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Ronald B. Narode

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"Math in a Can": Teaching Mathematics and Engineering Design

Ronald B. Narode

Portland State University

Abstract

Using an apparently simple problem, "Design a cylindrical can that will hold a liter of milk," this paper demonstrates how engineering design may facilitate the teaching of the following ideas to secondary students: linear and non-linear relationships; basic geometry of circles, rectangles, and cylinders; unit measures of area and volume; solving systems of equations with at least two variables; minimization of area to control materials costs and to prevent heat exchange; packing geometry to minimize space for transportation and storage and for controlling for heat exchange; golden ratio as a design aesthetic; ergonomic factors in design including considerations of comfort of handling and safety; and strength of design for stacking and handling as well as for the prevention of accidental tipping. This interdisciplinary curriculum uses engineering design challenges to engage students with meaningful and fun group activities and discussions that also teach a multitude of diverse and powerful mathematical concepts.

Keywords: engineering design, mathematics, geometry, algebra, problem-solving

Breaking "The Mold" of Mathematics Problem Solving

As a mathematics teacher and teacher educator, I am mindful of my profession's recommendations that important mathematics concepts be taught in the context of real-world problem-solving (National Mathematics Advisory Panel, 2008). One possible adaptation and lesson design is suggested here.

A typical problem that engages students in the learning of area and volume of right cylinders often looks like this: *"Find the volume and surface area of a right cylinder with radius 5 cm and height 20 cm."* To make it more "applied," I referred to the cylinder as a "can." Of course, the appellation changed little if anything in the students' work or attitude. They simply identify the appropriate formula and perform the calculations – hardly a more challenging or more engaging curriculum for our troubles.

There is another way to build a challenging curriculum – by making the problem less constrained but more focused on student understanding through "generative thinking" (thinking that requires the construction of new ideas) (Carpenter & Lehrer, 1999), students could do more and learn more mathematics. The original problem was amended as follows: *"Find the dimensions of a can that will hold a liter of milk."* Not only is this a more challenging problem for the students, but it also adds significantly more advanced mathematical content. It is also an application that appears truer to the purpose of designing a can where the volume is known and the shape may or may not be assumed. Student responses to this problem also informed me of the gaps in their understanding of geometry, algebra, and the conversion of units. More importantly, their frustration at having to work a problem so different from the type of problem they usually see (as in the case where the dimensions are specified), demonstrated how little confidence students have in their abilities to solve a problem that requires them to think beyond applying rehearsed algorithms and procedures. While mathematics offers an impressive curriculum of clever and proven procedures and ideas, the teaching of these often leads to the most superficial understanding for our students and to a habit of schooling and learning that misses many opportunities for teaching for student understanding (Kilpatrick, 2001; Carpenter & Lehrer, 1999; Romberg & Kaput, 1999).

"Ill-structured" and "Well-structured" Problem Solving

The distinctions between expert and novice problem-solvers as those who can work in an unstructured problem space and those who cannot was reported almost 40 years ago by Newell and Simon (1972). They described the characteristics of two types of problems: the "well-structured" problem is the typical math text problem that appears on most standardized tests as well as in classroom lessons and assignments, and the "ill-structured" problem that seldom appears in school but is ubiquitous outside of school, the type of problems usually encountered in the world.

The well-structured problem is a problem that contains all of the information needed to solve it with no extraneous or redundant information; and it usually has only one unique solution even if there are multiple solution paths (seldom taught). The solution to the well-structured problem is also "well-structured" in that it is usually linear and proscribed. Mathematics textbooks are careful to present step-by-step procedures that teachers demonstrate for their students and that students are expected to remember and recreate on assessments, often with points given or removed based upon the completeness and accuracy of all of the steps. Success or failure is usually determined for each individual student working independently.

