inch}}^2 \ast \lambda_{\text{inch TPR}}^2 \ast w_{\text{inch}}^4 - (4 \ast \omega_{\text{inch TPR}}^4 - 17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + ... \\
4 \ast \mu_{\text{inch}}^2 \ast \omega_{\text{inch TPR}}^2 \ast \omega_{\text{inch TPR}}^4 - (2 + ... \\
17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + 4 \ast w_{\text{inch}}^{-4})^2 \ast w_{\text{inch}}^2 + ... \\
4 \ast \omega_{\text{inch TPR}}^4 + 3 \ast \lambda_{\text{inch TPR}}^2 \ast N_{\text{stiff 1}} \ast \omega_{\text{inch TPR}}^4 - (204 \ast \omega_{\text{inch TPR}}^4 - 92 \ast \omega_{\text{inch TPR}}^2 + 240 \ast w_{\text{inch}}^2 + ... \\
\omega_{\text{inch TPR}}^2) \ast (1 + 876 \ast \omega_{\text{inch TPR}}^2))}^{(2/3)})^{(2/3)})^{(2/3)} \ast \sqrt{(... \\
4 \ast \omega_{\text{inch TPR}}^{-2} \ast (-9 \ast N_{\text{stiff 1}} \ast \omega_{\text{inch TPR}}^2 \ast (-4 \ast \omega_{\text{inch TPR}}^2 - w_{\text{inch}}^2) + ... \\
\kappa_{\text{inch}}^2 \ast (4 - 48 \ast \omega_{\text{inch TPR}}^2 - 12 \ast w_{\text{inch}}^2))}^{-2} + (4 \ast \omega_{\text{inch TPR}}^2 + w_{\text{inch}}^2) + ... \\
4 \ast w_{\text{inch}}^{-4} \ast (4 - 48 \ast \omega_{\text{inch TPR}}^2 + 12 \ast w_{\text{inch}}^2))^{-2} - (4 \ast \omega_{\text{inch TPR}}^2 + w_{\text{inch}}^2) + ... \\
3 \ast \omega_{\text{inch TPR}}^8 \ast (2 + 17 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^2 + ... \\
3 \ast \omega_{\text{inch TPR}}^2 \ast (256 + 4349 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^2 + ... \\
19556 \ast \omega_{\text{inch TPR}}^4 \ast w_{\text{inch}}^4 - (128 + 3264 \ast \omega_{\text{inch TPR}}^2 + ... \\
30708 \ast \omega_{\text{inch TPR}}^2 + 106043 \ast \omega_{\text{inch TPR}}^6) \ast w_{\text{inch}}^6 + ... \\
12 \ast (66 + 1052 \ast \omega_{\text{inch TPR}}^2 + 5339 \ast \omega_{\text{inch TPR}}^4) \ast w_{\text{inch}}^8 - ... \\
48 \ast (20 + 443 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^{10} + 2752 \ast w_{\text{inch}}^{12} + ... \\
12 \ast \kappa_{\text{inch}}^4 \ast \lambda_{\text{inch TPR}}^4 \ast w_{\text{inch}}^2 - 17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + ... \\
4 \ast \omega_{\text{inch TPR}}^4 \ast (-16 \ast \mu_{\text{inch}}^2 \ast \omega_{\text{inch TPR}}^2 \ast \omega_{\text{inch TPR}}^4 - 2 + ... \\
17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + 4 \ast \omega_{\text{inch TPR}}^{-4})^2 \ast w_{\text{inch}}^2 + ... \\
4 \ast \omega_{\text{inch TPR}}^4 \ast (127 + 1100 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^2 + (128 + ... \\
2144 \ast \omega_{\text{inch TPR}}^2 + 10677 \ast \omega_{\text{inch TPR}}^4) \ast w_{\text{inch}}^4 - ... \\
8 \ast (56 + 661 \ast \omega_{\text{inch TPR}}^2) \ast w_{\text{inch}}^6 + 848 \ast w_{\text{inch}}^8) - ... \\
12 \ast \kappa_{\text{inch}}^2 \ast \lambda_{\text{inch TPR}}^2 \ast w_{\text{inch}}^4 - (4 \ast \omega_{\text{inch TPR}}^4 - 17 \ast \omega_{\text{inch TPR}}^2 \ast w_{\text{inch}}^2 + ... \\

\[ 4 \cdot w_0_1inch^4 - 4 \cdot \text{mu}_1inch^2 \cdot w_0_1inch \cdot N_{\text{stiff}_1} \cdot (4 \cdot \omega_1inch_{\text{TPR}}^4 - (2 + 17 \cdot \omega_1inch_{\text{TPR}}^2) \cdot w_0_1inch^2 + 4 \cdot \omega_1inch_{\text{TPR}}^4) + 3 \cdot \text{Lam}_1inch_{\text{TPR}}^2 \cdot \omega_1inch_{\text{TPR}} \cdot (204 \cdot \omega_1inch_{\text{TPR}}^4 - 92 \cdot w_0_1inch_{\text{TPR}}^2 + 240 \cdot w_0_1inch^4 - \omega_1inch_{\text{TPR}}^4 \cdot (1 + 876 \cdot w_0_1inch^2)) \cdot (1/3)); \]