In contrast, there is the "ill-structured" problem – with missing, irrelevant, or redundant information; often with multiple solutions with varying utility that requires argument and judgment. Engineering design problems are typical example of "ill-structured" problems in that many designs are possible, and the "best" design must be judged on the basis of many product specifications. The success of the designer(s) often derives from a combination of creativity, communication, understanding of design principles as well as an understanding of the basic science applied to the purpose of a mechanism or process, and almost always requires a critical understanding of "systems" instead of discreet and independent objects and ideas. Furthermore, the design process is seldom individual so that multiple perspectives add to a design's utility based upon more and more varied criteria and creativity. Contrary to the linear prescriptions in most mathematics instruction, engineering design is not linear, nor is it individual. In the national report by the Committee on K-12 Engineering Education (Katehi, Pearson, & Feder, 2009) the distinguishing attributes of engineering design classifies it clearly as an "ill-structured" problem-solving activity that is largely absent from most mathematics instruction. Among the characteristics described in the study, engineering design begins with a clear goal; it is shaped by "specifications and constraints"; it is systematic and well-organized; and it is usually performed in collaborative teams where good communication is critically important. The authors also describe the creative and iterative processes in design. They caution that one may never know the "true" answer to the problem, but rather the answer that one commits to, where personal factors play a prominent role in addition to technical factors (Katehi, Pearson, & Feder, 2009).

Although the preponderance of "ill-structured" problems in the world would lead an educator to want to focus largely on this type of problem, it is also clear that our success in solving these types of problems often requires that we convert them from "ill-structured" into "well-structured" problems. When the problem question is ill-defined, we attempt to define it within the parameters that we believe have the most utility. When information is missing, we attempt to find it, and when we cannot find it, we often make assumptions that we believe are "safe" or "realistic." Although there may be many solution paths, we make a determination to commit to one and then constantly re-evaluate and revise. Solving "ill-structured" problems requires that we understand how "well-structured" problems and processes function, and that we continually work to transform the former into the latter. This requires that educators make sure that students have experience and instruction solving both types of problems (Jonassen, 1997, 2000; Jonassen, Strobel, & Lee, 2006; Xun & Land, 2004).

The Usefulness of a "Can" in Teaching Geometry and Algebra

Among all of the constructions easily available to mathematics teachers of geometry and algebra, the right cylinder must count as one of the most familiar geometric shapes to students. They are readily available as cans in all sizes, easily manufactured from construction paper, and incorporate most of the important concepts that students are expected to know for district and state assessments. In geometry: the distinction between linear dimensions (length and width), area (circles and rectangles), and volume (cylinders and cubes). In algebra: conversion of units for length, area and volume, and the respective formulae for the various geometric shapes. For more advanced students in algebra, the topic may be expanded to include the solution of linear equations and quadratic equations as well as the methods and conditions for solving simultaneous equations. For the even more advanced students of algebra, there are graphical representations of functions and composite functions with determination of maxima and minima that may yield the can whose area is a minimum for a given volume or alternatively, the shape of a cylindrical can whose volume is maximized given a fixed area. This final problem is also typical for students taking calculus. For all of these reasons, right cylinders figure prominently in nearly all mathematical curricula at the middle school, high school, and introductory college levels.

While the application of these ideas may be turned toward an investigation of the dimensions and volume of "real cans," these applications usually occur at the end of a unit, if at all, and are often overlooked in the interest of saving time (Jonassen, 2000). These are usually in the form of well-structured problems like the one stated at the beginning of this article: *"Find the volume and surface area of a 'can' with radius 5 cm and height 20 cm."* Greater understanding of mathematics is possible by offering a curriculum that begins with the ill-structured problem of designing a can that contains one liter of milk, and furthermore, that this can is somehow the "best" can for the job. To facilitate learning, the students will be guided by their teacher in learning all the math concepts indicated above in addition to others that consider numerous other factors. The curriculum is fashioned as a series of investigations that the teacher will facilitate through questions, discussions, research, and experimentation, and a lot of "just-in-time" teaching where the teacher might have to instruct the students about a concept or procedure that they may need along the way.

Mathematics Content

Since all states currently have mathematics standards that list the most important content for the curriculum, it is helpful for teachers to be able to identify the mathematics content of these lessons in conjunction with those standards (National Council of Teachers of Mathematics (NCTM), 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). Of course, all of these topics either require prerequisite knowledge or are themselves prerequisite to each other. The level of investigation depends on the developmental readiness of the class. Here is a partial list of mathematics content that may be included in the unit:

Area and Volume: Area and volume are non-linear relationships that may be taught in distinction to linear relationships as seen with the volume of a cylinder that varies linearly with the height of the can and quadratically with the length of the diameter.

*Measuremen*t: To understand the above distinction, students need to know how a cylinder is constructed with rectangles and circles and how these shapes are defined and calculated. Among the concepts required are the dimensions of length, area, and volume, and the invention of units for their measurement. All students ought to first construct a prism or "cuboid" box that will contain a liter of volume, but without using any instruments other than rulers, paper, tape, and rigid sticks or straws. No other measuring devices will be used (such as a liter container).