\[ b_{25inch} = (2^{1/3} \cdot \omega_{25inch} \cdot (-2 \cdot \omega_{25inch} - 2 \cdot w_0_{25inch}) \cdot (2 \cdot \omega_{25inch} + 2 \cdot w_0_{25inch} \cdot (-9 \cdot N_{\text{stiff}_25} \cdot w_0_{25inch}^2 \cdot (-4 \cdot \omega_{25inch}^2 + w_0_{25inch}^2) + \kappa_{25inch}^2 \cdot (4 - 48 \cdot \omega_{25inch}^2 + 12 \cdot w_0_{25inch}^2)) \cdot (\mu_{25inch}^6 - 72 \cdot \text{Lam}_{25inch_{\text{TPR}}}^2 \cdot \mu_{25inch}^4 \cdot N_{\text{stiff}_25}^2 + 432 \cdot \text{Lam}_{25inch_{\text{TPR}}}^4 \cdot \mu_{25inch}^2 \cdot N_{\text{stiff}_25}^3 - 945 \cdot \text{Lam}_{25inch_{\text{TPR}}}^6 \cdot N_{\text{stiff}_25}^4) \cdot w_0_{25inch}^6 \cdot (4 \cdot \omega_{25inch}^4 - 17 \cdot \omega_{25inch}^2 \cdot w_0_{25inch}^2 + 4 \cdot w_0_{25inch}^4)^3 - 16 \cdot \kappa_{25inch}^6 \cdot \text{Lam}_{25inch_{\text{TPR}}}^6 \cdot (1024 \cdot \omega_{25inch}^{12} - 768 \cdot \omega_{25inch}^8 \cdot (2 + 17 \cdot \omega_{25inch}^2) \cdot w_0_{25inch}^2 + 3 \cdot \omega_{25inch}^4 \cdot (256 + 4349 \cdot \omega_{25inch}^2 + 19556 \cdot \omega_{25inch}^4) \cdot w_0_{25inch}^4 - (128 + 3264 \cdot \omega_{25inch}^2 + 10677 \cdot \omega_{25inch}^4 + 106043 \cdot \omega_{25inch}^6) \cdot w_0_{25inch}^6 + 8 \cdot (56 + 661 \cdot \omega_{25inch}^2) \cdot w_0_{25inch}^8 + 848 \cdot w_0_{25inch}^{10}) + 2752 \cdot w_0_{25inch}^{12}) + \ldots \]

\[ 3 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot (256 + 4349 \cdot \omega_{25inch_{\text{TPR}}}^2 + \ldots \]

\[ 19556 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot w_0_{25inch}^4 - (128 + 3264 \cdot \omega_{25inch}^2 + 10677 \cdot \omega_{25inch}^4 + 106043 \cdot \omega_{25inch}^6) \cdot w_0_{25inch}^6 + 8 \cdot (56 + 661 \cdot \omega_{25inch}^2) \cdot w_0_{25inch}^8 + 848 \cdot w_0_{25inch}^{10}) + 2752 \cdot w_0_{25inch}^{12}) + \ldots \]

\[ 12 \cdot \kappa_{25inch}^2 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot (4 \cdot \omega_{25inch_{\text{TPR}}}^4 - 17 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot w_0_{25inch}^2 + \ldots \]

\[ 4 \cdot w_0_{25inch}^4 \cdot (-16 \cdot \mu_{25inch}^2 \cdot (4 \cdot \omega_{25inch_{\text{TPR}}}^4 - (2 + 17 \cdot \omega_{25inch_{\text{TPR}}}^2) \cdot w_0_{25inch}^2 + 4 \cdot w_0_{25inch}^2) \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot w_0_{25inch}^2 + \ldots \]

\[ 3 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot N_{\text{stiff}_25}^2 \cdot (512 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot (127 + 1100 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot \omega_{25inch_{\text{TPR}}}^6 \cdot \omega_{25inch_{\text{TPR}}}^6 \cdot \omega_{25inch_{\text{TPR}}}^6 \cdot 2 + (128 \ldots \]

\[ 2144 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot 10677 \cdot \omega_{25inch_{\text{TPR}}}^4 \cdot w_0_{25inch}^4 - (56 + 661 \cdot \omega_{25inch_{\text{TPR}}}^2 \cdot w_0_{25inch}^8) + \ldots \]
\[
\begin{align*}
\kappa_{25\text{inch}}^2 & \times (4 - 48 \omega_{25\text{inch}}^2 + 12 w_{0_{25\text{inch}}}^2) \times (-3 \times \lambda_{25\text{inch}} TPR^2 - 4 \times w_{0_{25\text{inch}}}^2 - 2 w_{0_{25\text{inch}}}^2) \times \sqrt{4} \\
B_{4\text{inch}} & = (2^{-1/3} \times (2 \omega_{4\text{inch}}^2 - 2 w_{0_{4\text{inch}}}^2 - 2 w_{0_{4\text{inch}}}^2) \times (3 \times \lambda_{4\text{inch}} TPR^2 - 4 \times w_{0_{4\text{inch}}}^2 - 2 w_{0_{4\text{inch}}}^2) \times \sqrt{4} \\
\end{align*}
\]
\( \kappa_{4\text{inch}} \times (4 - 48 \omega_{4\text{inch}}^2 + 12 w_{0\text{inch}}^2) \times (4 - 48 \omega_{4\text{inch}}^2 + 12 w_{0\text{inch}}^2) \times \sqrt{\omega_{4\text{inch}}^2 - 4 w_{0\text{inch}}^2} \)
\[ 12 \cdot \kappa_{4\text{inch}}^{4} \cdot \Lambda_{4\text{inch}_{\text{TPR}}}^{4} \cdot w_{0\text{4inch}}^{2} \cdot (\omega_{4\text{inch}_{\text{TPR}}}^{4} - 17 \cdot \omega_{4\text{inch}_{\text{TPR}}}^{2} \cdot w_{0\text{4inch}}^{2} + ... + 4 \cdot w_{0\text{4inch}}^{4})^{2} \cdot \mu_{4\text{inch}}^{2} \cdot (\omega_{4\text{inch}_{\text{TPR}}}^{4} - 17 \cdot \omega_{4\text{inch}_{\text{TPR}}}^{2} \cdot w_{0\text{4inch}}^{2} + ... + 4 \cdot \omega_{4\text{inch}}^{4})^{2} + 10677 \cdot \omega_{4\text{inch}_{\text{TPR}}}^{4} \cdot \Lambda_{4\text{inch}_{\text{TPR}}}^{2} \cdot N_{\text{stiff}_{4}}^{2} - ... \]
Appendix B: System of equations for the ODE solver MATLAB code

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 25.0mm damaged panel
% with corrected coefficients at the primary resonance
function dy = rigid_1inch_PR(t,y)
F=8.5;
mass=0.0044659349267087;
w0=2937.3053530236;
omega=w0;
eps=0.42240283586376;
mu=1468.67773139705;
kappa=3495331361.73455;
N=3559322976884.92;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^7*mu*y(2)-kappa*eps^5*y(1)^2-N*eps^5*y(1)^3-(w0)^2*y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 63.5mm damaged panel
% with corrected coefficients at the primary resonance
function dy = rigid_25inch_PR(t,y)
F=8.5;
mass=0.00487829293676733;
w0=2258.72134212697;
omega=w0;
eps=0.490565315624642;
mu=1129.45310720352;
kappa=936970833.069940;
N=588922754392.767;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^7*mu*y(2)-kappa*eps^5*y(1)^2-N*eps^5*y(1)^3-(w0)^2*y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 101mm damaged panel
% with corrected coefficients at the primary resonance
function dy = rigid_4inch_PR(t,y)
F=8.5;
mass=0.00528364585758987;
w0=1859.96945858233;
omega=w0;
epsilon = 0.641017164096910;
mu = 930.021434671778;
kappa = 140191279.968064;
N = 93001354771.3662;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*epsilon^7*mu*y(2) - kappa*epsilon^5*y(1)^2 - N*epsilon^5*y(1)^3 - (w0)^2* y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 25.0mm damaged panel
% with corrected coefficients at one-half the primary resonance
function dy = rigid_1inch_HPR(t,y)
F=8.5;
mass=0.00446593449267087;
w0=2937.30535350236;
omega=w0/2;
epsilon=0.422402835986376;
mu=1468.67773139705;
kappa=3495331361.73455;
N=3559322976884.92;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*epsilon^7*mu*y(2) - kappa*epsilon^5*y(1)^2 - N*epsilon^5*y(1)^3 - (w0)^2* y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 63.5mm damaged panel
% with corrected coefficients at one-half the primary resonance
function dy = rigid_25inch_HPR(t,y)
F=8.5;
mass=0.00487829293676733;
w0=2258.72134212697;
omega=w0/2;
epsilon=0.490565315624642;
mu=1129.45310720352;
kappa=936970833.069940;
N=588922754392.767;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*epsilon^7*mu*y(2) - kappa*epsilon^5*y(1)^2 - N*epsilon^5*y(1)^3 - (w0)^2* y(1);
% This function defines the system of equations and the coefficients for numerically solving the equation of motion for the 101mm damaged panel with corrected coefficients at one-half the primary resonance

function dy = rigid_4inch_HPR(t,y)
F = 8.5;
mass = 0.00528364858758987;
w0 = 1859.96945858233;
omega = w0/2;
eps = 0.641017164096910;
mu = 930.021434671778;
kappa = 140191279.968064;
N = 93001354771.3662;

dy = zeros(2,1); % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*eps^7*mu*y(2) - kappa*eps^5*y(1)^2 - N*eps^5*y(1)^3 - (w0)^2*y(1);
end

% This function defines the system of equations and the coefficients for numerically solving the equation of motion for the 25.0mm damaged panel with corrected coefficients at one-third the primary resonance

function dy = rigid_1inch_TPR(t,y)
F = 8.5;
mass = 0.00446593449267087;
w0 = 2937.30535530236;
omega = w0/3;
eps = 0.422402835986376;
mu = 1468.6773139705;
kappa = 3495331361.73455;
N = 3559322976884.92;

dy = zeros(2,1); % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*eps^7*mu*y(2) - kappa*eps^5*y(1)^2 - N*eps^5*y(1)^3 - (w0)^2*y(1);
end
% This function defines the system of equations and the coefficients for % numerically solving the equation of motion for the 63.5mm damaged panel % with corrected coefficients at one-third the primary resonance
function dy = rigid_25inch_TPR(t,y)
F=8.5;
mass=0.0048782923676733;
w0=2258.72134212697;
omega=w0/3;
eps=0.490565315624642;
mu=1129.45310720352;
kappa=936970833.069940;
N=588922754392.767;

dy = zeros(2,1); % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^7*mu*y(2)-kappa*eps^5*y(1)^2-N*eps^5*y(1)^3-(w0)^2* y(1);
end