Units of Measure: Related to measurement, but a topic of study that is itself a challenge for many students, is the idea of using units to measure things. While many students may be familiar with the terms for units of measure, they are usually very unclear about the relative sizes of these units (the weight of a kilo compared to a gram or a pound, the millimeter to the kilometer, etc.), their appropriate dimensions (weight, area, volume, and length), and their conversion from one to the other (inches in a mile or a kilometer for example). They are also unfamiliar with the ideas of unit standards, for example, that a pint of water is equivalent to a one-pound weight and a liter of water to a kilogram. In this regard, it is revealing to ask students to demonstrate with their hands their initial approximation of the size of a can that will hold one liter of milk, and to estimate its weight.

Variables and Equations: Students will come to understand that an infinite number of cans may be constructed to contain a liter, but that specifying any and only one such can will involve fixing one variable so that the other may be found; for example finding the height of the can given a desired width. This process will help teach several concepts and procedures needed to find and solve a system of simultaneous equations, in addition to helping students understand the origins and meaning of the equations used to find the area and volume of a can.

Functions, Graphs, Maxima and Minima: There are other practical goals involved in the construction of the can; for example, which can uses the least amount of materials (minimize area); which can insures that the milk stays cold the longest (minimizes surface area); and which can of milk refrigerates the quickest (maximizes surface area)? Of course there are iterative experiments that students may use to try to answer these questions, but there are also theoretical considerations that lend themselves to mathematical analysis. By constructing a data table that considers varying values of height and diameter, students may approximate the solution to these questions. Alternatively, they may find a composite function that enables them to graph the area as a function of the dimensions of the one-liter can in order to identify the dimensions that yield the minimum area. In more advanced mathematics classes, calculus may be used to find the solution to this problem very quickly.

Tesselations: In consideration of economy of space for packing and transportation, students may investigate which shape of can may be packed with minimum loss of space and are strong enough to be stacked for ease of storage and transport.

Measures of Central Tendency: Mean, Median, and Mode: In consideration of ergonomic design – which can feels best in the hand (wide enough to hold comfortably and long enough so that my lips clear my fingers), students may consider whose hand will be holding the can for pouring and how wide and long should the can be to accommodate the greatest number of these people's hands. This is a wonderful opportunity to consider measures of central tendency in engineering design.

Ratio and Proportion, the "Golden Ratio": The appearance of a can is important to its commercial success. In this regard, the "Golden Ratio" is an aesthetic artifact and cultural icon that students may not be consciously aware of, yet may find ubiquitous in their lives when they learn what it is and how to identify it (Herz-Fischler, 1998) as a ratio: length/width = width/(length – width) where the length is simply the longer side of a rectangle; using the quadratic formula, it can be shown to be approximately 1.618. This ratio is very close to the shape of a face that is considered "well-proportioned" and beautiful as in DaVinci's painting, the *Mona Lisa*. It is a familiar ratio in architecture, windows and doorways, and in the design of many familiar objects including this page.

These are some of the ideas that mathematics teachers may draw upon to teach the mathematics curriculum required within current state and national standards. It is also a wonderful opportunity to incorporate engineering design principles into an interdisciplinary curriculum that will engage students with a meaningful and fun group of activities and discussions. A series of questions and activities is offered below to aid teachers in guiding student investigations for designing the "best can to hold a liter of milk." Of course, these are only some suggestions that teachers will adjust and expand upon according to their needs and interests.

Class Project: Design and Build a Model for the "Best Can" to Hold One Liter of Milk

Introducing the Problem: Show-and-Tell Cans

Since everyone is familiar with cans in all shapes, sizes, and purposes, one fun way to begin the investigation into the "best can" is to consider the purpose of the can and the ways it is used. Bring to class a wide selection of cans with the labels removed and ask students to offer conjectures about what the cans may have inside them (without handling them). This is a challenge that leads to a discussion about the purpose and shapes of cans. For example: the shape of a can that holds tuna fish is similar to the shape of a can that contains catfood, yet few students would suggest that asparagus or fruit juice would be in that can. It is also helpful to bring in cans that are different shapes, for example, olive oil often comes in box-shaped cans, sardines in oval or semi-oval cans, ham in triangle-shaped cans, etc. Students should be encouraged to speculate the reasons for the shapes and the sizes of the cans and their contents. Numbering each can at the front of the room and asking students to work in pairs to write down what they believe each can contains is one way to have everyone engaged and accountable. Taking their conjectures afterward in whole-class discussion will facilitate the investigation. Opening one can each day of instruction on the unit may sustain student interest and prove quite entertaining.