% This function defines the system of equations and the coefficients for % numerically solving the equation of motion for the 101mm damaged panel % with corrected coefficients at one-third the primary resonance
function dy = rigid_4inch_TPR(t,y)
F=8.5;
mass=0.00528354585758987;
w0=1859.96945858233;
omega=w0/3;
eps=0.641017164096910;
mu=930.021434671778;
kappa=140191279.968064;
N=93001354771.3662;

dy = zeros(2,1); % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^7*mu*y(2)-kappa*eps^5*y(1)^2-N*eps^5*y(1)^3-(w0)^2* y(1);
end

% This function defines the system of equations and the coefficients for % numerically solving the equation of motion for the 25.0mm damaged panel % with uncorrected coefficients at the primary resonance
function dy = rigid_1inch_PR2(t,y)
F=8.5;
mass=0.00788855641386146;
w0=2937.30535530236;
omega=w0;
eps=0.0309739087686158;
mu = 1468.67773139705;
kappa = 6423959450.58249;
N = 12352672807077.2;

\[
\begin{align*}
\text{dy} &= \text{zeros}(2,1); \quad \text{\% a column vector} \\
\text{dy}(1) &= y(2); \\
\text{dy}(2) &= F/mass \cdot \cos(\omega t) - 2 \cdot \epsilon^1 \cdot \mu \cdot y(2) - \kappa \cdot \epsilon^3 \cdot y(1)^2 - N \cdot \epsilon^5 \cdot y(1)^3 - (w_0)^2 \cdot y(1);
\end{align*}
\]

end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 63.5mm damaged panel
% with uncorrected coefficients at the primary resonance
function \( \text{dy} = \text{rigid}_25\text{inch}_\text{PR2}(t,y) \)
\[
\begin{align*}
F &= 8.5; \\
mass &= 0.00686858672591264; \\
w_0 &= 2258.72134212697; \\
\omega &= w_0; \\
\epsilon &= 0.199285778578922; \\
\mu &= 1129.45310720352; \\
\kappa &= 1304072507.67976; \\
N &= 1143186073624.58;
\end{align*}
\]

\[
\begin{align*}
\text{dy} &= \text{zeros}(2,1); \quad \text{\% a column vector} \\
\text{dy}(1) &= y(2); \\
\text{dy}(2) &= F/mass \cdot \cos(\omega t) - 2 \cdot \epsilon^1 \cdot \mu \cdot y(2) - \kappa \cdot \epsilon^3 \cdot y(1)^2 - N \cdot \epsilon^5 \cdot y(1)^3 - (w_0)^2 \cdot y(1);
\end{align*}
\]

end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 101mm damaged panel
% with uncorrected coefficients at the primary resonance
function \( \text{dy} = \text{rigid}_4\text{inch}_\text{PR2}(t,y) \)
\[
\begin{align*}
F &= 8.5; \\
mass &= 0.001614277875453698; \\
w_0 &= 1859.96945858233; \\
\omega &= w_0; \\
\epsilon &= 0.173505531748448; \\
\mu &= 930.021434671778; \\
\kappa &= 185896983.762469; \\
N &= 129120583443.497;
\end{align*}
\]

\[
\begin{align*}
\text{dy} &= \text{zeros}(2,1); \quad \text{\% a column vector} \\
\text{dy}(1) &= y(2); \\
\text{dy}(2) &= F/mass \cdot \cos(\omega t) - 2 \cdot \epsilon^1 \cdot \mu \cdot y(2) - \kappa \cdot \epsilon^3 \cdot y(1)^2 - N \cdot \epsilon^5 \cdot y(1)^3 - (w_0)^2 \cdot y(1);
\end{align*}
\]
% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 25.0mm damaged panel
% with uncorrected coefficients at one-half the primary resonance
function dy = rigid_1inch_HPR2(t,y)
F=8.5;
mass=0.0078885641386146;
w0=2937.3055630236;
omega=w0/2;
eps=0.0309739087686158;
mu=1468.67773139705;
kappa=6423959450.58249;
N=12352672807077.2;

dy = zeros(2,1);   % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^1*mu*y(2)-eps^1*kappa*y(1)^2-eps^2*N*y(1)^3-(w0)^2*
y(1);
%
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 63.5mm damaged panel
% with uncorrected coefficients at one-half the primary resonance
function dy = rigid_25inch_HPR2(t,y)
F=8.5;
mass=0.00686858672591264;
w0=2258.72134212697;
omega=w0/2;
eps=0.199285778578922;
mu=1129.45310720352;
kappa=1304072507.67976;
N=1143186073624.58;

dy = zeros(2,1);   % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^2*mu*y(2)-kappa*eps^2*y(1)^2-N*eps^4*y(1)^3-(w0)^2*
y(1);
end
% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 101mm damaged panel
% with uncorrected coefficients at one-half the primary resonance
function dy = rigid_4inch_HPR2(t,y)
F=8.5;
mass=0.00614277875453698;
w0=1859.96945858233;
omega=w0/2;
eps=0.173505531748448;
mu=930.021434671778;
kappa=185896983.762469;
N=129120583443.497;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^2*mu*y(2)-kappa*eps*2*y(1)^2-N*eps*4*y(1)^3-(w0)^2*y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 25.0mm damaged panel
% with uncorrected coefficients at one-third the primary resonance
function dy = rigid_1inch_TPR2(t,y)
F=8.5;
mass=0.00788855641386146;
w0=2937.30535530236;
omega=w0/3;
eps=0.0309739087686158;
mu=1468.67773139705;
kappa=6423959450.58249;
N=12352672807077.2;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t)-2*eps^1*mu*y(2)-kappa*eps*1*y(1)^2-N*eps*2*y(1)^3-(w0)^2*y(1);
end

% This function defines the system of equations and the coefficients for
% numerically solving the equation of motion for the 63.5mm damaged panel
% with uncorrected coefficients at one-third the primary resonance
function dy = rigid_25inch_TPR2(t,y)
F=8.5;
mass=0.00686858672591264;
w0=2358.72130212697;
omega=w0/3;
eps=0.199285778578922;
mu = 1129.45310720352;
kappa = 1304072507.67976;
N = 1143186073624.58;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*eps^2*mu*y(2) - kappa*eps^2*y(1)^2 - N*eps^4*y(1)^3 - (w0)^2* y(1);

end

% This function defines the system of equations and the coefficients for % numerically solving the equation of motion for the 101mm damaged panel % with uncorrected coefficients at one-third the primary resonance
function dy = rigid_4inch_TPR2(t,y)
F = 8.5;
mass = 0.00614277875453698;
w0 = 1859.96945858233;
omega = w0/3;
eps = 0.173505531748448;
mu = 930.021434671778;
kappa = 185896983.762469;
N = 129120583443.497;

dy = zeros(2,1);  % a column vector
dy(1) = y(2);
dy(2) = F/mass*cos(omega*t) - 2*eps^2*mu*y(2) - kappa*eps^2*y(1)^2 - N*eps^4*y(1)^3 - (w0)^2* y(1);

end
Appendix C: Amplitude Tracking and Plotting MATLAB code

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %%%%%%%%%%%%%%%%%%%%%%%%
% This code loads the experimental data from all three panels when excited
% at the primary resonance. It then tracks the amplitudes of the
% response peaks at known frequencies and plots these peaks, either as raw
% displacement data or normalized.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %%%%%%%%%%%%%%%%%%%%%%%%
clear all;
clc;

%%
for ii = 1:31
    cnt = num2str(ii);
    datafile = ['C:\Users\editman\Documents\MATLAB\Data Aq\Apr_25_25inch_360_amp\data' ... 
                ' cnt '.mat'];
    scan_data2(ii) = load(datafile);
    temp = abs(fft(scan_data2(1, ii).data(:,1),640000));
    Dft2(:,ii) = temp;
    Force_Dft2(:,ii) = abs(fft(scan_data2(1, ii).data(:,2),640000));
    [C,I] = max(Dft2(:,1));
    omega2 = I/640000*6400;
    peak_360(ii) = max(Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii)) ... 
                  /3200*9.81/(omega2*2*pi)^2;
    peak_719(ii) = max(Dft2((2*omega2-10)/6400*640000:(2*omega2+10)/6400*640000,ii)) ... 
                  /3200*9.81/(omega2*2*pi)^2;
    peak_1080(ii) = max(Dft2((3*omega2-10)/6400*640000:(3*omega2+10)/6400*640000,ii)) ... 
                   /3200*9.81/(omega2*3*2*pi)^2;
    force_peak2(ii) = max(Force_Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii)) ... 
                     /3200*4.44822162;        
end

for ii = 1:31
    cnt = num2str(ii);
    datafile = ['C:\Users\editman\Documents\MATLAB\Data Aq\Apr_25_1inch_468_amp\data' ... 
                ' cnt '.mat'];
    scan_data3(ii) = load(datafile);
    temp = abs(fft(scan_data3(1, ii).data(:,1),640000));
    Dft3(:,ii) = temp;
    Force_Dft3(:,ii) = abs(fft(scan_data3(1, ii).data(:,2),640000));
    [C,I] = max(Dft3(:,1));
    omega3 = I/640000*6400;
    peak_468(ii) = max(Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii)) ... 
                  /3200*9.81/(omega3*2*pi)^2;
peak_936(ii)=max(Dft3((2*omega3-10)/6400*640000:(2*omega3+10)/6400*640000,ii))/3200*9.81/(omega3*2*2*pi)^2;

peak_1404(ii)=max(Dft3((3*omega3-10)/6400*640000:(3*omega3+10)/6400*640000,ii))/3200*9.81/(omega3*3*2*pi)^2;

force_peak3(ii)=max(Force_Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii))/3200*4.44822162;

for ii = 1:46
  cnt=num2str(ii);
  datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Run_8_283_amp\data\' cnt '.mat'];
  scan_data1(ii)=load(datafile);
  temp = abs(fft(scan_data1(ii).data(:,1),640000));
  Dft1(:,ii)=temp;
  Force_Dft1(:,ii)=abs(fft(scan_data1(1,ii).data(:,2),640000));
  [C,I]=max(Dft1(:,1));
  omega1=I/640000*6400;

  peak_294(ii)=max(Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*pi)^2;
  peak_590(ii)=max(Dft1((2*omega1-10)/6400*640000:(2*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*2*pi)^2;
  peak_885(ii)=max(Dft1((3*omega1-10)/6400*640000:(3*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*3*2*pi)^2;
  force_peak1(ii)=max(Force_Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*4.44822162;
end