Invitation to Wider Explorations and Research

Each pair of students should have a research project about cans that they may access in the school's computer lab or in the school library. Here are some questions that could be posed prior to their getting into the lab, but that they may take with them for their research project. The students may have more of their own questions.

a) What is the purpose, importance, and history of cans and canning?
b) What materials could be used and why?
c) What are the different shapes of cans and to what purpose? (Look at lots of different cans brought into class without labels).
d) What are the biggest, smallest, heaviest cans that were ever made?
e) How long can food last in cans?
f) Is a jar a can? a barrel?
g) How is a can made?
h) How is canning done?

Competition as Incentive

Depending on the age of the students, a competition and prizes for the "first," "second," and "third" place designs for the "best can" could garner a lot of sustained student interest, especially if there is a possibility for "honorable mentions" that help all students that remain engaged to succeed. This may not be necessary for older students, but then again… The size of the group of students working together should be at least two and no more than four. This will insure that everyone is accountable for their products, and yet have the advantage of the diversity of ideas from a group (Lochhead, 2001).

Defining Questions for Investigation and Discussion in Assigned Groups

It is preferable that students attempt to invent their own criteria for success in designing the best can for the job by identifying and defining the variables that are most important to them (Zohar, 1995). Their questions will hopefully approximate and exceed those offered here, however, the following questions may be given individually and discussed in groups prior to discussing with the class and prior to the students jumping into their design work. They should be handed out to the students only *after* they have had an opportunity to invent or surpass them. Most importantly, the students should attend to the discussion and not to the sheet of paper while the discussion is supposed to occur.

What is the best shape can for holding, drinking, and pouring? – Note, if the students come up with a right cylinder, ask how many cans of this shape and volume are possible, and also to demonstrate with their hands the sizes they propose?

What is the best shape for transporting many cans? – Note, students should consider weight for handling: how heavy is the can when it is filled? How many would fit into a case and what would the case weight? What is the best way to pack the cans so that they waste the least space, note that the topic of tessellations is important here since rectilinear box-shaped cans fit together with little wasted space, but cylinders don't. What if the can had an octagonal shape or hexagonal? Would these cans "tessellate"?

What is the best shape for 'stability in use'? – Students should consider stacking, carrying, resting on a table, tipping, cup holders, etc.

What is the best shape for comfort and appearance? – Students should consider ergonomic design features so that the can may be held and handled as well as whether it may be drunk from without dribbling or poured without spilling. They should also consider the aesthetics of shape and labeling. This is a great opportunity to teach about ratio and proportion as it relates to aesthetics, especially in consideration of the "Golden Ratio."

What is the best shape for economy of materials used in construction? – Students should consider which dimensions yield the least surface area.

What is the best shape for economy of heat exchange? – How might a can of milk be designed to refrigerate to cold quickly, or alternatively, so that it remains cold for as long as possible? Students should consider volume to surface area ratios in evaluating relative heat exchange with different shapes.

What are the dimensions of this "best" can? – Students need to show the math, especially with regard to the use of simultaneous equations and graphical representations of the appropriate functions to determine the most economical can.

Final Presentations: The "Best Can" and the "Math in the Can"

Having completed their investigations, the respective student groups will present their "best can" as constructed from paper and decorated. They will also assemble a poster that describes the factors they considered and their reasons for thinking their can the "best." They should then be judged within categories and confidentially with ballots cast by the class. The categories of judgment ought to be discussed with the students with a general consensus that hopefully will grade the following: how well the criteria for design fit the purpose, how well the design fit the selected criteria, the quality of the mathematics used in the design, quality of oral presentation, quality of the poster, and the quality of the overall final construction.

Opportunities for prizes within categories in addition to the overall first, second and third place prizes for the best can, ought to give ample opportunity for every team to succeed in earning some recognition for their efforts.

The most important prize is the understanding of mathematics that occurs when instruction is motivated by the creativity of students working together to design something new and useful.

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