%%
clc;

freq=(0:6400/65536:6399.999);
numb=1:0.001:6;

lin=numb;
quad=numb.^2;

%%
figure;
plot(peak_294,force_peak1,'-xk',peak_360,force_peak2,'-ob',
     peak_468,force_peak3,'-+g');
ylabel('Force in N');
xlabel('Displacement in m');
legend('101 mm damage','63.5 mm damage','25.0 mm damage');
grid on;

%%
% This code loads the experimental data from all three panels when excited at one-half the primary resonance. It then tracks the amplitudes of the response peaks at known frequencies and plots these peaks, either as raw displacement data or normalized.

clear all;

figure;
plot(force_peak1, peak_294, 'k-', force_peak2, force_peak3, 'b-', ...
    peak_360, peak_468, 'g-');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage', '63.5 mm damage', '25.0 mm damage');
gr on;

figure;
plot(force_peak1, peak_590, 'k-', force_peak2, force_peak3, 'b-', ...
    peak_719, peak_936, 'g-');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage', '63.5 mm damage', '25.0 mm damage');
gr on;

figure;
plot(force_peak1, peak_885, 'k-', force_peak2, force_peak3, 'b-', ...
    peak_1080, peak_1404, 'g-');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage', '63.5 mm damage', '25.0 mm damage');
gr on;

figure;
plot(peak_885, force_peak1, 'k-', peak_1080, force_peak2, 'b-', ...
    peak_1404, force_peak3, 'g-');
ylabel('Force in N');
xlabel('Displacement in m');
legend('101 mm damage', '63.5 mm damage', '25.0 mm damage');
gr on;

figure;
plot(peak_590, force_peak1, 'k-', peak_719, force_peak2, 'b-', ...
    peak_936, force_peak3, 'g-');
ylabel('Force in N');
xlabel('Displacement in m');
legend('101 mm damage', '63.5 mm damage', '25.0 mm damage');
gr on;
clc;

for ii = 1:26
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_25_25inch_180_amp\data' cnt '.mat'];
    scan_data2(ii) = load(datafile);
    temp = abs(fft(scan_data2(1, ii).data(:, 1), 640000));
    Dft2(:, ii) = temp;
    Force_Dft2(:, ii) = abs(fft(scan_data2(1, ii).data(:, 2), 640000));
    [C, I] = max(Dft2(:, 1));
    omega2 = I / 640000 * 6400;
    peak_180(ii) = max(Dft2((omega2 - 10) / 640000: (omega2 + 10) / 640000, ii)) / 3200 * 9.81 / (omega2 * 2 * pi)^2;
    peak_360(ii) = max(Dft2((2 * omega2 - 10) / 640000: (2 * omega2 + 10) / 640000, ii)) / 3200 * 9.81 / (omega2 * 2 * 2 * pi)^2;
    peak_720(ii) = max(Dft2((4 * omega2 - 10) / 640000: (4 * omega2 + 10) / 640000, ii)) / 3200 * 9.81 / (omega2 * 4 * 2 * pi)^2;
    force_peak2(ii) = max(Force_Dft2((omega2 - 10) / 640000: (omega2 + 10) / 640000, ii)) / 3200 * 4.44822162;
end

for ii = 1:6
    cnt = num2str(ii);
    datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_24_1inch_234_amp\data' cnt '.mat'];
    scan_data3(ii) = load(datafile);
    temp = abs(fft(scan_data3(1, ii).data(:, 1), 640000));
    Dft3(:, ii) = temp;
    Force_Dft3(:, ii) = abs(fft(scan_data3(1, ii).data(:, 2), 640000));
    [C, I] = max(Dft3(:, 1));
    omega3 = I / 640000 * 6400;
    peak_234(ii) = max(Dft3((omega3 - 10) / 640000: (omega3 + 10) / 640000, ii)) / 3200 * 9.81 / (omega3 * 2 * pi)^2;
    peak_468(ii) = max(Dft3((2 * omega3 - 10) / 640000: (2 * omega3 + 10) / 640000, ii)) / 3200 * 9.81 / (omega3 * 2 * 2 * pi)^2;
    peak_936(ii) = max(Dft3((4 * omega3 - 10) / 640000: (4 * omega3 + 10) / 640000, ii)) / 3200 * 9.81 / (omega3 * 4 * 2 * pi)^2;
    force_peak3(ii) = max(Force_Dft3((omega3 - 10) / 640000: (omega3 + 10) / 640000, ii)) / 3200 * 4.44822162;
end

for ii = 1:21
cnt = num2str(ii);  
datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\May_2_4inch_147_amp\data'  
cnt '.mat'];  
scan_data1(ii)=load(datafile);  
temp=abs(fft(scan_data1(ii).data(:,1),640000));  
Dft1(:,ii)=temp;  
Force_Dft1(:,ii)=abs(fft(scan_data1(1,ii).data(:,2),640000));  
[C,I]=max(Dft1(:,1));  
omega1=1/640000*6400;  
peak_147(ii)=max(Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*pi)^2;  
peak_294(ii)=max(Dft1((2*omega1-10)/6400*640000:(2*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*2*2*pi)^2;  
peak_590(ii)=max(Dft1((4*omega1-10)/6400*640000:(4*omega1+10)/6400*640000,ii))/3200*9.81/(omega1*4*2*pi)^2;  
force_peak1(ii)=max(Force_Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii))/3200*4.44822162;  
end  
%%  
clear;  

cfreq=(0:6400/65536:6399.999);  
numb=1:0.001:6;  
lin= numb;  
quad= numb.^2;  
quart= numb.^4;  
%%  
figure;  
plot(peak_147,force_peak1,'-xk',peak_180,force_peak2,'-ob',peak_234,force_peak3,'-+g');  
ylabel('Force in N');  
xlabel('Displacement in m');  
legend('101 mm damage','63.5 mm damage','25.0 mm damage');  
grid on;  
%%  
figure;  
plot(force_peak1/force_peak1(1),peak_147/peak_147(1),'-xk',force_peak2/force_peak2(1)  
,peak_180/peak_180(1),'-ob',...,  
force_peak3/force_peak3(1),peak_234/peak_234(1),'-+g',numb,lin,'-r');  
xlabel('Normalized Forcing Amplitude N/N');  
ylabel('Normalized Response Amplitude m/m');  
legend('101 mm damage','63.5 mm damage','25.0 mm damage');  
axis([1 6 1 6]);  
grid on;  
%%  
figure;
\texttt{plot} (force\_peak1 / force\_peak1 (1) , peak\_294 / peak\_294 (1) , '-xk' , force\_peak2 / force\_peak2 (1) 
, peak\_360 / peak\_360 (1) , '-ob' , ...
force\_peak3 / force\_peak3 (1) , peak\_468 / peak\_468 (1) , '-+g' , numb , quad , '-r');
\texttt{xlabel} ('Normalized Forcing Amplitude N/N');
\texttt{ylabel} ('Normalized Response Amplitude m/m');
\texttt{legend} ('101 mm damage', '63.5 mm damage', '25.0 mm damage');
\texttt{axis} ([1 6 1 36]);
\texttt{grid on};

$$
\texttt{figure;}
\texttt{plot} (peak\_590 , force\_peak1 , '-xk', peak\_720 , force\_peak2 , '-ob', peak\_936 , force\_peak3 , '+g');
\texttt{ylabel} ('Force in N');
\texttt{xlabel} ('Displacement in m');
\texttt{legend} ('101 mm damage', '63.5 mm damage', '25.0 mm damage');
\texttt{grid on};
$$

\texttt{figure;}
\texttt{plot} (force\_peak1 / force\_peak1 (1) , peak\_590 / peak\_590 (1) , '-xk' , force\_peak2 / force\_peak2 (1) 
, peak\_720 / peak\_720 (1) , '-ob' , ...
force\_peak3 / force\_peak3 (1) , peak\_936 / peak\_936 (1) , '+g' , numb , quart , '-r');
\texttt{xlabel} ('Normalized Forcing Amplitude N/N');
\texttt{ylabel} ('Normalized Response Amplitude m/m');
\texttt{legend} ('101 mm damage', '63.5 mm damage', '25.0 mm damage');
\texttt{axis} ([1 6 1 36]);
\texttt{grid on};

$\%$ This code loads the experimental data from all three panels when excited at one-third the primary resonance. It then tracks the amplitudes of the response peaks at known frequencies and plots these peaks, either as raw displacement data or normalized.
$\%$

\texttt{clear all;}
\texttt{clc;}

\texttt{for ii = 1:26}
\texttt{cnt=num2str(ii);}
\texttt{datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_25_25inch_120_amp\data' cnt '.mat'];}
\texttt{scan\_data2 (ii)=load(datafile);}\texttt{;}
\texttt{temp=abs(fft(scan\_data2 (1,ii).data(:,1),65536));}
\texttt{Dft2(:,ii)=temp;}
\texttt{Force\_Dft2(:,ii)=abs(fft(scan\_data2 (1,ii).data(:,2),65536));}
\texttt{[C,I]=max(Dft2(:,1));}
\texttt{omega2=1/640000*6400;}
\texttt{peak\_120 (ii)=max(Dft2((omega2-10)/64000*640000:(omega2+10)/64000*640000,ii)) /3200*9.81/(omega2*2*pi)^2;}\texttt{;}
\texttt{peak\_360 (ii)=max(Dft2((3*omega2-10)/64000*640000:(3*omega2+10)/64000*640000,ii)) /3200*9.81/(omega2*3*2*pi)^2;}\texttt{;}

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %%%%%%%%%%%%%%%%%%%%%%%%
% This code loads the experimental data from all three panels when excited at one-third the primary resonance. It then tracks the amplitudes of the response peaks at known frequencies and plots these peaks, either as raw displacement data or normalized.
% This code loads the experimental data from all three panels when excited at one-third the primary resonance. It then tracks the amplitudes of the response peaks at known frequencies and plots these peaks, either as raw displacement data or normalized.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %%%%%%%%%%%%%%%%%%%%%%%%
peak_1080(ii)=max(Dft2((9*omega2-10)/6400*640000:(9*omega2+10)/6400*640000,ii)) /3200*9.81/(omega2+9*2*pi)^2;

force_peak2(ii)=max(Force_Dft2((omega2-10)/6400*640000:(omega2+10)/6400*640000,ii ))/3200*4.44822162;

end

for ii = 1:19
cnt=num2str(ii);

datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Oct_2_4inch\99_amp\data' cnt '.mat '];

scan_data1(ii)=load(datafile);

temp=abs(fft(scan_data1(1,ii).data(:,1),65536));
Dft1(:,ii)=temp;
Force_Dft1(:,ii)=abs(fft(scan_data1(1,ii).data(:,2),65536));

[C,I]= max(Dft1(:,1));
omega1=I/640000*6400;

peak_98(ii)=max(Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii)) /3200*9.81/(omega1+2*pi)^2;

peak_295(ii)=max(Dft1((3*omega1-10)/6400*640000:(3*omega1+10)/6400*640000,ii)) /3200*9.81/(omega1+3*2*pi)^2;

peak_985(ii)=max(Dft1((9*omega1-10)/6400*640000:(9*omega1+10)/6400*640000,ii)) /3200*9.81/(omega1+9*2*pi)^2;

force_peak1(ii)=max(Force_Dft1((omega1-10)/6400*640000:(omega1+10)/6400*640000,ii ))/3200*4.44822162;

end

for ii = 2:26
cnt=num2str(ii);

datafile = ['C:\Users\edittman\Documents\MATLAB\Data Aq\Apr_29_1inch_156_amp\data' cnt '.mat '];

scan_data3(ii-1)=load(datafile);

temp=abs(fft(scan_data3(1,ii-1).data(:,1),65536));
Dft3(:,ii-1)=temp;
Force_Dft3(:,ii-1)=abs(fft(scan_data3(1,ii-1).data(:,2),65536));

[C,I]= max(Dft3(:,1));
omega3=I/640000*6400;

peak_156(ii-1)=max(Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000,ii-1)) /3200*9.81/(omega3+2*pi)^2;

peak_468(ii-1)=max(Dft3((3*omega3-10)/6400*640000:(3*omega3+10)/6400*640000,ii-1)) /3200*9.81/(omega3+3*2*pi)^2;

peak_1404(ii-1)=max(Dft3((9*omega3-10)/6400*640000:(9*omega3+10)/6400*640000,ii -1))/3200*9.81/(omega3+9*2*pi)^2;

force_peak3(ii-1)=max(Force_Dft3((omega3-10)/6400*640000:(omega3+10)/6400*640000, ii-1))/3200*4.44822162;

end
clc;

freq=(0:6400/65536:6399.999);
numb=1:0.001:6;

lin=numb;
quad=numb.^2;
cube=numb.^3;

figure;
plot(peak_98,force_peak1,'-xk',peak_120,force_peak2,'-ob',peak_156,force_peak3,'-+g');
ylabel('Force in N');
xlabel('Displacement in m');
legend('101 mm damage','63.5 mm damage','25.0 mm damage');
grid on;

figure;
plot(force_peak1/force_peak1(1),peak_98/peak_98(1),'-xk',force_peak2/force_peak2(1),
     peak_120/peak_120(1),'-ob',... 
     force_peak3/force_peak3(1),peak_156/peak_156(1),'-+g',numb,lin,'-r');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage','63.5 mm damage','25.0 mm damage');
axis([1 6 1 6]);
grid on;

figure;
plot(force_peak1/force_peak1(1),peak_295/peak_295(1),'-xk',force_peak2/force_peak2(1),
     peak_360/peak_360(1),'-ob',... 
     force_peak3/force_peak3(1),peak_468/peak_468(1),'-+g',numb,cube,'-r');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage','63.5 mm damage','25.0 mm damage');
axis([1 6 1 10]);
grid on;

figure;
plot(force_peak1/force_peak1(1),peak_985/peak_985(1),'-xk',force_peak2/force_peak2(1),
     peak_1080/peak_1080(1),'-ob',... 
     force_peak3/force_peak3(1),peak_1404/peak_1404(1),'-+g');
xlabel('Normalized Forcing Amplitude N/N');
ylabel('Normalized Response Amplitude m/m');
legend('101 mm damage','63.5 mm damage','25.0 mm damage');
axis([1 6 1 10]);
grid on;
VITA
VITA

Eric Richard Dittman was born in Northridge, California where he lived for 14 years. His family made the difficult, but promising, decision to move to New Jersey in pursuit of better employment. It was in New Jersey that Eric graduated from West Morris Central High School in 1998 and left in pursuit of a career as a pilot at the United States Air Force Academy. After taking two years off from the Academy to serve a church mission in Brazil, Eric returned to graduate from the Air Force Academy in June of 2004 with a Bachelors of Science in Engineering Mechanics, but shifted his focus to engineering instead of flying. Having graduated in the top 10% academically from USAFA, he was selected to work on a Masters degree in Aeronautical Engineering at the Air Force Institute of Technology, which he completed in March of 2006. He then was employed as an Accident Investigator and Failure Analyst in the Air Force Research Lab’s Materials and Manufacturing Directorate. Eric was then hired on as an Instructor at the United States Air Force Academy where he was able to rise to the position of Assistant Professor. With further teaching in mind, he was selected to pursue a PhD, which he is doing under the guidance and direction of Dr. Douglas E. Adams. He is focusing on nonlinear vibrations and how they can be used in detecting and understanding damage in composites, and hopes to bring more of this knowledge back to the Air Force.